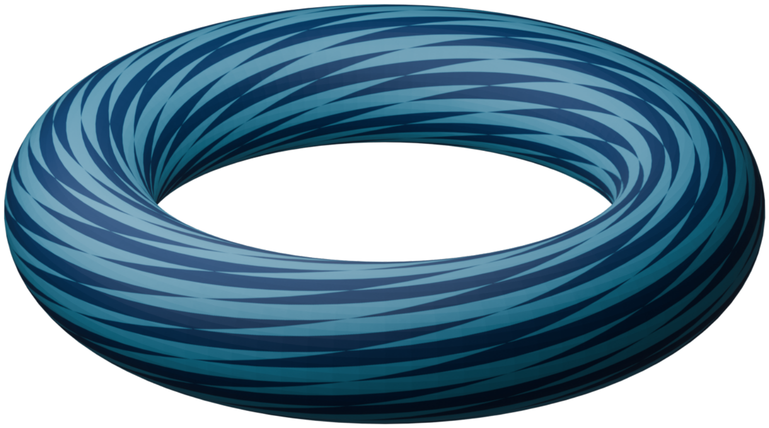
KU LEUVEN

ARENBERG DOCTORAL SCHOOL
Faculty of Science

Which closed manifolds admit an Anosov diffeomorphism?



Thomas Witdouck

Supervisors:
Prof. dr. Jonas Deré
Prof. dr. Marco Zambon

Dissertation presented in partial fulfillment of the requirements for the degree of Doctor of Science (PhD): Mathematics

September 2024

Which closed manifolds admit an Anosov diffeomorphism?

Thomas WITDOUCK

Examination committee:
Prof. dr. David Dudal, chair
Prof. dr. Jonas Deré, supervisor
Prof. dr. Marco Zambon, supervisor
Prof. dr. Karel Dekimpe
Prof. dr. Joeri Van der Veken
Prof. dr. Wendy Goemans
Prof. dr. Tracy Payne
(Idaho State University)
Prof. dr. Ric Wade
(University of Oxford)

Dissertation presented in partial fulfillment of the requirements for the degree of Doctor of Science (PhD): Mathematics

September 2024

Uitgegeven in eigen beheer, Thomas Witdouck, Etienne Sabbelaan 53, BE-8500 Kortrijk (Belgium)

Preface

Mijn doctoraat is tot stand gekomen dankzij de onmisbare steun en inspiratie van velen om mij heen. Daarom wil ik graag mijn oprechte dank uitspreken aan allen die een rol hebben gespeeld in deze bijzondere reis.

Eerst en vooral wil ik mijn promotor bedanken. **Jonas**, bedankt voor je begeleiding gedurende de afgelopen vier jaar. Je gaf me de vrijheid die ik nodig had om een zelfstandige onderzoeker te worden, terwijl je deur altijd openstond voor een gesprek, of het nu over onderzoek ging of over andere zaken. Daarnaast heb je me aangemoedigd om naar het buitenland te gaan in het kader van mijn doctoraat en om wiskundigen van over de hele wereld te leren kennen. Vooral het bezoek aan Argentinië heeft een blijvende indruk op mij gemaakt, zoals je waarschijnlijk hebt gemerkt aan mijn maté-consumptie.

Vervolgens wil ik mijn co-promotor, **prof. dr. Marco Zambon**, bedanken. Bedankt voor de waardevolle gesprekken die we hebben gehad. Hoewel mijn onderzoek enigszins afweek van jouw eigen werk, weerhield dat je er niet van om je in mijn project te verdiepen en me nieuwe ideeën aan te reiken.

Additionally, I would like to thank the other members of the jury: **Prof. Dr. Karel Dekimpe, Prof. Dr. David Dudal, Prof. Dr. Wendy Goemans, Prof. Dr. Tracy Payne, Prof. Dr. Joeri van der Veken, and Prof. Dr. Ric Wade**. Thank you for taking the time to travel to Kortrijk, thoroughly reviewing my thesis, giving useful suggestions and helping to eliminate spelling errors and inaccuracies. I greatly appreciated your efforts.

Dan wil ik mijn collega's bedanken. Beginnen doe ik met mijn bureaugenoten. **Sam**, lang hebben we niet in hetzelfde kantoor gewerkt, maar ik heb in die tijd veel aan jou gehad om wegwijs te worden als personeelslid op de kulak. **Pieter**, je hebt me vaak geholpen in mijn onderzoek. Zonder jouw hulp had ik nooit mijn paper over de R_∞-eigenschap kunnen schrijven. **Maarten**, als ik vragen had over koken, kon ik altijd bij jou terecht. Het paellareceptje was dan ook echt een succes! De drag show waar we als gringos naartoe gingen in Brazilië zal ik niet snel vergeten. En onthoud: 'you are soy interesting!'. **Lukas**, bedankt om met mij weer een West-Vlaamse meerderheid te vormen op kantoor.

Dan zijn er nog de collega's die ik vooral zag tijdens de koffie- en lunchpauzes: **Arne**, bedankt om altijd je enthousiaste zelf te zijn; **Benjamin**, bedankt om het sociaal aanvaard te maken je koffietas nooit uit te wassen; **David**, als ik met iemand over sport wilde praten, was jij de persoon bij uitstek. Zo zijn we ook ooit eens naar Kortrijk Koerse gaan kijken toen die aan de Kulak passeerde. De regels van Kubb ken je nog niet helemaal, maar dat is je bij deze vergeven; **Emil**, thank you for the NBA talks; **Erik**, bedankt voor de broodjesservice. Ik denk dat iedereen het enorm apprecieert dat jij je daarvoor engageert; **Jens**, bedankt voor alle 10-uurtjes/ontbijtjes; **Joren**, bedankt om last-minute met mij mee te gaan naar het congres in Münster, waar je je ontpopte tot spraakwaterval. De Pinkus Müller-avond zal ik nooit vergeten; **Karel**, eerst en vooral wil ik je bedanken om van Benno's presentatie een legendarisch moment in mijn studententijd op de Kulak te maken. Tijdens mijn doctoraat stond je als diensthoofd altijd klaar om te antwoorden op eender welke vraag die ik had; **Kevin**, ik kijk ernaar uit om opnieuw collega's te worden. Eens zien of mijn tekening daar effectief hangt; **Koen**, bedankt voor de kansen die je mij gaf als jobstudent aan de Kulak; **Charlotte** en **Laura**, bedankt voor het organiseren van de teambuildings; **Lisa**, bedankt voor de heerlijke kaas tijdens de masterproefpresentaties; **Lore**, jou een kilo yoghurt zien eten was een van de betere dingen die ik al zag tijdens de koffiepauze; **Louise**, bedankt om samen met mij wat meer mindful te worden tijdens mijn eerste jaar; **Michiel**, bedankt voor alle gekke bekken die je trok toen je aan onze deur passeerde. Het hielp om de dag te breken; **Michiel**, bedankt voor het ritje in je oldtimer; **Stijn**, ik herinner me nog dat we in COVID-tijden af en toe een gesprek hadden in de online koffiekamer. Ik wil je daarvoor bedanken, want het is niet eenvoudig om een doctoraat te starten tijdens die COVID-periode; **Thomas**, Ice-T, bedankt om me af en toe nog eens wat bij te leren over theoretische fysica; **Wouter**, jij weet dit waarschijnlijk niet, maar ik heb koffie leren drinken door jou tijdens mijn eerste vakantiejob op de Kulak. Verder maakte je mij toen ook wegwijs als jobstudent, waarvoor ik je wil bedanken; **Brecht**, **Eef**, **Herbert**, **Iris**, **Jan**, **Jarne**, **Jens**, **Jorik**, **Marie**, **Mieke**, **Nathan**, **Sebbe**, **Shakir**, **Stijn**, **Ursula**, **Wim**, bedankt voor de leuke gesprekken tijdens de pauzes.

Finally, there are two colleagues who deserve special mention, as we shared a house for nearly a year. **Lore** and **Davide**, thank you for being such wonderful housemates. You both helped me take my mind off research when I needed it. Lore, thank you for being so open. I feel like I can share anything with you. Davide, I learned a lot from you: how to make ragu, how to swear in Italian, how to properly dispose of pineapple pizza and how to do 'the butterfly'. Dank je wel, tot ziens! I would also like to thank Lore's pets. Sokka, you have the power to make me instantly happy. One of my biggest accomplishments in life is definitely teaching you to do 'pootje'. Kali and Osiris, thank you for your occasional epic cat battles, although I must admit, I don't miss your constant meowing.

Marcos, thank you for picking me up at the airport and showing me around Córdoba. I will never forget the moment I drank my first cup of mate at your apartment and saw the surprise on your face when I said I actually really liked it. You taught me so much about Argentine culture, for which I am forever grateful. I still use the mate cup you gave me. I hope to visit Córdoba again sometime, and when I do, I hope Lomitos 348 is still open.

Ava, Benno, Frederik, Jérôme en **Thomas**, bedankt om mijn studententijd aan de Kulak zo plezierig te maken. Van stunten op dinsdag tot 'Umklaps' tijdens Vaste-stoffysica, het waren drie fantastische jaren. Tijdens mijn doctoraat werd ik nog vaak herinnerd aan die tijd, zoals wanneer ik de W&T-dixit in LaTeX opmaak zag liggen of wanneer ik een uitnodiging voor een 'Wijn- en Kaasavond' zag passeren. Ook na de studententijd zijn we elkaar blijven zien en ik hoop dat dit zo mag blijven.

Arne, Hobie, Jarit, Lars en **Tibo**, wij zijn van ploeggenoten geëvolueert naar een hechte vriendenbende. Woorden schieten te kort om te beschrijven wat jullie voor mij betekend hebben tijdens mijn doctoraat. Van weekendjes en reizen tot gezellige feestjes, jullie waren er altijd. Merci!

Bavo, Henri, Nico, Robin, Sander, Tony, Wiebe en **Wout**, a.k.a. de 'Gullegemcrew', bedankt voor de avonden in Pand A, de tripjes naar Mont Ventoux en Athene, en de vele malen dat we nieuwjaar samen vierden. Het was altijd een groot plezier met jullie samen te zijn. Ik mag mezelf gelukkig prijzen met zulke vrienden als jullie.

Ten slotte wil ik mijn familie bedanken, in het bijzonder mijn ouders en zus. **Mama en papa**, jullie hebben mij altijd gesteund in alle keuzes die ik maakte en mij alle kansen gegeven. Nooit heb ik iets tekortgekomen. Zonder jullie had ik dit doctoraat nooit kunnen voltooien, daar ben ik zeker van. **Sara**, het is bewonderenswaardig om te zien hoe je naast je werk nog steeds de energie vindt om voor mij en de rest van de familie te zorgen. Ik ben echt trots dat jij mijn kleine zus bent. Dank jullie wel!

Dank jullie allemaal!
Thank you all!
Grazie a tutti!
Dziękuję wam wszystkim!
¡Gracias a todos!

Abstract

Anosov diffeomorphisms on closed manifolds form an important class of dynamical systems characterized by global hyperbolic behaviour. These systems exhibit sensitivity to initial conditions and are conjectured to have both a dense set of periodic points and a dense forward orbit. In the sixties, Dmitri Anosov proved that these systems are structurally stable, which is why they where named after him.

The most elementary example of an Anosov diffeomorphism is Arnold's cat map, which is the diffeomorphism on the torus $\mathbb{R}^2/\mathbb{Z}^2$ induced by the matrix

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Similar examples can be constructed on all higher-dimensional tori. Stephen Smale provided the first example of a non-toral Anosov diffeomorphism on the quotient of a direct product of two Heisenberg groups by a suitable cocompact lattice. Smale's construction can be generalized to yield examples in the class of infra-nilmanifolds, also known as almost-flat manifolds. However, not every infra-nilmanifold admits an Anosov diffeomorphism.

A major unresolved problem in this field is the classification of all closed manifolds that admit an Anosov diffeomorphism. A long-standing conjecture states that such manifolds must be homeomorphic to infra-nilmanifolds. This thesis focuses on addressing this problem within the class of (infra-)nilmanifolds, where there are algebraic techniques at hand to decide whether a given manifold can admit an Anosov diffeomorphism. This effectively transforms the problem into a classification problem on rational nilpotent Lie algebras.

For abelian and free nilpotent Lie algebras, the problem has been resolved. A significant portion of this thesis is devoted to extending these results to the class of nilpotent partially commutative Lie algebras, which can be seen as an interpolation between the abelian and free nilpotent cases. First, a classification of the rational forms of these Lie algebras is made using the theory of Galois cohomology. Subsequently, a characterization of those forms corresponding to a nilmanifold admitting an Anosov diffeomorphism is proven. The proof combines Dirichlet's unit theorem with the theory of linear algebraic groups and the theory of partially commutative Lyndon words.

A closely related problem is determining which (nilpotent) partially commutative groups possess the R_∞-property, a notion originating from Nielsen fixed point theory. To conclude the thesis, several results concerning this problem are proven, and the relationship with abstract commensurability is explored.

Beknopte samenvatting

Anosov diffeomorfismen op gesloten variëteiten vormen een belangrijke klasse van dynamische systemen, gekenmerkt door globaal hyperbolisch gedrag. Deze systemen vertonen gevoeligheid voor beginvoorwaarden en men beweert dat ze zowel een dichte verzameling van periodieke punten als een dichte voorwaartse baan hebben. In de jaren zestig toonde Dmitri Anosov aan dat deze systemen structureel stabiel zijn, wat de reden is dat ze naar hem vernoemd zijn.

Het meest fundamentele voorbeeld van een Anosov diffeomorfisme is 'Arnold's cat map', het diffeomorfisme op de torus $\mathbb{R}^2/\mathbb{Z}^2$ geïnduceerd door de matrix

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Gelijkaardige voorbeelden kunnen worden geconstrueerd op alle hoger-dimensionale tori. Stephen Smale gaf het eerste voorbeeld van een Anosov diffeomorfisme dat niet op een torus werkt, maar op het quotiënt van een direct product van twee Heisenberg groepen met een goed gekozen cocompact rooster. Smale's constructie kan worden veralgemeend om voorbeelden te verkrijgen in de klasse van infra-nilvariëteiten, ook bekend als bijna-platte variëteiten. Echter, niet elke infra-nilvariëteit laat een Anosov diffeomorfisme toe.

Een onopgelost probleem tot op heden is het classificeren van alle gesloten variëteiten die een Anosov diffeomorfisme toelaten. Er wordt beweerd dat zulke variëteiten altijd homeomorf zijn aan een infra-nilvariëteit. Deze thesis richt zich op dit probleem binnen de klasse van (infra-)nilvariëteiten, waar algebraïsche technieken beschikbaar zijn om te bepalen of een Anosov diffeomorfisme op een gegeven variëteit bestaat. Dit vertaalt het probleem naar een classificatieprobleem van rationale nilpotente Lie-algebra's.

Voor abelse en vrije nilpotente Lie-algebra's is dit probleem reeds opgelost. Een groot deel van deze thesis probeert deze resultaten uit te breiden naar de familie van nilpotente partieel commutatieve Lie-algebra's, die gezien kunnen worden als een interpolatie tussen het abelse en vrije nilpotente geval. Eerst wordt een classificatie gemaakt van de rationale vormen van deze Lie-algebra's met behulp van de theorie van Galois cohomologie. Vervolgens bewijzen we een karakterisatie van de rationale vormen die overeenkomen met een nilvariëteit die een Anosov diffeomorfisme toelaat. Het bewijs combineert de eenhedenstelling van Dirichlet met de theorie van lineair algebraïsche groepen en de theorie van partieel commutatieve Lyndon woorden.

Een sterk gerelateerd probleem is het bepalen van welke (nilpotente) partieel commutatieve groepen de R_∞-eigenschap hebben, een eigenschap afkomstig van Nielsen vastepuntstheorie. Om de thesis af te sluiten, worden enkele resultaten hieromtrent aangetoond en wordt het verband met abstracte commensurabiliteit verkend.

List of Symbols

$\mathbf{Alg}_K$	The category of associative, commutative algebras over the field K with 1
$\overline{\mathcal{G}}$	The quotient graph of the graph $\mathcal{G}$
$\mathbb{N}$	The natural numbers including 0
$\mathbb{N}_0$	The natural numbers excluding 0
π_{ab}	The projection onto the abelianization
$\mathrm{Fix}(f)$	The set of fixed points of the map f
$\mathrm{NW}(f)$	The set of non-wandering points of the map f
$\mathrm{Per}(f)$	The set of periodic points of the map f
$E(L/K, \mathfrak{g})$	The set of isomorphism classes of K-forms of the Lie algebra $\mathfrak{g}$ over L
$H^1(L/K, G)$	The first Galois cohomology set for the Galois extension L/K and the $\mathrm{Gal}(L/K)$-group G
$\gamma_i(\mathfrak{g})$	The i-th ideal in the lower central series of the Lie algebra $\mathfrak{g}$
$\gamma_i(G)$	The i-th subgroup in the lower central series of the group G
$\mathfrak{n}_G^{\mathbb{Q}}$	The rational Mal'cev Lie algebra associated to a finitely generated torsion-free nilpotent group G
$G^{\mathbb{Q}}$	The rational Mal'cev completion of a finitely generated torsion-free nilpotent group G
$\mathrm{Aut}_g(\mathfrak{g})$	The group of graded automorphisms of a graded Lie algebra $\mathfrak{g}$

$\mathrm{BW}(S)$	The set of bracket words (or non-associative words) on the alphabet S
$\mathfrak{g}^K(\mathcal{G})$	The partially commutative Lie algebra associated to the graph $\mathcal{G}$, over the field K
$\mathrm{gr}(\mathfrak{g})$	The graded Lie algebra associated to the Lie algebra $\mathfrak{g}$ by use of the lower central series
Grp	The category of groups
$\Lambda_{\mathcal{G}}$	The set of coherent components of the graph $\mathcal{G}$
$\mathrm{LE}(\mathcal{G})$	The set of Lyndon elements for a graph $\mathcal{G}$
$\mathrm{LE}(\mathcal{G}, c)$	The set of Lyndon elements of length at most c for a graph $\mathcal{G}$
$\mathbf{LieAlg}_K$	The category of Lie algebras over the field K
LieRing	The category of Lie rings
$\mathfrak{n}^K(\mathcal{G}, c)$	The c-step nilpotent partially commutative Lie algebra associated to the graph $\mathcal{G}$, over the field K
$\mathrm{Perm}(S)$	The group of permutations on the set S
$\mathrm{W}(S)$	The set of words on S
$A(\mathcal{G})$	The partially commutative group associated to the graph $\mathcal{G}$
$A(\mathcal{G}, c)$	The c-step nilpotent partially commutative group associated to the graph $\mathcal{G}$
$L^K(G)$	The graded Lie algebra over K associated to the group G by use of the lower central series
$M(\mathcal{G})$	The partially commutative monoid associated to the graph $\mathcal{G}$
$N(v)$	The open neighbourhood of a vertex v
$N[v]$	The closed neighbourhood of a vertex v
$R(\varphi)$	The Reidemeister number of a group automorphism φ

Contents

Chapter 1

Introduction

1.1 Historical background

In 1687 Newton wrote down his law of gravity. Ever since, it has been of great interest to mathematicians and physicists to describe the dynamics of the solar system. The motion of two bodies under mutual gravitational forces is well-understood. Introducing a third body, on the contrary, turned out to be significantly more difficult. Given the positions and velocities of the bodies, it is very hard to predict whether, over time, a collision will happen or one of the bodies will be catapulted out of the system. This is known as the *three body problem*. In 1890 Poincaré published his paper titled *Sur le problème des trois corps et les équations de la dynamique* for which he won the King Oscar prize. This was the beginning of a new mathematical domain now known as *chaos theory*.

A fundamental idea of chaos theory is that even deterministic systems can be hard to predict if the system is highly sensitive to initial conditions. Let us formulate this in a precise way. A *dynamical system* is an action of $\mathbb{R}$ on a smooth manifold M such that the defining map

$$\Phi : \mathbb{R} \times M \to M : (t, p) \mapsto t \cdot p$$

is smooth. This action tells us how the system evolves in time, namely if the system is in a state $p \in M$, then after a period of time t, the system is in state $t \cdot p \in M$. The action completely determines the behaviour of the system and thus we call such a system deterministic.

Assume that there is also a metric d present on the manifold M. We say that the dynamical system is *sensitive to initial conditions* if there exists some constant $\delta > 0$ such that for any point $x \in M$ and any neighborhood U of p, there exists a point $y \in U$ and a non-negative real number $t \geq 0$ such that $d(t \cdot x, t \cdot y) \geq \delta$. Intuitively, this means that there exist arbitrarily small changes in the initial condition, which, over time, result in a change of behaviour of magnitude δ.

An important class of dynamical systems which exhibit this behaviour and in some sense maximize it, are *Anosov flows*. These systems are characterized by the fact that at each point $p \in M$, the tangent space T_pM has a decomposition into three components: a stable component E^s, an unstable component E^u and a one-dimensional component E^0 in the direction of the flow of the system. Varying the initial condition along the stable component gives trajectories which converge exponentially over time, while varying the initial condition along the unstable component gives trajectories which diverge exponentially over time. This is illustrated in Figure 1.1 below. An important example of such a system is the geodesic flow on a negatively curved manifold [AS67]. In the sixties, Dmitri Anosov published several papers on the properties of these systems, which is why they carry his name [Ano62] [Ano63] [Ano69]. One of the things he proved is that Anosov flows are *structurally stable*, which means that small perturbations of the system do not affect the topological dynamics of the system.

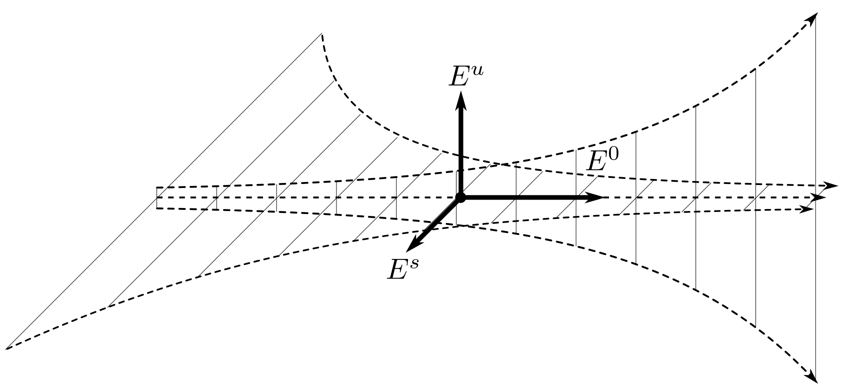


Figure 1.1: The behaviour of trajectories close to a fixed trajectory in an Anosov flow. The dashed lines represent the trajectories.

Another important concept in dynamical systems is that of a *Poincaré section*. This is a codimension one submanifold $N \subset M$ with tangent space transversal to the flow direction and such that any trajectory that starts at N intersects

N again after moving some finite period of time both forwards and backwards. The Poincaré section N is said to be *global* if any trajectory of the dynamical system intersects N. When a global Poincaré section exists, the dynamics of the system are completely determined by the iterations of a diffeomorphism $f : N \to N$ called the *Poincaré map*. This diffeomorphism f is determined by mapping a point p to $t \cdot p$, where $t > 0$ is the smallest positive real number such that $t \cdot p \in N$.

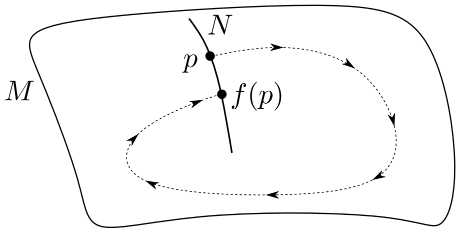


Figure 1.2: Illustration of a global Poincaré section and the associated Poincaré map.

This brings us to the protagonist of this thesis, the *Anosov diffeomorphism*. Anosov diffeomorphisms are the Poincaré maps associated to Anosov flows which admit a global Poincaré section. They are an interesting object of study on their own, without the context of Anosov flows, and can be defined as follows.

Definition 1.1.1. A diffeomorphism $f : M \to M$ on a Riemannian manifold M is said to be an *Anosov diffeomorphism* if there exists a df-invariant continuous splitting of the tangent bundle $TM = E^u \oplus E^s$ and constants $c > 0$, $\lambda > 1$ such that

$$\forall v \in E^u : \ \forall k \in \mathbb{N} : \|df^k v\| \geq c\lambda^k \|v\|$$

$$\forall v \in E^s : \ \forall k \in \mathbb{N} : \|df^k v\| \leq \frac{1}{c\lambda^k} \|v\|.$$

For a more complete discussion on Anosov diffeomorphisms, their dynamics and the relation with Anosov flows, we refer the reader to Chapter 3. In this thesis, we will mainly be interested in Anosov diffeomorphisms on compact manifolds. In this case, the definition does not depend on the choice of Riemannian metric. The most fundamental and well-known example of an Anosov diffeomorphism on a compact manifold is the diffeomorphism on the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ defined by

$$\begin{pmatrix} x \\ y \end{pmatrix} + \mathbb{Z}^2 \ \mapsto \ \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \mathbb{Z}^2.$$

This map is known as *Arnold's cat map* and is named after Russian mathematician Vladimir Arnold who illustrated the behaviour of this map by applying it to the image of a cat (see Figure 1.3 below).

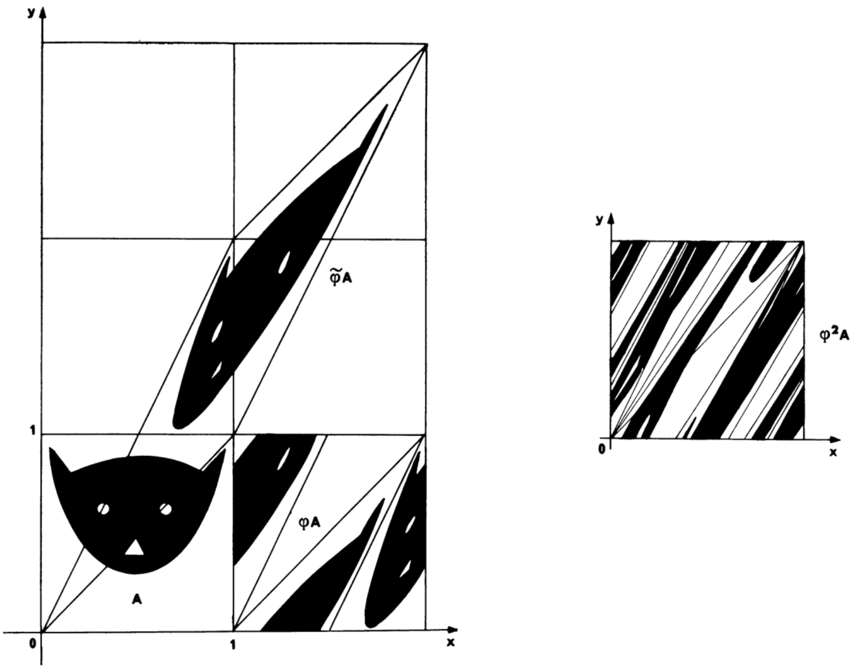


Figure 1.3: A copy of Figure 1.17. from the monograph 'Problèmes ergodiques de la mécanique classique' [AA67].

Similar examples can be constructed on all tori $T^n = \mathbb{R}^n/\mathbb{Z}^n$ for $n \geq 2$. In 1966, Anosov asked whether there exist non-toral examples of Anosov diffeomorphisms on compact manifolds. In his survey on differentiable dynamical systems, Stephen Smale answers this question positively by giving an explicit example [Sma67]. In the same work, he asks the following more general question which is also the title of this thesis.

Question 1.1.2. Which closed manifolds admit an Anosov diffeomorphism?

Smale's example is constructed on the quotient of a direct product of two real Heisenberg groups by a nicely chosen cocompact lattice. It is a special case of a general construction on *infra-nilmanifolds*, which are quotients of simply connected nilpotent Lie groups by the action of an almost-Bieberbach group. An extensive introduction to this class of manifolds is given in Chapter 2. The following conjecture remains open until today.

Conjecture 1.1.3. All Anosov diffeomorphisms are topologically conjugate to hyperbolic affine infra-nilmanifold automorphisms.

Several other classes of closed manifolds have been proven not to admit Anosov diffeomorphisms, including: certain closed manifolds with a virtually polycyclic fundamental group [Hir71], spheres and projective spaces [Shi73] and closed negatively curved manifolds [Yan83], thus strengthening the above conjecture. On the other hand, even if this conjecture turns out to be true, it does not give a complete answer to Question 1.1.2, since not every infra-nilmanifold admits an Anosov diffeomorphism. In fact, the existence is rather rare, which leads us to the following question.

Question 1.1.4. Which (infra-)nilmanifolds admit an Anosov diffeomorphism?

There exist algebraic tools to check the admittance of an Anosov diffeomorphism on infra-nilmanifolds. Let us state this characterization for the smaller class of nilmanifolds as they will be the main focus throughout this thesis. A nilmanifold is a quotient of a simply connected nilpotent Lie group N by a cocompact lattice Γ. We write $\mathfrak{n}$ for the real Lie algebra associated to N. By the work of Mal'cev [Mal49a], any cocompact lattice $\Gamma \leq N$ corresponds to a rational Lie algebra $\mathfrak{n}_\Gamma^\mathbb{Q}$ which is a *rational form* of the Lie algebra $\mathfrak{n}$.

Theorem 1.1.5 ([Man74], [Dek01])**.** *The nilmanifold N/Γ admits an Anosov diffeomorphism if and only if the associated rational Lie algebra $\mathfrak{n}_\Gamma^\mathbb{Q}$ admits an automorphism which has a characteristic polynomial with integer coefficients, has determinant equal to ± 1 and has no eigenvalues of absolute value 1.*

Automorphisms of a rational Lie algebra that satisfy the properties in the above theorem will be called *Anosov automorphisms*. If a rational Lie algebra admits an Anosov automorphism, we say it is an *Anosov Lie algebra*. The above theorem is the beginning of a research program focussed on the following questions.

Question 1.1.6. Given a real (or complex) Lie algebra, what are its rational forms up to isomorphism? Which of those rational forms admit an Anosov automorphism? Which real Lie algebras admit at least one rational form which has an Anosov automorphism?

A classification of rational Anosov Lie algebras up to dimension eight was completed in [LW09]. For some classes of Lie algebras, the answer to the above questions is known.

Abelian Lie algebras have only one rational form, up to isomorphism. This rational form is Anosov if and only if the dimension is at least two, which corresponds with the existence of Anosov diffeomorphisms on tori. More general, if an Anosov Lie algebra has a non-trivial abelian factor, this factor must have dimension at least two [LW08].

For free nilpotent Lie algebras, there is again only one rational form, up to isomorphism, and it is Anosov if and only if the nilpotency class is strictly less than the rank [DV09].

In [DM05] the standard rational form of a 2-step nilpotent Lie algebra associated to a graph was considered and the admittance of an Anosov automorphism on this form was characterized through properties of the graph. One of the main results of this thesis is a generalization of this result in two directions: all rational forms are considered (contrary to only the standard one) and higher nilpotency classes are allowed. Lie algebras associated to a graph are part of a more general theme in algebra known as *partially commutative structures*, to which we give an introduction in Chapter 4.

1.2 Overview of the main results

As we try to understand which rational Lie algebras admit an Anosov automorphism, it is important to find relations with other Lie algebra properties. Such a property of interest is the existence of a *positive grading*. All low dimensional examples of Anosov Lie algebras exhibit a positive grading, thus raising the question whether this holds in general. J. Deré answered this question negatively by giving the first example of a rational Lie algebra which admits an Anosov automorphism, but which does not admit a positive grading [Der17]. The dimension of the example, however, is high and not explicitly known due to the method of construction. The first main result of this thesis, which can be found in Section 6.5, is a concrete example of such a Lie algebra which is of minimal dimension, proving the following theorem.

Theorem A. *There exists a rational Lie algebra of dimension 12 which admits an Anosov automorphism, but which is not positively graded and this is the smallest possible dimension for such an example.*

As a tool for proving the above theorem, we show the following result in the preceding Section 6.4 on decompositions of Anosov Lie algebras, which is also of interest on its own.

Theorem B. *Let* $\mathfrak{n}$ *be a rational nilpotent Lie algebra with a decomposition*

$$\mathfrak{n} = \mathfrak{a} \oplus \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$$

into non-abelian indecomposable ideals $\mathfrak{g}_i$ *and an abelian ideal* $\mathfrak{a}$*. The Lie algebra* $\mathfrak{n}$ *is Anosov if and only if* $\dim(\mathfrak{a}) \neq 1$ *and* $\mathfrak{g}_i$ *is Anosov for all* $i \in \{1, \ldots, k\}$*.*

A second and substantial part of this thesis is devoted to answering Question 1.1.6 within the class of *free nilpotent partially commutative Lie algebras.* These Lie algebras are defined using a simple undirected graph $\mathcal{G} = (V, E)$. First, one defines the free partially commutative Lie algebra by the presentation

$$\mathfrak{g}^K(\mathcal{G}) = \langle V \mid [v, w] = 0; \{v, w\} \notin E \rangle ,$$

where K denotes the base field. The c-step nilpotent partially commutative Lie algebra is then obtained by taking the quotient with the $(c+1)$-th ideal of the lower central series:

$$\mathfrak{n}^K(\mathcal{G}, c) = \mathfrak{g}^K(\mathcal{G}) / \gamma_{c+1}(\mathfrak{g}^K(\mathcal{G})).$$

To classify the rational forms of $\mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c)$, we define a *quotient graph* $\overline{\mathcal{G}}$ from the graph $\mathcal{G}$. The vertex set of $\overline{\mathcal{G}}$ is the set of *coherent components* of $\mathcal{G}$ which we write as $\Lambda_{\mathcal{G}}$ and is a partition of the vertices of $\mathcal{G}$. The classification is proven in Section 5.6 by use of Galois cohomolgy and can be formulated as follows.

Theorem C. *The rational forms of* $\mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c)$*, up to isomorphism, are in one-to-one correspondence with faithful actions of finite Galois groups* $\mathrm{Gal}(L/\mathbb{Q})$ *on the quotient graph* $\overline{\mathcal{G}}$*, up to conjugacy of actions.*

Next, in Section 6.6, we use this classification result to determine which rational forms admit an Anosov automorphism. To formulate the result we define for any action $\rho : \mathrm{Gal}(L/\mathbb{Q}) \to \mathrm{Aut}(\overline{\mathcal{G}})$ a function $z_\rho : \Lambda_{\mathcal{G}} \to \{1/2, 1\}$ which takes the value 1 on the orbits containing a fixed point under the action of the complex conjugation automorphism and 1/2 otherwise.

Theorem D. *Let* $\rho : \mathrm{Gal}(L/\mathbb{Q}) \to \mathrm{Aut}(\overline{\mathcal{G}})$ *be a faithful action of a finite Galois group and* $\tau \in \mathrm{Gal}(L/\mathbb{Q})$ *the complex conjugation automorphism. The corresponding rational form of* $\mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c)$ *admits an Anosov automorphism if and only if for any non-empty connected subset* $A \subset \Lambda_{\mathcal{G}}$ *such that* $\rho_\tau(A) \cup A$ *is* ρ*-invariant, it holds that*

$$c < \sum_{\lambda \in A \cup \rho_\tau(A)} z_\rho(\lambda) \cdot |\lambda|.$$

The study of Anosov automorphisms on rational Lie algebras is one where eigenvalues play an important role. In particular, we do not allow eigenvalues of absolute value 1. A related problem is that of determining which groups have the *R_∞-property*, a property related to fixed points of self homeomorphisms of topological spaces. In particular, for cocompact lattices in nilpotent Lie groups, proving that such a group has the R_∞-property boils down to proving that any induced automorphism on the Lie algebra has an eigenvalue equal to 1. In the last part of this thesis, results on the R_∞-property are proven using techniques developed in our study of Anosov automorphisms on partially commutative Lie algebras.

Analogous to Lie algebras, one defines the partially commutative group associated to the graph $\mathcal{G}$ by the group presentation

$$A(\mathcal{G}) = \langle V \mid [v, w] = 1; \{v, w\} \notin E \rangle .$$

These groups are better known as *right-angled Artin groups.* The c-step nilpotent partially commutative group is defined as

$$A(\mathcal{G}, c) = A(\mathcal{G})/\gamma_{c+1}(\mathcal{G}).$$

By a result from [GW09], a group G has the R_∞-property if $G/\gamma_i(G)$ has the R_∞-property for some positive integer i. This motivates the definition of the *R_∞-nilpotency index* of a group G as the smallest positive integer c such that $G/\gamma_{c+1}(G)$ has the R_∞-property. In Section 7.2, we introduce numbers $\xi(\mathcal{G}), \Xi(\mathcal{G})$ associated to a graph $\mathcal{G}$ and prove the following theorem.

Theorem E. *Every non-abelian right-angled Artin group $A(\mathcal{G})$ has the R_∞-property with R_∞-nilpotency index c satisfying*

$$\xi(\mathcal{G}) \leq c \leq \Xi(\mathcal{G}).$$

By Theorem 1.1.5, the existence of an Anosov diffeomorphism on a nilmanifold N/Γ depends only on the abstract commensurability class of the lattice Γ. Concerning the R_∞-property for finitely generated torsion-free nilpotent groups, one can similarly ask whether it is an abstract commensurability invariant. In Section 7.3 we answer this question negatively and prove the following theorem.

Theorem F. *There exist finitely generated torsion-free nilpotent groups G, H that are abstractly commensurable and such that G has the R_∞-property, but H does not have the R_∞-property.*

Chapter 2

Infra-nilmanifolds

In this chapter we introduce a class of manifolds that will be used throughout this thesis. They can be characterized both in a geometric way and in an algebraic way. To state the geometric characterizations, we need some basic notions from Riemannian geometry of which we give a brief overview in the first section. For a more thorough introduction to Riemannian geometry, we refer the reader to [dC92]. The discussion on the algebraic characterization of these manifolds is mainly based on [Dek18] and [Dek96].

2.1 Preliminaries from Riemannian geometry

All manifolds in this thesis are assumed to be Hausdorff, second countable and smooth. We write $\mathfrak{X}(M)$ for the vector space of smooth vector fields on M.

Definition 2.1.1. A *Riemannian manifold* is a manifold M equipped with for each $p \in M$ an inner product $g_p : T_pM \times T_pM \to \mathbb{R}$ such that for any $X, Y \in \mathfrak{X}(M)$ the assignment $p \mapsto g_p(X_p, Y_p)$ is a smooth map on M. The map g is called a *Riemannian metric* on M.

The term 'metric' refers to the fact that g induces a distance between points on M. For any smooth curve $\gamma : [0,1] \to M$ one defines its length with respect to g as

$$l(\gamma) := \int_0^1 \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))}dt.$$

For any $p, q \in M$ the distance between p and q is defined as

$$d(p,q) := \inf \{l(\gamma) \mid \gamma : [0,1] \to M \text{ is smooth}, \gamma(0) = p, \gamma(1) = q\}.$$

This distance function turns M into a metric space.

Definition 2.1.2. The *diameter* of a Riemannian manifold (M, g) is defined as the supremum of all distances between points on M

$$d(M, g) = \sup\{d(p,q) \mid p, q \in M\}.$$

Note that we allow the diameter to be possibly infinite. For compact Riemannian manifolds, the diameter is always finite.

Let (M, g) be a Riemannian manifold. There exists a unique bilinear map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M) : (X, Y) \mapsto \nabla_X Y$$

which satisfies for any $X, Y, Z \in \mathfrak{X}(M)$ the *Koszul identity*:

$$\begin{aligned} 2g(\nabla_X Y, Z) = X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) \\ - g([Y,X],Z) - g([X,Z],Y) - g([Y,Z],X). \end{aligned}$$

This map ∇ is called the *Levi-Civita connection on* (M, g). The term 'connection' comes from the fact that it gives us a way of identifying the tangent spaces $T_{\gamma(t)}M$ along a smooth curve γ in the manifold. In particular it allows one to differentiate a vector field along a curve. The following object that we define measures in some sense the commutativity of the Levi-Civita connection.

Definition 2.1.3. Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. The *Riemann curvature tensor* is defined as the multilinear map

$$R : \mathfrak{X}(M)^3 \to \mathfrak{X}(M) : (X, Y, Z) \mapsto \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The term 'tensor' refers to the fact that R can be evaluated in tangent vectors at a point in the manifold. More specific, for any point $p \in M$ and any vector fields $X_1, X_2, X_3, Y_1, Y_2, Y_3 \in \mathfrak{X}(M)$ with $X_i(p) = Y_i(p)$ for all $i \in \{1, \ldots, 3\}$ it holds that $R(X_1, X_2, X_3)_p = R(Y_1, Y_2, Y_3)_p$. The expression $R(u, v, w)$ for any $p \in M$ and $u, v, w \in T_pM$ thus has a well-defined value in T_pM.

Definition 2.1.4. Let (M, g) be a Riemannian manifold with Riemann curvature tensor R. For any point $p \in M$ and plane $\pi \subset T_pM$ one defines the *sectional curvature* of π as

$$K(\pi) = g_p(R(u, v, v), u)$$

where u, v is any orthonormal basis for π.

Remark 2.1.5. Let g be a Riemannian metric on a manifold M and $\lambda > 0$ a real constant. One can define a new metric $\tilde{g}$ on M by letting $\tilde{g}_p = \lambda g_p$ for any $p \in M$. The Levi-Civita connection, Riemann curvature tensor, sectional curvature and diameter of the two metrics can be related as follows:

$$\tilde{\nabla} = \nabla, \quad \tilde{R} = R, \quad \tilde{K} = \frac{1}{\lambda}K, \quad d(M, \tilde{g}) = \sqrt{\lambda}\, d(M, g).$$

2.2 Flat manifolds

We start the discussion with the class of flat manifolds. Since the goal of this thesis is to determine the existence of Anosov diffeomorphisms on closed manifolds, we will mostly be interested in the closed flat manifolds. As the name suggests, the definition of a flat manifold is a geometric one. In the closed case we will show that the definition can be made purely algebraic.

Definition 2.2.1. A Riemannian manifold (M, g) is called a *space form* if it has constant sectional curvature. If in addition the sectional curvature is zero, we call the manifold *flat*.

A first step in trying to understand the space forms (or the flat manifolds in particular) is by determining what their universal cover looks like. Recall that if $p : \tilde{M} \to M$ is a covering projection between manifolds and g is a Riemannian metric on M, then there exists a unique Riemannian metric $\tilde{g}$ on $\tilde{M}$ such that $p : (\tilde{M}, \tilde{g}) \to (M, g)$ is a local isometry. As a consequence, the universal cover of a Riemannian manifold automatically carries a Riemannian structure as well. Since the metric on the covering space is locally isometric to the metric on the base space, it follows that the universal cover of a space form is again a space form. In particular, the universal cover of a flat manifold is again a flat manifold.

Example 2.2.2. There are three standard examples of simply connected space forms:

1. n-dimensional Euclidean space $\mathbb{R}^n$ with metric at $x \in \mathbb{R}^n$ given by

   $$g_x(v, w) = \langle v, w \rangle$$

   for any tangent vectors $v, w \in T_x\mathbb{R}^n \cong \mathbb{R}^n$ and where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^n$. The sectional curvature of $\mathbb{R}^n$ is equal to zero and thus $\mathbb{R}^n$ is a flat manifold.

2. The n-dimensional sphere $S^n(R) = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \ldots + x_{n+1}^2 = R^2\}$ of radius R. It has a natural embedding $i : S^n(R) \hookrightarrow \mathbb{R}^{n+1}$:

$x \mapsto x$. The metric on $S^n(R)$ can then be defined for all $x \in S^n(R)$ and $v, w \in T_x S^n(R)$ by

$$g_x(v, w) = \langle \mathrm{d}i(v), \mathrm{d}i(w) \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^{n+1} \cong T_{i(x)}\mathbb{R}^{n+1}$. The sectional curvature of $S^n(R)$ is equal to $1/R^2$.

3. n-dimensional hyperbolic space $\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 > 0\}$ with metric at $x = (x_1, \dots, x_n)$ given by

$$g_x(v, w) = \frac{1}{x_1^2} \langle v, w \rangle$$

for any tangent vectors $v, w \in T_p\mathbb{H}^n \cong \mathbb{R}^n$. The sectional curvature of $\mathbb{H}^n$ is equal to -1. By Remark 2.1.5, we can rescale this metric with the positive constant $R^2 > 0$ to obtain a Riemannian manifold $\mathbb{H}^n(R)$ with constant negative sectional curvature $-1/R^2$.

In [Hop26][Kil91] Killing and Hopf proved that, up to isometry, S^n, $\mathbb{R}^n$ and $\mathbb{H}^n$ are also the only possible simply connected space forms within the class of the complete manifolds. Recall that a metric space is called *complete* if any Cauchy sequence converges.

Theorem 2.2.3 (Killing-Hopf)**.** *Let (M, g) be an n-dimensional complete space form and let K denote the (constant) sectional curvature of M.*

1. *If $K > 0$, then the universal cover of M is $S^n(R)$, where $R = 1/\sqrt{K}$.*
2. *If $K = 0$, then the universal cover of M is $\mathbb{R}^n$.*
3. *If $K < 0$, then the universal cover of M is $\mathbb{H}^n(R)$, where $R = 1/\sqrt{-K}$.*

Note that a closed (or compact) Riemannian manifold is always complete. Thus given an n-dimensional closed flat manifold (M, g), we know that its universal cover is Euclidean space $\mathbb{R}^n$ and we have a covering projection $p : \mathbb{R}^n \to M$, which is a local isometry. Let $\varphi \in \mathrm{Diff}(\mathbb{R}^n)$ be a covering transformation of $p : \mathbb{R}^n \to M$, i.e. $\varphi \circ p = p$. Let $\tilde{g}$ denote the Euclidean metric on $\mathbb{R}^n$. Then we have for any $x \in \mathbb{R}^n$ and $v, w \in T_p\mathbb{R}^n$:

$$\begin{aligned} \tilde{g}_{\varphi(x)}(\mathrm{d}\varphi\, v, \mathrm{d}\varphi\, w) &= g_{p(\varphi(x))}(\mathrm{d}p\, \mathrm{d}\varphi\, v, \mathrm{d}p\, \mathrm{d}\varphi\, w) \\ &= g_{p(x)}(\mathrm{d}p\, v, \mathrm{d}p\, w) \\ &= \tilde{g}_x(v, w). \end{aligned}$$

As a consequence φ is an isometry of $\mathbb{R}^n$. If we let Γ denote the group of all covering transformations or Deck transformations of $p : \mathbb{R}^n \to M$, then this exactly means that Γ is a subgroup of $\operatorname{Iso}(\mathbb{R}^n)$. Note that the isometry group of $\mathbb{R}^n$ can be written as a semi-direct product

$$\operatorname{Iso}(\mathbb{R}^n) \cong \mathbb{R}^n \rtimes \mathrm{O}_n(\mathbb{R})$$

where an element (b, A) of the semi-direct product $\mathbb{R}^n \rtimes \mathrm{O}_n(\mathbb{R})$ corresponds to the isometry of $\mathbb{R}^n$ which sends x to $Ax + b$ for all $x \in \mathbb{R}^n$. Equivalently, we can say that $\operatorname{Iso}(\mathbb{R}^n)$ fits in the short exact sequence

$$1 \longrightarrow \mathbb{R}^n \xrightarrow{t} \operatorname{Iso}(\mathbb{R}^n) \xrightarrow{r} \mathrm{O}_n(\mathbb{R}) \longrightarrow 1 \tag{2.1}$$

where for any $b \in \mathbb{R}^n$, the isometry $t(b) : \mathbb{R}^n \to \mathbb{R}^n : x \mapsto x + b$ is *the translation over* b and for any isometry $f \in \operatorname{Iso}(\mathbb{R}^n)$, the linear map $r(f) : \mathbb{R}^n \to \mathbb{R}^n : x \mapsto f(x) - f(0)$ is *the linear part of* f. Analogously, we can define the Affine transformation group $\operatorname{Aff}(\mathbb{R}^n)$ of $\mathbb{R}^n$ as the semi-direct product $\mathbb{R}^n \rtimes \mathrm{GL}_n(\mathbb{R})$ and get a short exact sequence

$$1 \longrightarrow \mathbb{R}^n \xrightarrow{t} \operatorname{Aff}(\mathbb{R}^n) \xrightarrow{r} \mathrm{GL}_n(\mathbb{R}) \longrightarrow 1.$$

Note that $\operatorname{Iso}(\mathbb{R}^n)$ is a subgroup of $\operatorname{Aff}(\mathbb{R}^n)$.

If we endow $\mathbb{R}^n$, $\mathrm{O}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{R}^n)$ with their usual smooth structure then the smooth product structure on $\mathbb{R}^n \rtimes \mathrm{O}_n(\mathbb{R})$ and $\mathbb{R}^n \rtimes \mathrm{GL}_n(\mathbb{R})$ turns $\operatorname{Iso}(\mathbb{R}^n)$ and $\operatorname{Aff}(\mathbb{R}^n)$ into Lie groups.

2.2.1 Crystallographic and Bieberbach groups

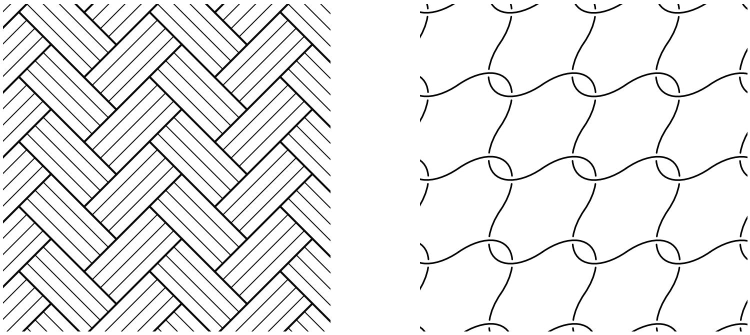
In crystallography, the group $\operatorname{Iso}(\mathbb{R}^n)$ plays a central role since the symmetries of a crystal form a subgroup of $\operatorname{Iso}(\mathbb{R}^n)$. The following definition makes this concrete. Let us recall that a subgroup H of a topological group G is called *discrete* if the subspace topology on H inherited from G is the discrete topology and is called *cocompact* if the space of left cosets G/H is compact for the quotient topology. A group is called *torsion-free* if there are no elements with finite order except for the neutral element.

Definition 2.2.4. A subgroup Γ of $\operatorname{Iso}(\mathbb{R}^n)$ is called a *crystallographic group* if it is discrete and cocompact. If in addition Γ is torsion-free, we call Γ a *Bieberbach group.*

The *dimension* of a crystallographic or Bieberbach group is defined as the dimension n of the Euclidean space $\mathbb{R}^n$ that it acts on. The two-dimensional crystallographic groups are sometimes referred to as the ‘wallpaper groups’ whereas the three-dimensional crystallographic groups are sometimes referred to as the ‘space groups’.

Example 2.2.5. Let us give some examples of crystallographic and Bieberbach groups. Recall that for any $b \in \mathbb{R}^n$, $t(b) : \mathbb{R}^n \to \mathbb{R}^n : x \mapsto x + b$ denotes the translation over b.

1. The subgroup $\{t(b) \mid b \in \mathbb{Z}^n\}$ of $\mathrm{Iso}(\mathbb{R}^n)$ consisting of all integer translations is a Bieberbach group. This group is clearly free abelian of rank n.
2. Consider the subgroup of $\mathrm{Iso}(\mathbb{R}^2) \cong \mathbb{R}^2 \rtimes \mathrm{O}_2(\mathbb{R})$ generated by
$$\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\right) \quad \text{and} \quad t\begin{pmatrix}0\\1\end{pmatrix}.$$
This is a Bieberbach group. It is isomorphic to the fundamental group of the Klein bottle, which is no coincidence as we will see. It can also be seen as the group of symmetries of the pattern drawn in Figure 2.1a. This group and the free abelian group of rank two exhibit all the Bieberbach groups of dimension two, up to isomorphism.
3. Consider the subgroup of $\mathrm{Iso}(\mathbb{R}^2) \cong \mathbb{R}^2 \rtimes \mathrm{O}_2(\mathbb{R})$ generated by
$$\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}\right), \quad t\begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad t\begin{pmatrix}0\\1\end{pmatrix}.$$
This is an example of a crystallographic group which is not a Bieberbach group. Indeed, the first generator has order two, thus implying the group is not torsion-free. The group can also be seen as the symmetries of the pattern drawn in Figure 2.1b. In total, there are 17 crystallographic groups of dimension 2, up to isomorphism.
4. Consider the subgroup of $\mathrm{Iso}(\mathbb{R}^3)$ generated by the elements
$$\left(\begin{pmatrix}-1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 1\end{pmatrix}, \begin{pmatrix}1/2\\1/2\\1/2\end{pmatrix}\right) \quad \text{and} \quad \left(\begin{pmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & -1\end{pmatrix}, \begin{pmatrix}1/2\\0\\0\end{pmatrix}\right).$$
This is a Bieberbach group of dimension 3. In total there are, up to isomorphism, 10 Bieberbach groups of dimension 3.



(a) The herringbone pattern. (b) The pattern of a wire fence.

Figure 2.1: Some patterns in the plane giving rise to crystallographic groups.

Note how the crystallographic group from Example 2.2.5.3 has an element which leaves the origin of $\mathbb{R}^2$ fixed, while the Bieberbach group from Example 2.2.5.2 has no elements with fixed points except for the identity. It turns out this is exactly what characterises the Bieberbach groups within the crystallographic groups. To make this more concrete, let us recall some definitions regarding group actions.

Definition 2.2.6. Let G be a group acting on a set X and let us write this action with $g{\cdot}x$ for any $g \in G$ and $x \in X$. If G is a topological group and X a topological space, we say the action is *continuous* if the map $G \times X \to X : (g, x) \mapsto g \cdot x$ is continuous. Moreover we say the action is:

(i) *properly discontinuous* if for any compact subset $K \subseteq X$ the set

$$\{g \in G \mid K \cap g \cdot K \neq \emptyset\}$$

is finite,

(ii) *cocompact* if the quotient space $G \backslash X$ is compact and

(iii) *free* if for any $g \in G, x \in X$, it holds that $g \cdot x$ implies g is equal to the neutral element of G.

Note that $\mathrm{Iso}(\mathbb{R}^n)$ is a topological group since it satisfies the stronger condition of a Lie group and thus any subgroup of $\mathrm{Iso}(\mathbb{R}^n)$ is a topological group with the subspace topology. Subgroups of $\mathrm{Iso}(\mathbb{R}^n)$ naturally act on $\mathbb{R}^n$ and as one can check these actions are continuous. The following proposition now gives us characterizations of crystallographic groups and Bieberbach groups in terms of the action on $\mathbb{R}^n$.

Proposition 2.2.7. *Let* Γ *be a subgroup of* $\mathrm{Iso}(\mathbb{R}^n)$*, then*

(i) Γ *is a discrete subgroup of* $\mathrm{Iso}(\mathbb{R}^n)$ *if and only if* Γ *acts properly discontinuously on* $\mathbb{R}^n$*,*

(ii) Γ *is a cocompact subgroup of* $\mathrm{Iso}(\mathbb{R}^n)$ *if and only if* Γ *acts cocompactly on* $\mathbb{R}^n$*,*

(iii) if Γ *is a discrete subgroup, it is torsion-free if and only if* Γ *acts freely on* $\mathbb{R}^n$*.*

Proof. We will only prove part (iii). Let Γ be a discrete subgroup of $\mathrm{Iso}(\mathbb{R}^n)$ and first assume Γ is torsion-free. Say we have $\gamma \in \Gamma$ and $x \in \mathbb{R}^n$ such that $\gamma \cdot x = x$ or equivalently $\gamma(x) = x$. Let us write $\gamma = (b, A)$ with $b \in \mathbb{R}^n$ and $A \in \mathrm{O}_n(\mathbb{R})$. Then consider the element $\tilde{\gamma} = t(x)^{-1} \circ \gamma \circ t(x)$. Note that $\tilde{\gamma}(0) = 0$ and thus that $\tilde{\gamma} \in \{0\} \times \mathrm{O}_n(\mathbb{R}^n)$ which is a compact subgroup of $\mathbb{R}^n \rtimes \mathrm{O}_n(\mathbb{R}^n)$. As a consequence, the discrete subgroup generated by $\tilde{\gamma}$ must be finite. Thus there exists an integer $n \in \mathbb{Z}$ such that $\tilde{\gamma}^n = 1$ which in turn implies as well that $\gamma^n = 1$. Since Γ was assumed to be torsion-free, it follows that $\gamma = 1$. This proves that Γ acts freely on $\mathbb{R}^n$.

Conversely, assume Γ acts freely on $\mathbb{R}^n$ and let γ be an element of finite order in Γ, say n. Then define

$$x := \sum_{i=0}^{n-1} \frac{1}{n} \gamma^i(0).$$

As one can check, we now have that $\gamma(x) = x$ or equivalently $\gamma \cdot x = x$. Since we assumed that the action was free, it follows that $\gamma = 1$. This proves that Γ is torsion-free. □

The Bieberbach theorems. In 1911-1912, Ludwig Bieberbach published two papers with title 'Über die Bewegungsgruppen der Euklidischen Räume' [Bie11][Bie12], in which he proved three important theorems about the structure of crystallographic groups. In this paragraph, we state these theorems. Recall the notation of the group morphisms r and t from equation (2.1).

Theorem 2.2.8 (First Bieberbach)**.** *Let* $\Gamma \subseteq \mathrm{Iso}(\mathbb{R}^n)$ *be a crystallographic group of dimension* n*. Then* $r(\Gamma)$ *is a finite group and* $\Gamma \cap t(\mathbb{R}^n)$ *is a free abelian group of rank* n*.*

The subgroup $\Gamma \cap t(\mathbb{R}^n)$ will be called the *subgroup of pure translations* and the group $r(\Gamma)$ the *holonomy group.* We have a couple of observations about the subgroup of pure translations.

- First, note that it is normal in Γ. This follows from the more general fact that $t(\mathbb{R}^n)$ is normal in $\mathrm{Iso}(\mathbb{R}^n)$.

- Second, it is a maximal abelian subgroup of Γ, i.e. it is not strictly contained in another abelian subgroup of Γ. Indeed, assume $\gamma = (b, A)$ is an element in Γ which commutes with every element of $t(\mathbb{R}^n) \cap \Gamma$. Since $t(\mathbb{R}^n) \cap \Gamma$ is free abelian of rank n, there exists a basis $v_1, \ldots, v_n$ of $\mathbb{R}^n$ such that the image of this basis under t is a basis for $t(\mathbb{R}^n) \cap \Gamma$ as a $\mathbb{Z}$-module. The commutation relations

$$1 = [\gamma, t(v_i)] = \gamma\, t(v_i)\, \gamma^{-1}\, t(v_i)^{-1} = t\big((A - I)v_i\big)$$

for all $1 \leq i \leq n$ imply that A is equal to the identity I. This shows that $\gamma \in t(\mathbb{R}^n) \cap \Gamma$ and thus that $t(\mathbb{R}^n) \cap \Gamma$ is indeed maximal abelian in Γ.

- Third, it is the only subgroup of Γ which is both normal and maximal abelian. Indeed, assume S is a normal abelian subgroup of Γ and let $\gamma = (b, A)$ be an element of S. Let $v_1, \ldots, v_n$ be the basis of $\mathbb{R}^n$ as above. From the normality of S, it follows that $[\gamma, t(v_i)]$ lies in S. Since S is also abelian, we have for all $1 \leq i \leq n$, the commutation relations

$$1 = [\gamma, [\gamma, t(v_i)]] = t\big((A - I)^2 v_i\big)$$

which implies that $(A - I)^2 = 0$. Using that $A \in \mathrm{O}_n(\mathbb{R})$, we can from this deduce that $A = I$ and thus that $\gamma \in t(\mathbb{R}^n) \cap \Gamma$. This shows that $S \subset t(\mathbb{R}^n) \cap \Gamma$ and thus that the subgroup of pure translations is indeed the only normal and maximal abelian subgroup of Γ. As a consequence we also have that it is a characteristic subgroup of Γ, i.e. it is preserved under any automorphism of Γ.

The first Bieberbach theorem thus implies that every crystallographic group has a (unique) normal torsion-free maximal abelian subgroup of finite index. Due to Zassenhauss [Zas48], we know this is also a sufficient condition for an abstract group to be isomorphic to a crystallographic group. This is captured in the following theorem with the use of short exact sequences.

Theorem 2.2.9 (Zassenhauss). *A group Γ is isomorphic to a crystallographic group of dimension n if and only if it fits into a short exact sequence*

$$1 \xrightarrow{i} \mathbb{Z}^n \longrightarrow \Gamma \longrightarrow F \longrightarrow 1$$

where $i(\mathbb{Z})$ is maximal abelian in Γ and F is a finite group.

Now let us state the second Bieberbach theorem, which tells us what an isomorphism between two crystallographic groups in $\mathrm{Iso}(\mathbb{R}^n)$ looks like.

Theorem 2.2.10 (Second Bieberbach). *Let* $\Gamma, \Gamma' \subseteq \mathrm{Iso}(\mathbb{R}^n)$ *be crystallographic groups of dimension* n*. Let* $f : \Gamma \to \Gamma'$ *be a group isomorphism. Then there exists an element* $\alpha \in \mathrm{Aff}(\mathbb{R}^n)$ *such that for all* $\gamma \in \Gamma$ *it holds that* $f(\gamma) = \alpha\gamma\alpha^{-1}$*. In other words, every isomorphism between crystallographic groups can be realized as an affine transformation of coordinates.*

Note that two crystallographic groups of different dimensions can never be isomorphic as abstract groups. Indeed, as discussed above, a crystallographic group has a unique normal maximal abelian subgroup which is free abelian of rank equal to the dimension of the crystallographic group. If two crystallographic groups are isomorphic, then their normal maximal abelian subgroups must be isomorphic and thus they must have the same dimension.

At last, we state the third Bieberbach theorem.

Theorem 2.2.11 (Third Bieberbach). *Given* n*, there are, up to isomorphism, only finitely many crystallographic subgroups of* $\mathrm{Iso}(\mathbb{R}^n)$*.*

In low dimensions, the number of crystallographic and Bieberbach groups up to isomorphism have been calculated [BBN+78] [PS00] [CS01]. These numbers are given in Table 2.1.

dimension	crystallographic	Bieberbach
1	2	1
2	17	2
3	219	10
4	4 783	74
5	222 018	1 060
6	28 927 922	38 746

Table 2.1: The number of crystallographic groups and Bieberbach groups in low dimensions.

2.2.2 Algebraic characterization of flat manifolds

Let (M, g) be a closed flat manifold. As discussed before, its universal cover is given by n-dimensional Euclidean space $\mathbb{R}^n$ and the group of covering transformations Γ, is a subgroup of $\mathrm{Iso}(\mathbb{R}^n)$. In general, the group of covering transformation of a covering projection acts properly continuous and freely on the cover if we assume the spaces to be connected, locally connected and locally compact. Using Proposition 2.2.7 we thus immediately find that Γ is a Bieberbach group. This proves the following theorem which gives an algebraic characterisation of closed flat manifolds.

Theorem 2.2.12. *If (M, g) is an n-dimensional closed flat manifold, then there exists a Bieberbach group $\Gamma \subset \mathbb{R}^n$ such that (M, g) is isometric to $\Gamma\backslash\mathbb{R}^n$ and the fundamental group of M is isomorphic to Γ.*

Using the above theorem, we can now give geometric interpretations to the Bieberbach theorems, but in order to do so we need the notion of an affine equivalence between Riemannian manifolds.

Definition 2.2.13. Let (M, g) and $(N, \tilde{g})$ be Riemannian manifolds with Levi-Civita connections ∇ and $\tilde{\nabla}$, respectively. A diffeomorphism $f : M \to N$ is called an *affine equivalence* if $\nabla = f^*\tilde{\nabla}$ where $f^*\tilde{\nabla}$ is the pull-back of $\tilde{\nabla}$ along f. In case such a diffeomorphism exists, we say (M, g) and $(N, \tilde{g})$ are *affinely equivalent*.

Theorem 2.2.14 (Geometric Bieberbach)**.** *Let (M, g) and $(N, \tilde{g})$ be n-dimensional closed flat manifolds. Then the following hold:*

(i) M is covered by a flat n-dimensional torus and the covering projection is a local isometry.

(ii) The fundamental groups of M and N are isomorphic if and only if M and N are affinely equivalent.

(iii) Up to affine equivalence, there are only finitely many flat manifolds in each dimension.

Proof. For (i), there exists by Theorem 2.2.12 a Bieberbach group $\Gamma \subset \mathrm{Iso}(\mathbb{R}^n)$ such that M is isometric to $\Gamma\backslash\mathbb{R}^n$. As one can check, the quotient map $p : \Gamma \cap t(\mathbb{R}^n)\backslash\mathbb{R}^n \to \Gamma\backslash\mathbb{R}^n$ is a covering projection and also a local isometry. The first Bieberbach theorem now exactly tells us that p has finite fibers and that $\Gamma \cap t(\mathbb{R}^n)\backslash\mathbb{R}^n$ is isometric to a flat torus.

For (ii), note that the 'if' direction is trivial since a diffeomorphism between manifolds induces an isomorphism between their fundamental groups. For the 'only if' direction, let $\Gamma, \tilde{\Gamma} \subset \mathrm{Iso}(\mathbb{R}^n)$ be Bieberbach groups such that M and N are isometric to $\Gamma\backslash\mathbb{R}^n$ and $\tilde{\Gamma}\backslash\mathbb{R}^n$, respectively. Since Γ and $\tilde{\Gamma}$ are isomorphic to the fundamental groups of M and N, respectively, the assumption gives us an isomorphism $F : \Gamma \to \tilde{\Gamma}$. By the second Bieberbach theorem, there exists an $\alpha \in \mathrm{Aff}(\mathbb{R}^n)$ such that $F(\gamma) = \alpha\gamma\alpha^{-1}$ for all $\gamma \in \Gamma$. As one can check, the affine map $\alpha : \mathbb{R}^n \to \mathbb{R}^n$ induces a map $\overline{\alpha} : \Gamma\backslash\mathbb{R}^n \to \tilde{\Gamma}\backslash\mathbb{R}^n : \Gamma \cdot x \mapsto \Gamma \cdot \alpha(x)$ which is in fact an affine equivalence.

For (iii), combine the third Bieberbach theorem with part (ii). $\square$

Remark 2.2.15. Note that we cannot expect part (iii) of the above theorem to hold up to isometry. Indeed, for any closed flat manifold (M, g) and any positive real constant $\lambda > 0$, the manifold $(M, \lambda g)$ is also flat, but is not isometric to (M, g) if $\lambda \neq 1$. Indeed they would have different diameters. Note that this diameter is finite since the manifolds are assumed to be compact. This shows that there are infinitely many non isometric closed flat manifolds in each dimension.

Note that part (ii) of Theorem 2.2.14 implies that in the class of closed flat manifolds the notions 'homeomorphic', 'isomorphic fundamental group' and 'affinely equivalent' are interchangeable. In Table 2.1 we can thus read under Bieberbach groups, the number of closed flat manifolds in low dimensions, up to homeomorphism. In dimension one, there is in the first place only one closed smooth manifold, which is the circle. In dimension two, we find two manifolds, namely the torus and the Klein bottle.

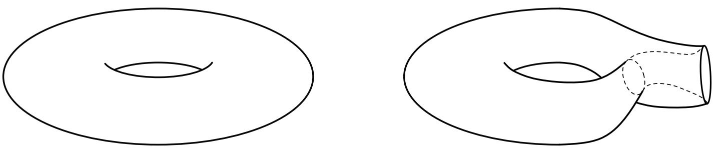


Figure 2.2: Up to homeomorphism, there are only two closed flat manifolds in dimension two: the torus (left) and the Klein-bottle (right).

2.3 Almost-flat manifolds

Definition 2.3.1. Let (M, g) be a Riemannian manifold with finite diameter. We say the metric g is *ϵ-flat* if its sectional curvature is bounded in terms of the diameter by

$$|K(v, w)| \leq \frac{\epsilon}{d(M, g)^2}$$

for any unit tangent vectors $v, w \in T_pM$ and any $p \in M$. A compact manifold is called *almost flat* if it admits ϵ-flat metrics for $\epsilon > 0$ arbitrarily small.

Note that the definition of a flat manifold is one on differentiable manifolds and not on Riemannian manifolds.

Remark 2.3.2. Note that being ϵ-flat is independent of scaling the metric with a constant. Indeed, let g be a Riemannian metric on a manifold M, $\lambda > 0$

a real constant and define a new metric $\tilde{g}$ on M by letting $\tilde{g}_p = \lambda g_p$ for any $p \in M$. By Remark 2.1.5 we have that

$$|\tilde{K}(v,w)| \cdot d(M,\tilde{g})^2 = \frac{1}{|\lambda|} \cdot |K(v,w)| \cdot \sqrt{\lambda}^2 \cdot d(M,g)^2 = |K(v,w)| \cdot d(M,g)^2.$$

This implies that g is ϵ-flat if and only if $\tilde{g}$ is ϵ-flat.

Every manifold that admits a flat metric is also an almost-flat manifold. As a consequence, for any Bieberbach group $\Gamma \leq \mathrm{Iso}(\mathbb{R}^n)$, we have that $\Gamma\backslash\mathbb{R}^n$ is an almost-flat manifold. As we will see later on, the simplest example of an almost-flat manifold that does not admit a flat metric is the quotient $H(\mathbb{Z})\backslash H(\mathbb{R})$ (see Example 2.3.4 below). The key difference between $\Gamma\backslash\mathbb{R}^n$ and $H(\mathbb{Z})\backslash H(\mathbb{R})$ is that $\mathbb{R}^n$ is an abelian Lie group and $H(\mathbb{R})$ is not. The group $H(\mathbb{R})$ is, however, nilpotent and this will turn out to be necessary as well.

2.3.1 Torsion-free finitely generated nilpotent groups

In what follows we give an overview of some results due to Mal'cev, which help describe the lattices in simply connected nilpotent Lie groups. First, for sake of completeness, let us recall the definition of a nilpotent group.

Definition 2.3.3. A group N is called *nilpotent* if it has a *central series* of finite length, i.e. a finite sequence of subgroups of N:

$$\{1\} = N_0 \trianglelefteq N_1 \trianglelefteq \ldots \trianglelefteq N_{n-1} \trianglelefteq N_n = N$$

such that for all $1 \leq i \leq n$ we have $[N, N_i] \leq N_{i-1}$. The smallest possible length n for a central series of a nilpotent group N will be called the *nilpotency class* of N and is denoted by c. We also say that N is *c-step nilpotent.*

Note that in a central series, for all $1 \leq i \leq n$ it holds that $[N, N_i] \leq N_{i-1} \leq N_i$ and thus that N_i is normal in N. As a consequence, we can equivalently characterize a central series as a sequence $\{1\} = N_0 \trianglelefteq N_1 \trianglelefteq \ldots \trianglelefteq N_{n-1} \trianglelefteq N_n = N$ of normal subgroups of N such that $N_i/N_{i-1} \leq Z(N_i/N_{i-1})$ for all $1 \leq i \leq n$.

Example 2.3.4. Let us give some examples of nilpotent groups.

1. Every abelian group A is 1-step nilpotent since it has a trivial central series $\{1\} = A_0 \triangleleft A_1 = A$.

2. Let R be a commutative ring with 1. The multiplicative group of upper triangular matrices with coefficients in R and 1's on the diagonal:

$$UT_n(R) = \left\{ \begin{pmatrix} 1 & r_{12} & r_{13} & \cdots & r_{1n} \\ 0 & 1 & r_{23} & \cdots & r_{2n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & r_{n-1n} \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix} \middle| \, r_{ij} \in R, 1 \leq i < j \leq n \right\}$$

is nilpotent of class $n-1$. As a consequence, every subgroup of $UT_n(R)$ is nilpotent of class at most $n-1$. In the special case where $n = 3$ we call this group the *Heisenberg group* (with entries in R) and write it as $H(R) := UT_3(R)$. Note that the group $UT_n(R)$ is torsion-free if R is an integral domain of characteristic 0. From this it follows that any subgroup of $UT_n(\mathbb{Z})$ is torsion-free. Moreover, since $UT_n(\mathbb{Z})$ is finitely generated and nilpotent, it follows that any of its subgroups are finitely generated as well [Seg83, Ch. 1, Prop. 4]. We thus have that any subgroup of $UT_n(\mathbb{Z})$ is a finitely generated torsion-free nilpotent group (of class at most $n-1$).

3. As a special case of the previous examples, consider for any positive integer $n > 0$ the following subgroup of the integral Heisenberg group $H(\mathbb{Z})$:

$$H_n(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & na & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \middle| \, a, b, c \in \mathbb{Z} \right\}.$$

These groups are two-step nilpotent and pairwise non-isomorphic for different positive integers n. For $n = 1$, we recover the Heisenberg group $H(\mathbb{Z})$.

Now suppose N is a finitely generated torsion-free nilpotent group. It will be very useful to consider a central series for which all the quotients N_{i+1}/N_i are torsion-free as well. This is the case for the so called *upper central series*

$$\{1\} = Z_0(N) \trianglelefteq Z_1(N) \trianglelefteq \ldots \trianglelefteq Z_{c-1}(N) \trianglelefteq Z_c(N) = N$$

defined inductively by $N_i/N_{i-1} = Z(N/N_{i-1})$. Intuitively, it is the central series which grows fastest. To see the quotients are torsion-free, note that the quotient of a torsion-free nilpotent group by its centre is still torsion-free [Seg83, Ch. 1, Cor. 5].

Dual to the upper central series, we can also consider the *lower central series*,

$$N = \gamma_1(N) \trianglerighteq \gamma_2(N) \trianglerighteq \ldots \trianglerighteq \gamma_c(N) \trianglerighteq \gamma_{c+1}(N) = \{1\}$$

defined inductively by $\gamma_{i+1}(N) = [N, \gamma_i(N)]$. Intuitively, this is the fastest shrinking central series. Note that both the upper and lower central series of a nilpotent group N have length equal to the nilpotency class of N. In contrary to the upper central series, the quotients $\gamma_{i+1}(N)/\gamma_i(N)$ of the lower central series of a finitely generated torsion-free nilpotent group N, are not generally torsion-free. To illustrate this, consider the groups $H_n(\mathbb{Z})$ from Example 2.3.4. Their commutator subgroup is given by

$$\gamma_2(H_n(\mathbb{Z})) = \left\{ \begin{pmatrix} 1 & 0 & nc \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \, c \in \mathbb{Z} \right\}.$$

From this it follows that $H_n(\mathbb{Z})/\gamma_2(H_n(\mathbb{Z})) \cong \mathbb{Z}^2 \oplus (\mathbb{Z}/n\mathbb{Z})$ and thus that its lower central series does not have torsion-free quotients if $n > 1$. We can solve this by adding the 'roots' to each subgroup in the lower central series.

Definition 2.3.5. Let G be a group and H a subgroup of G. We define the *isolator or root set of H in G* as the set

$$\sqrt[G]{H} = \{g \in G \mid \exists n \in \mathbb{N} : g^n \in H\}.$$

In general, the isolator of a subgroup does not need to be a subgroup itself, but the isolators of the subgroups in the lower central series are.

Proposition 2.3.6. *Let N be a torsion-free nilpotent group. Then $\sqrt[N]{\gamma_i(N)}$ is a subgroup for all $1 \leq i \leq c+1$ and*

$$N = \sqrt[N]{\gamma_1(N)} \trianglerighteq \sqrt[N]{\gamma_2(N)} \trianglerighteq \ldots \trianglerighteq \sqrt[N]{\gamma_c(N)} \trianglerighteq \sqrt[N]{\gamma_{c+1}(N)} = \{1\}$$

is a central series for N with torsion-free quotients. This central series is called the adapted lower central series.

Proof. We refer to [Pas77, p.473, Lemma 1.8]. □

Note that the subgroups in the upper, lower and adapted lower central series are all characteristic subgroups.

Let N be a finitely generated torsion-free nilpotent group and

$$N = N_1 \trianglerighteq N_2 \trianglerighteq \ldots \trianglerighteq N_n \trianglerighteq N_{n+1} = \{1\} \tag{2.2}$$

a central series for N with torsion-free quotients (and note it is written in descending order). We thus have $[N, N_i] \leq N_{i+1}$ and $N_i/N_{i+1} \cong \mathbb{Z}^{d_i}$ for some integers $d_1, \ldots, d_n \in \mathbb{Z}$. It can be shown that the sum of these integers

$D := \sum_{i=1}^{n} d_i$ is independent of the chosen central series. This number D is usually referred to as the *Hirsch length* or *Hirsch number* of N. By refining the series in (2.2), we can get a central series

$$N = \tilde{N}_1 \geq \tilde{N}_2 \geq \ldots \geq \tilde{N}_D \geq \tilde{N}_{D+1} = \{1\}$$

such that $N_i = \tilde{N}_{1+d_1+\ldots+d_{i-1}}$ for any $i \in \{1, \ldots, n+1\}$ and $\tilde{N}_i / \tilde{N}_{i+1} \cong \mathbb{Z}$ for any $i \in \{1, \ldots, D\}$. For any $i \in \{1, \ldots, D\}$, we can thus choose an element $a_i \in N_i$ such that the quotient $\tilde{N}_i / \tilde{N}_{i+1}$ is generated by $a_i \tilde{N}_{i+1}$. Any D-tuple $a = (a_1, \ldots, a_D)$ of elements of N obtained in this way will be called a *Mal'cev basis for* N. Fixing such a Mal'cev basis, every element $x \in N$ can be written uniquely as a product

$$x = a_1^{x_1} \cdot \ldots \cdot a_D^{x_D}$$

with $x_1, \ldots, x_D \in \mathbb{Z}$. This gives a bijection $\varphi_a : \mathbb{Z}^D \to N$ where

$$\varphi_a(x_1, \ldots, x_D) = a_1^{x_1} \cdot \ldots \cdot a_D^{x_D}.$$

An important fact about finitely generated torsion-free nilpotent groups is that the group operation has a polynomial structure with respect to a Mal'cev basis.

Proposition 2.3.7. *Let N be a finitely generated torsion-free nilpotent group of Hirsch length D and let $a = (a_1, \ldots, a_D)$ be a Mal'cev basis for N. For any $i \in \{1, \ldots, D\}$, there exists a rational polynomial*

$$p_i \in \mathbb{Q}[x_1, \ldots, x_{i-1}, y_1, \ldots, y_{i-1}]$$

which satisfies $p_i(\mathbb{Z}) \subset \mathbb{Z}$, such that for any two elements $x = a(x_1, \ldots, x_D)$ and $y = a(y_1, \ldots, y_D)$ in N, their product is given by $xy = a(z_1, \ldots, z_D)$ with

$$z_i = x_i + y_i + p_i(x_1, \ldots, x_{i-1}, y_1, \ldots, y_{i-1})$$

for any $i \in \{1, \ldots, D\}$.

Example 2.3.8. To illustrate the above, consider the groups $H_n(\mathbb{Z})$ from Example 2.3.4. A Mal'cev basis for $H_n(\mathbb{Z})$ can be given by $a = (a_1, a_2, a_3)$ with

$$a_1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad a_2 := \begin{pmatrix} 1 & n & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad a_3 := \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

An element of $H_n(\mathbb{Z})$ thus has coordinates with respect to this Mal'cev basis as

$$\varphi_a(x_1, x_2, x_3) = \begin{pmatrix} 1 & nx_2 & x_3 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{pmatrix}$$

for any $(x_1, x_2, x_3) \in \mathbb{Z}^3$. A computation shows that

$$\varphi_a(x_1, x_2, x_3) \cdot \varphi_a(y_1, y_2, y_3) = \begin{pmatrix} 1 & n(x_2 + y_2) & x_3 + y_3 + nx_2y_1 \\ 0 & 1 & x_1 + y_1 \\ 0 & 0 & 1 \end{pmatrix}.$$

The polynomials from Proposition 2.3.7 are thus given by

$$p_1 = 0, \quad p_2 = 0, \quad p_3 = nx_2y_1.$$

As a consequence of Proposition 2.3.7, we can naturally extend a torsion-free finitely generated nilpotent group N to bigger groups. Let $a = (a_1, \ldots, a_D)$ be a Mal'cev basis for N and $p_1, \ldots, p_D$ the associated polynomials from Proposition 2.3.7. Define the groups $N^{\mathbb{Q}}$ and $N^{\mathbb{R}}$ as the sets $\mathbb{Q}^D$ and $\mathbb{R}^D$, respectively, equipped with the group operation $(x_1, \ldots, x_D) \cdot (y_1, \ldots, y_D) = (z_1, \ldots, z_D)$ where $z_i = x_i + y_i + p_i(x_1, \ldots, x_{i-1}, y_1, \ldots, y_{i-1})$ for any $x_1, \ldots, x_D, y_1, \ldots, y_D$ in $\mathbb{Q}$ or $\mathbb{R}$, respectively. We thus get a sequence of injective group morphisms

$$N \quad \xrightarrow{\varphi_a^{-1}} \quad N^{\mathbb{Q}} \quad \hookrightarrow \quad N^{\mathbb{R}}.$$

By identifying elements we can thus view N as a subgroup of $N^{\mathbb{Q}}$. The groups $N^{\mathbb{Q}}$ and $N^{\mathbb{R}}$ are referred to as the *rational Mal'cev completion* and the *real Mal'cev completion*, respectively.

Example 2.3.9. The rational and real Mal'cev completions of $\mathbb{Z}^n, +$ are just $\mathbb{Q}^n, +$ and $\mathbb{R}^n, +$, respectively.

Example 2.3.10. Consider the groups $H_n(\mathbb{Z})$ from Examples 2.3.4 for positive integers $n > 0$. Their rational Mal'cev completions are given by $H_n(\mathbb{Z})^{\mathbb{Q}} \cong H(\mathbb{Q})$. We thus see that non-isomorphic torsion-free finitely generated nilpotent groups can have isomorphic Mal'cev completions. Note that we also have $H_n(\mathbb{Z})^{\mathbb{R}} \cong H(\mathbb{R})$.

Note that the Mal'cev completions have the property that all roots of an elements exist.

Definition 2.3.11. A group G is said to be *radicable* if for any $x \in G$ and positive integer $n > 0$, there exists a unique element $y \in G$ such that $y^n = x$. If H is a finitely generated torsion-free nilpotent group and G a radicable group such that H is a subgroup of G and for any $x \in G$ there exists a positive integer $n > 0$ such that $x^n \in H$, then we say G is a *radicable hull of H*.

A radicable hull of a finitely generated torsion-free nilpotent group H is uniquely determined up to isomorphism. This follows from the following property.

Proposition 2.3.12. *Let H be a finitely generated torsion-free nilpotent group and G a radicable hull of H. If $f : H \to M$ is an injective group morphism to a group M for which every element $x \in M$ has some positive power lying in $f(H)$, then there exists a unique morphism $g : M \to G$ which makes the following diagram commute:*

$$\begin{array}{ccc} G & \xleftarrow{\;g\;} & M \\ \uparrow & \nearrow_{f} & \\ H & & \end{array} \tag{2.3}$$

Proof. See [Seg83, Section 6.A., Exercise 1]. □

If N is a finitely generated torsion-free nilpotent group and $N^{\mathbb{Q}}$ its rational Mal'cev completion, then $N^{\mathbb{Q}}$ is a radicable hull of N. We thus see that the isomorphism type of $N^{\mathbb{Q}}$ does not depend on the choice of Mal'cev basis to construct it. As a consequence of the above proposition, we also have that injective maps between finitely generated torsion-free nilpotent groups extend uniquely to their rational Mal'cev completions.

Corollary 2.3.13. *If N_1, N_2 are two finitely generated torsion-free nilpotent groups and $f : N_1 \to N_2$ is an injective homomorphism, then there exists a unique homomorphism $\tilde{f} : N_1^{\mathbb{Q}} \to N_2^{\mathbb{Q}}$ such that $\tilde{f}|_{N_1} = f$. If $N_1 = N_2$ and f is an automorphism, then so is $\tilde{f}$.*

Let N be a finitely generated torsion-free nilpotent group. Since $N^{\mathbb{R}}$ is just $\mathbb{R}^D$ as a set, we can give it its standard differentiable manifold structure. Since multiplication is given by polynomials in these coordinates, it is straightforward to verify that multiplication and inversion are smooth maps. We thus have that $N^{\mathbb{R}}$ is simply connected nilpotent Lie group, an object on which we elaborate in the following section.

2.3.2 Nilpotent Lie groups and Lie algebras

Since it will be a central object in this thesis, let us recall the definition of a Lie algebra (over arbitrary fields) and the notion of homomorphisms between them.

Definition 2.3.14. Let K be a field. A *Lie algebra over the field K* is a K-vector space $\mathfrak{g}$ equipped with a bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} : (X, Y) \mapsto [X, Y]$, called the *Lie bracket of* $\mathfrak{g}$, which satisfies the following two conditions:

(i) $\forall X \in \mathfrak{g} : \; [X, X] = 0$,

(ii) $\forall X, Y, Z \in \mathfrak{g} : \ [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$

Definition 2.3.15. Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras. A linear map $\varphi : \mathfrak{g} \to \mathfrak{h}$ is said to be a *Lie algebra homomorphism* if for any $X, Y \in \mathfrak{g}$ it holds that $\varphi([X, Y]) = [\varphi(X), \varphi(Y)]$. If in addition φ is invertible, then we call φ a *Lie algebra isomorphism* and we say that $\mathfrak{g}$ and $\mathfrak{h}$ are *isomorphic*. The isomorphisms from $\mathfrak{g}$ to itself are called *Lie algebra automorphisms* and the set of all Lie algebra automorphisms of $\mathfrak{g}$ is written as $\mathrm{Aut}(\mathfrak{g})$.

An important result in the theory of Lie groups is that one can associate to any Lie group G an (up to isomorphism) unique Lie algebra $\mathfrak{g}$ over $\mathbb{R}$. The underlying vector space of $\mathfrak{g}$ is the vector space of left-invariant vector fields on G and the Lie bracket is the Lie bracket between vector fields from differential geometry. By evaluating the left invariant vector fields at the identity $e \in G$, one gets an identification of $\mathfrak{g}$ with T_eG, as vector spaces. It is thus clear that $\mathfrak{g}$ is finite dimensional. Moreover, this assignment of a Lie algebra to a Lie group is functorial. If G, H are Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively, and if $f : G \to H$ is a Lie group homomorphism, then the derivative at the identity $d_e f$ gives a Lie algebra homomorphism from $\mathfrak{g}$ to $\mathfrak{h}$. By the works of Sophus Lie and Elie Cartan we know that this functor assigning to a Lie group its Lie algebra is essentially surjective. For a detailed discussion, see for instance [DK00, Chapter 1].

Theorem 2.3.16 (Cartan, Lie)**.** *1. Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively, such that G is simply connected. For every Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{h}$ there exists a Lie group homomorphism $\psi : G \to H$ such that $\varphi = \mathrm{d}_e\psi$.*

2. For every real Lie algebra $\mathfrak{g}$, there exists a, up to isomorphism, unique simply connected Lie group G such that the Lie algebra of G is isomorphic to $\mathfrak{g}$.

For any $X \in \mathfrak{g} \cong T_eG$, there is a unique smooth homomorphism $\varphi_X : (\mathbb{R}, +) \to G$ such that $\varphi'_X(0) = X$. This gives a mapping from the Lie algebra to the Lie group, which will be referred to as *the exponential map* and is defined by

$$\exp : \mathfrak{g} \to G : X \mapsto \varphi_X(1).$$

It is also common to write e^X for $\exp(X)$.

The Lie bracket on $\mathfrak{g}$ and the product on G are related to each other by the *Baker-Campbell-Haussdorf (BCH) formula*:

$$\forall X, Y \in \mathfrak{g} : \ e^X e^Y = e^Z$$

with

$$Z = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + \dots$$

on condition that this series converges. This is guaranteed if X and Y are chosen small enough.

Example 2.3.17 (Heisenberg Lie algebra)**.** Consider the Heisenberg group $H(\mathbb{R})$, which is a Lie group. If we see $H(\mathbb{R})$ as a subset of $\mathbb{R}^{3\times 3}$, then the tangent space of $H(\mathbb{R})$ at the identity can be identified with the vector space of matrices spanned by

$$X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This is the underlying vector space of the Lie algebra of $H(\mathbb{R})$, which we write as $\mathfrak{h}_3$. The Lie bracket on $\mathfrak{h}_3$ is equal to the commutator of matrices and thus we get that

$$[X,Y] = Z \quad \text{and} \quad [X,Z] = [Y,Z] = 0.$$

In general, the 3-dimensional Lie algebra determined by the above relations over an arbitrary field K will be called the *Heisenberg Lie algebra over K*.

For the following definitions, let $\mathfrak{g}$ be an arbitrary Lie algebra over an arbitrary field K. For two sub vector spaces $\mathfrak{h}, \mathfrak{k} \subset \mathfrak{g}$, one defines

$$[\mathfrak{h}, \mathfrak{k}] = \operatorname{span}_K\{[X,Y] \mid X \in \mathfrak{h}, Y \in \mathfrak{k}\}.$$

Similar as with groups, the *lower central series of the Lie algebra* $\mathfrak{g}$ is defined as the descending sequence

$$\mathfrak{g} = \gamma_1(\mathfrak{g}) \supseteq \gamma_2(\mathfrak{g}) \supseteq \gamma_3(\mathfrak{g}) \supseteq \dots$$

where

$$\gamma_{i+1}(\mathfrak{g}) = [\mathfrak{g}, \gamma_i(\mathfrak{g})]$$

for any $i \in \mathbb{N} \setminus \{0\}$.

Definition 2.3.18. A Lie algebra $\mathfrak{g}$ (over an arbitrary field) is called *nilpotent* if there exists an integer $i \in \mathbb{N} \setminus \{0\}$ such that $\gamma_i(\mathfrak{g}) = \{0\}$. If such an integer exists, then the smallest such integer is called the *nilpotency class of* $\mathfrak{g}$ and this integer is denoted by c. In this case, we also say that $\mathfrak{g}$ is *c-step nilpotent.*

Example 2.3.19. 1. The Heisenberg Lie algebra from Example 2.3.17 is a 2-step nilpotent Lie algebra.

2. Let K be a field and V a finite dimensional vector space over K. Let $A \in \mathrm{End}(V)$ be a linear map which is nilpotent. Write $m \geq 0$ for the smallest integer such that $A^m = 0$. Define the Lie algebra $\mathfrak{g}_A = KX \oplus V$ with Lie bracket defined by

$$[X, Y] = AY, \quad [Y_1, Y_2] = 0$$

for any $Y, Y_1, Y_2 \in V$. The Lie algebra $\mathfrak{g}_A$ is an m-step nilpotent Lie algebra.

3. Let K be a field and define a Lie algebra with basis $X_1, \ldots, X_n$ and relations

$$[X_1, X_i] = X_{i+1} \quad \forall\, 2 \leq i \leq n-1$$

where all other brackets of basis vectors are zero. Note that this Lie algebra has a maximal nilpotency index, namely its dimension minus one. Lie algebras with this property are called *filiform Lie algebras.* The above Lie algebra is called the *Standard filiform Lie algebra.* Note that it is also isomorphic to the Lie algebra $\mathfrak{g}_A$ from the previous example, where A has matrix representation

$$\begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}.$$

We say a Lie group is nilpotent if it is so as an abstract group. The following theorem gives a connection between nilpotent Lie groups and nilpotent Lie algebras.

Theorem 2.3.20. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. If G is nilpotent (as an abstract group), then $\mathfrak{g}$ is nilpotent. If G is connected and $\mathfrak{g}$ is nilpotent, then G is nilpotent. Moreover, if G is simply-connected, we have for any $i > 0$ that*

$$\gamma_i(G) = \exp(\gamma_i(\mathfrak{g})).$$

Proof. See for instance [Bou72]. □

2.3.3 Cocompact lattices in nilpotent Lie groups

Let G be a Lie group. We say that a (group theoretic) subgroup of G is a *cocompact lattice*, if it is a discrete subset with respect to the topology on G and if the coset space G/Γ is cocompact for the quotient topology. Note that the latter condition is equivalent to the space of right cosets $\Gamma\backslash G$ being compact. This in turn agrees with the definition of a cocompact actions (Definition 2.2.6 (ii)) if we consider the action of Γ on G by left-translation.

Not every Lie group admits a cocompact lattice. Thus, the question arises: which Lie groups admit a cocompact lattice? And how does the set of cocompact lattices look like? In the class of nilpotent Lie groups, there is a nice theory on cocompact lattices of which we state the main results in this section. All results can be attributed to the work of Mal'cev in [Mal49a] and [Mal49b].

First, let us say something about what type of groups can occur as a cocompact lattice in a simply connected nilpotent Lie group.

Theorem 2.3.21. *Every cocompact lattice in a simply connected nilpotent Lie group is a torsion-free finitely generated nilpotent group and any torsion-free finitely generated nilpotent group occurs as a cocompact lattice in some simply connected nilpotent Lie group.*

Proof. We leave the first statement as an exercise to the reader. For the second statement, recall that in Section 2.3.1 we constructed the so called real Mal'cev completion of a torsion-free finitely generated group. This is a simply connected nilpotent Lie group and contains the original group as a cocompact lattice. □

Next, we relate cocompact lattices in simply connected nilpotent Lie groups to certain objects in the associated Lie algebra. Let N be a simply connected nilpotent Lie group and write $\mathfrak{n}$ for the real Lie algebra associated to N. It is well-known that in this setting, $\exp : \mathfrak{n} \to N$ is a diffeomorphism. We write $\log : N \to \mathfrak{n}$ for its inverse. Let Γ be a cocompact lattice of N and define the set

$$\mathfrak{n}_\Gamma^{\mathbb{Q}} = \operatorname{span}_{\mathbb{Q}} (\log(\Gamma)) .$$

One can show that this subset of $\mathfrak{n}$ satisfies the following definition.

Definition 2.3.22. Let $\mathfrak{g}$ be a real Lie algebra. A subset $\mathfrak{h} \subset \mathfrak{g}$ is said to be a *rational form* of $\mathfrak{g}$ if

(i) $\mathfrak{h}$ is closed under taking $\mathbb{Q}$-linear combinations,

(ii) $\mathfrak{h}$ is closed under the Lie bracket of $\mathfrak{g}$ and

(iii) $\mathfrak{h}$ (as a vector space over $\mathbb{Q}$) has a basis, which is also a basis for $\mathfrak{g}$ (as a vector space over $\mathbb{R}$).

In Section 5.2 of Chapter 5, we give a more general definition of (rational) forms of Lie algebras. Note that a rational form has itself the structure of a Lie algebra defined over the field $\mathbb{Q}$.

We say that two rational forms $\mathfrak{h}_1$, $\mathfrak{h}_2 \subset \mathfrak{g}$ of a real Lie algebra $\mathfrak{g}$ are *isomorphic* if they are isomorphic as Lie algebras over $\mathbb{Q}$. This is equivalent with the existence of an automorphism $\varphi \in \operatorname{Aut}(\mathfrak{g})$ such that $\varphi(\mathfrak{h}_1) = \mathfrak{h}_2$.

Example 2.3.23. 1. Consider the real Heisenberg Lie algebra $\mathfrak{h}_3$ with basis X, Y, Z and defining relations $[X, Z] = [Y, Z] = 0$ and $[X, Y] = Z$. Up to isomorphism, this Lie algebra has only one rational form, given by

$$\operatorname{span}_{\mathbb{Q}}\{X, Y, Z\}.$$

2. Consider the direct sum of two real Heisenberg Lie algebras with bases X_1, Y_1, Z_1 for the first copy and X_2, Y_2, Z_2 for the second copy. Up to isomorphism, this Lie algebra has infinitely many rational forms. For any integer $k \geq 1$, we have a rational form $\mathfrak{n}_k^{\mathbb{Q}}$ which is the $\mathbb{Q}$-span of the vectors

$$\overline{X_1} = X_1 + X_2, \qquad \overline{Y_1} = Y_1 + Y_2, \qquad \overline{Z_1} = Z_1 + Z_2,$$

$$\overline{X_2} = \sqrt{k}(X_1 - X_2), \quad \overline{Y_2} = \sqrt{k}(Y_1 - Y_2), \quad \overline{Z_2} = \sqrt{k}(Z_1 - Z_2).$$

Two such rational forms $\mathfrak{n}_k^{\mathbb{Q}}$ and $\mathfrak{n}_{k'}^{\mathbb{Q}}$ are isomorphic if and only if there exists an integer $l > 0$ such that $k = l^2 k'$. As rational Lie algebras, one can calculate the non-zero bracket relations of $\mathfrak{n}_k^{\mathbb{Q}}$ as

$$[\overline{X_1}, \overline{Y_1}] = \overline{Z_1}, \qquad [\overline{X_1}, \overline{Y_2}] = \overline{Z_2},$$

$$[\overline{X_2}, \overline{Y_1}] = \overline{Z_2}, \qquad [\overline{X_2}, \overline{Y_2}] = k\overline{Z_1}.$$

We refer the reader to [Lau08] for a proof that these are indeed all rational forms up to isomorphism. This follows as well from a more general statement that we will prove in Chapter 5, see Example 5.6.5.

Definition 2.3.24. Two groups G and H are said to be *abstractly commensurable* if there exist finite index subgroups $G_1 \leq G$ and $H_1 \leq H$ such that G_1 is isomorphic to H_1.

The following theorem establishes a correspondence between cocompact lattices and rational forms, given a simply connected nilpotent Lie group.

Theorem 2.3.25. *Let N be a simply connected nilpotent Lie group with associated real Lie algebra $\mathfrak{n}$. The assignment $\Gamma \mapsto \mathfrak{n}_\Gamma^{\mathbb{Q}}$ gives a one-to-one correspondence between the cocompact lattices in N up to abstract commensurability and the rational forms of $\mathfrak{n}$ up to isomorphism.*

In particular, we have that a simply connected nilpotent Lie group admits a cocompact lattice if and only if its associated real Lie algebra admits a rational form. This is again equivalent with the existence of a basis with respect to which all structural constants of the Lie algebra are rational numbers. We recall that the *structural constants* of a Lie algebra $\mathfrak{g}$ with respect to a basis $X_1, \ldots, X_n$ for $\mathfrak{g}$ are the numbers $c_{ij}^k \in \mathbb{R}$, $1 \leq i, j, k \leq n$ such that

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k X_k$$

for all $i, j \in \{1, \ldots, n\}$.

2.3.4 Almost-crystallographic and almost-Bieberbach groups

In our discussion of flat manifolds, we saw how Bieberbach groups played an essential role in classifying them. In what follows, we will generalize the notion of Bieberbach groups by considering a more general ambient group of isometries. Indeed, we will replace our Euclidean space $\mathbb{R}^n$, which is an abelian Lie group, with a nilpotent Lie group.

Let us fix a simply connected nilpotent Lie group N. For any choice of inner product $\langle \cdot, \cdot \rangle$ on the tangent space at the identity $e \in N$, we define a closed subgroup of $\mathrm{Aut}(N)$ by

$$\mathrm{Aut}\left(N, \langle \cdot, \cdot \rangle\right) = \left\{\varphi \in \mathrm{Aut}(N) \mid \forall v, w \in T_e N : \langle d\varphi(v), d\varphi(w) \rangle = \langle v, w \rangle \right\}.$$

Note that these subgroups of $\mathrm{Aut}(N)$ are compact and that any compact subgroup of $\mathrm{Aut}(N)$ is contained in such a subgroup. Using the natural action of $\mathrm{Aut}(N, \langle \cdot, \cdot \rangle)$ on N, one defines the semi-direct product

$$\mathrm{Iso}(N, \langle \cdot, \cdot \rangle) = N \rtimes \mathrm{Aut}(N, \langle \cdot, \cdot \rangle).$$

Note that this is a Lie group as well and, in particular, it is a topological group. It fits in the short exact sequence:

$$1 \longrightarrow N \xrightarrow{t} \mathrm{Iso}(N, \langle \cdot, \cdot \rangle) \xrightarrow{r} \mathrm{Aut}(N, \langle \cdot, \cdot \rangle) \longrightarrow 1.$$

Definition 2.3.26. Let N be a simply connected Lie group and $\langle \cdot, \cdot \rangle$ an inner product on the tangent space at the identity of N. A subgroup Γ of $N \rtimes \mathrm{Aut}(N, \langle \cdot, \cdot \rangle)$ is called an *almost-crystallographic group (modelled on N)* if it is discrete and cocompact. If in addition Γ is torsion-free, we call Γ an *almost-Bieberbach group (modelled on N).*

The *dimension* of an almost-crystallographic or almost-Bieberbach group is defined as the dimension of the Lie group N.

Note that the almost-crystallographic and almost-Bieberbach groups modelled on the abelian Lie group $\mathbb{R}^n$ are exactly the crystallographic and Bieberbach groups of dimension n, respectively. Therefore the almost-crystallographic and almost-Bieberbach groups are the natural generalization from the abelian setting to the nilpotent setting.

Example 2.3.27. Consider the real Heisenberg group $H(\mathbb{R})$ which is a simply connected nilpotent Lie group. Define the automorphism $\varphi \in \mathrm{Aut}(H(\mathbb{R}))$ by

$$\varphi\left(\begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & -y & -z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}.$$

Let X, Y, Z be the basis for the Lie algebra of $H(\mathbb{R})$ as described in Example 2.3.17 and let $\langle \cdot, \cdot \rangle$ be the inner product on $\mathfrak{h}_3$ which makes X, Y, Z an orthonormal basis. As one can check, $\mathrm{d}\varphi X = X$, $\mathrm{d}\varphi Y = -Y$, $\mathrm{d}\varphi Z = -Z$ and thus $\varphi \in \mathrm{Aut}(H(\mathbb{R}), \langle \cdot, \cdot \rangle)$. Next, define the element

$$\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} \in H(\mathbb{R}),$$

and the subgroup of $\mathrm{Iso}(H(\mathbb{R}), \langle \cdot, \cdot \rangle)$:

$$\Gamma = \Big\langle H(\mathbb{Z}),\ (\alpha, \varphi) \Big\rangle.$$

where we used the embedding $t : H(\mathbb{Z}) \to \mathrm{Iso}(H(\mathbb{R}), \langle \cdot, \cdot \rangle) : x \mapsto (x, \mathrm{Id})$. As one can check, the group Γ is an almost-Bieberbach group modelled on $H(\mathbb{R})$.

The notation of the group $\mathrm{Iso}(N, \langle \cdot, \cdot \rangle)$ is no coincidence. This group acts smoothly on N by the law:

$$(n, \varphi) \cdot x = n\varphi(x).$$

This action induces a map $\mathrm{Iso}(N, \langle \cdot, \cdot \rangle) \to \mathrm{Diff}(N)$ of which the image is exactly equal to the isometry group of the Riemannian manifold one obtains by left-translating the inner product $\langle \cdot, \cdot \rangle$ on the Lie group N. The almost-crystallographic and almost-Bieberbach groups can now also be characterized in terms of their actions on the Lie group N.

Proposition 2.3.28. *Let* Γ *be a subgroup of* $\mathrm{Iso}(N, \langle \cdot, \cdot \rangle)$*, then* *(i)* Γ *is a discrete subgroup of* $\mathrm{Iso}(N, \langle\cdot,\cdot\rangle)$ *if and only if* Γ *acts properly discontinuously on* N*,*

(ii) Γ *is a cocompact subgroup of* $\mathrm{Iso}(N, \langle\cdot,\cdot\rangle)$ *if and only if* Γ *acts cocompactly on* N*,*

(iii) if Γ *is a discrete subgroup, it is torsion-free if and only if* Γ *acts freely on* N*.*

The generalized Bieberbach theorems. As with crystallographic or Bieberbach groups, there are theorems describing the structure of almost-crystallographic or almost-Bieberbach groups. They are referred to as the generalized Bieberbach theorems. The first one was proven by Auslander.

Theorem 2.3.29 ([Aus60])**.** *Let* $\Gamma \subseteq \mathrm{Iso}(N, \langle\cdot,\cdot\rangle)$ *be an almost-crystallographic group of dimension* n*. Then* $r(\Gamma)$ *is a finite group and* $\Gamma \cap t(N)$ *is a cocompact lattice of* N*.*

For any almost-crystallographic group $\Gamma \leq \mathrm{Iso}(N, \langle\cdot,\cdot\rangle)$, let us call $r(\Gamma)$ the *holonomy group* of Γ and $\Gamma \cap t(N)$ the *group of pure translations* of Γ.

A generalization of the second Bieberbach theorem was proven by Lee and Raymond.

Theorem 2.3.30 ([LR85])**.** *Let* $\Gamma, \Gamma' \subseteq \mathrm{Iso}(N)$ *be almost-crystallographic groups. Let* $f : \Gamma \to \Gamma'$ *be a group isomorphism. Then there exists an element* $\alpha \in \mathrm{Aff}(N) = N \rtimes \mathrm{Aut}(N)$ *such that for all* $\gamma \in \Gamma$ *it holds that* $f(\gamma) = \alpha\gamma\alpha^{-1}$*. In other words, every isomorphism between almost-crystallographic groups can be realized as an affine transformation of coordinates on* N*.*

2.3.5 Infra-nilmanifolds

Definition 2.3.31. Let N be a simply connected nilpotent Lie group. An *infra-nilmanifold (modelled on* N*)* is a quotient manifold $\Gamma\backslash N$ where Γ is an almost Bieberbach group. In case $r(\Gamma) = 1$, the manifold $\Gamma\backslash N$ is called a *Nilmanifold.*

Example 2.3.32 (The Heisenberg nilmanifold)**.** Consider the nilpotent Lie group $H(\mathbb{R})$ and its cocompact lattice $H(\mathbb{Z})$. The Heisenberg nilmanifold is the quotient

$$M = H(\mathbb{Z})\backslash H(\mathbb{R}).$$

Let us identify $H(\mathbb{R})$ with $\mathbb{R}^3$ by

$$(x,y,z) = \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \tag{2.4}$$

Every element of $H(\mathbb{Z})\backslash H(\mathbb{R})$ can be uniquely represented by an element of the form

$$H(\mathbb{Z}) \cdot (x,y,z) \quad \text{with} \quad 0 < x,y,z < 1.$$

One also has the identities

$$H(\mathbb{Z}) \cdot (0,y,z) = H(\mathbb{Z}) \cdot (1,y,z)$$

$$H(\mathbb{Z}) \cdot (x,0,z) = H(\mathbb{Z}) \cdot (x,1,x+z)$$

$$H(\mathbb{Z}) \cdot (x,y,0) = H(\mathbb{Z}) \cdot (x,y,1).$$

Topologically, we can thus construct $H(\mathbb{Z})\backslash H(\mathbb{R})$ by considering the closed unit cube and identifying points on opposite sides according to certain maps. This is illustrated in Figure 2.3 below. In fact, the nilmanifold $M = H(\mathbb{Z})\backslash H(\mathbb{R})$ is homeomorphic to a mapping torus

$$M \cong \frac{[0,1] \times T^2}{(0,x) \sim (1,f(x))},$$

where $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ is the two-torus and $f : T^2 \to T^2$ is the homeomorphism induced by the matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

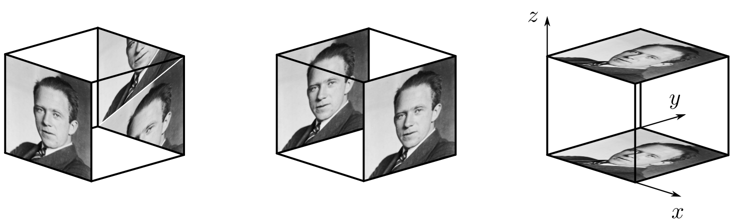


Figure 2.3: Representation of the nilmanifold $H(\mathbb{Z})\backslash H(\mathbb{R})$ as the closed unit cube with identifications between opposite sides. Image credit: Bundesarchiv, Bild 183-R57262, under licence: CC-BY-SA 3.0.

Example 2.3.33. Consider the almost-Bieberbach group $\Gamma \leq \mathrm{Iso}(H(\mathbb{R}), \langle\cdot,\cdot\rangle)$ from Example 2.3.27 and the associated infra-nilmanifold

$$M = \Gamma\backslash H(\mathbb{R}).$$

We use the identification of $H(\mathbb{R})$ with $\mathbb{R}^3$ from (2.4). Every element of $\Gamma\backslash H(\mathbb{R})$ can be uniquely represented by an element of the form

$$\Gamma \cdot (x, y, z) \quad \text{with} \quad 0 < x, z < \frac{1}{2}, \quad 0 < y < 1.$$

One also has the identities

$$\Gamma \cdot (0, y, z) = \Gamma \cdot \left(\frac{1}{2}, -y, -z\right)$$

$$\Gamma \cdot (x, 0, z) = \Gamma \cdot (x, 1, x + z)$$

$$\Gamma \cdot (x, y, 0) = \Gamma \cdot \left(x, y, \frac{1}{2}\right).$$

Topologically, we can thus construct $\Gamma\backslash H(\mathbb{R})$ by considering the closed unit cube (after rescaling the x and z directions) and identifying points on opposite sides according to certain maps. This is illustrated in Figure 2.3 below.

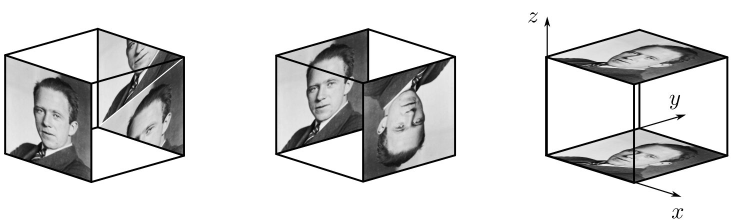


Figure 2.4: Representation of the infra-nilmanifold $\Gamma\backslash H(\mathbb{R})$ as the closed unit cube with identifications between opposite sides. Image credit: Bundesarchiv, Bild 183-R57262, under licence: CC-BY-SA 3.0.

2.3.6 Algebraic characterization of almost-flat manifolds

At last, we arrive at the celebrated Gromov-Ruh theorem, which characterizes almost-flat manifolds in an algebraic way.

Theorem 2.3.34 ([Gro78], [Ruh82])**.** *A compact manifold M is almost flat if and only if it is diffeomorphic to an infra-nilmanifold. In particular, the fundamental group of an almost-flat manifold is an almost-Bieberbach group.*

In the following paragraph, we sketch the proof of one direction of the equivalence in the above theorem, namely, we construct on any infra-nilmanifold an ϵ-flat metric for ϵ arbitrarily small. The discussion is based on section 7.7 in [Gro78].

An ϵ-flat metric on an infra-nilmanifold. First, let us recall the definition of a left-invariant metric on a Lie group.

Let G be a Lie group and $\mathfrak{g} = T_1 G$ its Lie algebra. Let $L_x : G \to G : y \mapsto xy$ denote the diffeomorphism on G given by left multiplication by $x \in G$. For any inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ one can define a metric g on G by

$$g_x(v, w) = \langle dL_x^{-1} v, dLx^{-1} w \rangle$$

for any $x \in G$ and $v, w \in T_x G$. This metric is called the *left-invariant metric on G associated to $\langle \cdot, \cdot \rangle$*.

Since the metric g is fully determined by the inner product on the Lie algebra, it is no surprise that the curvatures of (G, g) can be calculated from the inner product $\langle \cdot, \cdot \rangle$ and the Lie bracket $[\cdot, \cdot]$ on $\mathfrak{g}$. In particular, the curvature can be bounded by the norm of the Lie bracket.

For any finite dimensional vector space V endowed with an inner product $\langle \cdot, \cdot \rangle$ and a multilinear map $T : V^n \to V$ we define the norm of T as

$$\|T\| := \max\{\|T(v_1, \ldots, v_n)\| \mid v_i \in V, \|v_i\| = 1\}.$$

Note that if $u_1, \ldots, u_n$ is an orthonormal basis for $V, \langle \cdot, \cdot \rangle$, then we can also calculate $\|T\|$ as

$$\|T\| = \max\{\|T(u_{i_1}, \ldots, u_{i_n})\| \mid i_1, \ldots, i_n \in \{1, \ldots, n\}\}. \tag{2.5}$$

Proposition 2.3.35. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, $\langle \cdot, \cdot \rangle$ an inner product on $\mathfrak{g}$ and g the associated left-invariant metric on G. Let $R : \mathfrak{g}^3 \to \mathfrak{g}$ be the curvature tensor of g, evaluated at the identity of G. Then R is bounded by the Lie bracket by*

$$\|R\| \leq 6 \| \, [\cdot, \cdot] \, \|.$$

Proof. See [BK81, Proposition 7.7.1]. □

As a consequence we can also bound the sectional curvature by the Lie bracket. Take any plane $\pi \subset T_1 G$ and let $u, v \in T_1 G$ be an orthonormal basis for π. Then we have

$$\begin{aligned} |K(\pi)| &= |\langle R(u, v, v), u \rangle| \\ &\leq \|R(u, v, v)\| \|u\| \\ &\leq \|R\| \|u\|^2 \|v\|^2 \\ &= \|R\| \leq 6 \| \, [\cdot, \cdot] \, \|. \end{aligned}$$

Proposition 2.3.36. *Let N be a connected nilpotent Lie group with Lie algebra $\mathfrak{n}$ endowed with an inner product $\langle \cdot, \cdot \rangle$. There exists a sequence of inner products $\langle \cdot, \cdot \rangle_k$, $k \in \mathbb{N}_0$ on $\mathfrak{n}$ with corresponding Riemann curvature tensors R_k at the identity of N, such that*

(i) there exists a $C > 0$ and $k_0 \in \mathbb{N}$ with $\|R_k\|_k \leq C$ for all $k \geq k_0$,

(ii) for any $v \in \mathfrak{n}$ it holds that $\lim\limits_{k \to \infty} \|v\|_k = 0$ and

(iii) $\operatorname{Aut}(N, \langle \cdot, \cdot \rangle) = \operatorname{Aut}(N, \langle \cdot, \cdot \rangle_k)$.

Proof. Let c be the nilpotency class of $\mathfrak{n}$. For any $1 \leq i \leq c$, write $\mathfrak{n}_i$ for the orhogonal complement of $\gamma_{i+1}(\mathfrak{n})$ inside $\gamma_i(\mathfrak{n})$ (with respect to the given inner product $\langle \cdot, \cdot \rangle$). This gives an orthogonal decomposition of $\mathfrak{n}$:

$$\mathfrak{n} = \mathfrak{n}_1 \overset{\perp}{\oplus} \mathfrak{n}_2 \overset{\perp}{\oplus} \ldots \overset{\perp}{\oplus} \mathfrak{n}_c.$$

For any $k \in \mathbb{N}_0$, define the positive real numbers $a_1(k), \ldots, a_c(k)$ backwards inductively by

$$a_c(k) := \frac{1}{k}, \qquad a_i(k) := \sqrt[4]{\sum_{j=i+1}^{c} a_j(k)^2}.$$

We define the new inner product $\langle \cdot, \cdot \rangle_k$ by scaling the inner product $\langle \cdot, \cdot \rangle$ on each component $\mathfrak{n}_i$ according to the numbers $a_i(k)$. Thus, we have for any $1 \leq i, j \leq c$ and any $X \in \mathfrak{n}_i, Y \in \mathfrak{n}_j$ that

$$\langle X, Y \rangle_k = \begin{cases} a_i(k)\langle X, Y \rangle & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Note that any automorphism of $\mathfrak{n}$ preserves the ideals in the lower central series. It follow that any element of $\operatorname{Aut}(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ must preserve the subspaces $\mathfrak{n}_i$. The same is true for the elements of $\operatorname{Aut}(\mathfrak{n}, \langle \cdot, \cdot \rangle_k)$. Since on each $\mathfrak{n}_i$, the inner product only differs up to scaling, it is not hard to see that then $\operatorname{Aut}(\mathfrak{n}, \langle \cdot, \cdot \rangle) = \operatorname{Aut}(\mathfrak{n}, \langle \cdot, \cdot \rangle_k)$ and as a consequence also $\operatorname{Aut}(N, \langle \cdot, \cdot \rangle) = \operatorname{Aut}(N, \langle \cdot, \cdot \rangle_k)$. This proves part (iii) of the statement.

Next, note that $a_c(k) \to 0$ as $k \to \infty$. By induction on i, one can check that $a_i(k) \to 0$ as $k \to \infty$ for any $i \in \{1, \ldots, c\}$. As a consequence, we have that $\lim_{k \to \infty} \|v\|_k = 0$ for any $v \in \mathfrak{n}$. This proves part (ii) of the statement. As another consequence, there is a $k_0 \in \mathbb{N}_0$ such that $a_i(k) < 1$ for all $k \geq k_0$ and

$i \in \{1, \ldots, c\}$. For such k we thus have that

$$\begin{aligned} a_i(k) &= \sqrt[4]{\sum_{j=i+1}^{c} a_j(k)^2} \\ &\geq \sqrt[4]{a_{i+1}(k)^2} \\ &= \sqrt{a_{i+1}(k)} \geq a_{i+1}(k), \end{aligned}$$

which gives $a_1(k) \geq \ldots \geq a_c(k)$.

Finally, we prove part (i) of the statement. Let us take a $k \geq k_0$. Take any $1 \leq i \leq j \leq c$ and $X \in \mathfrak{n}_i, Y \in \mathfrak{n}_j$. Note that there exist unique vectors $X_{j+1} \in \mathfrak{n}_{j+1}, \ldots, X_c \in \mathfrak{n}_c$ such that

$$[X, Y] = X_{j+1} + \ldots + X_c.$$

Let us write $B = \|[\cdot,\cdot]\|$ and note that

$$B^2\|X\|^2\|Y\|^2 \geq \|[X,Y]\|^2 = \|X_{j+1}\|^2 + \ldots + \|X_c\|^2.$$

In particular, we also have for any $j+1 \leq l \leq c$ that $\|X_l\|^2 \leq B^2\|X\|^2\|Y\|^2$. Using this, we find that

$$\begin{aligned} \|[X,Y]\|_k^2 &= \|X_{j+1}\|_k^2 + \ldots \|X_c\|_k^2 \\ &= a_{j+1}(k)^2\|X_{j+1}\|^2 + \ldots + a_c(k)^2\|X_c\|^2 \\ &\leq \left(a_{j+1}(k)^2 + \ldots + a_c(k)^2\right) B^2\|X\|^2\|Y\|^2 \\ &= a_j(k)^4 B^2 \frac{\|X\|_k^2}{a_i(k)^2}\frac{\|Y\|_k^2}{a_j(k)^2} \\ &\leq B^2\|X\|_k^2\|Y\|_k^2 \end{aligned}$$

Thus, we find that $\|[\cdot,\cdot]\|_k \leq B$ for any $k \geq k_0$. Using proposition 2.3.35 we can conclude that

$$\|R_k\|_k \leq 6\|[\cdot,\cdot]\|_k \leq 6B =: C.$$

□

Proposition 2.3.37. *Every infra-nilmanifold is almost-flat.*

Proof. Let N be a simply connected nilpotent Lie group, $\mathfrak{n}$ its Lie algebra and $\langle\cdot,\cdot\rangle$ an inner product on $\mathfrak{n}$. Let $\Gamma \leq \mathrm{Iso}(N, \langle\cdot,\cdot\rangle)$ be an almost-Bieberbach group.

We have to prove that $M := \Gamma\backslash N$ is almost flat. Let $\langle\cdot,\cdot\rangle_k$ be the sequence of inner products on $\mathfrak{n}$ constructed in Proposition 2.3.36. From property (iii), it follows that $\mathrm{Iso}(N, \langle\cdot,\cdot\rangle_k) = \mathrm{Iso}(N, \langle\cdot,\cdot\rangle)$ and thus that Γ is also a subgroup of $\mathrm{Iso}(N, \langle\cdot,\cdot\rangle_k)$. As a consequence each $\langle\cdot,\cdot\rangle_k$ gives rise to a left-invariant metric on N which descends to a well-defined metric g_k on the quotient $\Gamma\backslash N$. Next, define the the numbers

$$A_k = \max\{\|v\|_k \mid v \in \mathfrak{n}, \|v\|_1 = 1\}.$$

Using property (ii) and that $\mathfrak{n}$ is finite dimensional, it is an easy exercise to check that $\lim_{k\to\infty} A_k = 0$. Using that the metrics g_k are left-invariant, we find that their associated distance functions satisfy $d_k(x,y) \leq A_k d_1(x,y)$ for all $k \in \mathbb{N}$. As a consequence we find that for the associated diameters, it holds that $d(M, g_k) \leq A_k d(M, g_1)$ and thus that $\lim_{k\to\infty} d(M, g_k) = 0$. If we write K_k for the sectional curvature of g_k, we find that $K_k(\pi) \leq \|R_k\|_k \leq C$ for all $k \geq k_0$, $p \in M$ and any plane $\pi \subset T_pM$. We thus find that

$$\lim_{k\to\infty} |K_k(\pi)| \cdot d(M, g_k) = 0,$$

which proves that the sequence of metrics g_k become ϵ-flat for ϵ arbitrarily small and thus that M is an almost-flat manifold. □

Chapter 3

Anosov diffeomorphisms

In this chapter, we will define the protagonist of this thesis: the Anosov diffeomorphism. Since Anosov diffeomorphisms have very interesting dynamical properties, we first give a short introduction on dynamical systems which is based on the books [HK03] [KH95]. All manifolds in this chapter will be assumed to be connected unless stated otherwise.

3.1 Dynamical systems and chaos

Let M be a differentiable manifold and $f : M \to M$ a diffeomorphism. Together, these two objects give an action of the integers on M by

$$\forall k \in \mathbb{Z}, \forall p \in M : \quad k \cdot p = f^k(p).$$

In this section we will study the dynamical properties of actions obtained in this way.

Definition 3.1.1. A *(time-Discrete differentiable) Dynamical System* (DDS for short) is an action of $\mathbb{Z}, +$ on a differentiable manifold M by diffeomorphisms.

Note that any DDS also has a defining diffeomorphism $f : M \to M : p \mapsto 1 \cdot p$. We will often write down the DDS as a tuple (M, f). The dynamical properties of a DDS will be studied from a topological point of view. This means that we will forget about the differentiable structure on the manifold when looking at the dynamics.

Definition 3.1.2. Let (M, f) be a DDS. A point $p \in M$ is said to:

(i) be *fixed* if $f(p) = p$. The set of all fixed points in M is written as $\mathrm{Fix}(f)$.

(ii) be *periodic* if there exists an integer $n > 0$ such that $f^n(p) = p$. The minimal such integer n is called the *period* of p. The set of all periodic points in M is written as $\mathrm{Per}(f)$.

(iii) have a *dense forward orbit* if the set $\{f^k(x) \mid k \in \mathbb{N}\}$ is dense in M.

(iv) be *wandering* if there exists a neighbourhood U of p such that

$$U \cap \bigcup_{n=1}^{\infty} f^n(U) = \emptyset.$$

A point is said to be *non-wandering* if it is not wandering. The set of all non-wandering points in M is written as $\mathrm{NW}(f)$.

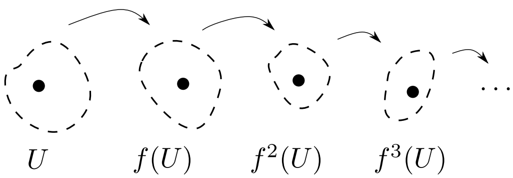


Figure 3.1: Illustration of a wandering point.

Remark 3.1.3. Note that one has the inclusions

$$\mathrm{Fix}(f) \subseteq \mathrm{Per}(f) \subseteq \mathrm{NW}(f)$$

and that each of these sets is f-invariant. Additionally, the set $\mathrm{NW}(f)$ is closed in M.

Example 3.1.4. 1. Consider the circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ and the diffeomorphism

$$f_\theta : S^1 \to S^1 : z \mapsto e^{i\theta} z$$

which rotates the circle through an angle θ. The dynamics of (S^1, f_θ) depend on the angle θ. If $\theta/2\pi$ is a rational number, then all points are periodic with the same period. In particular, all points are non-wandering. If $\theta/2\pi$ is an irrational number, then no point is periodic, but still all points are non-wandering. In fact, every point has a dense forward orbit in S^1.

2. Consider the sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + (z-1)^2 = 1\}$. Let us call $N = (0, 0, 2)$ the 'north pole' and $S = (0, 0, 0)$ the 'south pole'. Using the stereographic projection:

$$\pi : S^2 \setminus \{N\} \to \mathbb{R}^2 : (x, y, z) \mapsto \left(\frac{2x}{2-z}, \frac{2y}{2-z} \right),$$

one can identify $S^2 \setminus \{N\}$ with $\mathbb{R}^2$.

One then defines the diffeomorphism f on S^2 by

$$f(p) = \begin{cases} \pi^{-1}(2\pi(p)) & \text{if } p \neq N \\ N & \text{else,} \end{cases}$$

as illustrated in Figure 3.2. All points of S^2 are then wandering except for the north pole and south pole, so

$$\mathrm{NW}(f) = \mathrm{Fix}(f) = \{S, N\}.$$

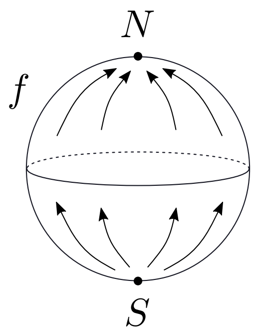


Figure 3.2: A diffeomorphism on the sphere with a lot of wandering points.

Other (topological) dynamical properties of a dynamical system can be formulated as follows.

Definition 3.1.5. A DDS (M, f) is called:

1. *topologically transitive* if for any two open subsets $U, V \subseteq M$, there exists a non-negative integer $k \geq 0$ such that $f^k(U) \cap V \neq \emptyset$.
2. *topologically mixing* if for any two open subsets $U, V \subseteq M$, there exists a non-negative integer $N \geq 0$ such that for all $k > N$ it holds that $f^k(U) \cap V \neq \emptyset$.

Remark 3.1.6. Clearly, if a system is topologically mixing, it must also be topologically transitive. The other implication is not true. An example of a system that illustrates this is the rotation of the circle from Example 3.1.4 with θ an irrational multiple of 2π. Such a system is topologically transitive, but not topologically mixing.

Lemma 3.1.7. *Let M be a compact and connected differentiable manifold. A DDS (M, f) is topologically transitive if and only if there exists a point $p \in M$ with a dense forward orbit.*

The following property of a DDS is not purely topological but requires a choice of metric on the space compatible with the topology.

Definition 3.1.8. Let (M, f) be a DDS and d a metric on M which induces its topology. The system (M, f) is said to be *sensitive to initial conditions with respect to the metric d* if there exists a constant $\delta > 0$ such that for any $\epsilon > 0$ and any $x \in M$, there exists a $y \in M$ with $0 < d(x, y) < \epsilon$ and a positive integer k such that $d(f^k(x), f^k(y)) \geq \delta$.

However, if M is compact, the definition is independent from the chosen metric and thus becomes a topological property. This is due to the following lemma.

Lemma 3.1.9. *Let X be a compact topological space and d, d' two metrics on X which induce its topology. For any $\delta > 0$, there exists a $\delta' > 0$ such that for any points $x, y \in X$ with $d(x, y) \geq \delta$ it holds that $d'(x, y) \geq \delta'$.*

Proof. Consider the set $K = \{(x, y) \in X \times X \mid d(x, y) \geq \delta\}$. It is closed in the compact space $X \times X$ and thus compact itself. Since $d' : X \times X \to \mathbb{R} : (x, y) \mapsto d'(x, y)$ is continuous, it follows that $\delta' = \min d'(K)$ exists. Since the diagonal of $X \times X$ does not intersect K it follows that $d'(K)$ does not contain 0 and thus that $\delta' > 0$. This concludes the proof. □

Corollary 3.1.10. *Let (M, f) be a DDS with M compact. If (M, f) is sensitive to initial conditions with respect to some metric d which induces the topology on M, then it is sensitive to initial conditions with respect to any metric on M which induces its topology.*

Definition 3.1.11. A DDS (M, f) is said to be *chaotic* if

(i) it is topologically transitive,

(ii) it is sensitive to initial conditions,

(iii) $\mathrm{Per}(f)$ is dense in M.

The following is a well-known example of a chaotic DDS.

Example 3.1.12 (Arnold's cat map)**.** Consider the linear map

$$g : \mathbb{R}^2 \to \mathbb{R}^2 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{3.1}$$

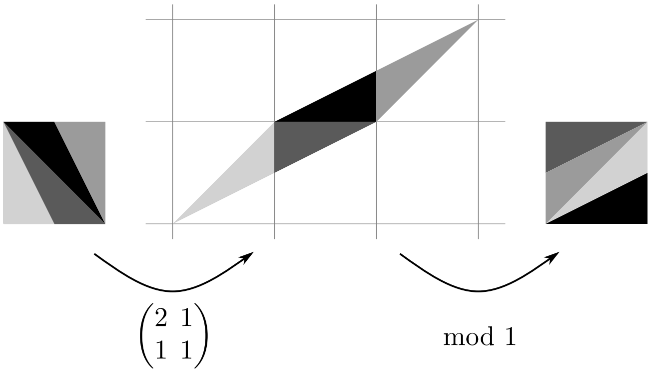


Figure 3.3: An illustration of Arnold's cat map. On the unit square, the map can be seen as first applying a matrix and subsequently reducing each coordinate modulo 1.

Since its matrix has integer coefficients, it follows that $g(\mathbb{Z}^2) \subseteq \mathbb{Z}^2$. Moreover, since the determinant is equal to 1, its inverse satisfies the same property and thus we get that $g(\mathbb{Z}) = \mathbb{Z}$. Therefore one gets an induced bijective map on the torus

$$f : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2 : v + \mathbb{Z}^2 \mapsto g(v) + \mathbb{Z}^2.$$

As one can check, f is a diffeomorphism. If we consider the torus as the closed unit square with opposite sides identified, Figure 3.3 illustrates how the map f works.

Proposition 3.1.13. *Arnold's cat map is chaotic.*

Proof. We only argue why the periodic points are dense in $\mathbb{R}^2/\mathbb{Z}^2$. For any positive integer $n > 0$, consider set

$$Q_n = \left(\frac{1}{n}\mathbb{Z}^2\right) \Big/ \mathbb{Z}^2$$

as a subset of $\mathbb{R}^2/\mathbb{Z}^2$. Note that Q_n is finite and that $f(Q_n) = Q_n$. As a consequence, f restricted to Q_n must have finite order and thus all points in Q_n are periodic points of f. The union

$$\bigcup_{n>0} Q_n = \mathbb{Q}^2/\mathbb{Z}^2$$

thus gives a set of periodic points of f which is dense in $\mathbb{R}^2/\mathbb{Z}^2$. □

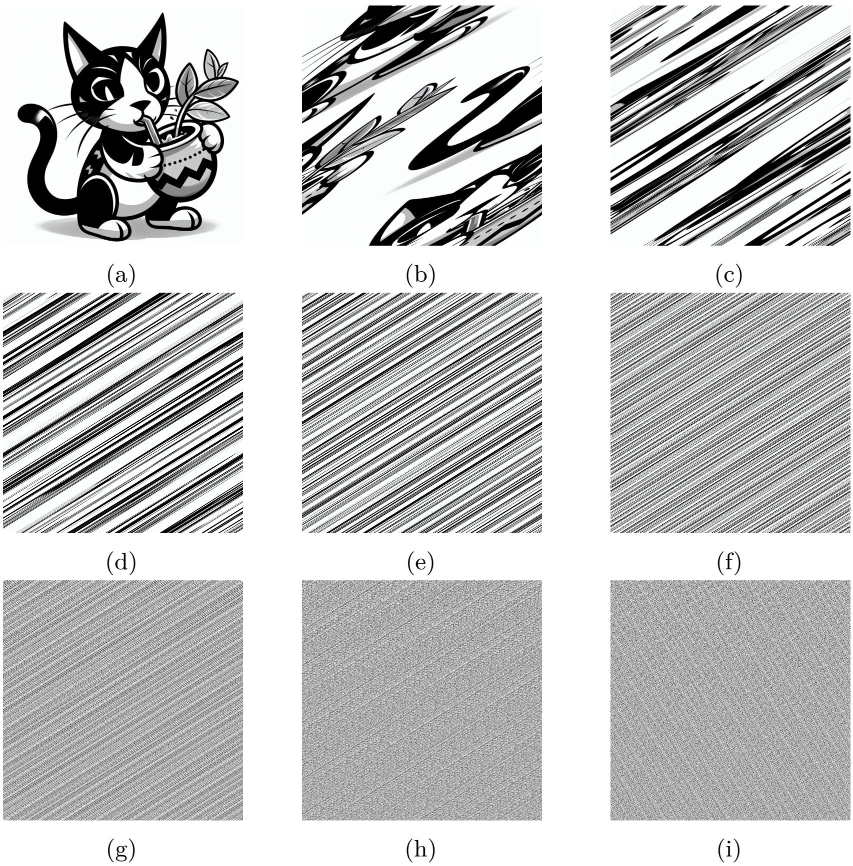


Figure 3.4: Arnold's cat map being applied iteratively to an image showing its chaotic behaviour. The image was generated by dall-e with input: 'A cat drinking mate'.

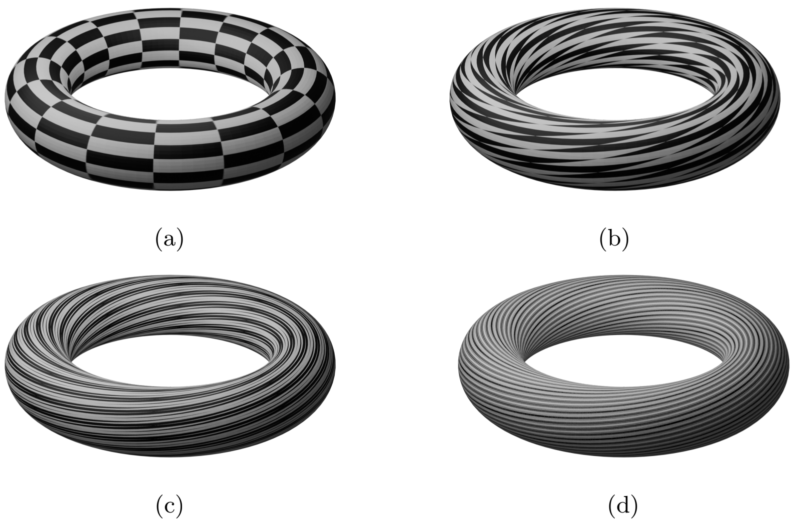


Figure 3.5: Illustration of Arnold's cat map being applied iteratively to a chequerboard pattern on a torus embedded in three dimensional space.

We can ask ourselves the question: 'When do two DDSs exhibit the same topological dynamics?'.

Definition 3.1.14. Two DDSs (M, f) and (N, g) are said to be *topologically conjugate* if there exists a homeomorphism $h : M \to N$ such that $h \circ f = g \circ h$.

Let (M, f) and (N, g) be two topologically conjugate DDSs and let $h : M \to N$ be the homeomorphism realizing the topological conjugation. The following can be said about the dynamical properties of both systems:

- $h(\text{Fix}(f)) = \text{Fix}(g)$,
- $h(\text{Per}(f)) = \text{Per}(g)$,
- $h(\text{NW}(f)) = \text{NW}(g)$,
- (M, f) is topologically transitive if and only if (N, g) is topologically transitive,
- (M, f) is topologically mixing if and only if (N, g) is topologically mixing,
- (M, f) is chaotic if and only if (N, g) is chaotic.

At last, we arrive at the notion of *structural stability*. Intuitively, a structurally stable DDS is one for which the dynamics of the system do not change under small perturbations of the system. To make the notion of small perturbations concrete, we equip $\mathrm{Diff}(M)$ with the *Whitney C^1-topology*.

Definition 3.1.15. A DDS (M, f) is called *structurally stable* if there exists a neighbourhood U of p in the C^1-topology of $\mathrm{Diff}(M)$, such that for any $g \in U$ it holds that f and g are topologically conjugate.

Example 3.1.16. Consider the rotation of the circle f_θ from Example 3.1.4 for some angle $\theta \in \mathbb{R}$. The system (S^1, f_θ) is not structurally stable. In any neighbourhood of f_θ there will be also other rational and irrational rotations of the circle. Since rational and irrational rotations of the circle can not be topologically conjugate (they exhibit different dynamics), it follows that (S^1, f_θ) can not be structurally stable.

Time-continuous dynamics

In this section, we describe the relation between time-discrete dynamics, which we discussed in previous section, and a specific class of time-continuous dynamical systems. The discussion is based on [Sma63].

Definition 3.1.17. A *time-Continuous differentiable Dynamical System* (CDS for short) is an action of $\mathbb{R}, +$ on a differentiable manifold M such that its defining map $\Phi : \mathbb{R} \times M \to M : (t, p) \mapsto t \cdot p$ is differentiable. We write the CDS as the tuple (M, Φ).

Any DDS gives rise to a CDS on a mapping torus of which we now give the construction. Let (M, f) be a DDS. Note that $\mathbb{Z}, +$ acts on the product $\mathbb{R} \times M$ by

$$k \cdot (s, p) = (s + k, f^k(p)).$$

The mapping torus M_f associated to (M, f) is then given by the orbit space of this action

$$M_f = \mathbb{Z} \backslash (\mathbb{R} \times M).$$

Let us write the elements of M_f as $\overline{(s, p)} = \mathbb{Z} \cdot (s, p)$. One defines an action of $\mathbb{R}, +$ on M_f by

$$t \cdot \overline{(s, p)} = \overline{(s + t, p)}.$$

As one can check, the defining map $\Phi : \mathbb{R} \times M_f \to M_f$ for this action is differentiable, thus turning M_f into a CDS. This CDS is called the *Poincaré suspension*.

On the other hand, there is also a way to go from a CDS to a DDS, which is in some sense an inverse to the construction of the Poincaré suspension. However, some conditions have to be satisfied on the CDS. Note that every CDS (M, Φ) naturally has an associated smooth vectorfield $X \in \mathfrak{X}(M)$, defined by

$$X(p) = \frac{d}{dt}\bigg|_0 \Phi(t, p).$$

We say a submanifold $N \subset M$ is *transversal to* Φ if $X(p) \notin T_pN$ for all $p \in M$.

Definition 3.1.18. Let (M, Φ) be a CDS. A submanifold $N \subset M$ is called a *global Poincaré section* if

(i) it has codimension one and is closed in M,

(ii) it is transversal to Φ,

(iii) $\forall p \in N : \exists t > 0 : \Phi(t, p) \in N$,

(iv) $\forall p \in N : \exists t < 0 : \Phi(t, p) \in N$,

(v) every orbit in M intersects N.

Remark 3.1.19. If M is assumed to be compact and connected, conditions (iv) and (v) of Definition 3.1.18 follow automatically from the others.

Consider a CDS (M, Φ) with a global Poincaré section $N \subset M$. One can associate to this a map $\tau : N \to \mathbb{R}$ by letting $\tau(p)$ be the minimal positive real number $\tau(p) > 0$ such that $\Phi(\tau(p), p) \in N$. This gives rise to another map

$$f : N \to N : p \mapsto \Phi(\tau(p), p),$$

which is called the *Poincaré map* associated to the global Poincaré section N and is illustrated in Figure 3.6.

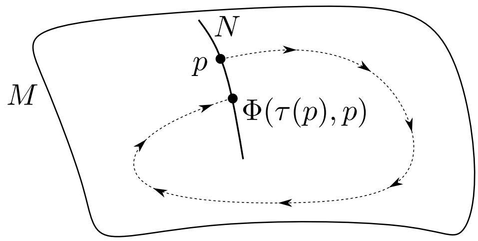


Figure 3.6: Illustration of a global Poincaré section and the associated Poincaré map.

It is not hard to show that this Poincaré map is a diffeomorphism. Therefore one obtains a DDS (N, f). This association is a reduction in the following sense: many of the dynamical properties of the CDS (M, Φ) are characterized by a dynamical property of the associated DDS (N, f). We illustrate this with an example.

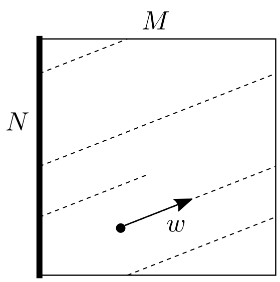
Example 3.1.20. Let M be the torus $\mathbb{R}^2/\mathbb{Z}^2$ and consider the action

$$\Phi : \mathbb{R} \times M \to M : (t, v + \mathbb{Z}^2) \mapsto v + tw + \mathbb{Z}^2$$

for some fixed vector $w = (a, b) \in \mathbb{R}^2$ with $a \neq 0$. A global Poincaré section for (M, Φ) can be given by the submanifold

$$N = \{(0, y) + \mathbb{Z}^2 \mid y \in \mathbb{R}\}$$

which is diffeomorphic to the circle $S^1 = \mathbb{R}/\mathbb{Z}$ by $S^1 \to N : y + \mathbb{Z} \mapsto (0, y) + \mathbb{Z}^2$.



The map $\tau : N \to \mathbb{R}$ is constant and takes the value $1/a$. The corresponding Poincaré map is given by

$$f : S^1 \to S^1 : y + \mathbb{Z} \mapsto y + \frac{b}{a} + \mathbb{Z}.$$

This is a rotation of the circle as discussed in Example 3.1.4. If b/a is rational, all points of (S^1, f) are periodic and as a result, all points of (M, Ψ) will be periodic as well. (For a CDS a point $p \in M$ is called periodic if there exists $t > 0$ such that $t \cdot p = p$.) If b/a is irrational, every point in (S^1, f) has a dense forward orbit and as a result, all points of (M, Ψ) will have a dense forward orbit as well. (For a CDS a point $p \in M$ is said to have a dense forward orbit if the set $\{t \cdot p \mid t > 0\}$ is dense in M.)

The following definition on CDSs is the analogue of topological conjugacy for DDSs.

Definition 3.1.21. Two CDSs (M, Φ) and (N, Ψ) are said to be *topologically equivalent* if there exists a homeomorphism $h : M \to N$ which maps orbits of (M, Φ) to orbits of (N, Ψ), i.e.

$$\forall p \in M : h(\{\Phi(t,p) \mid t \in \mathbb{R}\}) = \{\Psi(t, h(p)) \mid t \in \mathbb{R}\}$$

and such that the orientation of the orbits is preserved.

At last, we give the following lemma which says that from a topological point of view, the Poincaré suspension and the Poincaré map are inverse operations.

Lemma 3.1.22. *Let (M, Φ) be a CDS with a global Poincaré section $N \subset M$ and let $f : N \to N$ denote the Poincaré map. Then (M, Φ) is topologically equivalent to the Poincaré suspension $(N_f, \tilde{\Phi})$ under the homeomorphism*

$$h : N_f \to M : \overline{(s,p)} \mapsto \Phi(s\tau(p), p) \quad 0 \leq s \leq 1.$$

3.2 Anosov diffeomorphisms

In Example 3.1.12 we gave a chaotic DDS called Arnold's cat map. This example came in the form of a map f on the torus $\mathbb{R}^2/\mathbb{Z}^2$, induced by the linear map $g : \mathbb{R}^2 \to \mathbb{R}^2$ given by equation (3.1). Note that the tangent space at each point of the torus $\mathbb{R}^2/\mathbb{Z}^2$ can be naturally identified with $\mathbb{R}^2$. It is clear that, under this identification, for any point p, the derivative $d_pf : \mathbb{R}^2 \to \mathbb{R}^2$ is equal to the linear map g. Note that this linear map has two real eigenvalues, $\lambda = \dfrac{3+\sqrt{5}}{2}$ and $\lambda^{-1} = \dfrac{3-\sqrt{5}}{2}$, which satisfy $\lambda > 1$ and $0 < \lambda^{-1} < 1$. The corresponding eigenvectors are

$$v = \begin{pmatrix} 1+\sqrt{5} \\ 2 \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} 1-\sqrt{5} \\ 2 \end{pmatrix},$$

respectively. If we apply df consecutively to these eigenvectors v_p, w_p (now as tangent vectors at some point p on the torus), we see how v_p grows exponentially while w_p shrinks exponentially (with respect to some norm on $\mathbb{R}^2$). This kind of global hyperbolic behaviour turns out to give very interesting dynamics in general and will be exactly the defining property of an Anosov diffeomorphism.

Before we state the definition of an Anosov diffeomorphism, we need the notion of a *continuous splitting of the tangent bundle.* Let M be a differentiable manifold of dimension n with tangent bundle TM. A continuous splitting of the tangent bundle is for each $p \in M$ a choice of vector subspaces $E_p, F_p \subset T_pM$

such that $T_pM = E_p \oplus F_p$ and such that for each $p \in M$, there exists a neighbourhood $U \subset M$ of p and *continuous* vector fields $X_1, \ldots, X_n$ on U such that for each $q \in U$ it holds that $E_q = \operatorname{span}\{X_1(q), \ldots, X_m(q)\}$ and $F_q = \operatorname{span}\{X_{m+1}(q), \ldots, X_n(q)\}$ for a certain $m \in \{1, \ldots, n\}$. Note that we do not ask these vector fields to be differentiable. As one can show, the unions $E = \bigcup_{p \in M} E_p$ and $F = \bigcup_{p \in M} F_p$ together with their natural projection maps $\pi : E \to M$ and $\pi : F \to M$ carry the structure of a *(topological) vector bundle* over M. Notationally, we write the continuous splitting of TM as $TM = E \oplus F$.

If $f : M \to M$ is a diffeomorphism, we say that a continuous splitting $TM = E \oplus F$ is *f-invariant* if for any $p \in M$ it holds that $df(E_p) = E_{f(p)}$ and $df(F_p) = F_{f(p)}$.

Definition 3.2.1. Let (M, g) be a Riemannian manifold. A diffeomorphism $f : M \to M$ is said to be an *Anosov diffeomorphism* if there exists a df-invariant continuous splitting of the tangent bundle $TM = E^u \oplus E^s$ and constants $c > 0$, $\lambda > 1$ such that

$$\forall v \in E^u : \ \forall k \in \mathbb{N} : \|df^k v\| \geq c\lambda^k \|v\|$$

$$\forall v \in E^s : \ \forall k \in \mathbb{N} : \|df^k v\| \leq \frac{1}{c\lambda^k} \|v\|.$$

Note that as of now, the definition of an Anosov diffeomorphism depends on the chosen metric on M. For compact manifolds, however, the choice of metric will not be important due to the following lemma.

Lemma 3.2.2. *Let M be a compact manifold and g_1, g_2 two Riemannian metrics. There exists a real constant $A > 0$ such that for each $v \in TM$:*

$$\frac{1}{A} g_1(v, v) \leq g_2(v, v) \leq A g_1(v, v).$$

Proof. Consider the subset

$$X = \{w \in TM \mid g_1(w, w) = 1\} \subset TM$$

which is a compact subset since M is compact. Then define the real numbers

$$A_1 = \min_{w \in X} g_2(w, w) \quad \text{and} \quad A_2 = \max_{w \in X} g_2(w, w)$$

which exist since X is compact. Clearly, A_1 is non-zero since every $w \in X$ is non-zero. Thus, there exists a real number $A > 0$ such that $\frac{1}{A} \leq A_1 \leq A_2 \leq A$. Next, for any $v \in TM$, we can write $v = \sqrt{g_1(v, v)} \cdot w$ for some $w \in X$. For w, we have

$$\frac{1}{A} \leq g_2(w, w) \leq A.$$

Multiplying the inequalities with $g_1(v, v)$, we find the desired inequality. □

Using this lemma, it is then not hard to show the following:

Corollary 3.2.3. *Let M be a compact manifold. If a diffeomorphism is Anosov with respect to some Riemannian metric on M, then it is Anosov with respect to any Riemannian metric on M.*

For an Anosov diffeomorphism $f : M \to M$ on a compact manifold, define for each $p \in M$ the following subsets:

$$W^s(p) = \left\{ q \in M \;\middle|\; \lim_{k\to\infty} d(f^k(p), f^k(q)) = 0 \right\},$$

$$W^u(p) = \left\{ q \in M \;\middle|\; \lim_{k\to\infty} d(f^{-k}(p), f^{-k}(q)) = 0 \right\}$$

where d is the metric induced by some Riemannian metric on M (it does not matter which one since M is compact). These subsets are called the *stable and unstable manifolds*, respectively. The following theorem is known as the *stable manifold theorem* and is here stated in the specific case of an Anosov diffeomorphism. It justifies why we call these subsets manifolds.

Theorem 3.2.4. *Let $f : M \to M$ be an Anosov diffeomorphism on a compact manifold M. For any $p \in M$, the subsets $W^s(p)$ and $W^u(p)$ are C^∞ immersed submanifolds with*

$$T_x W^s(p) = E^s_x \quad \text{and} \quad T_y W^u(p) = E^u_y$$

for any $x \in W^s(p)$ and $y \in W^u(p)$.

Note that the collections $\{W^s(p) \mid p \in M\}$ and $\{W^u(p) \mid p \in M\}$ are partitions of the manifold M.

Definition 3.2.5. Let M be an n-dimensional manifold. A *C^k-foliation with C^l-leaves* is a collection $\mathcal{F}$ of C^l immersed connected m-dimensional submanifolds (called the *leaves*), such that $\mathcal{F}$ is a partition of M and such that for any $p \in M$, there is a C^k map

$$\varphi : U \to \mathbb{R}^n : q \mapsto (\varphi_1(q), \dots, \varphi_n(q))$$

from a neighbourhood U of p such that every leaf of $\mathcal{F}$ intersects U either in the empty set or in a countable union of subsets of the form

$$\{q \in U \mid \varphi_{m+1}(q) = c_{m+1}, \dots, \varphi_n(q) = c_n\}$$

with $c_{m+1}, \dots, c_n \in \mathbb{R}$.

Theorem 3.2.6. *Let M be a compact manifold and $f : M \to M$ an Anosov diffeomorphism. The collection of stable manifolds and the collection of unstable manifolds both give a C^0-foliation of M with C^∞ leaves.*

It is from these foliations and the fact that they give rise to a 'local product structure' (see [Sma67, Theorem 7.4]), that the following dynamical property (see Definition 3.1.8) of Anosov diffeomophisms can be proven.

Theorem 3.2.7. *An Anosov diffeomorphism on a compact manifold is sensitive to initial conditions.*

The following theorem was proven by John Franks.

Theorem 3.2.8 ([Fra70])**.** *For any Anosov diffeomorphism f on a compact manifold M it holds that* $\mathrm{Per}(f)$ *is dense in* $\mathrm{NW}(f)$.

It leaves open the following question.

Question 3.2.9. Is the non-wandering set of an Anosov diffeomorphism on a compact manifold equal to the whole manifold?

A related unanswered question is the following.

Question 3.2.10. Does every Anosov diffeomorphism on a compact manifold have a fixed point?

Recall Definition 3.1.15. The last theorem of this section is the celebrated result of Dmitri Anosov and is the reason why the diffeomorphisms carry his name.

Theorem 3.2.11 ([Ano69])**.** *Anosov diffeomorphisms are structurally stable. Moreover, the set of Anosov diffeomorphisms is an open subset of* $\mathrm{Diff}(M)$.

3.3 Anosov flows

In this section, we introduce the time-continuous analogue of an Anosov diffeomorphism. They are related to one another by the Poincaré suspension. The discussion is based on [AS67].

We first introduce a CDS which one can naturally associate to any Riemannian manifold (M, g). The *unit tangent bundle* is the subset $UTM \subset TM$ defined by

$$UTM = \{v \in TM \mid g(v, v) = 1\}.$$

It is an embedded submanifold of TM. Note that there is a canonical choice of Riemannian metric on TM called the *Sasaki metric* (see [Sas58] and [Sas62]). We thus also have a natural induced Riemannian metric on UTM.

A smooth curve $\gamma : [a, b] \to M$ in a Riemannian manifold (M, g) is said to be a *geodesic* if it satisfies the equation $\nabla_{\gamma'}\gamma' = 0$ on the interval $[a, b]$. In the particular case of a compact manifold M, the Hopf-Rinow theorem tells us that for any $v \in TM$, there exists a unique geodesic $\gamma_v : \mathbb{R} \to M$ such that $\gamma_v'(0) = v$. Since the velocity of geodesics is constant, we also have that $\|\gamma_v'(t)\| = \|v\|$ for all $t \in \mathbb{R}$.

Thus, if (M, g) is a compact Riemannian manifold, we obtain a canonical CDS (UTM, Φ) called *the geodesic flow*, defined by

$$\Phi : \mathbb{R} \times UTM \to UTM : (t, v) \mapsto \gamma_v'(t).$$

As it turns out, if (M, g) is of strictly negative sectional curvature, this CDS satisfies some remarkable properties with respect to the Sasaki metric on UTM. Let us formulate these properties for an arbitrary CDS with a compact underlying manifold. For a CDS (M, Φ), let us write Φ_t for the diffeomorphism $\Phi_t : M \to M : p \mapsto \Phi(t, p)$.

Definition 3.3.1. Let (M, Φ) be a CDS on a compact Riemannian manifold (M, g). We say that (M, Φ) is an *Anosov flow* if there exists a continuous splitting of the tangent bundle $TM = E^0 \oplus E^u \oplus E^s$ and constants $a > 0, c > 0$ such that:

(i) the vector field $X \in \mathfrak{X}(M)$ defined by $X(p) = \left.\frac{d}{dt}\right|_0 \Phi(t, p)$ is non-vanishing,

(ii) for all $t \in \mathbb{R}$ the splitting $TM = E^0 \oplus E^u \oplus E^s$ is $d\Phi_t$-invariant,

(iii) at each $p \in M$, the vector space E^0_p is spanned by $X(p)$ and

$$\forall v \in E^u : t \geq 0 : \; \|d\Phi_t(v)\| \geq ce^{at}\|v\|,$$

$$\forall v \in E^u : t \geq 0 : \; \|d\Phi_t(v)\| \leq \frac{1}{ce^{at}}\|v\|.$$

Remark 3.3.2. Since the manifold is assumed to be compact, the metric is not of importance in the definition of an Anosov flow, similar to the case of Anosov diffeomorphisms. When going to another Riemannian metric, only the constant c might change.

There is also a relation between Anosov flows and Anosov diffeomorphisms by use of a Poincaré section.

Proposition 3.3.3. *Let (M, Φ) be a CDS with a compact global Poincaré section $N \subset M$ and associated Poincaré map $f : N \to N$. Then (M, Φ) is an Anosov flow if and only if f is an Anosov diffeomorphism. In particular, for any Anosov diffeomorphism, the associated Poincaré suspension is an Anosov flow.*

The following result is proven in section 3 of [AS67]. It gives us a large class of Anosov flows and thus, by the above proposition, a natural occurence of Anosov diffeomorphisms when a Poincaré section is present.

Theorem 3.3.4. *Let (M, g) be a compact Riemannian manifold of strictly negative sectional curvature. The geodesic flow on M is an Anosov flow.*

Let us illustrate this with an example.

Example 3.3.5 (The hyperbolic plane)**.** Consider the hyperbolic plane $\mathbb{H}^2$ as introduced in Example 2.2.2 in Chapter 2. The geodesics in $\mathbb{H}^2$ are either contained in circles with center on the line $y = 0$ or are contained in vertical lines. Note that $\mathbb{H}^2$ is not compact, but we can make it compact by modding out by a suitable cocompact discrete subgroup of isometries (see *non-Euclidean crystallographic groups* and *Fushian groups* [Lyn74]). Since $\mathbb{H}$ has constant strictly negative sectional curvature, its geodesic flow is Anosov. Let us illustrate, by use of figures, how the splitting $T(UT\mathbb{H}^2) = E^0 \oplus E^u \oplus E^s$ looks like. Since tangent vectors in $UT\mathbb{H}^2$ are difficult to draw, we will give typical curves in $UT\mathbb{H}^2$ which are tangent to one of the bundles E^0, E^u or E^s. Note that a curve in $UT\mathbb{H}^2$ is a parametrization of unit tangent vectors in $\mathbb{H}^2$.

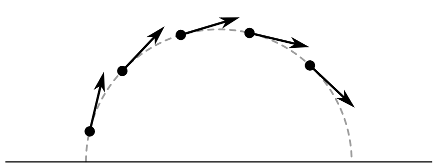


Figure 3.7: A typical curve in $UT\mathbb{H}^2$ tangent to E^0. The curve is drawn as a sequence of tangent vectors. The dashed line represents a geodesic in $\mathbb{H}^2$.

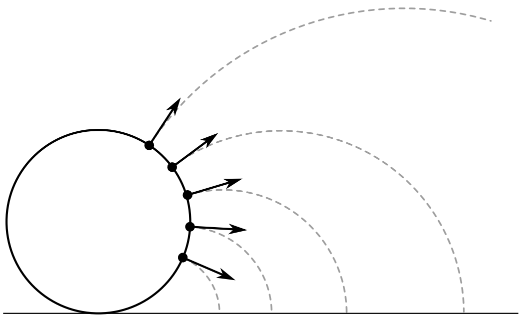


Figure 3.8: A typical curve in $UT\mathbb{H}^2$ tangent to E^u. The curve is drawn as a sequence of tangent vectors. The dashed lines represent geodesics in $\mathbb{H}^2$.

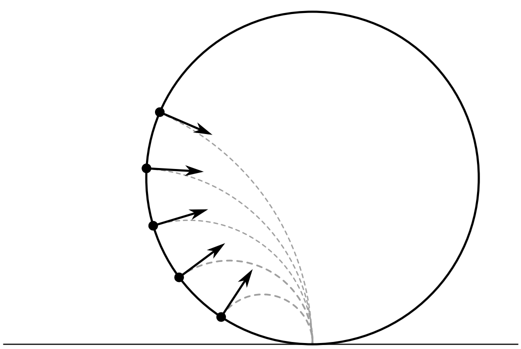


Figure 3.9: A typical curve in $UT\mathbb{H}^2$ tangent to E^s. The curve is drawn as a sequence of tangent vectors. The dashed lines represent geodesics in $\mathbb{H}^2$.

3.4 Closed manifolds admitting an Anosov diffeomorphism

This section gives an introductory discussion on the following unsolved problem in mathematics, which is also the title of this thesis.

Question 3.4.1. Which closed manifolds admit an Anosov diffeomorphism?

From here on all manifolds on which we consider an Anosov diffeomorphism are assumed to be compact. An immediate consequence of the manifold being compact is the following:

Lemma 3.4.2. *Let $f : M \to M$ be an Anosov diffeomorphism on a compact manifold M with associated continuous splitting $TM = E^u \oplus E^s$. It holds that* $\dim E^u_p > 0$ *and* $\dim E^s_p > 0$ *and thus that* $\dim M \geq 2$.

Proof. If this were not the case, then, without loss of generality, we can assume $TM = E^u$ (otherwise, consider the Anosov diffeomorphism f^{-1}). Let g be a Riemannian metric on M and $c > 0, \lambda > 1$ the constants for which f is Anosov with respect to g. Since f^k is a diffeomorphism, it induces for any $p, q \in M$ a bijection from the set of smooth curves from p to q to the set of smooth curves from $f^k(p)$ to $f^k(q)$. Under this bijection, it maps a curve of length a to a curve of length at least $c\lambda^k a$. By taking infimums of the lengths of curves in both sets, we find that

$$d(f^k(p), f^k(q)) \geq c\lambda^k d(p, q)$$

for any $p, q \in M$. Thus, this implies that distances between points on M can become arbitrarily large. This is of course impossible since a compact Riemannian manifold has finite diameter. □

As a consequence, we see that the circle S^1 does not admit an Anosov diffeomorphism. In general, the existence of Anosov diffeomorphisms is rather rare. There are several non-existence results in the literature. We list some of them here.

- Any differentiable manifold with virtually polycyclic fundamental group, universal cover with finite dimensional rational homology and first integral cohomology isomorphic to $\mathbb{Z}$, does not admit an Anosov diffeomorphism [Hir71]. In particular, the mapping torus of a hyperbolic toral automorphism does not admit an Anosov diffeomorphism.
- Every m-dimensional differentiable manifold of which the homology with coefficients is $\mathbb{Q}$ is isomorphic to that of the m-dimensional sphere does not

admit an Anosov diffeomorphism [Shi73]. In particular, all spheres, real projective spaces and lens spaces do not admit an Anosov diffeomorphism. For the latter two, one uses that the lift of an Anosov diffeomorphism to the universal cover is also an Anosov diffeomorphism.

- Closed negatively curved manifolds do not admit Anosov diffeomorphisms [Yan83] [GL16, Section 4].
- Certain products of manifolds with spheres do not admit Anosov diffeomorphisms [GRH14]. In particular, if M is a product of spheres such that there exists an odd k with S^k appearing exactly once in the product, then M does not admit an Anosov diffeomorphism. Another application is that the product of a torus with an odd dimensional sphere of dimension at least 3, does not admit an Anosov diffeomorphism.
- Certain products of infra-nilmanifolds with aspherical manifolds do not admit Anosov diffeomorphisms [GL16].

On the other hand, from Example 3.1.12, we know that the torus $\mathbb{R}^2/\mathbb{Z}^2$ does admit an Anosov diffeomorphism, namely Arnold's cat map. Recall that this map was induced by a matrix of which the eigenvalues were not of absolute value one. In the following section we discus some elementary result on these kind of linear maps. In Section 3.4.2, we will generalize the example of Arnold's cat map to give the most general known way of constructing Anosov diffeomorphisms on compact manifolds.

3.4.1 Hyperbolic linear maps

In this section we give some elementary results on invertible linear maps which do not have eigenvalues of modulus one. These results are then used in the next section to construct Anosov diffeomorphisms on compact manifolds.

Definition 3.4.3. Let V be a finite dimensional vector space over a subfield of the complex numbers $\mathbb{C}$. A linear map $A : V \to V$ is called *hyperbolic* if it is invertible and all of its eigenvalues have modulus different from 1.

A hyperbolic linear map on a real vector space imposes a canonical decomposition of this vector space.

Lemma 3.4.4. *If V is a real finite dimensional vector space equipped with a norm $\| \cdot \|$ and $A \in \mathrm{GL}(V)$ is a hyperbolic linear map, then there exists an*

A-invariant decomposition $V = V^u \oplus V^s$ and constants $c > 0, \lambda > 1$ such that

$$\forall v \in V^u : \ \forall k \in \mathbb{N} : \|A^k v\| \geq c\lambda^k \|v\|$$

$$\forall v \in V^s : \ \forall k \in \mathbb{N} : \|A^k v\| \leq \frac{1}{c\lambda^k} \|v\|.$$

Moreover, the subspaces V^u and V^s are unique and can alternatively be characterized as

$$V^u = \left\{ v \in V \,\middle|\, \lim_{k\to\infty} A^{-k} v = 0 \right\},$$

$$V^s = \left\{ v \in V \,\middle|\, \lim_{k\to\infty} A^{k} v = 0 \right\}.$$

Proof. See for example Proposition 1.2.8 (p.24) of [KH95]. □

For a hyperbolic linear map $A : V \to V$ the above decomposition can be constructed as follows. Let us write $V^{\mathbb{C}}$ for the complexification $V \otimes_{\mathbb{R}} \mathbb{C}$ of the real vector space V and note that A naturally extends to an invertible linear map on $V^{\mathbb{C}}$. Let us write for any $\alpha \in \mathbb{C}$ its associated generalized eigenspace of A as

$$V_\alpha = \left\{ v \in V^{\mathbb{C}} \,\middle|\, \exists k \in \mathbb{N} : (A - \alpha \operatorname{Id})^k v = 0 \right\}.$$

This gives a direct sum decomposition

$$V^{\mathbb{C}} = \bigoplus_{\alpha \in \operatorname{Spec}(A)} V_\alpha$$

where $\operatorname{Spec}(A)$ is the set of eigenvalues of A. Now define

$$\tilde{V^u} = \bigoplus_{|\alpha|>1} V_\alpha \quad \text{and} \quad \tilde{V^s} = \bigoplus_{|\alpha|<1} V_\alpha.$$

Then since A is hyperbolic, it follows that $V^{\mathbb{C}} = \tilde{V^u} \oplus \tilde{V^s}$. Moreover, since A was originally defined over the real vector space V and complex conjugation does not change the modulus of a complex number, it follows that the real vector spaces

$$V^u = V \cap \tilde{V^u} \qquad V^s = V \cap \tilde{V^s}$$

have the same dimension as the complex vector spaces $\tilde{V^u}$ and $\tilde{V^s}$, respectively. It follows that we have a direct sum decomposition $V = V^u \oplus V^s$, which, as one can check, agrees with the direct sum decomposition from the lemma above.

Lemma 3.4.5. *Let V be a real finite dimensional vector space equipped with a norm $\|\cdot\|$. Consider a hyperbolic linear map $A \in \mathrm{GL}(V)$ and a subgroup $H \leq \mathrm{GL}(V)$ such that $\forall h \in H, v \in V : \|hv\| = \|v\|$ and $AHA^{-1} = H$. If $V = V^u \oplus V^s$ is the unique decomposition from the above lemma w.r.t. A, then $h(V^u) = V^u$ and $h(V^s) = V^s$ for any $h \in H$.*

Proof. Take any $v \in V^s$, $k \in \mathbb{N}$ and $h \in H$. Note that there exists a $\tilde{h} \in H$ such that $A^k h = \tilde{h} A^k$. As a consequence, we find that

$$\|A^k hv\| = \|\tilde{h} A^k v\| = \|A^k v\|$$

and thus that $\lim_{k\to\infty} \|A^k hv\| = 0$. This implies that $hv \in V^s$. By applying the same argument to h^{-1}, we find that $h(V^s) = V^s$. Analogously, we find that $h(V^u) = V^u$. $\square$

3.4.2 Constructing Anosov diffeomorphisms

In what follows, we give the most general known way of constructing Anosov diffeomorphisms on compact manifolds.

Take any connected and simply connected Lie group G together with a choice of inner product $\langle \cdot, \cdot \rangle$ on its Lie algebra $\mathfrak{g} = T_e G$ at the identity $e \in G$. Let g denote the associated left-invariant Riemannian metric on G. Let Γ be a subgroup of $G \rtimes \mathrm{Aut}(G, \langle \cdot, \cdot \rangle)$ where $\mathrm{Aut}(G, \langle \cdot, \cdot \rangle)$ are those automorphisms of G which preserve the inner product at the identity of G (and are therefore also isometries of (G, g)). We also require the action of Γ on G to be cocompact, properly discontinuous and free, such that the quotient $\Gamma \backslash G$ is a compact manifold. Note that since every element of Γ acts on G by isometries with respect to g, there is also an induced Riemannian metric on $\Gamma \backslash G$.

Next, consider an affine map $f = (b, \varphi) \in G \rtimes \mathrm{Aut}(G)$ such that:

(i) $d_e \varphi : \mathfrak{g} \to \mathfrak{g}$ is hyperbolic,

(ii) $(b, \varphi)\Gamma(b, \varphi)^{-1} = \Gamma$.

Due to condition (ii), f induces a diffeomorphism

$$\overline{f} : \Gamma \backslash G \to \Gamma \backslash G : \Gamma g \mapsto \Gamma b \varphi(g).$$

Lemma 3.4.6. *The map $\overline{f}$ is an Anosov diffeomorphism.*

Proof. Let $\mathfrak{g} = V^u \oplus V^s$ be the canonical decomposition from Lemma 3.4.4. For any $g \in G$, write $L_g : G \to G : x \mapsto gx$ for the left translation on G by g. Define the following subspaces of T_gG:

$$E^u_g = (d_e L_g)(V^u) \quad \text{and} \quad E^s_g = (d_e L_g)(V^s).$$

As one can check, this gives a continuous splitting of the tangent bundle $TG = E^u \oplus E^s$. Write $\pi : G \rtimes \mathrm{Aut}(G) \to \mathrm{Aut}(G)$ for the projection morphism. From condition (i) it follows that $\varphi\pi(\Gamma)\varphi^{-1} = \pi(\Gamma)$. Therefore Lemma 3.4.5 tells us that

$$\pi(\gamma)(V^s) = V^s \quad \text{and} \quad \pi(\gamma)(V^u) = V^u$$

for any $\gamma \in \Gamma$. It is not hard to show then that the splitting $TG = E^u \oplus E^s$ is γ-invariant for any $\gamma \in \Gamma$. As a consequence, the splitting descends to a continuous splitting on the quotient $\Gamma\backslash G$ which we, with abuse of notation, also write as $T(\Gamma\backslash G) = E^u \oplus E^s$.

One easily verifies that for any $g \in G$ it holds that

$$L^{-1}_{b\varphi(g)} \circ (b, \varphi) \circ L_g = \varphi$$

and thus that we have the commutative diagram:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{dL_g} & T_gG \\ {\scriptstyle d\varphi}\downarrow & & \downarrow{\scriptstyle df} \\ \mathfrak{g} & \xrightarrow[dL_{f(g)}]{} & T_{f(g)}G \end{array} \tag{3.2}$$

As a consequence, the behaviour of the derivative $df : TG \to TG$ is completely determined by $d_e\varphi : \mathfrak{g} \to \mathfrak{g}$ (using left-translation to identify tangent spaces). It is now easily verified that $f : G \to G$ satisfies the definition of an Anosov diffeomorphism with respect to the previously defined continuous splitting $TG = E^u \oplus E^s$ and the constants $c > 0, \lambda > 1$ that we obtained from applying Lemma 3.4.4 to $d_e\varphi$. At last, it follows that the induced map $\overline{f}$ is also an Anosov diffeomorphism for the splitting $T(\Gamma\backslash G) = E^u \oplus E^s$ and the same constants $c > 0, \lambda > 1$. □

Remark 3.4.7. Note that when Γ contains only left-translations (i.e. $\Gamma \subset G$) condition (ii) reduces to $\varphi(\Gamma) = \Gamma$.

Example 3.4.8 (Tori)**.** Let $G = \mathbb{R}^n$ be the simply connected abelian Lie group of dimension n. Consider the cocompact discrete subgroup $\Gamma = \mathbb{Z}^n$. The group of automorphisms of G is equal to $\mathrm{GL}_n(\mathbb{R})$ and the condition $\varphi(\mathbb{Z}^n) = \mathbb{Z}^n$ for any $\varphi \in \mathrm{GL}_n(\mathbb{R})$ is equivalent to $\varphi \in \mathrm{GL}_n(\mathbb{Z})$. Thus, to construct an Anosov diffeomorphism on the n-torus $\mathbb{Z}^n \backslash \mathbb{R}^n$ we need a hyperbolic matrix in $\mathrm{GL}_n(\mathbb{Z})$. Such matrices can be found by, for example, combining the matrices

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Indeed, both these matrices are hyperbolic and in $\mathrm{GL}_n(\mathbb{Z})$ and thus so are the block diagonal matrices

$$\begin{pmatrix} A & & \\ & \ddots & \\ & & A \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} B & & & \\ & A & & \\ & & \ddots & \\ & & & A \end{pmatrix},$$

which give examples in any even and odd dimension ≥ 2, respectively.

Example 3.4.9 (Smale's non-toral example)**.** Consider the direct product of two real Heisenberg groups $H(\mathbb{R}) \times H(\mathbb{R})$. Its Lie algebra is the direct sum of two Heisenberg Lie algebras $\mathfrak{h}_3 \oplus \mathfrak{h}_3$. Let X_1, Y_1, Z_1 be the basis as described in Example 2.3.17 for the first copy of $\mathfrak{h}_3$ and X_2, Y_2, Z_2 this basis for the second copy of $\mathfrak{h}_3$. Let $\Gamma_0 \subset \mathfrak{h}_3 \oplus \mathfrak{h}_3$ be the subset given by the $(2\mathbb{Z})$-span of the vectors

$$\begin{aligned} X_1 + X_2, &\qquad \sqrt{3}(X_1 - X_2), \\ Y_1 + Y_2, &\qquad \sqrt{3}(Y_1 - Y_2), \\ Z_1 + Z_2, &\qquad \sqrt{3}(Z_1 - Z_2). \end{aligned}$$

Let φ_0 be the Lie algebra automorphism on $\mathfrak{h}_3 \oplus \mathfrak{h}_3$ given by

$$\begin{aligned} X_1 \mapsto \zeta X_1 &\qquad X_2 \mapsto \zeta^{-1} X_2 \\ Y_1 \mapsto \zeta^2 Y_1 &\qquad Y_2 \mapsto \zeta^{-2} Y_2 \\ Z_1 \mapsto \zeta^3 Z_1 &\qquad Z_2 \mapsto \zeta^{-3} Z_2 \end{aligned}$$

where $\zeta = 2 + \sqrt{3}$. Next, define $\varphi = \exp(\varphi_0)$ and $\Gamma = \exp(\Gamma_0)$. It is not hard to show that $\varphi_0(\Gamma_0) = \Gamma_0$ and thus that $\varphi(\Gamma) = \Gamma$. Using the BCH-formula, one can check that Γ is a subgroup and thus, using that exp is a diffeomorphism, a cocompact lattice in $H_3(\mathbb{R}) \times H_3(\mathbb{R})$. We also have that $d_e\varphi = \varphi_0$ is hyperbolic. As a consequence, φ induces an Anosov diffeomorphism on the nilmanifold

$$\Gamma \backslash \left(H_3(\mathbb{R}) \times H_3(\mathbb{R}) \right).$$

This example is the first example of an Anosov diffeomorphism not on a torus and was given originally by Stephen Smale in [Sma67].

Note that in the above examples, all Lie groups G are nilpotent. This is no coincidence. As it turns out, condition (i) on page 61 has strong implications on the structure of the underlying Lie algebra. This is one of the results of the work of N. Jacobson on automorphisms and derivations of Lie algebras.

Theorem 3.4.10 ([Jac55])**.** *If a finite dimensional Lie algebra (over any field) admits an automorphism of which all eigenvalues are not roots of unity, then the Lie algebra must be nilpotent.*

In particular, a hyperbolic automorphism does not have eigenvalues which are roots of unity. Thus, in the above construction, this implies that $\mathfrak{g}$ is nilpotent and thus that G is a simply connected nilpotent Lie group, which we now write with N. Consequently, we also have that the group $\Gamma \leq N \rtimes \operatorname{Aut}(N, \langle \cdot, \cdot \rangle)$, which is assumed to act cocompact, properly discontinuously and freely on N, is an almost-Bieberbach group modelled on N. This follows from Proposition 2.3.28. Finally, the manifold $\Gamma \backslash N$ must be an infra-nilmanifold.

Definition 3.4.11. Let $\Gamma \backslash N$ be an infra-nilmanifold and $f = (b, \varphi) \in N \rtimes \operatorname{Aut}(N)$ an affine map such that $f \Gamma f^{-1} = \Gamma$. The induced map $\overline{f} : \Gamma \backslash N \to \Gamma \backslash N : \Gamma x \mapsto \Gamma f(x)$ is called an *affine automorphism* on the infra-nilmanifold $\Gamma \backslash N$. In particular, when b is trivial, we say $\overline{f}$ is an *automorphism* of the infra-nilmanifold $\Gamma \backslash N$. We say $\overline{f}$ is *hyperbolic* if the Lie algebra automorphism associated to φ is hyperbolic.

The above construction only provides examples of Anosov diffeomorphisms which are affine automorphisms on infra-nilmanifolds. So far, no one has been able to construct an example which is not topologically conjugate to one of this kind, leading to the following conjecture which has been open since the '60s.

Conjecture 3.4.12. Every Anosov diffeomorphism on a compact manifold is topologically conjugate to a hyperbolic affine automorphism on an infra-nilmanifold.

In [Man74], it was claimed that the conjecture must be true within the class of infra-nilmanifolds. However, in [Dek12], some problems with the proof were identified. For nilmanifolds, the statement does remain true.

Theorem 3.4.13 ([Man74], [Dek12])**.** *Every Anosov diffeomorphism on a nilmanifold is topologically conjugate to a hyperbolic automorphism on that nilmanifold.*

In the case of infra-nilmanifolds, we can only say something about the existence.

Theorem 3.4.14 ([Dek12]). *Let $\Gamma\backslash N$ be an infra-nilmanifold. The following are equivalent:*

(i) $\Gamma\backslash N$ *admits an Anosov diffeomorphism*

(ii) $\Gamma\backslash N$ *admits a hyperbolic affine automorphism*

(iii) $\Gamma\backslash N$ *admits a hyperbolic automorphism*

The implication (ii) to (iii) can be argued by combining Lemma 3.4.15 and Lemma 3.4.16 from the next section.

3.4.3 Periodic points of hyperbolic affine automorphisms

In Section 3.2 we discussed the dynamics of Anosov diffeomorphisms on closed manifolds and arrived at the open question which asks whether the non-wandering set must be the whole manifold and thus whether the periodic points are dense in the manifold. For those Anosov diffeomorphisms which come in the form of a hyperbolic affine automorphism on an infra-nilmanifold, the question has a positive answer.

First, we give the following elementary result which answers Question 3.2.10 in the case of hyperbolic affine automorphisms on infra-nilmanifolds.

Lemma 3.4.15. *Let N be a simply connected nilpotent Lie group and $(b,\varphi) \in N \rtimes \operatorname{Aut}(N)$. If $d_e\varphi$ does not have eigenvalue 1, then the affine map $f : N \to N : x \mapsto b\varphi(x)$ has a fixed point.*

Proof. We use induction on the nilpotency class of N. For the base case, assume that N is abelian and thus isomorphic to the Lie group $(\mathbb{R}^n, +)$. A point $x \in \mathbb{R}^n$ is then a fixed point of $f = (b, \varphi)$ with $\varphi \in \mathrm{GL}_n(\mathbb{R})$ if and only if

$$\varphi x + b = x \quad \Leftrightarrow \quad b = (\operatorname{Id} - \varphi)x.$$

Since φ does not have eigenvalue 1, it follows that $\operatorname{Id} - \varphi$ is invertible and thus that $x = (\operatorname{Id} - \varphi)^{-1} b$ is a fixed point.

For the general case, let N be of nilpotency class c and assume the lemma holds for lower nilpotency classes. One has an induced affine map on $N/\gamma_c(N)$:

$$x\gamma_c(N) \mapsto b\varphi(x)\gamma_c(N).$$

Since $N/\gamma_c(N)$ has nilpotency class $c-1$, we can use the induction hypothesis and find an $x_0 \in N$ such that

$$x_0\gamma_c(N) = b\varphi(x_0)\gamma_c(N).$$

Thus, there exists a $y \in \gamma_c(N)$ such that $b\varphi(x_0) = x_0 y$. Let $\psi : \gamma_c(N) \to \gamma_c(N)$ be the restriction of φ to $\gamma_c(N)$. Since $\gamma_c(N)$ is isomorphic to $(\mathbb{R}^k, +)$ for some k, ψ is a linear map, which, by the assumption, also does not have an eigenvalue 1 (otherwise so does $d_e\varphi$). Thus, we can consider the element $z \in \gamma_c(N)$ defined by

$$z = (\operatorname{Id} - \psi)^{-1} y.$$

As one can check, it thus holds that $\varphi(z) = y^{-1}z$. As a consequence, the element $x = x_0 z$ satisfies

$$b\varphi(x) = b\varphi(x_0)y^{-1}z = x_0 y y^{-1} z = x_0 z = x$$

which means it is a fixed point of f. □

The following lemma is an adapted version of Theorem 2.9. from [Der16b].

Lemma 3.4.16. *An affine infra-nilmanifold automorphism with a fixed point is topologically conjugate to an infra-nilmanifold automorphism.*

Proof. Let $\Gamma\backslash N$ be an infra-nilmanifold with affine automorphism $\overline{f} : \Gamma\backslash N \to \Gamma\backslash N$, where $f = (b, \varphi) \in N \rtimes \operatorname{Aut}(N)$ and $f\Gamma f^{-1} = \Gamma$. Let Γy be the given fixed point of $\overline{f}$. Define a new almost-Bieberbach group

$$\Gamma' = (y, 1)^{-1}\Gamma(y, 1)$$

and the maps

$$h : \Gamma\backslash N \to \Gamma'\backslash N : \Gamma x \mapsto \Gamma' y^{-1} x$$

and

$$\overline{g} : \Gamma'\backslash N \to \Gamma'\backslash N : \Gamma' x \mapsto \Gamma'\varphi(x).$$

As one can check, the maps above are well defined, h is a homeomorphism, g is an infra-nilmanifold automorphism and $h \circ \overline{f} = \overline{g} \circ h$. □

Theorem 3.4.17. *Every hyperbolic affine automorphism on an infra-nilmanifold has a fixed point and the set of periodic points is dense in the manifold.*

Proof. Let $\Gamma\backslash N$ be an infra-nilmanifold with affine automorphism $\overline{f} : \Gamma\backslash N \to \Gamma\backslash N$, where $f = (b, \varphi) \in N \rtimes \operatorname{Aut}(N)$ and $f\Gamma f^{-1} = \Gamma$. By Lemma 3.4.15, we know that $f : N \to N$ has a fixed point $y \in N$ and thus Γy is a fixed point of $\overline{f}$. By Lemma 3.4.16, we can thus assume that $b = 1$ and thus that $\overline{f}$ is a hyperbolic automorphism on $\Gamma\backslash N$. Let $H = \Gamma \cap N$ be the group of pure translations of Γ. Then we have that $f(H) = \varphi(H) = H$. Now, define for any integer $n > 0$ the group

$$H^{\frac{1}{n}} = \langle x \in N \mid x^n \in H \rangle.$$

As argued in [Der16b], we have $[H^{\frac{1}{n}} : H] < \infty$. If we write $p : N \to \Gamma\backslash N$ for the projection, it follows that the set $Q_n = p(H^{\frac{1}{n}})$ is finite. Moreover, since $\varphi(H) = H$, we have $\varphi(H^{\frac{1}{n}}) = H^{\frac{1}{n}}$ and thus that $\overline{f}(Q_n) = Q_n$. It follows that all points in Q_n are periodic points of $\overline{f}$. Write $H^{\mathbb{Q}}$ for the radicable hull of H (see Definition 2.3.11). Then we can write $H \subset H^{\mathbb{Q}} \subset N$. Clearly, we have

$$p(H^{\mathbb{Q}}) = p\left(\bigcup_{n>0} H^{\frac{1}{n}}\right) = \bigcup_{n>0} Q_n$$

and thus all points in this set are periodic points of $\overline{f}$. Since $H^{\mathbb{Q}}$ is a dense set of N, it follows that $p(H^{\mathbb{Q}})$ is dense in $\Gamma\backslash N$. Thus, we proved that the periodic points of $\overline{f}$ are dense in $\Gamma\backslash N$. □

3.4.4 A characterization for existence on the Lie algebra

In this section we will give characterization for the existence of Anosov diffeomorphisms on nilmanifolds on the level of the associated Lie algebra. The idea is that this characterization is easier to check since on Lie algebras one can use tools from linear algebra.

We start the discussion with the definition of a property of invertible linear maps.

Definition 3.4.18. Let V be a finite dimensional vector space over a subfield of the complex numbers $\mathbb{C}$. An invertible linear map $A : V \to V$ is called *integer-like* if its characteristic polynomial has integral coefficients and constant term equal to 1 or -1.

Remark 3.4.19. Using the *Frobenius normal form* or *rational canonical form*, it is not hard to see that if $A : V \to V$ is integer-like, then there exists a basis of V with respect to which A is represented by a matrix with integer coefficients and determinant equal to 1 or -1.

Let N be a simply connected nilpotent Lie group and $\mathfrak{n}$ the associated real Lie algebra. Recall from Section 2.3.3 of Chapter 2 that for any cocompact Lattice $\Gamma \leq N$ we have an associated rational form $\mathfrak{n}_\Gamma^\mathbb{Q} \subset \mathfrak{n}$. Using the theory of polynomial permutations, K. Dekimpe proved the following theorem.

Theorem 3.4.20 ([Dek01])**.** *Let Γ be a cocompact lattice in a simply connected nilpotent Lie group N with Lie algebra $\mathfrak{n}$ and write $\mathfrak{n}_\Gamma^\mathbb{Q} \subset \mathfrak{n}$ for the associated rational form of $\mathfrak{n}$. If $\varphi \in \mathrm{Aut}(N)$ satisfies that*

(i) $d_e\varphi$ is integer-like and

(ii) $d_e\varphi(\mathfrak{n}_\Gamma^\mathbb{Q}) = \mathfrak{n}_\Gamma^\mathbb{Q}$,

then there exists a positive integer $k > 0$ such that $\varphi^k(\Gamma) = \Gamma$.

Combining this result with Theorem 3.4.13, one obtains the algebraic characterization on the level of the Lie algebra.

Theorem 3.4.21 ([Dek01])**.** *A nilmanifold $\Gamma\backslash N$ admits an Anosov diffeomorphism if and only if the rational Lie algebra $\mathfrak{n}_\Gamma^\mathbb{Q}$ admits an integer-like hyperbolic automorphism.*

This characterization leads us to a research program which is focussed on answering the following two questions.

Question 3.4.22. Given a real (or complex) Lie algebra, what are its rational forms up to isomorphism?

Question 3.4.23. Which rational Lie algebras admit an integer-like hyperbolic automorphism?

In chapters 5 and 6 we focus on questions 3.4.22 and 3.4.23, respectively.

3.5 Anosov diffeomorphisms versus expanding maps

Another type of map on a manifold which exhibits interesting dynamics under iteration is the so called *expanding map*.

Definition 3.5.1. Let (M, g) be a Riemannian manifold. A differentiable map $f : M \to M$ is said to be *expanding* if there exist constants $c > 0$, $\lambda > 1$ such that

$$\forall v \in TM : \forall k \in \mathbb{N} : \|df^k v\| \geq c\lambda^k \|v\|.$$

Analogous to the case of Anosov diffeomorphisms (see Corollary 3.2.3), on compact manifolds, the definition of an expanding map does not depend on the choice of Riemannian metric.

Analogous to hyperbolic linear maps (Definition 3.4.3) and affine infra-nilmanifold automorphisms (Definition 3.4.11), we make the following two definitions.

Definition 3.5.2. Let V be a finite dimensional vector space over a subfield of the complex numbers $\mathbb{C}$. A linear map $A : V \to V$ is called *expanding* if all of its eigenvalues have modulus strictly bigger than 1.

Definition 3.5.3. Let $\Gamma\backslash N$ be an infra-nilmanifold and $f = (b, \varphi) \in N \rtimes \mathrm{Aut}(N)$ an affine map such that $f\Gamma f^{-1} \leq \Gamma$. The induced map $\overline{f} : \Gamma\backslash N \to \Gamma\backslash N : \Gamma x \mapsto \Gamma f(x)$ is called an *affine endomorphism* on the infra-nilmanifold $\Gamma\backslash N$. In particular, when b is trivial, we say $\overline{f}$ is an *endomorphism* of the infra-nilmanifold $\Gamma\backslash N$. We say $\overline{f}$ is *expanding* if the Lie algebra automorphism associated to φ is expanding.

By the work of J. Franks, M. Shub and M. Gromov, all expanding maps on compact manifolds are classified up to topological conjugacy. We note that the original statement used ‘expanding endomorphism’ instead of ‘expanding affine endomorphism’ which is a false claim. This was corrected by K. Dekimpe in [Dek12].

Theorem 3.5.4 ([Fra70] [Shu70] [Gro81]). *Every expanding map on a compact manifold is topologically conjugate to an expanding affine endomorphism on an infra-nilmanifold.*

Although the above result restricts the class of compact manifolds admitting an expanding map to the class of infra-nilmanifolds, as with Anosov diffeomorphisms, it is far from true that every infra-nilmanifold admits an expanding map. Thus, there is a need for a characterization on the Lie algebra. For such a characterization, we make the following definition.

Definition 3.5.5. A Lie algebra $\mathfrak{g}$ is said to have a *positive grading* if it can be written as a direct sum $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$ such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$ for all $i, j \in \{1, \ldots, k\}$.

By the work of K. Dekimpe and J. Deré, we have the following characterization.

Theorem 3.5.6 ([DD16] [Der17]). *Let* $\Gamma\backslash N$ *be an infra-nilmanifold and write* $\mathfrak{n}$ *for the real Lie algebra of* N*. Then the following are equivalent:*

(i) $\Gamma\backslash N$ *admits an expanding map,*

(ii) $\mathfrak{n}$ *admits an expanding automorphism,*

(iii) $\mathfrak{n}$ *has a positive grading.*

In particular, we see that the existence of an expanding map on $\Gamma\backslash N$ does not depend on the choice of almost-Bieberbach group Γ.

It is clear that Anosov diffeomorphisms and expanding maps have similar definitions. One can ask whether there is some relation between the existence of the two on infra-nilmanifolds. Does one imply the other? One direction has an easy counterexample: the circle $\mathbb{R}/\mathbb{Z}$ admits an expanding map (namely $x + \mathbb{Z} \mapsto 2x + \mathbb{Z}$), but does not admit an Anosov diffeomorphism. For the other direction, it is much harder to come up with a counterexample. In [Der17] J. Deré constructs such a counterexample, but the dimension of the nilmanifold is far from minimal. This raises the following question.

Question 3.5.7. What is the minimal dimension of an infra-nilmanifold which admits an Anosov diffeomorphism, but no expanding map?

In Section 6.5 of Chapter 6, we give an answer to this question.

Chapter 4

Partially commutative structures

In this chapter, we introduce groups and Lie algebras which are defined using a graph. They are free (nilpotent) up to the fact that some of the generators commute. The information of which generators commute is captured by the graph. In the literature these objects are also referred to as graph groups or graph algebras. In Section 4.6, we discus the automorphisms of the Lie algebra. In particular, we prove that, for a fixed graph, the projection of the automorphism group of the Lie algebra onto the abelianization does not depend on the chosen nilpotentcy class of the Lie algebra (see Corollary 4.6.4). In Section 4.6.3, we prove a result on the eigenvalues of vertex-diagonal automorphisms (see Proposition 4.6.14). Both results will be used in the following chapters.

4.1 Graphs

We start by specifying what we mean by a graph.

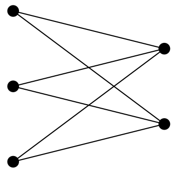
Definition 4.1.1. A *graph* is a pair $\mathcal{G} = (V, E)$ where V is a finite set, which is called the set of *vertices*, and E is a subset of $\{\{v, w\} \mid v, w \in V, v \neq w\}$, which is called the set of *edges*.

Two distinct vertices v and w for which $\{v, w\}$ is an edge will be called *adjacent*. We will call a graph *empty* if its set of edges is empty and we will call it *complete* if any two distinct vertices are adjacent.

Example 4.1.2. 1. A *complete bipartite graph* is a graph for which the vertices V can be written as a disjoint union of subsets $V = S_1 \sqcup S_2$ such that the edge set is equal to

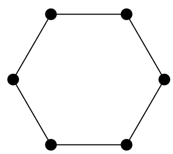
$$E = \{\{v, w\} \mid v \in S_1, w \in S_2\}.$$

With other words every vertex from S_1 is adjacent to every vertex in S_2 and these constitute all edges in the graph. An example with $|S_1| = 3$ and $|S_2| = 2$ is drawn below.



2. Consider a set of vertices $V = \{1, \dots, n\}$ with the set of edges $E = \{\{i, i+1\} \mid 1 \leq i \leq n-1\}$. The resulting graph is called the *path graph* on n vertices and is drawn below for $n = 5$.

3. Consider a set of vertices $V = \{1, \dots, n\}$ with the set of edges $E = \{\{i, i+1\} \mid 1 \leq i \leq n-1\} \cup \{\{n, 1\}\}$. The resulting graph is called the *cycle graph* on n vertices and is drawn below for $n = 6$.



For any set S, we will write $\mathrm{Perm}(S)$ for the group of permutations on S. The automorphisms of a graph $\mathcal{G} = (V, E)$ are the permutations on the vertices V that preserve the edges. They form a group under composition which we will write as

$$\mathrm{Aut}(\mathcal{G}) = \{\varphi \in \mathrm{Perm}(V) \mid \forall e \in E : \varphi(e) \in E\}.$$

A tuple of vertices $(v_1, \dots, v_n) \in V^n$ will be called a *walk (of length n)* in $\mathcal{G}$ if for any $i \in \{1, \dots, n-1\}$ it holds that $\{v_i, v_{i+1}\} \in E$. A walk $(v_1, \dots, v_n)$ for which all vertices $v_1, \dots, v_n$ are distinct is called a *path.* A walk $(v_1, \dots, v_n)$ for which only the vertices v_1 and v_n are equal is called a *cycle.* This brings us to an important notion in graph theory.

Definition 4.1.3. Let $\mathcal{G} = (V, E)$ be a graph. A subset $X \subset V$ is called *connected in* $\mathcal{G}$ if for any two vertices $v, w \in X$ there exists a walk $(v_1, \ldots, v_n)$ in $\mathcal{G}$ such that $v_1 = v$, $v_n = w$ and $v_i \in X$ for all $i \in \{1, \ldots, n\}$. The graph $\mathcal{G}$ is called *connected* if V is connected in $\mathcal{G}$.

We can now also define an important subclass of graphs.

Definition 4.1.4. A graph $\mathcal{G} = (V, E)$ is called a *forest* if $\mathcal{G}$ has no cycles. If in addition $\mathcal{G}$ is connected, we call $\mathcal{G}$ a *tree*.

In what follows, we give some definitions which play a central role in the study of automorphisms of partially commutative structures. See for instance [Lau95], [DM23] and Section 4.6.2.

Let $\mathcal{G} = (V, E)$ be a graph. The *(open) neighbourhood* of a vertex v is defined as

$$N(v) = \{w \in V \mid \{v, w\} \in E\}.$$

Note that v is not contained in its open neighbourhood $N(v)$. The *closed neighbourhood* of v is defined as

$$N[v] = N(v) \cup \{v\}.$$

This allows us to define a relation $\prec$ on the vertices by

$$v \prec w \quad \Leftrightarrow \quad N(v) \subset N[w].$$

We also say w *dominates* v. This relation is a *preorder*, i.e. a relation which is reflexive and transitive but not necessarily anti-symmetric. One way to resolve the lack of anti-symmetry is to identify vertices which mutually dominate each other. We define the equivalence relation $\sim$ on V by

$$v \sim w \quad \Leftrightarrow \quad v \prec w \ \wedge \ v \succ w.$$

Equivalently, we have $v \sim w$ if and only if $N(v) = N(w)$ or $N[v] = N[w]$. Again equivalent to that is saying that $v \sim w$ if and only if the transposition of v with w (leaving all other vertices fixed) defines a graph automorphism of $\mathcal{G}$.

An equivalence class $[v] = \{w \in V \mid w \sim v\}$ is called a *coherent component* and

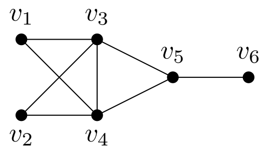
$$\Lambda_{\mathcal{G}} = V/\sim$$

denotes the set of all coherent components. Note that the relation $\prec$ on V induces a relation $\overline{\prec}$ on the coherent components $\Lambda_{\mathcal{G}}$ defined by

$$\begin{aligned} \lambda \overline{\prec} \mu \quad &\Leftrightarrow \quad \exists v \in \lambda,\, w \in \mu : v \prec w \\ &\Leftrightarrow \quad \forall v \in \lambda,\, w \in \mu : v \prec w. \end{aligned}$$

This relation is reflexive, transitive and anti-symmetric and thus gives a partial order on $\Lambda_{\mathcal{G}}$.

Example 4.1.5. Consider the graph as drawn below:



The coherent components are given by $\{v_1, v_2\}, \{v_3, v_4\}, \{v_5\}, \{v_6\}$ and satisfy the relations $\{v_1, v_2\}\overline{\prec}\{v_3, v_4\}$, $\{v_1, v_2\}\overline{\prec}\{v_5\}$ and $\{v_6\}\overline{\prec}\{v_5\}$.

Remark 4.1.6. Let $\lambda, \mu \in \Lambda_{\mathcal{G}}$ be two (not necessarily distinct) coherent components of $\mathcal{G}$. The following observation is an easy exercise: if there exists a vertex in λ which is adjacent to a vertex in μ, then all vertices in λ are adjacent to all vertices in μ.

Following the above remark, it is natural to define a new graph on the coherent components. The *quotient graph* of a graph $\mathcal{G} = (V, E)$ is defined as the triple:

$$\overline{\mathcal{G}} = (\Lambda_{\mathcal{G}}, \overline{E}, \Phi)$$

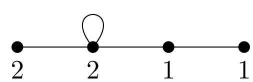
where

$$\overline{E} = \{\ \{[v], [w]\}\ \mid v, w \in V, \{v, w\} \in E\}$$

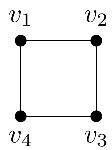
and

$$\Phi : \Lambda_{\mathcal{G}} \to \mathbb{N} : \lambda \mapsto |\lambda|.$$

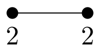
Example 4.1.7. Consider the graph from Example 4.1.5. The quotient graph can be represented by the drawing below. The numbers denote the cardinalities of the coherent components. A loop is drawn at those coherent components λ for which $\{\lambda\} \in \overline{E}$.



Example 4.1.8. Consider the cycle graph on 4 vertices. The vertices are given by $V = \{v_1, v_2, v_3, v_4\}$ and the edges are given by $E = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_1, v_4\}\}$.



The coherent components are then given by $\Lambda_{\mathcal{G}} = \{\{v_1, v_3\}, \{v_2, v_4\}\}$ and the quotient graph can be represented by the drawing below.



The reason that we incorporate the map Φ into the definition of the quotient graph is to define its automorphism group in a suitable way:

$$\mathrm{Aut}\left(\overline{\mathcal{G}}\right) = \{\varphi \in \mathrm{Perm}(\Lambda_{\mathcal{G}}) \mid \forall e \in \overline{E} : \varphi(e) \in \overline{E}, \Phi \circ \varphi = \Phi\}.$$

Note that for any graph automorphism $\varphi \in \mathrm{Aut}(\mathcal{G})$ and any vertices $v, w \in V$ it holds that $v \sim w \Leftrightarrow \varphi(v) \sim \varphi(w)$. As a consequence, φ induces an automorphism on the quotient graph $\overline{\varphi} \in \mathrm{Aut}(\overline{\mathcal{G}})$ by $\overline{\varphi}([v]) = [\varphi(v)]$ for any $v \in V$. Thus, we get a group morphism from $\mathrm{Aut}(\mathcal{G})$ to $\mathrm{Aut}(\overline{\mathcal{G}})$, which fits in the short exact sequence:

$$1 \longrightarrow \prod_{\lambda \in \Lambda_{\mathcal{G}}} \mathrm{Perm}(\lambda) \longrightarrow \mathrm{Aut}(\mathcal{G}) \longrightarrow \mathrm{Aut}(\overline{\mathcal{G}}) \longrightarrow 1. \tag{4.1}$$

Remark 4.1.9. The above short exact sequence is right-split, i.e. there exists a group morphism $r : \mathrm{Aut}(\overline{\mathcal{G}}) \to \mathrm{Aut}(\mathcal{G})$ such that $\overline{r(\varphi)} = \varphi$ for all $\varphi \in \mathrm{Aut}(\overline{\mathcal{G}})$. One can explicitly construct such a morphism as follows. Order the vertices in each coherent component $\lambda \in \Lambda_{\mathcal{G}}$ as

$$\lambda = \{v_{\lambda,1}, v_{\lambda,2}, \ldots, v_{\lambda,|\lambda|}\}.$$

Using these orders, define for any $\varphi \in \mathrm{Aut}(\overline{\mathcal{G}})$, $\lambda \in \Lambda_{\mathcal{G}}$ and $1 \leq i \leq |\lambda|$:

$$r(\varphi)(v_{\lambda,i}) = v_{\varphi(\lambda),i}.$$

We end this section with a note on the asymptotic behaviour of a property of graphs with respect to the number of vertices. Two graphs $\mathcal{G}_1 = (V_1, E_1)$ and $\mathcal{G}_2 = (V_2, E_2)$ are said to be *isomorphic* if there exists a bijection $\phi : V_1 \to V_2$ such that the assignment $\{v, w\} \mapsto \{\phi(v), \phi(w)\}$ defines a bijection from E_1 to E_2. Let $\mathcal{P}$ be a property of graphs, invariant under isomorphism. Let $N(k)$ denote the total number of isomorphism classes of graphs on k vertices and $N(\mathcal{P}, k)$ the total number of isomorphism classes of graphs on k vertices which have property $\mathcal{P}$.

Definition 4.1.10. The property $\mathcal{P}$ is said *to hold for almost all unlabelled graphs* if

$$\lim_{k\to\infty} \frac{N(\mathcal{P},k)}{N(k)} = 1.$$

The term ‘unlabelled’ refers to the fact that we count the graphs up to isomorphism and thus we do not use labels on the vertices. As a consequence of a classical result of Erdös and Rényi on labelled graphs [ER63], the following statement is argued in section 1.6 of [Bab95].

Theorem 4.1.11. *Almost all unlabelled graphs have a trivial automorphism group.*

In the next section, we associate several algebraic structures to a graph $\mathcal{G}$. Later in this thesis, when we prove properties of these associated algebraic structures, we will use the above theorem to find a corollary regarding the asymptotic behaviour of that property.

4.2 The associated algebraic structures

In this section we associate algebraic structures to a graph. The graph encodes commutation relations among the generators. As a result, some generators commute, while others do not. This is the reason why these structures are often referred to as *(free) partially commutative.*

The monoid Let $\mathcal{G} = (V, E)$ be a graph. Consider the monoid $\mathrm{W}(V)$ of all words on the alphabet V. This set also contains the empty word, which we denote by $\varnothing$. We define the relation $\leftrightarrow$ on $\mathrm{W}(V)$ by saying for any $w_1, w_2 \in \mathrm{W}(V)$ that $w_1 \leftrightarrow w_2$ if and only if $w_1 = u_1v_1v_2u_2$ and $w_2 = u_1v_2v_1u_2$ for some words $u_1, u_2 \in \mathrm{W}(V)$ and some vertices $v_1, v_2 \in V$ with $\{v_1, v_2\} \notin E$. This relation is reflexive and symmetric, but not transitive. From this relation, we define the equivalence relation $\sim$ on $\mathrm{W}(V)$ by:

$$w_1 \sim w_2 \quad \Leftrightarrow \quad \exists \tilde{w}_1, \ldots, \tilde{w}_n \in \mathrm{W}(V) : w_1 \leftrightarrow \tilde{w}_1 \leftrightarrow \ldots \leftrightarrow \tilde{w}_n \leftrightarrow w_2.$$

One can check that this relation behaves well with respect to the product on $\mathrm{W}(V)$. Indeed, when $w_1 \sim w_2$ and $u_1 \sim u_2$, then also $w_1w_2 \sim u_1u_2$. As a consequence, we can define the monoid

$$M(\mathcal{G}) = \mathrm{W}(V)/\sim$$

which we refer to as the *(free) partially commutative monoid* associated to $\mathcal{G}$.

The group Let $\mathcal{G} = (V, E)$ be a graph. The *(free) partially commutative group* (or *right-angled Artin group* or *graph group*) associated to the graph $\mathcal{G}$ is the group defined by the presentation

$$A(\mathcal{G}) = \langle V \mid [v, w] = 1; \{v, w\} \notin E \rangle.$$

where $[v, w] = vwv^{-1}w^{-1}$ is the commutator of two group elements. We can make this group nilpotent by modding out a subgroup from the lower central series. The *free c-step nilpotent partially commutative group* associated to a non-empty graph $\mathcal{G}$ is then defined as

$$A(\mathcal{G}, c) = A(\mathcal{G})/\gamma_{c+1}(A(\mathcal{G})).$$

These groups are torsion-free finitely generated nilpotent groups, see for instance Theorem 6.4. in [Wad15].

The Lie algebra Let $\mathcal{G} = (V, E)$ be a graph and K a field. Write $\mathfrak{f}(V)$ for the free Lie algebra on V over the field K. Define $\mathfrak{i}(\mathcal{G})$ as the Lie ideal of $\mathfrak{f}(V)$ generated by the Lie brakets $[v, w]$ for $v, w \in V$ and $\{v, w\} \notin E$. The *(free) partially commutative Lie algebra over K* associated to $\mathcal{G}$ is then defined by

$$\mathfrak{g}^K(\mathcal{G}) = \mathfrak{f}(V)/\mathfrak{i}(\mathcal{G}).$$

We can make this Lie algebra nilpotent by modding out an ideal from its lower central series. The *free c-step nilpotent partially commutative Lie algebra over K* associated to a non-empty graph $\mathcal{G}$ is defined as

$$\mathfrak{n}^K(\mathcal{G}, c) = \mathfrak{g}^K(\mathcal{G})/\gamma_{c+1}(\mathfrak{g}^K(\mathcal{G})).$$

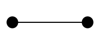
Example 4.2.1.

1. If $\mathcal{G}$ is the complete graph on n vertices, then $M(\mathcal{G})$, $A(\mathcal{G})$ and $\mathfrak{g}^K(\mathcal{G})$ are the free monoid, free group and free Lie algebra, respectively, each on n generators.

2. If $\mathcal{G}$ is the empty graph on n vertices, then $M(\mathcal{G})$ is isomorphic to the abelian monoid $(\mathbb{N}^n, +)$, $A(\mathcal{G})$ is isomorphic to the free abelian group $(\mathbb{Z}^n, +)$ and $\mathfrak{g}^K(\mathcal{G})$ is isomorphic to the abelian Lie algebra K^n.

3. If $\mathcal{G}$ is a complete bipartite graph on $n + m$ vertices, then $A(\mathcal{G})$ is a free product of two free abelian groups of rank n and m.

4. If $\mathcal{G}$ is the graph with two vertices and one edge, as drawn below, then the group $A(\mathcal{G}, 2)$ is isomorphic to the Heisenberg group $H(\mathbb{Z})$ from Example 2.3.4. The Lie algebra $\mathfrak{n}^K(\mathcal{G}, 2)$ is the so called Heisenberg Lie algebra (over the field K).



Remark 4.2.2. We note here that in the literature, the opposite convention is often used to define these algebraic structures, namely that two vertices v, w commute if and only if $\{v, w\} \in E$.

Remark 4.2.3 (Universal property)**.** Note that the above algebraic structures all satisfy the following universal property. Let $\mathcal{G} = (V, E)$ be a graph and K a field. Let $X(\mathcal{G})$ be one of the algebraic structures: $M(\mathcal{G})$, $A(\mathcal{G})$ or $\mathfrak{g}^K(\mathcal{G})$, and let Y be a monoid, a group or a Lie algebra over K, respectively. For any injection $i : V \to Y$ such that $i(v)$ and $i(w)$ commute whenever $\{v, w\} \notin E$, there exists a unique homomorphism $f : X(\mathcal{G}) \to Y$ which makes the following diagram commutative:

$$\begin{array}{ccc} X(\mathcal{G}) & \xrightarrow{\;f\;} & Y \\ \uparrow & \nearrow_{i} & \\ V & & \end{array}$$

Moreover, the algebraic structures $M(\mathcal{G})$, $A(\mathcal{G})$ and $\mathfrak{g}^K(\mathcal{G})$ are uniquely determined by this property, up to isomorphism.

4.3 A connection between group and Lie algebra

In this section we show how the partially commutative Lie algebra can be obtained from the corresponding partially commutative group by use of a general construction on groups. First, let us define gradings on Lie algebras.

4.3.1 Gradings on Lie algebras

The discussion in this section is partially based on [Cor16].

Definition 4.3.1. Let $\mathfrak{g}$ be a Lie algebra and G a group. A *G-grading* for $\mathfrak{g}$ is a direct sum decomposition of $\mathfrak{g}$ as a vector space

$$\mathfrak{g} = \bigoplus_{x \in G} \mathfrak{g}_x$$

such that $[\mathfrak{g}_x, \mathfrak{g}_y] \subset \mathfrak{g}_{xy}$ for any $x, y \in G$. In the particular case where G is equal to $\mathbb{Z}, +$, we say the grading is

(i) a *non-negative grading* if $\mathfrak{g}_i = \{0\}$ for any $i < 0$,

(ii) a *positive grading* if $\mathfrak{g}_i = \{0\}$ for any $i \leq 0$,

(iii) a *Carnot grading* if $\mathfrak{g}_1$ generates $\mathfrak{g}$ as a Lie algebra.

Remark 4.3.2. Note that if $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$ is a Carnot grading, then $\mathfrak{g}_1$ generates $\mathfrak{g}$ and thus

$$\mathfrak{g} = \mathfrak{g}_1 + [\mathfrak{g}_1, \mathfrak{g}_1] + [\mathfrak{g}_1, [\mathfrak{g}_1, \mathfrak{g}_1]] + [\mathfrak{g}_1, [\mathfrak{g}_1, [\mathfrak{g}_1, \mathfrak{g}_1]]] + \dots$$

Additionally, one has the inclusions

$$[\mathfrak{g}_1, \mathfrak{g}_1] \subset \mathfrak{g}_2, \quad [\mathfrak{g}_1, [\mathfrak{g}_1, \mathfrak{g}_1]] \subset \mathfrak{g}_3, \quad [\mathfrak{g}_1, [\mathfrak{g}_1, [\mathfrak{g}_1, \mathfrak{g}_1]]] \subset \mathfrak{g}_4, \dots \tag{4.2}$$

Hence we see that the above inclusions must be equalities and thus we get

$$\forall i \leq 0 : \ \mathfrak{g}_i = \{0\} \quad \text{and} \quad \forall i > 0 : \ [\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}.$$

From this it follows that we have the implications

$$\text{Carnot grading} \quad \Rightarrow \quad \text{positive grading} \quad \Rightarrow \quad \text{non-negative grading}.$$

The free Lie algebra is a prime example of a Lie algebra having a Carnot grading. Indeed if $\mathfrak{f}(X)$ is the free Lie algebra on the set X over K, then the inductively defined subspaces

$$\mathfrak{f}_1(X) = \operatorname{span}_K(X), \qquad \mathfrak{f}_{i+1}(X) = [\mathfrak{f}_1(X), \mathfrak{f}_i(X)]$$

give a Carnot grading of $\mathfrak{f}(X)$. In this thesis, this will always be the grading considered on the free Lie algebra.

Definition 4.3.3. Let G be a group and $\mathfrak{g} = \bigoplus_{x\in G} \mathfrak{g}_x$ and $\mathfrak{h} = \bigoplus_{x\in G} \mathfrak{h}_x$ two Lie algebras with a G-grading.

(i) A homomorphism $\varphi : \mathfrak{g} \to \mathfrak{h}$ will be called a *graded homomorphism* if $\varphi(\mathfrak{g}_x) \subset \mathfrak{h}_x$ for any $x \in G$.

(ii) An automorphism φ of $\mathfrak{g}$ will be called a *graded automorphism* if $\varphi(\mathfrak{g}_x) = \mathfrak{g}_x$ for any $x \in G$. The subgroup of $\operatorname{Aut}(\mathfrak{g})$ consisting of all graded automorphisms of $\mathfrak{g}$ will be written as $\operatorname{Aut}_g(\mathfrak{g})$ (whenever it is clear what the chosen grading on $\mathfrak{g}$ is).

(iii) An ideal $\mathfrak{i}$ of $\mathfrak{g}$ will be called a *graded ideal* if

$$\mathfrak{i} = \bigoplus_{x\in G} \mathfrak{i} \cap \mathfrak{g}_x.$$

The proof of the following lemma is left as an exercise to the reader.

Lemma 4.3.4. *Let $\mathfrak{g}$ be a Lie algebra with a G-grading $\mathfrak{g}_x$, $x \in G$. Let $\mathfrak{i}$ be a graded ideal of $\mathfrak{g}$ and write $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{i}$ for the projection morphism. The Lie algebra $\mathfrak{g}/\mathfrak{i}$ is also G-graded with grading given by $\pi(\mathfrak{g}_x)$, $x \in G$. Moreover, if the grading on $\mathfrak{g}$ is non-negative, positive or Carnot, then the grading on $\mathfrak{g}/\mathfrak{i}$ is also non-negative, positive or Carnot, respectively.*

Let $\mathcal{G} = (V, E)$ be a graph. The ideal $\mathfrak{i}(\mathcal{G})$ of $\mathfrak{f}(V)$ (as defined in Section 4.2) is clearly a graded ideal of $\mathfrak{f}(V)$. By the above lemma, we obtain a Carnot grading on $\mathfrak{g}^K(\mathcal{G})$. For the remainder of the thesis, this will always be the grading considered on $\mathfrak{g}^K(\mathcal{G})$ and we write it as

$$\mathfrak{g}^K(\mathcal{G}) = \bigoplus_{i=1}^{\infty} \mathfrak{g}_i^K(\mathcal{G}).$$

As one can check, we have for the lower central series of $\mathfrak{g}^K(\mathcal{G})$ that

$$\gamma_{c+1}(\mathfrak{g}^K(\mathcal{G})) = \bigoplus_{i>c} \mathfrak{g}_i^K(\mathcal{G})$$

for any $c > 0$. Thus, the ideal $\gamma_{c+1}(\mathfrak{g}^K(\mathcal{G}))$ is a graded ideal of $\mathfrak{g}^K(\mathcal{G})$. Again by the above lemma, we obtain a Carnot grading on $\mathfrak{n}^K(\mathcal{G}, c)$, which we will write as

$$\mathfrak{n}^K(\mathcal{G}, c) = \bigoplus_{i=1}^{\infty} \mathfrak{n}_i^K(\mathcal{G}, c). \tag{4.3}$$

Note that this grading satisfies $\mathfrak{n}_i^K(\mathcal{G}, c) = \{0\}$ for all $i > c$.

Note that one can construct from any Lie algebra $\mathfrak{g}$ a positively graded Lie algebra $\mathrm{gr}(\mathfrak{g})$. This Lie algebra is defined as the direct sum

$$\mathrm{gr}(\mathfrak{g}) = \bigoplus_{i>0} \gamma_i(\mathfrak{g})/\gamma_{i+1}(\mathfrak{g})$$

equipped with the Lie bracket

$$[X + \gamma_{i+1}(\mathfrak{g}), Y + \gamma_{j+1}(\mathfrak{g})] = [X, Y] + \gamma_{i+j+1}(\mathfrak{g})$$

for any $X \in \gamma_i(\mathfrak{g})$ and $Y \in \gamma_j(\mathfrak{g})$ and extending this definition linearly to all of $\mathrm{gr}(\mathfrak{g})$. In fact, when $\mathfrak{g}$ is nilpotent, $\mathfrak{g}$ having a Carnot grading is equivalent to $\mathfrak{g} \cong \mathrm{gr}(\mathfrak{g})$.

The proof of the following lemma is left as an exercise to the reader.

Lemma 4.3.5. *Let $\mathfrak{g}$ be a Lie algebra with Carnot grading $\mathfrak{g} = \bigoplus_{i=1}^{\infty} \mathfrak{g}_i$, β a basis for $\mathfrak{g}_1$ and $\mathfrak{f}(\beta)$ the free Lie algebra on β, defined over the same field as $\mathfrak{g}$. The induced homomorphism*

$$f : \mathfrak{f}(\beta) \to \mathfrak{g}$$

is surjective and satisfies $f(\mathfrak{f}_i(\beta)) \subset \mathfrak{g}_i$ *for any* $i > 0$*, or equivalently,* $\ker(f)$ *is a graded ideal of* $\mathfrak{f}(\beta)$*. Moreover,* $\mathfrak{g} \cong \mathfrak{f}(\beta)/\ker(f)$*.*

Remark 4.3.6. Note that Lemma 4.3.4 and Lemma 4.3.5 combined tell us that the Carnot Lie algebras are essentially all Lie algebras obtained by taking the quotient of a free Lie algebra by one of its graded ideals.

4.3.2 The Lie ring construction

In this paragraph we recall a well-known construction which associates to any group G a Lie ring $L(G)$, see [Laz54]. First let us recall the definition of a Lie ring.

Definition 4.3.7. A *Lie ring* is an abelian group $(L, +)$ together with a $\mathbb{Z}$-bilinear map $[\cdot,\cdot] : L \times L \to L : (x, y) \mapsto [x, y]$ which satisfies

(i) $\forall x \in L : [x, x] = 0$,

(ii) $\forall x, y, z \in L : [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

Remark 4.3.8. Notice the similarity with the definition of a Lie algebra in Definition 2.3.14. It is therefore not surprising that for any Lie ring L and field K, the tensor product $L \otimes_{\mathbb{Z}} K$ is a Lie algebra over K.

Let G be any group. Note that for any $i > 0$ one can define the abelian group

$$L_i(G) = \gamma_i(G)/\gamma_{i+1}(G).$$

We define the Lie ring associated to G as the following direct sum of these abelian groups

$$L(G) = \bigoplus_{i=1}^{\infty} L_i(G)$$

together with a bracket defined by

$$\begin{aligned}[x\,\gamma_{i+1}(G), y\,\gamma_{j+1}(G)] &= [x, y]\,\gamma_{i+j+1}(G)\\ &= xyx^{-1}y^{-1}\,\gamma_{i+j+1}(G)\end{aligned}$$

for any $i, j > 0$, $x\gamma_{i+1}(G) \in L_i(G)$, $y\gamma_{j+1}(G) \in L_j(G)$ and extending this bilinearly to all of $L(G)$. A proof that $L(G)$ indeed defines a Lie ring as well as a more general approach using any central series of G, can be found in [Wad15].

Note that L defines a functor

$$L : \mathbf{Grp} \to \mathbf{LieRing}$$

Composing L with the tensor product functor (restricted to **LieRing**)

$$K\otimes_{\mathbb{Z}} : \mathbf{LieRing} \to \mathbf{LieAlg}_K$$

one obtains a functor

$$L^K = K \otimes_{\mathbb{Z}} L : \mathbf{Grp} \to \mathbf{LieAlg}_K.$$

Clearly, since $L(G)$ was defined as a direct sum, so is $L^K(G)$:

$$L^K(G) = \bigoplus_{i=1}^{\infty} L_i^K(G), \quad \text{with} \quad L_i^K(G) \cong K \otimes_{\mathbb{Z}} L_i(G).$$

In fact, from the definition of $L(G)$, it follows that this direct sum is a positive grading for the Lie algebra $L^K(G)$. Moreover, if G is a finitely generated group, it follows from [Wad15, Proposition 2.5] that $L^K(G)$ is generated by $L_1^K(G)$ and thus that $L^K(G) = \bigoplus_{i=1}^{\infty} L_i^K(G)$ is a Carnot grading. The following lemma is left as an exercise to the reader (see also Remark 4.3.2).

Lemma 4.3.9. *Let G be a finitely generated group and K a field. For any $i > 0$, we have*

$$\gamma_i(L^K(G)) = \bigoplus_{j=i}^{\infty} L_j^K(G).$$

Moreover, we have a natural isomorphism

$$\frac{L^K(G)}{\gamma_{i+1}(L^K(G))} \cong L^K\left(\frac{G}{\gamma_{i+1}(G)}\right).$$

Note that by the universal property of $\mathfrak{g}^K(\mathcal{G})$ (see Remark 4.2.3), we get from the injection

$$V \to L^K(A(\mathcal{G})) : v \mapsto v\gamma_2(A(\mathcal{G})) \otimes 1,$$

a Lie algebra homomorphism

$$\alpha : \mathfrak{g}^K(\mathcal{G}) \to L^K(A(\mathcal{G})). \tag{4.4}$$

Note that α maps representatives of vertices in $\mathfrak{g}^K(\mathcal{G})$ to its corresponding representatives of vertices in $L^K(A(\mathcal{G}))$. Moreover, we have the following theorem.

Theorem 4.3.10 ([DK92])**.** *Let $\mathcal{G}$ be a graph, K a field and $A(\mathcal{G})$, $\mathfrak{g}^K(\mathcal{G})$ its associated group and Lie algebra, respectively. The morphism α, as defined in (4.4), is an isomorphism which preserves the gradings on both Lie algebras.*

Combining this theorem with Lemma 4.3.9 from above, we find for all $c > 1$ the natural isomorphism:

$$\mathfrak{n}^K(\mathcal{G}, c) = \frac{\mathfrak{g}^K(\mathcal{G})}{\gamma_{c+1}(\mathfrak{g}^K(\mathcal{G}))} \cong \frac{L^K\left(A(\mathcal{G})\right)}{\gamma_{c+1}\left(L^K\left(A(\mathcal{G})\right)\right)} \cong L^K\left(A(\mathcal{G}, c)\right). \tag{4.5}$$

Remark 4.3.11. Note that any automorphism on $A(\mathcal{G})$ induces an automorphism on $L^K(A(\mathcal{G}))$. By the isomorphism in (4.4), we now also get an induced automorphism on $\mathfrak{g}^K(\mathcal{G})$. The same holds for the group $A(\mathcal{G}, c)$ and the Lie algebra $\mathfrak{n}^K(\mathcal{G}, c)$.

4.4 A basis for the Lie algebra

In this section we give a method for writing down a basis for the partially commutative Lie algebra. The method and notation are based on [Wad15], but the original work on these bases can be found in [KL93], [Lal93] and [Lal95].

Lyndon elements Let $\mathcal{G}$ be a graph. Recall from Section 4.2 the definition of the partially commutative monoid $M(\mathcal{G})$ as the quotient of $\mathrm{W}(V)$ by the equivalence relation $\sim$. Let us write π for the associated projection map

$$\pi : \mathrm{W}(V) \to M(\mathcal{G}) : w \mapsto \overline{w}.$$

From here we fix a total order $\leq$ on the vertices. Note that this induces a total order on $\mathrm{W}(V)$ known as the *lexicographical order*. It is the unique total order $\leq$ on $\mathrm{W}(V)$ satisfying the following axioms:

(i) $\forall w \in \mathrm{W}(V) : \varnothing \leq w$.

(ii) $\forall w_1, w_2 \in \mathrm{W}(V)$, $v \in V$: if $w_1 \leq w_2$ then $vw_1 \leq vw_2$.

(iii) $\forall w_1, w_2 \in \mathrm{W}(V)$, $v_1, v_2 \in V$: if $v_1 < v_2$ then $v_1w_1 \leq v_2w_2$.

This order on $\mathrm{W}(V)$ allows us to choose an element from each equivalence class of $M(\mathcal{G})$.

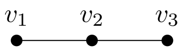
Definition 4.4.1. For any $m \in M(\mathcal{G})$, we define the *standard representative* $\mathrm{std}(m)$ of m as the maximum of the set $\pi^{-1}(m)$ with respect to the lexicographical order on $\mathrm{W}(V)$.

Using this map std : $M(\mathcal{G}) \to \mathrm{W}(V)$, we can define a total order on $M(\mathcal{G})$ by

$$x \leq y \quad \Leftrightarrow \quad \mathrm{std}(x) \leq \mathrm{std}(y)$$

for any $x, y \in M(\mathcal{G})$.

Example 4.4.2. Consider the path graph on 3 vertices $\mathcal{G} = (V, E)$ with $V = \{v_1, v_2, v_3\}$ and $E = \{\{v_1, v_2\}, \{v_2, v_3\}\}$ as drawn below.



Let the total order on V be the one induced by the indices: $v_1 < v_2 < v_3$. We have

$$\mathrm{std}(\overline{v_1 v_2 v_3}) = v_1 v_2 v_3, \qquad \mathrm{std}(\overline{v_1 v_1 v_3}) = v_3 v_1 v_1.$$

Thus, we see that $\overline{v_1 v_2 v_3} \leq \overline{v_1 v_1 v_3}$.

Next, we will define conjugacy classes in $M(\mathcal{G})$. We say two elements m_1, m_2 are *preconjugate*, written $m_1 P m_2$, if there exist elements $x, y \in M(\mathcal{G})$ such that $m_1 = xy$ and $m_2 = yx$. Note that this relation is reflexive and symmetric, but not transitive. We say two elements m_1, m_2 are *conjugate*, written $m_1 C m_2$, if there exist elements $x_1, \ldots, x_n \in M(\mathcal{G})$ such that

$$m_1 P x_1, \ x_1 P x_2, \ \ldots, \ x_{n-1} P x_n, \ x_n P m_2.$$

Note that being conjugate is an equivalence relation on $M(\mathcal{G})$. We call its equivalence classes the *conjugacy classes* in $M(\mathcal{G})$.

At last, we say an element $m \in M(\mathcal{G})$ is *primitive* if for any $x, y \in M(\mathcal{G})$ such that $m = xy = yx$ it holds that $x = \overline{\varnothing}$ or $y = \overline{\varnothing}$.

Now we are ready to define the notion of the Lyndon elements in $M(\mathcal{G})$.

Definition 4.4.3. An element in $M(\mathcal{G})$ is said to be a *Lyndon element* if it is not equal to $\overline{\varnothing}$, it is primitive and it is minimal in its conjugacy class. The subset of Lyndon elements in $M(\mathcal{G})$ is written as $\mathrm{LE}(\mathcal{G})$.

Remark 4.4.4. Note that the set $\mathrm{LE}(\mathcal{G})$ depends on the chosen total order $\leq$ on the vertices V. If $\mathcal{G}$ is the complete graph, then $M(\mathcal{G})$ is isomorphic to $\mathrm{W}(V)$ and the resulting Lyndon elements in $M(\mathcal{G})$ correspond to the well-known *Lyndon words* (see chapter 5 in [Lot97]).

The *length* of a word $w \in \mathrm{W}(V)$ is defined inductively as follows. The length of an element of V is set to 1 and if w can be written as a product $w = w_1 w_2$ for non-empty words $w_1, w_2 \in \mathrm{W}(V)$, then the length of w is the sum of the lengths of w_1 and w_2. The *length* of an element $\overline{w} \in M(\mathcal{G})$ is defined as the length of its representative w. As one can check, this is well-defined.

Example 4.4.5. Consider the same graph and order on the vertices as in Example 4.4.2. The standard representatives of the Lyndon elements in $M(\mathcal{G})$ are given by

$$\begin{aligned}
&\text{Length 1:} && v_1,\ v_2,\ v_3,\\
&\text{Length 2:} && v_1v_2,\ v_2v_3\\
&\text{Length 3:} && v_1^2v_2,\ v_1v_2^2,\ v_2^2v_3,\ v_2v_3^2,\ v_1v_2v_3\\
&\text{Length 4:} && v_1^3v_2,\ v_1^2v_2^2,\ v_1v_2^3,\ v_2^3v_3,\ v_2^2v_3^2,\ v_2v_3^3,\\
& && v_1^2v_2v_3,\ v_1v_2^2v_3,\ v_1v_2v_3^2,\ v_1v_2v_3v_2.
\end{aligned}$$

The element $\overline{v_1v_3}$ is not a Lyndon element since it is not primitive. The element $\overline{v_1v_3v_2}$ is not a Lyndon element, since it is conjugate to $\overline{v_1v_2v_3}$, which is strictly smaller. Thus $\overline{v_1v_3v_2}$ is not minimal in its conjugacy class.

The Lyndon elements can also be characterized in different equivalent ways. In order to do so, we introduce some notation:

- For any $w \in \mathrm{W}(V)$, *the support* of w, written as $\mathrm{supp}(w)$, is the subset of V containing all letters that occur in the word w. This definition descends also to the monoid $M(\mathcal{G})$. Thus, $\mathrm{supp}(\overline{w}) = \mathrm{supp}(w)$.
- For any $m \in M(\mathcal{G})$, we write $\mathrm{init}(m)$ for the subset of V containing all letters that can occur as the first letter of a word representing the equivalence class m.
- For any $m \in M(\mathcal{G})$, we define

$$\zeta(m) = \bigcup_{v \in \mathrm{supp}(m)} N[v],$$

  where we recall the definition of the closed neighbourhood $N[v]$ of a vertex v from Section 4.1.

A proof of the following characterization can be found in [KL93, Propositions 3.5, 3.6 and 3.7].

Theorem 4.4.6. *Let $\mathcal{G} = (V, E)$ be a graph. For any $m \in M(\mathcal{G})$, the following are equivalent:*

(i) m is a Lyndon element.

(ii) *For all* $x, y \in M(\mathcal{G}) \setminus \{\overline{\varnothing}\}$*: if* $m = xy$*, then* $m < y$*.*

(iii) *Either* $|m| = 1$ *or there exist Lyndon elements* $x, y \in \mathrm{LE}(\mathcal{G})$ *such that* $x < y$*,* $\mathrm{init}(y) \subset \zeta(x)$ *and* $m = xy$*.*

(iv) $\mathrm{std}(m)$ *is a Lyndon word.*

Standard bracketing Let X be a set. We define the *bracket words* on X inductively as follows:

(i) There is one bracket word of length 0, namely the empty word $\varnothing$.

(ii) The bracket words of length 1 are the elements of X.

(iii) The bracket words of length $k > 1$ are the expressions of the form $[b_1, b_2]$, where b_1 and b_2 are bracket words of length k_1 and k_2, respectively and such that $k_1 + k_2 = k$.

The set of bracket words on X is written as $\mathrm{BW}(X)$. This set is also known as the set of *non-associative words on* X.

Let $\mathcal{G} = (V, E)$ be a graph and fix a total order $\leq$ on the vertices V. We write $\mathrm{LE}'(\mathcal{G})$ for the set of Lyndon elements of length at least 2. The *standard factorization* is a map $F : \mathrm{LE}'(\mathcal{G}) \to \mathrm{LE}(\mathcal{G}) \times \mathrm{LE}(\mathcal{G})$, defined by

$$F(m) = (x, y) \quad \text{where} \quad m = xy$$

and

$$y = \min\{y' \in \mathrm{LE}(\mathcal{G}) \mid x' \in \mathrm{LE}(\mathcal{G}),\, m = x'y'\}.$$

The standard factorization behaves well with respect to the standard representative. If $F(m) = (x, y)$, then

$$\mathrm{std}(m) = \mathrm{std}(x)\,\mathrm{std}(y).$$

By repeating this standard factorization on a Lyndon element, we get a map

$$\Theta : \mathrm{LE}(\mathcal{G}) \to \mathrm{BW}(V),$$

called the *standard bracketing*, defined inductively by

$$\Theta(m) = [\Theta(x), \Theta(y)] \quad \text{for} \quad F(m) = (x, y).$$

Example 4.4.7. The standard bracketing of the Lyndon elements of length at least 2 as given in Example 4.4.5, in the same order, is given by

Length 2: $[v_1, v_2],\ [v_2, v_3]$

Length 3: $[v_1, [v_1, v_2]],\ [[v_1, v_2], v_2],\ [v_2, [v_2, v_3]],\ [[v_2, v_3], v_3],\ [v_1, [v_2, v_3]]$

Length 4: $[v_1, [v_1, [v_1, v_2]]],\ [v_1, [[v_1, v_2], v_2]],\ [[[v_1, v_2], v_2], v_2],$

$[v_2, [v_2, [v_2, v_3]]],\ [v_2, [[v_2, v_3], v_3]],\ [[[v_2, v_3], v_3], v_3],$

$[v_1, [v_1, [v_2, v_3]]],\ [v_1, [v_2, [v_2, v_3]]],\ [v_1, [[v_2, v_3], v_3]],$

$[[v_1, [v_2, v_3]], v_2].$

The basis Let $\mathcal{G} = (V, E)$ be a graph and let $\mathfrak{g}$ be a Lie algebra together with an inclusion $i : V \to \mathfrak{g}$. Note that there is a canonical *evaluation* map

$$\mathrm{ev}_i : \mathrm{BW}(V) \to \mathfrak{g},$$

defined inductively on the length of the bracket words according to the rule $\mathrm{ev}_i([b_1, b_2]) = [\mathrm{ev}_i(b_1), \mathrm{ev}_i(b_2)]$ for any bracket words $b_1, b_2 \in \mathrm{BW}(V)$. For the Lie algebras $\mathfrak{g}^K(\mathcal{G})$ and $\mathfrak{n}^K(\mathcal{G}, c)$, there is a canonical inclusion of the vertices into these Lie algebras and thus we do not need to specify it. We thus write these maps as

$$\mathrm{ev} : \mathrm{BW}(V) \to \mathfrak{g}^K(\mathcal{G}).$$

and

$$\mathrm{ev}_c : \mathrm{BW}(V) \to \mathfrak{n}^K(\mathcal{G}, c).$$

At last, let us write $\mathrm{LE}(\mathcal{G}, c)$ for all Lyndon elements of length at most c. A proof of the following can be found in [Wad15, Corollary 5.24.], but note this is not the original proof.

Theorem 4.4.8. *Let $\mathcal{G}$ be a graph and K a field. The image of $\mathrm{LE}(\mathcal{G})$ under $\mathrm{ev} \circ \Theta$ is a basis for $\mathfrak{g}^K(\mathcal{G})$. For any integer $c > 1$, the image of $\mathrm{LE}(\mathcal{G}, c)$ under $\mathrm{ev}_c \circ \Theta$ is a basis for $\mathfrak{n}^K(\mathcal{G}, c)$.*

4.5 Linear algebraic groups

In this section we recall some elementary notions from algebraic geometry and linear algebraic groups that will be used throughout this thesis. We let K denote any field, unless otherwise specified. For an overview of some of the notions of field theory, we refer the reader to Section 5.1 of Chapter 5. For a detailed introduction to the theory of linear algebraic groups, we refer the reader to [Bor91], [Spr81] and [Hum81].

Definition 4.5.1. A set $V \subset K^n$ is called *algebraic* if there exists a set of polynomials $S \subset K[x_1, \dots, x_n]$ such that

$$V = \{(\alpha_1, \dots, \alpha_n) \in K^n \mid \forall f \in S : \ f(\alpha_1, \dots, \alpha_n) = 0\}.$$

If $K_0 \subset K$ is a subfield and we can choose S as a subset of $K_0[x_1, \dots, x_n]$ to define V, then we say V is *defined over* K_0.

The algebraic sets of K^n are the closed sets of a topology on K^n known as the *Zariski topology.*

To any algebraic set $V \subset K^n$ we associate an ideal of $K[x_1, \dots, x_n]$ by

$$I(V) = \{f \in K[x_1, \dots, x_n] \mid \forall (\alpha_1, \dots, \alpha_n) \in V : f(\alpha_1, \dots, \alpha_n) = 0\}.$$

The *coordinate algebra* of V is then defined as the quotient

$$A(V) = \frac{K[x_1, \dots, x_n]}{I(V)}.$$

Let us define $\mathbf{Alg}_K$ as the category of unital associative, commutative K-algebras. Let R be an object of $\mathbf{Alg}_K$. We define the *set of R-points* of an algebraic set $V \subset K^n$ as

$$V^R = \{(\alpha_1, \dots, \alpha_n) \in R^n \mid \forall f \in I(V) : f(\alpha_1, \dots, \alpha_n) = 0\}.$$

Note that, in particular, for any field extension L/K, the field L is an object of $\mathbf{Alg}_K$ and thus we have a set $V^L \subset L^n$.

For any algebraic set $V \subset K^n$ and any object R of $\mathbf{Alg}_K$, there is a canonical bijection

$$\mathrm{Hom}_{\mathbf{Alg}_K}(A(V), R) \to V^R : f \mapsto (f(\overline{x_1}), \dots, f(\overline{x_n})).$$

For the following definition, note that the general linear group $\mathrm{GL}_n(K)$ can be seen as a subset of K^{n^2}.

Definition 4.5.2. A *linear algebraic group* is a subgroup $G \leq \mathrm{GL}_n(K)$ which is given by the intersection of $\mathrm{GL}_n(K)$ with an algebraic subset of K^{n^2}. If this algebraic subset is defined over a subfield $K_0 \subset K$, we say G is *defined over* K_0.

This definition might feel odd at first, because $\mathrm{GL}_n(K)$ is not an algebraic set of K^{n^2} and thus a linear algebraic group might not be an algebraic subset of K^{n^2}. Nevertheless, we can always embed a linear algebraic group $G \leq \mathrm{GL}_n(K)$ as an algebraic set in K^{n^2+1} as follows:

$$G \hookrightarrow K^{n^2+1} : A \mapsto (A, \det(A)^{-1}).$$

If we use coordinates $(x_{11}, \ldots, x_{nn}, y)$ on K^{n^2+1}, it is not hard to check that the image of this embedding is an algebraic subset determined by the polynomials determining G in $\mathrm{GL}_n(K)$ and the additional polynomial $\det(x_{ij}) \cdot y - 1$. This embedding is known as *the Rabinowitsch trick*.

Example 4.5.3. Let K be a field.

1. The group $\mathrm{GL}_n(K)$ is a linear algebraic group as it is the intersection of $\mathrm{GL}_n(K)$ with the algebraic set K^{n^2}.
2. The group $\mathrm{SL}_n(K)$ is a linear algebraic group since it is equal to the algebraic set determined by the polynomial equation $\det(x_{ij}) - 1 = 0$.
3. The group $\mathrm{O}_n(K)$ is a linear algebraic group since the defining equation $A^t A = I$ are actually K^{n^2} polynomial equations with polynomials in $K[x_{11}, \ldots, x_{nn}]$.
4. The group $UT_n(K)$ from Example 2.3.4 is a linear algebraic group since it is determined by the polynomial equations $x_{ij} = 0$ for $i > j$ and $x_{ii} = 1$ for all i.

Definition 4.5.4. Let $G \leq \mathrm{GL}_n(K)$ be a linear algebraic group. A subgroup $H \leq G$ is said to be a *closed subgroup* if $H \leq \mathrm{GL}_n(K)$ is a linear algebraic group, i.e. H is the intersection of G with an algebraic subset of K^{n^2}.

Definition 4.5.5. A topological space X is said to be *irreducible* if it can not be written as a union $X = X_1 \cup X_2$ with X_1, X_2 proper closed subsets of X. A subset $Y \subset X$ is said to be irreducible if it is so for the subspace topology. A maximal irreducible subset is called an *irreducible component*.

It is not hard to check that any point of a topological space is contained in a (possibly non-unique) irreducible component. Note that a linear algebraic group $G \leq \mathrm{GL}_n(K)$ is a topological space with the Zariski topology inherited from K^{n^2}.

Theorem 4.5.6. *Let $G \leq \mathrm{GL}_n(K)$ be a linear algebraic group. There is a unique irreducible component containing the identity which we write as G_0. It is a closed normal finite index subgroup of G and its cosets are exactly all irreducible components of G.*

Let $g \in \mathrm{GL}_n(K)$. We say a vector subspace $W \subset K^n$ is *g-invariant* if $g(W) = W$.

Definition 4.5.7. Let K be a field. An element $g \in \mathrm{GL}_n(K)$ is said to be

(i) *semi-simple* if for any g-invariant subspace in K^n, there exists a complementary g-invariant subspace.

(ii) *unipotent* if there exists a positive integer $k > 0$ such that $(g - \mathrm{Id})^k = 0$.

The following theorem is the well-known multiplicative version of the Jordan-Chevalley decomposition. Moreover, it states that linear algebraic groups are closed under this decomposition.

Theorem 4.5.8. *Let K be a perfect field. For any $g \in \mathrm{GL}_n(K)$, there exist a unique semi-simple element $g_s \in \mathrm{GL}_n(K)$ and a unique unipotent element $g_u \in \mathrm{GL}_n(K)$ such that $g = g_s g_u = g_u g_s$. If $G \leq \mathrm{GL}_n(K)$ is a linear algebraic group and $g \in G$, then also $g_s \in G$ and $g_u \in G$.*

Definition 4.5.9. A linear algebraic group $G \leq \mathrm{GL}_n(K)$ is said to be *unipotent* if all its elements are unipotent.

Theorem 4.5.10. *Let K be a field and $G \leq \mathrm{GL}_n(K)$ a linear algebraic group. There exists a unique maximal closed normal unipotent subgroup of G.*

This group is called the *unipotent radical of G*.

Let $D_n(K)$ denote the subgroup of $\mathrm{GL}_n(K)$ of all diagonal matrices. Note that $D_n(K)$ is a linear algebraic group. For any field K, we write $\overline{K}$ for its algebraic closure (see Definition 5.1.10 and Theorem 5.1.11).

Definition 4.5.11. Let K be a field and $G \leq \mathrm{GL}_n(K)$ a linear algebraic group. We say that G is a *torus* if there exists a $g \in \mathrm{GL}_n(\overline{K})$ such that $gGg^{-1} \subset D_n(\overline{K})$.

Definition 4.5.12. Let K be a field and $G \leq \mathrm{GL}_n(K)$ a linear algebraic group. A closed subgroup $T \subset G$ is said to be a maximal torus if it is a torus and if there are no closed subgroups of G which are tori and contain T as a proper subset.

Theorem 4.5.13. *Let K be an algebraically closed field. Let $G \leq \mathrm{GL}_n(K)$ be a linear algebraic group. Any semi-simple element lies in some maximal torus of G. If T_1 and T_2 are two maximal tori of G, then there exists a $g \in G$ such that $gT_1g^{-1} = T_2$.*

Consider a group G and a representation $\rho : G \to \mathrm{GL}_n(K)$. A vector subspace $W \subset K^n$ is said to be ρ-invariant if $\rho_g(W) = W$ for any $g \in G$. The representation ρ is said to be *reducible* if for any ρ-invariant subspace $W \subset K^n$, there exists another ρ-invariant vector subspace which is complementary to W in K^n.

Definition 4.5.14. Let $G \leq \mathrm{GL}_n(K)$ be a linear algebraic group. A representation $\rho : G \to \mathrm{GL}_m(K)$ is called a *polynomial representation* if the coordinates of $\rho(g) \in \mathrm{GL}_m(K)$ can be expressed as polynomials in the coordinates of $g \in \mathrm{GL}_n(K)$. A linear algebraic group is said to be *linearly reductive* if every polynomial representation of it is reducible.

Example 4.5.15. For K a field of characteristic 0, the groups $\mathrm{GL}_n(K)$ and $\mathrm{O}_n(K)$ are examples of linearly reductive groups.

The following theorem can be found in [Hoc81, Chapter VIII.4, Theorem 4.3.]

Theorem 4.5.16. *Let K be a field of characteristic zero and G a linear algebraic group defined over K. There is a linearly reductive subgroup $P \leq G$ such that G is a semi-direct product $G_u \rtimes P$. If Q is any linearly reductive subgroup of G, then there exists a $g \in G$ such that $gQg^{-1} \subset P$.*

4.6 Automorphisms of the Lie algebra

In this section we discuss the automorphisms of the Lie algebra $\mathfrak{g}^K(\mathcal{G})$ and its nilpotent quotients.

4.6.1 Projection onto the abelianization

For any Lie algebra $\mathfrak{g}$, we will write π_{ab} for the projection map

$$\pi_{\mathrm{ab}} : \mathfrak{g} \to \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}].$$

Since $[\mathfrak{g}, \mathfrak{g}]$ is a characteristic ideal of $\mathfrak{g}$, i.e. an ideal preserved by any automorphism of the Lie algebra, the above projection induces a projection on the level of automorphisms, which we will write with the same symbol:

$$\pi_{\mathrm{ab}} : \mathrm{Aut}(\mathfrak{g}) \to \mathrm{GL}(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]).$$

Consider $\mathfrak{g}$ a finite dimensional Lie algebra over a field K and let $\beta = \{X_1, \dots, X_n\}$ be a basis for it. The condition for a linear map $g \in \mathrm{GL}(\mathfrak{g})$ to lie in $\mathrm{Aut}(\mathfrak{g})$ can be expressed by polynomial equations in the matrix entries of g with respect to β. As a consequence, $\mathrm{Aut}(\mathfrak{g})$ is a Zariski-closed subgroup of $\mathrm{GL}(\mathfrak{g})$ (with respect to the coordinates induced by the basis β). Thus, $\mathrm{Aut}(\mathfrak{g})$ carries the structure of a linear algebraic group. This will be exploited later on. Note as well that if all structural constants of $\mathfrak{g}$ with respect to the basis β lie in a subfield $L \subset K$, then $\mathrm{Aut}(\mathfrak{g})$ will be a linear algebraic group defined over the subfield L.

It is not hard to check that the image of $\mathrm{Aut}(\mathfrak{g})$ under π_{ab} will be a Zariski closed subgroup of $\mathrm{GL}(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])$ (with respect to coordinates induced by a basis of $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$). We will write this group as $\mathrm{Aut}_{\mathrm{ab}}(\mathfrak{g})$. One might wonder which linear algebraic groups can occur in this way. Over fields of characteristic 0, it was proven by Bryant and Groves that any Zariski-closed subgroup can occur and they can all be realized by nilpotent Lie algebras.

Theorem 4.6.1 ([BG86])**.** *Let K be a field of characteristic 0 and let H be a Zariski-closed subgroup of $\mathrm{GL}_n(K)$, where $n \geq 2$. Then there is an n-generator nilpotent Lie algebra $\mathfrak{n}$ over K and a basis for $\mathfrak{n}/[\mathfrak{n},\mathfrak{n}]$ such that $H = \mathrm{Aut}_{\mathrm{ab}}(\mathfrak{n})$. Furthermore, $\mathfrak{n}$ may be taken to be soluble of derived length at most 3.*

Note that in the above discussion, we require our Lie algebra $\mathfrak{g}$ to be finite dimensional. The Lie algebra $\mathfrak{g}^K(\mathcal{G})$ is not finite dimensional, except for when $\mathcal{G}$ has no edges. However, in what follows, we will show that the group $\mathrm{Aut}_{\mathrm{ab}}(\mathfrak{g}^K(\mathcal{G}))$ is equal to $\mathrm{Aut}_{\mathrm{ab}}(\mathfrak{n}^K(\mathcal{G},c))$ for any $c > 1$. The Lie algebras $\mathfrak{n}^K(\mathcal{G},c)$ are finite dimensional and thus we find that also the group $\mathrm{Aut}_{\mathrm{ab}}(\mathfrak{g}^K(\mathcal{G}))$ is a linear algebraic group.

First, we prove a result which holds in general for any Lie algebra with a Carnot grading.

Proposition 4.6.2. *If $\mathfrak{g}$ is a Lie algebra with a Carnot grading $\mathfrak{g} = \bigoplus_{i=1}^{\infty} \mathfrak{g}_i$, then for any automorphism φ of $\mathfrak{g}$ there exists a graded automorphism ψ of $\mathfrak{g}$ such that $\pi_{\mathrm{ab}}(\varphi) = \pi_{\mathrm{ab}}(\psi)$. Equivalently,*

$$\mathrm{Aut}_{\mathrm{ab}}(\mathfrak{g}) = \pi_{\mathrm{ab}}(\mathrm{Aut}_g(\mathfrak{g})).$$

Proof. Let β be a basis for $\mathfrak{g}_1$. By Lemma 4.3.5, we have an induced map $f : \mathfrak{f}(\beta) \to \mathfrak{g}$ which is surjective and such that its kernel $\ker(f)$ is a graded ideal in $\mathfrak{f}(\beta)$.

Take any $\varphi \in \mathrm{Aut}(\mathfrak{g})$. Note that using the basis β, we have natural vector space identifications

$$\mathfrak{f}_1(\beta) \cong \mathfrak{g}_1 \cong \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$$

and thus we can write $\pi_{\mathrm{ab}}(\varphi) \in \mathrm{GL}(\mathfrak{f}_1(\beta))$. It follows that there exists a graded automorphism $\psi \in \mathrm{Aut}_g(\mathfrak{f}(\beta))$ such that $\psi|_{\mathfrak{f}_1(\beta)} = \pi_{\mathrm{ab}}(\varphi)$.

Consider the diagram

$$\begin{array}{ccc} \mathfrak{f}(\beta) & \xrightarrow{\psi} & \mathfrak{f}(\beta) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ \mathfrak{g} & \xrightarrow{\varphi} & \mathfrak{g} \end{array}$$

Take k elements $X_1, X_2, \dots, X_k \in \mathfrak{f}_1(\beta)$. Note that for any $i \in \{1, \dots, k\}$, there exists an element $Y_i \in [\mathfrak{g}, \mathfrak{g}]$ such that

$$(f \circ \psi)(X_i) = (\varphi \circ f)(X_i) + Y_i.$$

Let us write

$$[X_1, \dots, X_k] = [X_1, [X_2, \dots, [X_{k-1}, X_k] \dots]].$$

Using that $(f \circ \psi)$ and $(\varphi \circ f)$ are homomorphisms, it is not hard to show that also

$$(f \circ \psi)([X_1, \dots, X_k]) = (\varphi \circ f)([X_1, \dots, X_k]) + Z$$

for some $Z \in \gamma_{k+1}(\mathfrak{g})$. Since the elements $[X_1, \dots, X_k]$ with X_i ranging in $\mathfrak{f}_1(\beta)$ span $\mathfrak{f}_k(\beta)$ as a vector space, it follows that for any $X \in \mathfrak{f}_k(\beta)$ it holds that $(f \circ \psi)(X) - (\varphi \circ f)(X) \in \gamma_{k+1}(\mathfrak{g})$. In particular, for any $X \in (\ker(f))_k$, we get that $(f \circ \psi)(X) \in \gamma_{k+1}(\mathfrak{g})$. Note that since the grading is Carnot, $\gamma_{k+1}(\mathfrak{g}) = \bigoplus_{i>k} \mathfrak{g}_i$ and that also $(f \circ \psi)(X) \in \mathfrak{g}_k$ since both ψ and f preserve the gradings. From $\gamma_{k+1}(\mathfrak{g}) \cap \mathfrak{g}_k = \{0\}$, we must thus conclude that $(f \circ \psi)(X) = 0$ and thus that $\psi(X) \in \ker(f)$. Since X was chosen arbitrarily in $(\ker(f))_k$ for an arbitrary $k > 0$, it follows that $\psi(\ker(f)) \subset \ker(f)$ and thus that ψ induces a graded homomorphism

$$\overline{\psi} : \mathfrak{g} \to \mathfrak{g}.$$

By applying the exact same construction to φ^{-1}, we get another graded homomorphism from $\mathfrak{g}$ to $\mathfrak{g}$ which can easily be seen to be the inverse of $\overline{\psi}$. Therefore $\overline{\psi}$ is a graded automorphism of $\mathfrak{g}$. At last, it follows by construction that $\pi_{\mathrm{ab}}(\overline{\psi}) = \pi_{\mathrm{ab}}(\varphi)$, which completes the proof. □

Proposition 4.6.3 ([DW23a]). *Let $\mathcal{G} = (V, E)$ be a graph and K a field. For any $c > 1$, we have*

$$\mathrm{Aut}_{\mathrm{ab}}(\mathfrak{g}^K(\mathcal{G})) = \mathrm{Aut}_{\mathrm{ab}}(\mathfrak{n}^K(\mathcal{G}, c)),$$

where we identify the abelianization of $\mathfrak{g}^K(\mathcal{G})$ with the abelianization of $\mathfrak{n}^K(\mathcal{G}, c)$ using the common basis of vertices V.

Proof. Note that every automorphism of $\mathfrak{g}^K(\mathcal{G})$ induces an automorphism on $\mathfrak{n}^K(\mathcal{G}, c)$ (since $\gamma_{c+1}(\mathfrak{g}^K(\mathcal{G}))$ is a characteristic ideal) which has the same projection onto the abelianization. Therefore, we are only left to prove the inclusion from right to left.

Let φ be an automorphism of $\mathfrak{n}^K(\mathcal{G}, c)$. By Proposition 4.6.2 above, we can assume that φ is a graded automorphism of $\mathfrak{n}^K(\mathcal{G}, c)$, with respect to its standard Carnot grading as given in Section 4.3.1. Note that using the basis of vertices, we have an identification of vector spaces $\mathfrak{f}_1(V) \cong \mathfrak{n}_1^K(\mathcal{G}, c)$. As a consequence, there exists a unique graded automorphism ψ of $\mathfrak{f}(V)$ such that the restriction of ψ to $\mathfrak{f}_1(V)$ coincides with the restriction of φ to $\mathfrak{n}_1(\mathcal{G}, c)$ under the aforementioned identification. This implies that we have the commutative diagrams

$$\begin{array}{ccc} \mathfrak{f}(V) & \xrightarrow{\psi} & \mathfrak{f}(V) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ \mathfrak{n}(\mathcal{G}, c) & \xrightarrow{\varphi} & \mathfrak{n}(\mathcal{G}, c) \end{array} \qquad \begin{array}{ccc} \mathfrak{f}(V) & \xrightarrow{\psi^{-1}} & \mathfrak{f}(V) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ \mathfrak{n}(\mathcal{G}, c) & \xrightarrow{\varphi^{-1}} & \mathfrak{n}(\mathcal{G}, c) \end{array} \tag{4.6}$$

where $f : \mathfrak{f}(V) \to \mathfrak{n}(\mathcal{G}, c)$ is the induced map from Lemma 4.3.5. We thus have that $\psi(\ker(f)) = \ker(f)$ and since $\ker(f)$ is a graded ideal, also that $\psi((\ker(f))_2) = (\ker(f))_2$. From the definition of $\mathfrak{n}^K(\mathcal{G}, c)$ and using that $c > 1$, we can see that $(\ker(f))_2$ is equal to the vector space span of $\{[v, w] \mid v, w \in V, \{v, w\} \notin E\}$ and thus that $(\ker(f))_2$ generates $\mathfrak{i}(\mathcal{G})$ as a Lie algebra ideal. Recall the definition of $\mathfrak{i}(\mathcal{G})$ from Section 4.2. It thus follows that $\psi(\mathfrak{i}(\mathcal{G})) = \mathfrak{i}(\mathcal{G})$ and that we have an induced automorphism $\overline{\psi} : \mathfrak{g}(\mathcal{G}) \to \mathfrak{g}(\mathcal{G})$. Clearly, we have that $\pi_{\mathrm{ab}}(\overline{\psi}) = \pi_{\mathrm{ab}}(\varphi)$ which completes the proof. □

Corollary 4.6.4 ([DW23a])**.** *For any graph $\mathcal{G} = (V, E)$, any field K and any integer $c > 1$, we have that the images of the groups*

$$\mathrm{Aut}(\mathfrak{g}^K(\mathcal{G})),\ \mathrm{Aut}(\mathfrak{n}^K(\mathcal{G}, c)),\ \mathrm{Aut}_g(\mathfrak{g}^K(\mathcal{G})),\ \mathrm{Aut}_g(\mathfrak{n}^K(\mathcal{G}, c)),$$

under the projection π_{ab} are equal, where we identify the abelianization of $\mathfrak{g}^K(\mathcal{G})$ with the abelianization of $\mathfrak{n}^K(\mathcal{G}, c)$ using the common basis of vertices V.

Let us write $G^K(\mathcal{G})$ for this 'common' projection onto the abelianization. Thus,

$$G^K(\mathcal{G}) = \mathrm{Aut}_{\mathrm{ab}}(\mathfrak{g}^K(\mathcal{G})).$$

As mentioned above, this is a linear algebraic group defined over K. Moreover, since all structural constants of $\mathfrak{g}^K(\mathcal{G})$ lie in $\mathbb{Z}$, if K is of characteristic 0, then $G^K(\mathcal{G})$ is actually defined over $\mathbb{Q}$. In the next section we describe the group $G^K(\mathcal{G})$.

Remark 4.6.5. Since the vertices V generate the whole Lie algebra $\mathfrak{n}^K(\mathcal{G}, c)$, it follows that a graded automorphism $\varphi \in \mathrm{Aut}_g(\mathfrak{n}^K(\mathcal{G}, c))$ is completely determined by its restriction $\varphi|_{\mathrm{span}_K(V)}$. Therefore, the morphism

$$\pi_{\mathrm{ab}} : \mathrm{Aut}_g(\mathfrak{n}^K(\mathcal{G}, c)) \to G^K(\mathcal{G})$$

is a bijection. In fact it is an isomorphism between linear algebraic groups.

4.6.2 Description of $G^K(\mathcal{G})$ for $K \subset \mathbb{C}$

In this section we describe the group $G^K(\mathcal{G})$. This is based on the work in [DM05] and [DM23].

First, let us introduce some new notation. Let $\mathcal{G} = (V, E)$ be a graph and W a vector space over a field K with basis V. Define for any two vertices $v, w \in V$ the linear endomorphism $E_{vw} : W \to W$ by

$$E_{vw}(u) = \begin{cases} v & \text{if } w = u \\ 0 & \text{else} \end{cases}$$

for any $u \in V$. Recall the relation $\prec$ on the vertices V defined in Section 4.1. We define the subgroup $U^K(\mathcal{G}) \leq \mathrm{GL}(W)$ by

$$U^K(\mathcal{G}) = \left\langle \mathrm{Id} + \alpha E_{vw} \,\middle|\, \alpha \in K,\, v, w \in V,\, v \prec w \wedge v \not\succ w \right\rangle.$$

Next, write $\mathrm{Perm}(V)$ for the group of permutations on the vertices V. For any $\sigma \in \mathrm{Perm}(V)$, we write $P(\sigma) \in \mathrm{GL}(W)$ for the linear map defined by

$$P(\sigma)(v) = \sigma(v)$$

for all $v \in V$. Note that P is a group morphism from $\mathrm{Perm}(V)$ to $\mathrm{GL}(W)$. At last, for any vector subspace $W' \subset W$, we identify $\mathrm{GL}(W')$ with its natural subgroup in $\mathrm{GL}(W)$.

We are now ready to formulate the main result of this section which describes the projection of the automorphism group of $\mathfrak{g}^K(\mathcal{G})$ onto the abelianization. It was first proven in [DM05] [DM23] for the two-step partially commutative Lie algebra $\mathfrak{n}^K(\mathcal{G}, 2)$, but by Corollary 4.6.4 we can extend their result to the partially commutative Lie algebra $\mathfrak{g}^K(\mathcal{G})$ (and thus also to any partially commutative nilpotent Lie algebra $\mathfrak{n}^K(\mathcal{G}, c)$).

Note that the abelianization of $\mathfrak{g}^K(\mathcal{G})$ is a vector space over K which has the vertices V of $\mathcal{G}$ as a basis (by identifying them with their projections). Thus, the notation as introduced above applies. Recall from Section 4.1 that $\Lambda_{\mathcal{G}}$ denotes the set of coherent components of the graph $\mathcal{G}$ and that $\mathrm{Aut}(\mathcal{G})$ denotes the automorphism group of the graph $\mathcal{G}$.

Theorem 4.6.6. *Let $\mathcal{G} = (V, E)$ be a graph K a subfield of $\mathbb{C}$. The linear algebraic group $G^K(\mathcal{G})$ is given by*

$$G^K(\mathcal{G}) = P(\mathrm{Aut}(\mathcal{G})) \cdot G_0^K(\mathcal{G}),$$

where $G_0^K(\mathcal{G})$ is the irreducible component of the identity given by

$$G_0^K(\mathcal{G}) = \left(\prod_{\lambda \in \Lambda_{\mathcal{G}}} \mathrm{GL}(\mathrm{span}_K(\lambda)) \right) \cdot U^K(\mathcal{G}).$$

Moreover, the subgroup $U^K(\mathcal{G})$ is equal to the unipotent radical of $G^K(\mathcal{G})$.

Note that

$$P(\mathrm{Aut}(\mathcal{G})) \cap G_0^K(\mathcal{G}) = P\left(\prod_{\lambda \in \Lambda_{\mathcal{G}}} \mathrm{Perm}(\lambda) \right).$$

We thus find that

$$\frac{G^K(\mathcal{G})}{G_0^K(\mathcal{G})} \cong \frac{P(\mathrm{Aut}(\mathcal{G}))}{P\left(\prod_{\lambda \in \Lambda_{\mathcal{G}}} \mathrm{Perm}(\lambda)\right)} \cong \mathrm{Aut}\left(\overline{\mathcal{G}}\right)$$

where the last isomorphism is induced by the exact sequence from (4.1). We thus also have a morphism

$$p : G^K(\mathcal{G}) \to \mathrm{Aut}\left(\overline{\mathcal{G}}\right) \tag{4.7}$$

with kernel $G_0^K(\mathcal{G})$.

Consider a morphism $r : \mathrm{Aut}(\overline{\mathcal{G}}) \to \mathrm{Aut}(\mathcal{G})$ as constructed in Remark 4.1.9 and write $\overline{P} = P \circ r : \mathrm{Aut}(\overline{\mathcal{G}}) \to G^K(\mathcal{G})$. As one can check, one has a 2-fold internal semi-direct product

$$\begin{aligned} G^K(\mathcal{G}) &= \overline{P}(\mathrm{Aut}(\overline{\mathcal{G}})) \cdot \left(\prod_{\lambda \in \Lambda_{\mathcal{G}}} \mathrm{GL}(\mathrm{span}_K(\lambda)) \right) \cdot U^K(\mathcal{G}) \\ &\cong \overline{P}(\mathrm{Aut}(\overline{\mathcal{G}})) \ltimes \left(\prod_{\lambda \in \Lambda_{\mathcal{G}}} \mathrm{GL}(\mathrm{span}_K(\lambda)) \right) \ltimes U^K(\mathcal{G}), \end{aligned} \tag{4.8}$$

where the order in which we take the two semi-direct products does not matter.

Remark 4.6.7. Recall that $\overline{\prec}$ defines a partial order on $\Lambda_{\mathcal{G}}$. Therefore, we can linearly order the coherent components as $\Lambda_{\mathcal{G}} = \{\lambda_1, \ldots, \lambda_r\}$ such that if $\lambda_i \overline{\prec} \lambda_j$ then $i \leq j$. This total order on $\Lambda_{\mathcal{G}}$ can be refined to a total order on V, i.e. a total order $\leq$ on V such that for all $v \in \lambda_i$, $w \in \lambda_j$ with $i < j$ it holds that $v < w$. Theorem 4.6.6 implies that with respect to this ordered basis of vertices, any element of $G^K(\mathcal{G})$ has a matrix representation of the form (where the blocks correspond with the ordered coherent components $\lambda_1, \ldots, \lambda_r$):

$$P(\varphi) \cdot \begin{pmatrix} A_1 & A_{12} & \ldots & A_{1r} \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & A_{r-1r} \\ 0 & \ldots & 0 & A_r \end{pmatrix}$$

where $\varphi \in \mathrm{Aut}(\mathcal{G})$, $A_i \in \mathrm{GL}_{|\lambda_i|}(K)$, $A_{ij} \in K^{|\lambda_i| \times |\lambda_j|}$ and $A_{ij} = 0$ if $\lambda_i \not\overline{\prec} \lambda_j$. The automorphism φ is not necessarily unique, but the induced automorphism on the quotient graph $\overline{\mathcal{G}}$ is unique, as follows from (4.7).

Example 4.6.8. Consider the graph $\mathcal{G}$ from Example 4.1.5. The corresponding groups as matrices with respect to the basis of vertices $\{v_1, \ldots, v_6\}$ are then given by

$$U^K(\mathcal{G}) = \left\{ \begin{pmatrix} 1 & & a & b & & c \\ & 1 & d & e & & f \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & g \\ & & & & & 1 \end{pmatrix} \middle| \; a, b, c, d, e, f, g \in K \right\},$$

$$\prod_{\lambda \in \Lambda_{\mathcal{G}}} \mathrm{GL}(\mathrm{span}_K(\lambda)) = \left\{ \begin{pmatrix} A & & & \\ & B & & \\ & & c & \\ & & & d \end{pmatrix} \middle| \; \begin{matrix} A, B \in \mathrm{GL}_2(K), \\ c, d \in K^* \end{matrix} \right\},$$

$$G_0^K(\mathcal{G}) = \left\{ \begin{pmatrix} A & E & & F \\ & B & & \\ & & c & g \\ & & & d \end{pmatrix} \middle| \; \begin{matrix} A, B \in \mathrm{GL}_2(K), \\ c, d \in K^*, g \in K \\ E \in K^{2\times 2}, F \in K^{2 \times 1} \end{matrix} \right\}$$

and

$$P(\mathrm{Aut}(\mathcal{G})) = \left\langle \begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 0 & 1 & & \\ & & 1 & 0 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} \right\rangle.$$

4.6.3 Vertex-diagonal automorphisms

This section is partially based on section 3.1 from [DW24]. Let $\mathcal{G} = (V, E)$ be a graph, K a field and $\mathfrak{n}^K(\mathcal{G}, c)$ the associated nilpotent Lie algebra over K.

Definition 4.6.9. We say an automorphism f of $\mathfrak{n}^K(\mathcal{G}, c)$ is *vertex diagonal* if there exists a map $\Psi : V \to K^*$ such that

$$f(v) = \Psi(v)v$$

for any vertex $v \in V$.

Note that for any map $\Psi : V \to K^*$, one can define a unique automorphism on the free Lie algebra $\mathfrak{f}^K(V)$ determined by $v \mapsto \Psi(v)v$ for all $v \in V$. As one can check, this map induces an automorphism on the quotient $\mathfrak{n}^K(\mathcal{G}, c)$ and we write this induced automorphism with f_Ψ. It is clear that this automorphism is vertex-diagonal (with associated function $\Psi : V \to K^*$) and we refer to f_Ψ as *the* vertex diagonal automorphism determined by Ψ.

In what follows, we will give a description of all eigenvalues of a vertex-diagonal automorphism.

We define the *weight* of a word $w \in \mathrm{W}(V)$ as the map $e_w : V \to \mathbb{N}$ which assigns to each $v \in V$ the number of times it occurs in the word w. Note that these notions descend nicely to the equivalence classes in $M(\mathcal{G})$ where the *weight* of an element $m = \overline{w} \in M$ is then defined as $e_m := e_w$.

The following lemma shows how one can construct Lyndon elements from connected subsets of the vertices of a graph $\mathcal{G}$. Recall the definition of connected subsets of vertices from Definition 4.1.3.

Lemma 4.6.10. *Let $\mathcal{G} = (V, E)$ be a graph with a total order $\leq$ on V and let $v_1, \ldots, v_k \in V$ be k distinct vertices with $k \geq 2$. If the set $\{v_1, \ldots, v_k\}$ is connected, then there exists a permutation $\sigma \in S_k$ such that for any positive integers $e_1, \ldots, e_k > 0$ the element*

$$\overline{v_{\sigma(1)}^{e_1} v_{\sigma(2)}^{e_2} \cdots v_{\sigma(k)}^{e_k}} \in M(\mathcal{G})$$

is a Lyndon element with respect to the ordering on V.

Proof. First we construct the permutation σ. Define $\sigma(1)$ by $v_{\sigma(1)} = \min\{v_1, \ldots, v_k\}$. Then define the other images of σ inductively by choosing for $\sigma(i)$ an element from $\{1, \ldots, k\} \setminus \{\sigma(1), \ldots, \sigma(i-1)\}$ such that $\{v_{\sigma(1)}, \ldots, v_{\sigma(i)}\}$ is connected. As one can check, this is always possible by using the assumption that $\{v_1, \ldots, v_k\}$ is connected.

Next, we prove that $m = \overline{v_{\sigma(1)}^{e_1} v_{\sigma(2)}^{e_2} \cdots v_{\sigma(k)}^{e_k}}$ is a Lyndon element. To do so, we use part (iv) of Theorem 4.4.6 on the characterization of Lyndon elements. First, note that by the way we constructed σ, it follows that $\operatorname{std}(m) = v_{\sigma(1)}^{e_1} w$ for some word w in the letters $v_{\sigma(2)}, \ldots, v_{\sigma(k)}$. Since $v_{\sigma(1)} < v_{\sigma(i)}$ for all $2 \leq i \leq k$, it follows $\operatorname{std}(m)$ is minimal in its conjugacy class. To see $\operatorname{std}(m)$ is primitive, we use the assumption that $k \geq 2$ and thus that w is not the empty word. This proves that $\operatorname{std}(m)$ is a Lyndon word and as a consequence that m is a Lyndon element. □

On the other hand, for disconnected subsets of vertices, we have the following.

Lemma 4.6.11. *Let $\mathcal{G} = (V, E)$ be a graph with a total order $\leq$ on V and take any non-empty $w \in \mathrm{W}(V)$. If $\operatorname{supp}(w) \subset V$ is disconnected, then the element $\overline{w}$ is not a Lyndon element.*

Proof. Since $\operatorname{supp}(w)$ is disconnected, we can find a partition $\operatorname{supp}(w) = V_1 \sqcup V_2$ with V_1, V_2 non-empty, such that there is no vertex in V_1 which is adjacent to a vertex in V_2. It follows that there exist non-empty words $u_1, u_2 \in \mathrm{W}(V)$ with $\operatorname{supp}(u_1) = V_1$ and $\operatorname{supp}(u_2) = V_2$ such that in $M(\mathcal{G})$, it holds that

$$\overline{w} = \overline{u_1 u_2} = \overline{u_2 u_1}.$$

Since, V_1, V_2 are non-empty, it follows that u_1, u_2 are not the empty word and thus we have shown that $\overline{w}$ is not primitive. By definition, $\overline{w}$ is not a Lyndon element. □

Combining the two lemmas above, we can characterize for a graph $\mathcal{G}$ those weight functions $V \to \mathbb{N}$ that occur as the weight function of a Lyndon element in $\mathrm{LE}(\mathcal{G})$. For any function $f : X \to \mathbb{N}$ we define its *support* as the set $\operatorname{supp}(f) = \{x \in X \mid f(x) \neq 0\}$. For any vertex $v \in V$, recall that its weight is the function $e_v : V \to \mathbb{N}$ which maps v to 1 and maps all other vertices to 0.

Proposition 4.6.12. *Let $\mathcal{G} = (V, E)$ be a graph with a total order $\leq$ on V. The set of all weights of Lyndon elements in $\mathrm{LE}(\mathcal{G})$ is given by*

$$\{e_v \mid v \in V\} \cup \{e : V \to \mathbb{N} \mid 2 \leq |\operatorname{supp}(e)|,\ \operatorname{supp}(e) \textit{ is connected}\}.$$

Proof. This follows from combining Lemma 4.6.10 and Lemma 4.6.11 and by observing that an element of the form $\overline{v^k}$ for any vertex $v \in V$ and any $k > 1$ is not a Lyndon element. Therefore the weights of the form $k \cdot e_v$ (which send v to k and any other vertex to 0) do not occur as the weight of a Lyndon element. □

At last, we characterize the eigenvalues of a vertex-diagonal automorphism. At the heart lies the following fundamental observation of which we omit the proof.

Lemma 4.6.13. *Let $\mathfrak{g}$ be a Lie algebra over a field K and f an automorphism of $\mathfrak{g}$. If x and y are eigenvectors of f with eigenvalues α and β, respectively, then $[x, y]$ is also an eigenvector of f with eigenvalue $\alpha\beta$.*

Let $\mathcal{G} = (V, E)$ be a graph, K a field and $c > 1$ an integer. Let f_Ψ be the vertex-diagonal automorphism on $\mathfrak{n}^K(\mathcal{G}, c)$ determined by the map $\Psi : V \to K^*$. From Theorem 4.4.8 we know that the set $\mathrm{LE}(\mathcal{G}, c)$ maps to a basis in $\mathfrak{n}^K(\mathcal{G}, c)$. Moreover, by using the lemma above, it is easily checked that this is a basis of eigenvectors for the automorphism f_Ψ. It follows from Lemma 4.6.13 that if $m \in \mathrm{LE}(\mathcal{G}, c)$ is a Lyndon element with weight $e : V \to \mathbb{N}$, then the eigenvalue of $(\mathrm{ev}_c \circ \Theta)(m) \in \mathfrak{n}^K(\mathcal{G}, c)$ can be expressed as the product

$$\prod_{v \in V} \Psi(v)^{e(v)}.$$

Together with Proposition 4.6.12 this immediately proves the main result of this section.

Proposition 4.6.14 ([DW24])**.** *Let $\mathcal{G} = (V, E)$ be a graph, $c > 1$ an integer and K a field. Let f_Ψ be the vertex-diagonal automorphism of $\mathfrak{n}^K(\mathcal{G}, c)$ determined by the map $\Psi : V \to K^*$. The set of eigenvalues of f_Ψ is given by*

$$\Psi(V) \cup \left\{ \prod_{v \in V} \Psi(v)^{e(v)} \;\middle|\; \begin{array}{c} e : V \to \mathbb{N},\ \mathrm{supp}(e) \textit{ is connected} \\ |\mathrm{supp}(e)| \geq 2,\ \sum_{v \in V} e(v) \leq c \end{array} \right\}.$$

Let us write $D^K(\mathcal{G}, c)$ for the subgroup of all vertex-diagonal automorphisms in $\mathrm{Aut}(\mathfrak{n}^K(\mathcal{G}, c))$.

Lemma 4.6.15. *The subgroup $D^K(\mathcal{G}, c)$ is a maximal torus of the linear algebraic group $\mathrm{Aut}(\mathfrak{n}^K(\mathcal{G}, c))$.*

Proof. Let us fix a basis for $\mathfrak{n}^K(\mathcal{G}, c)$ determined by $\mathrm{LE}(\mathcal{G}, c)$ (using Theorem 4.4.8) with respect to some total order $\leq$ on V. Following Lemma 4.6.13, every vertex-diagonal automorphism will be represented by a diagonal matrix. As a consequence $D^K(\mathcal{G}, c)$ is equal to the intersection of $\mathrm{Aut}(\mathfrak{n}^K(\mathcal{G}, c))$ with all diagonal matrices with respect to the chosen basis. Thus, it follows that $D^K(\mathcal{G}, c)$ is a torus in the linear algebraic group $\mathrm{Aut}(\mathfrak{n}^K(\mathcal{G}, c))$.

To see that it is a maximal torus, let D be any torus containing $D^K(\mathcal{G}, c)$. Clearly, D must be contained in the centralizer of $D^K(\mathcal{G}, c)$. Let us prove that this centralizer is exactly equal to $D^K(\mathcal{G}, c)$. Take any φ in the centralizer of $D^K(\mathcal{G}, c)$. Let $v_1, \ldots, v_n$ be a list of the vertices according to the ordering on V and let $p_1, \ldots, p_n$ be the first n prime numbers. Define the map $\Psi : V \to K^* : v_i \mapsto p_i$ and let f_Ψ be the associated vertex diagonal automorphism. Since φ and f_Ψ commute, φ must map the vertex v_i to a vector with the same eigenvalue, being p_i. Following Proposition 4.6.14 we see that the only eigenvectors with eigenvalue p_i will be multiples of v_i itself. Thus, φ has to be vertex diagonal, which concludes the proof. $\square$

Chapter 5

Forms of Lie algebras

In this chapter we will deal with Question 3.4.22 from Chapter 3 that reads: given a real (or complex) Lie algebra, what are its rational forms up to isomorphism? In particular, we will investigate this for the family of nilpotent partially commutative Lie algebras from Chapter 4. In Section 5.5, we prove a result on the Galois cohomology of a certain type of semi-direct product (see Theorem 5.5.1). This result is used in Section 5.6 to prove a characterization of the rational (or real) forms of a partially commutative nilpotent Lie algebra (see Theorem 5.6.3). At last, in Section 5.6.5, we determine which of these rational forms are indecomposable (see Theorem 5.6.22).

5.1 Galois theory

In this section we recall some definitions and results from field theory and Galois theory. For a complete introduction on this matter, we refer the reader to [Rot98], [Rom06] and [Mil22]. Recall that a *field* is a commutative ring in which every non-zero element is a unit.

Definition 5.1.1. A *field extension* consists of a pair of fields K and L such that K is a subfield of L and is written as L/K. A *tower of fields* is a sequence of subsequent field extensions K_n/K_{n-1}, $K_{n-1}/K_{n-2}, \dots, K_1/K_0$ and is written as $K_n/K_{n-1}/\dots/K_0$. We say M is an *intermediate field* of L/K if $L/M/K$ is a tower of fields.

Let L/K be a field extension. For any elements $\alpha_1, \dots, \alpha_k \in L$ we will write $K(\alpha_1, \dots, \alpha_k)$ for the smallest subfield of L containing both K and the elements

$\alpha_1, \ldots, \alpha_k$. Obviously, $K(\alpha_1, \ldots, \alpha_k)$ is an intermediate field of L/K.

Note that if L/K is a field extension, then L has the canonical structure of a vector space over K. The *degree of* L/K is defined as the dimension of this vector space and is written as $[L : K]$. Note that the degree of a field extension can be infinite.

For any field K, we let $\mathrm{Aut}(K)$ denote the group of field automorphisms of K. For any subgroup H of $\mathrm{Aut}(K)$, one defines the set

$$K^H = \{\alpha \in K \mid \forall \sigma \in H : \sigma(\alpha) = \alpha\}.$$

It is not hard to see that K^H is a subfield of K. We call K^H the *fixed field of* H. For any two field extensions L/K and L'/K, we say a field morphism $\sigma : L \to L'$ is a K*-morphism* from L to L' if $\sigma(\alpha) = \alpha$ for any $\alpha \in K$. In particular, when $L = L'$, we say σ is a K-automorphism of L. The set of all K-automorphisms of L is a subgroup of $\mathrm{Aut}(L)$ and is written as $\mathrm{Aut}(L/K)$. An element $\alpha \in L$ is called *algebraic over* K if there exists a polynomial $p(X) \in K[X]$ such that $p(\alpha) = 0$. A field extension L/K is called *algebraic* if all elements of L are algebraic over K. A classical result states that all finite degree field extensions are algebraic.

Example 5.1.2.

1. $\mathbb{C}/\mathbb{R}/\mathbb{Q}$ is a tower of fields. The extension $\mathbb{C}/\mathbb{R}$ has degree two and hence is algebraic, while $\mathbb{R}/\mathbb{Q}$ has infinite degree and is not algebraic.

2. For any field K and any irreducible polynomial $f(X) \in K[X]$, the quotient ring $K[X]/(f(X))$ is a field. Moreover, it is a extension field of K using the natural embedding of K in $K[X]/(f(X))$ as classes of constant polynomials.

3. As a special case of the previous example we can consider for any square-free non-zero integer $d \in \mathbb{Z}$ the field $\mathbb{Q}[X]/(X^2 - d)$ which is an extension field of $\mathbb{Q}$. For positive d, this field is isomorphic to the intermediate field $\mathbb{Q}(\sqrt{d})$ of the extension $\mathbb{R}/\mathbb{Q}$. It can be written as the set $\{a + b\sqrt{d} \mid a, b \in \mathbb{Q}\}$. For negative d, this field is isomorphic to the intermediate field $\mathbb{Q}(i\sqrt{|d|})$ of the extension $\mathbb{C}/\mathbb{Q}$. It can be written as the set $\{a + bi\sqrt{|d|} \mid a, b \in \mathbb{Q}\}$. These are in fact all the degree 2 field extensions of $\mathbb{Q}$, up to $\mathbb{Q}$-isomorphism.

Definition 5.1.3. A field extension L/K is said to be *Galois* if it is algebraic and $L^{\mathrm{Aut}(L/K)} = K$. In this case we write $\mathrm{Gal}(L/K)$ for $\mathrm{Aut}(L/K)$.

An equivalent condition for an algebraic extension L/K to be Galois is that for any $\alpha \in L \setminus K$, there exists a $\sigma \in \mathrm{Aut}(L/K)$ such that $\sigma(\alpha) \neq \alpha$.

Definition 5.1.4. Let K be a field and $f \in K[X]$ a polynomial of degree k. The polynomial f is said to *split over* K if there exist $\alpha_0, \alpha_1, \ldots, \alpha_k \in K$ such that

$$f(X) = \alpha_0(X - \alpha_1) \cdot \ldots \cdot (X - \alpha_k).$$

A *splitting field of* f over K is an extension field L of K such that $f \in K[X] \subset L[X]$ splits over L and L is generated by K and the roots of f.

The splitting field of f over K is unique up to K-isomorphism and is written as $K(f)$.

Definition 5.1.5. Let K be a field. A polynomial $f \in K[X]$ is called *separable* if all its roots in some splitting field are distinct. An extension L/K is called *separable* if for any $\alpha \in L$, the minimal polynomial of α over K is separable.

There is an alternative characterization of separable polynomials using the derivative of a polynomial. A polynomial $f \in K[X]$ is separable if and only if it is coprime with its derivative f'. This characterizations leads to the following observation. If $f \in K[X]$ is irreducible and separable, then f divides its derivative f', but since f' has strictly smaller degree, we must have that $f' = 0$. If in addition the field K is assumed to be of characteristic 0, this is not possible and thus all irreducible polynomials over K are separable.

Remark 5.1.6. If K is a field of characteristic zero, then every extension of K is separable.

Definition 5.1.7. A field K is said to be *perfect* if any irreducible polynomial in $K[X]$ is separable.

Example 5.1.8. Fields of characteristic 0, finite fields and algebraically closed fields are perfect.

Note that the definition of a Galois extension (Definition 5.1.3) does not require L/K to have finite degree. If we do make this assumption, there is a nice characterization of Galois extensions using splitting fields.

Theorem 5.1.9. *A field extension L/K of finite degree is Galois if and only if L is a splitting field of some separable polynomial over K.*

Proof. See Theorem 81 in [Rot98]. □

Definition 5.1.10. A field L is said to be *algebraically closed* if every polynomial in $L[X]$ is split over L. *An algebraic closure* of a field K is an algebraic field extension L/K such that L is algebraically closed.

Theorem 5.1.11. *Every field K has an algebraic closure which is unique up to a field isomorphism which fixes the elements of K.*

Proof. See Theorem 2.7.2 and Corollary 2.8.5 in [Rom06]. □

Since an algebraic closure is essentially unique, we will speak of *the* algebraic closure of a field K and write it as $\overline{K}$.

The following theorem, which is also known as the *fundamental theorem of Galois theory*, can also be found in [Rot98, Theorem 84].

Theorem 5.1.12 (Galois correspondence)**.** *Let L/K be a finite degree Galois extension. The assignment $H \mapsto L^H$ sets up an inclusion reversing one-to-one correspondence between the subgroups of $\mathrm{Gal}(L/K)$ and the intermediate fields of L/K. The inverse of this correspondence is given by $M \mapsto \mathrm{Gal}(L/M)$. Moreover, it holds that:*

(i) For any subgroup H of $\mathrm{Gal}(L/K)$, we have $|H| = [L : L^H]$.

(ii) A subgroup H of $\mathrm{Gal}(L/K)$ is normal if and only if L^H/K is a Galois extension. If H is normal, the map

$$\mathrm{Gal}(L/K)/H \to \mathrm{Gal}(L^H/K) : \sigma H \mapsto \sigma|_{L^H}$$

is an isomorphism of groups.

Galois theory in infinite degree

To generalize the Galois correspondence to extensions of infinite degree, one needs to define a topology on the Galois group.

Definition 5.1.13. Let L/K be a Galois extension. We define the basis of open sets in $\mathrm{Gal}(L/K)$ to be the collection of sets of the form $\sigma\,\mathrm{Gal}(L/M)$, where $\sigma \in \mathrm{Gal}(L/K)$ and M is an intermediate field of L/K such that M/K is Galois of finite degree. The topology induced by this basis of open sets is called the *Krull topology*.

Note that in case L/K is a finite degree Galois extension, the Krull topology on $\mathrm{Gal}(L/K)$ is simply the discrete topology.

For the remainder of this thesis, any Galois group will be assumed to be endowed with the Krull topology.

Theorem 5.1.14. *Let L/K be a Galois extension. The assignment $H \mapsto L^H$ sets up an inclusion reversing one-to-one correspondence between the closed subgroups of* $\mathrm{Gal}(L/K)$ *and the intermediate fields of L/K. The inverse of this correspondence is given by $M \mapsto \mathrm{Gal}(L/M)$. Moreover, it holds that:*

(i) For any closed subgroup H of $\mathrm{Gal}(L/K)$*, we have $[L^H : K] < \infty$ if and only if H is open.*

(ii) A closed subgroup H of $\mathrm{Gal}(L/K)$ *is normal if and only if L^H/K is a Galois extension. If H is normal, the map*

$$\mathrm{Gal}(L/K)/H \to \mathrm{Gal}(L^H/K) : \sigma H \mapsto \sigma|_{L^H}$$

is an isomorphism of groups.

5.2 Forms of Lie algebras

The discussion in this section is partially based on section 3 from [DW23a]. In this section, all Lie algebras are assumed to be finite dimensional.

Let L/K be an extension of fields. Consider a Lie algebra $\mathfrak{h}$ defined over the smaller field K. We can define a new Lie algebra $\mathfrak{h}^L$ over the field L by the tensor product

$$\mathfrak{h}^L := \mathfrak{h} \otimes_K L.$$

Clearly, $\mathfrak{h}^L$ is now a vector space over L. The Lie bracket on $\mathfrak{h}^L$ is defined by

$$[v \otimes \alpha, w \otimes \beta] = [v, w] \otimes \alpha\beta$$

for any $v, w \in \mathfrak{h}$ and $\alpha, \beta \in L$ and extending this additively to all of $\mathfrak{h}^L$.

Remark 5.2.1. Note that any basis $v_1, \ldots, v_n$ of $\mathfrak{h}$ (as vector space over K) gives a basis $v_1 \otimes 1, \ldots, v_n \otimes 1$ for $\mathfrak{h}^L$ (as vector space over L). Thus we have that $\dim_K(\mathfrak{h}) = \dim_L(\mathfrak{h}^L)$.

Definition 5.2.2. Let L/K be a field extension and $\mathfrak{g}$ a Lie algebra over L. A Lie algebra $\mathfrak{h}$ defined over the field K is called a *K-form* of $\mathfrak{g}$ if the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}^L = \mathfrak{h} \otimes_K L$ are isomorphic (as Lie algebras over L).

For any field extension L/K and Lie algebra $\mathfrak{g}$ over L, the above definition gives a category $\mathbf{Form}_K(\mathfrak{g})$ of which the objects are the K-forms of $\mathfrak{g}$ and the morphisms are Lie algebra morphisms (over the small field K). Clearly $\mathbf{Form}_K(\mathfrak{g})$ is a subcategory of $\mathbf{LieAlg}_K$. Note that $\mathbf{Form}_K(\mathfrak{g})$ is not a small category, i.e. a category of which the objects and the morphisms are both sets. However, the isomorphism classes do form a set. We write the set of isomorphism classes of $\mathbf{Form}_K(\mathfrak{g})$ as

$$E(L/K, \mathfrak{g}).$$

We write the elements of $E(L/K, \mathfrak{g})$ as $[\mathfrak{h}]$ where $\mathfrak{h}$ is an object of $\mathbf{Form}_K(\mathfrak{g})$. To see why $E(L/K, \mathfrak{g})$ is actually a set, note that if $\mathfrak{g}$ has dimension n, so does every Lie algebra in $\mathbf{Form}_K(\mathfrak{g})$ (see Remark 5.2.1). Now consider $(K^n)^3$ with elements written as $(a_{ij}^k)_{1 \leq i,j,k \leq n}$. The group $\mathrm{GL}_n(K)$ acts on $(K^n)^3$ by

$$(A \cdot x)_{ij}^k = \sum_{r,s,t} (A^{-1})_{kr}\, A_{si}\, A_{tj}\, a_{st}^r$$

where $x = (a_{ij}^k)_{ijk}$ and $A \in \mathrm{GL}_n(K)$. Any Lie algebra in $\mathbf{Form}_K(\mathfrak{g})$ now gives an element in $(K^n)^3$ by considering its structure constants with respect to some basis. Since one chooses a basis, this correspondence is not uniquely defined, but it does induce a well-defined injective map from $E(L/K, \mathfrak{g})$ to $\mathrm{GL}_n(K) \backslash (K^n)^3$. It follows that $E(L/K, \mathfrak{g})$ is a set.

Remark 5.2.3. In particular, the above discussion shows that if K is countable, then $E(L/K, \mathfrak{g})$ is also countable.

In this chapter, the goal is to determine what the set $E(L/K, \mathfrak{g})$ looks like for certain Lie algebras $\mathfrak{g}$. In what follows, the Lie algebra $\mathfrak{g}$ will also be defined over the small field K and thus we will try to determine the set $E(L/K, \mathfrak{g}^L)$ for a field extension L. If $L/M/K$ is a tower of fields, then we have a canonical inclusion

$$E\left(M/K, \mathfrak{g}^M\right) \hookrightarrow E\left(L/K, \mathfrak{g}^L\right). \tag{5.1}$$

Thus, bigger field extensions give bigger sets of isomorphism classes of K-forms. Due to Hilbert's Nulstellensatz, we know that the algebraic closure of $\overline{K}$ is the biggest field that one needs to consider.

Theorem 5.2.4 (Weak Nullstellensatz)**.** *If K is an algebraically closed field and I is a proper ideal of $K[X_1, \ldots, X_n]$ then $V(I) \neq \emptyset$.*

Proof. See for instance [Bum98, Theorem 1.1]. □

From this theorem we deduce the following lemma and corollary. For a field extension L/K and Lie algebras $\mathfrak{g}$, $\mathfrak{h}$ defined over K, we say $\mathfrak{g}$ *and* $\mathfrak{h}$ *are isomorphic over* L if the Lie algebras $\mathfrak{g}^L$ and $\mathfrak{h}^L$ are isomorphic.

Lemma 5.2.5. *Let L/K be a field extension and $\mathfrak{g}$, $\mathfrak{h}$ Lie algebras defined over K. If $\mathfrak{g}$ and $\mathfrak{h}$ are isomorphic over L, then they are also isomorphic over the algebraic closure of K. Moreover, there exists a finite degree field extension M/K such that $\mathfrak{g}$ and $\mathfrak{h}$ are isomorphic over M.*

Proof. The following is based on the proof of Lemma 3.5. of [Sul23]. Assume that both $\mathfrak{g}$ and $\mathfrak{h}$ are of dimension n. Let $a_{ij}^k, b_{ij}^k \in (K^n)^3$ be structure constants with respect to some basis for $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Then $\mathfrak{g}$ and $\mathfrak{h}$ are isomorphic if and only if there exists a matrix $A_{ij} \in \mathrm{GL}_n(K)$ such that

$$\forall i,j,k \in \{1,\dots,n\} : \quad \sum_{r=1}^{n} a_{ij}^r A_{kr} - \sum_{s,t=1}^{n} b_{st}^k A_{si} A_{tj} = 0. \tag{5.2}$$

If we consider the ideal I in the polynomial ring $R = K[d, A_{ij} \mid 1 \le i,j \le n]$ generated by the polynomials given in (5.2) and the polynomial $\det(A_{ij})d - 1$, then $\mathfrak{g}$ and $\mathfrak{h}$ are isomorphic if and only if the polynomials in I have a common zero in K^{n^2+1}. If we assume that $\mathfrak{g}$ and $\mathfrak{h}$ are isomorphic over L, then this means that the polynomials in I have a common zero in L^{n^2+1}. In particular, the ideal I is a proper ideal of R. Hilbert's weak Nullstellensatz thus tells us that the polynomials in I also have a common zero in $\overline{K}^{n^2+1}$, with $\overline{K}$ the algebraic closure of K. Since $\overline{K}$ is a union of intermediate finite degree extensions of K, there exists a finite degree extension M/K such that this common zero lies in M^{n^2+1}. □

Corollary 5.2.6. *Let $L/\overline{K}/K$ be a tower of fields where $\overline{K}$ is the algebraic closure of K and let $\mathfrak{g}$ be a Lie algebra defined over K. The inclusion*

$$E\left(\overline{K}/K, \mathfrak{g}^{\overline{K}}\right) \hookrightarrow E\left(L/K, \mathfrak{g}^L\right)$$

is surjective and thus one-to-one.

Remark 5.2.7. Recall from Section 2.3.3 of Chapter 2 that we defined *rational forms of real Lie algebras* as subsets of that Lie algebra satisfying certain axioms. Let us argue how that definition is consistent with the definition of forms from above. Let L/K be a field extension, $\mathfrak{g}$ a Lie algebra over L and $\mathfrak{h}$ a K-form of $\mathfrak{g}$. Then, by definition, there is a Lie algebra isomorphism $f : \mathfrak{h}^L \to \mathfrak{g}$. Composing this isomorphism with the inclusion

$$\mathfrak{h} \hookrightarrow \mathfrak{h}^L : v \mapsto v \otimes 1,$$

we obtain an inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$. Thus $\mathfrak{h}$ can be seen as a subset of $\mathfrak{g}$ in this way. As one can check, this subset is

(i) closed under K-linear combinations,

(ii) closed under the Lie bracket of $\mathfrak{g}$ and

(iii) has a basis (as vector space over K) which is also a basis for $\mathfrak{g}$ (as vector space over L).

These are exactly the three axioms that a subset of a real ($L = \mathbb{R}$) Lie algebra had to satisfy to be called a rational ($K = \mathbb{Q}$) form of $\mathfrak{g}$.

Conversely, given a Lie algebra $\mathfrak{g}$ over L and a subset $\mathfrak{h} \subset \mathfrak{g}$ satisfying the above three axioms, then $\mathfrak{h}$ inherits a Lie algebra structure from $\mathfrak{g}$ and we get an isomorphism of Lie algebras

$$\mathfrak{h}^L \to \mathfrak{g} : v \otimes \alpha \mapsto \alpha v.$$

Thus $\mathfrak{h}$ is a K-form of $\mathfrak{g}$ (following Definition 5.2.2).

5.3 Galois cohomology

In this section we define the first (non-abelian) Galois cohomology set. These sets are defined for groups equipped with a Galois action. Applying this to the automorphism group of a Lie algebra, Galois cohomology serves as a tool to classify forms of Lie algebras, as we will see in the next section. The discussion is partially based on Section 3 from [DW23a]. For a thorough introduction to the theory of Galois cohomology and Galois descent, we refer the reader to [Ser97] and [Ber10].

Recall from Definition 5.1.13 that any Galois group is equipped with the Krull topology.

Definition 5.3.1. Let L/K be a Galois extension and G a group. We say that G is a $\mathrm{Gal}(L/K)$*-group* if G is equipped with an action $f : \mathrm{Gal}(L/K) \times G \to G$ satisfying that

(i) f is continuous for the Krull topology on $\mathrm{Gal}(L/K)$ and the discrete topology on G and

(ii) the action is one by group automorphisms, i.e. $f(\sigma, g_1g_2) = f(\sigma, g_1)f(\sigma, g_2)$ for any $\sigma \in H, g_1, g_2 \in G$.

We will write such an action as $f(\sigma, g) = {}^{\sigma}g$.

Remark 5.3.2. As one can check, condition (i) is equivalent to all stabilizers of the $\mathrm{Gal}(L/K)$-action on G being open in $\mathrm{Gal}(L/K)$.

Remark 5.3.3. Whenever we have a map from $\mathrm{Gal}(L/K)$ to a set X (with possibly more structure), we will say it is *continuous* if it is so for the Krull topology on $\mathrm{Gal}(L/K)$ and the discrete topology on X, unless otherwise stated. Note that when L/K is a finite degree Galois extension, the Krull topology is the discrete topology and thus, in that case, any map starting from $\mathrm{Gal}(L/K)$ is considered continuous.

These $\mathrm{Gal}(L/K)$-groups can occur quite naturally as illustrated by the next examples.

Example 5.3.4. Consider a Galois extension L/K.

1. Write L_a for the additive group of the field L. The action ${}^{\sigma}(\alpha) = \sigma(\alpha)$ for any $\sigma \in \mathrm{Gal}(L/K)$ and $\alpha \in L$ turns L_a into a $\mathrm{Gal}(L/K)$-group.
2. Write $L^* = L \setminus \{0\}$ for the multiplicative group of the field L. With the same action as the previous example, L^* is a $\mathrm{Gal}(L/K)$-group.
3. The group $\mathrm{GL}_n(L)$ has a $\mathrm{Gal}(L/K)$-group structure by defining a coefficient-wise action:
$$ {}^{\sigma}\Big((a_{ij})_{ij}\Big) = (\sigma(a_{ij}))_{ij} $$
for any $\sigma \in \mathrm{Gal}(L/K)$ and $(a_{ij})_{ij} \in \mathrm{GL}_n(L)$.
4. In general, when $G \subset \mathrm{GL}_n(L)$ is a linear algebraic group defined over the small field K (see Section 4.5 from Chapter 4), then G is a $\mathrm{Gal}(L/K)$-group with the $\mathrm{Gal}(L/K)$-action defined by restricting the action on $\mathrm{GL}_n(L)$ from the previous example to G. Thus, other examples are $\mathrm{SL}_n(L), \mathrm{O}_n(L), \mathrm{SO}_n(L)$, ...

Definition 5.3.5. Let L/K be a Galois extension and G a $\mathrm{Gal}(L/K)$-group. A continuous map
$$ \rho : \mathrm{Gal}(L/K) \to G : \sigma \mapsto \rho_\sigma, $$
is called a *cocycle* if it satisfies the relation
$$ \rho_{\sigma\tau} = \rho_\sigma \, {}^{\sigma}\rho_\tau $$
for all $\sigma, \tau \in \mathrm{Gal}(L/K)$. The set of cocycles is denoted with $Z^1(L/K, G)$. Two cocycles $\rho, \eta \in Z^1(L/K, G)$ are said to be *equivalent* if there exists a $g \in G$ such that
$$ g\rho_\sigma ({}^{\sigma}g)^{-1} = \eta_\sigma $$

for all $\sigma \in \mathrm{Gal}(L/K)$. An equivalence class of a cocycle $\rho \in Z(L/K, G)$ is written as $[\rho]$ and the set of equivalence classes of cocycles is written as $H^1(L/K, G)$ and is called the *first Galois cohomology set.*

Remark 5.3.6. The first Galois cohomology set is never empty, as we always have the equivalence class of the *trivial cocycle* $\mathrm{Gal}(L/K) \to G : \sigma \mapsto e_G$. In fact this turns $H^1(L/K, G)$ into a pointed set, with the class of the trivial cocycle being the distinguished element. We will call $H^1(L/K, G)$ *trivial* if it only consists of this one element.

Now we give two classical theorems providing examples of $\mathrm{Gal}(L/K)$-groups which have trivial Galois cohomology.

Theorem 5.3.7. *Let L/K be a Galois extension and L_a the additive group of the field L. Then $H^1(L/K, L_a)$ is trivial.*

Proof. This is exactly [Ser97, p.72, Proposition 1]. □

Theorem 5.3.8 (Generalized Hilbert's theorem 90)**.** *Let L/K be a Galois extension and consider the general linear group $\mathrm{GL}_n(L)$. Then $H^1(L/K, \mathrm{GL}_n(L))$ is trivial.*

Proof. This is exactly [Ser97, p.122, Lemma 1]. □

Recall the definition of a unipotent linear algebraic group from Section 4.5.

Theorem 5.3.9. *Let L/K be a Galois extension with K a perfect field. Let $U \leq \mathrm{GL}_n(L)$ be a unipotent linear algebraic group defined over K. Then $H^1(L/K, U)$ is trivial.*

Proof. This is exactly [Ser97, p.128, Proposition 6]. □

Let L/K be a Galois extension, G_1, G_2 two $\mathrm{Gal}(L/K)$-groups and $f : G_1 \to G_2$ a $\mathrm{Gal}(L/K)$-equivariant group morphism, i.e.

$$f({}^\sigma g) = {}^\sigma f(g) \quad \text{and} \quad f(gh) = f(g)f(h)$$

for all $\sigma \in \mathrm{Gal}(L/K)$ and $g, h \in G_1$. There is a well-defined induced map on cohomology which we write as f_* and is defined by

$$f_* : H^1(L/K, G_1) \to H^1(L/K, G_2) : [\rho] \mapsto [f \circ \rho].$$

By using the above theorem on unipotent algebraic groups together with a twisting argument (as explained in chapter I, 5 of [Ser97]) one obtains the following result, which is also described in section 3 of [GS84].

Theorem 5.3.10. *Let L/K be a Galois extension with K a field of characteristic zero. Let $G \leq \mathrm{GL}_n(L)$ be a linear algebraic group defined over K and $U \leq G$ any closed normal unipotent subgroup also defined over K. Write $p : G \to G/U$ for the projection morphism. The induced map*

$$p_* : H^1(L/K, G) \to H^1(L/K, G/U)$$

is a bijection.

5.4 Galois descent for Lie algebras

The discussion in this section is partially based on section 3 from [DW23a]. Let L/K be a Galois extension and $\mathfrak{g}$ a Lie algebra defined over the small field K. There is a natural action of $\mathrm{Gal}(L/K)$ on $\mathfrak{h}^L = \mathfrak{h} \otimes_K L$ defined by

$$^{\sigma}(v \otimes \alpha) = v \otimes \sigma(\alpha) \tag{5.3}$$

for any $v \in \mathfrak{h}, \alpha \in L$ and $\sigma \in \mathrm{Gal}(L/K)$ and extending this additively to all of $\mathfrak{h}^L$. As one can check, this action satisfies for all $v, w \in \mathfrak{h}^L$, $\alpha \in L$ and $\sigma \in \mathrm{Gal}(L/K)$:

(i) $^{\sigma}(\alpha v) = \sigma(\alpha)\, {}^{\sigma}v$,

(ii) $^{\sigma}(v + w) = {}^{\sigma}v + {}^{\sigma}w$,

(iii) $^{\sigma}[v, w] = [{}^{\sigma}v, {}^{\sigma}w]$.

Conditions (i) and (ii) tell us that the map $v \mapsto {}^{\sigma}v$ is a so-called *semi-linear map* on $\mathfrak{h}^L$. We can thus say that $\mathrm{Gal}(L/K)$ has an action on $\mathfrak{h}^L$ by semi-linear maps. Note that, using this action, the image of the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{h}^L : v \mapsto v \otimes 1$ is characterized as the set

$$\{v \in \mathfrak{h}^L \mid \forall \sigma \in \mathrm{Gal}(L/K) :\ {}^{\sigma}v = v\}.$$

If $\mathfrak{g}$ and $\mathfrak{h}$ are two Lie algebras defined over the field K, we denote the set of Lie algebra homomorphisms (over L) from $\mathfrak{h}^L$ to $\mathfrak{m}^L$ by $\mathrm{Hom}(\mathfrak{g}^L, \mathfrak{h}^L)$. The actions of $\mathrm{Gal}(L/K)$ on $\mathfrak{g}^L$ and $\mathfrak{m}^L$ as described by equation (5.3) above, also induce an action of $\mathrm{Gal}(L/K)$ on $\mathrm{Hom}(\mathfrak{g}^L, \mathfrak{h}^L)$. This action is defined by

$$({}^{\sigma}\varphi)(v) := {}^{\sigma}\left(\varphi\left({}^{\sigma^{-1}}v\right)\right)$$

for any $\varphi \in \mathrm{Hom}(\mathfrak{g}^L, \mathfrak{h}^L), \sigma \in \mathrm{Gal}(L/K)$ and $v \in \mathfrak{g}^L$. From properties (i), (ii) and (iii) above, it follows that ${}^{\sigma}\varphi$ is indeed again a Lie algebra homomorphism.

Note that if $\mathfrak{k}$ is a third Lie algebra over K, the equality ${}^{\sigma}(\phi\varphi) = {}^{\sigma}\phi\,{}^{\sigma}\varphi$ holds for any $\varphi \in \mathrm{Hom}(\mathfrak{g}^L, \mathfrak{h}^L)$ and $\phi \in \mathrm{Hom}(\mathfrak{h}^L, \mathfrak{k}^L)$. In particular, the action of $\mathrm{Gal}(L/K)$ on the invariant subset $\mathrm{Aut}(\mathfrak{g}^L) \subset \mathrm{Hom}(\mathfrak{g}^L, \mathfrak{g}^L)$ is one by group automorphisms, where the group operation on $\mathrm{Aut}(\mathfrak{g}^L)$ is composition. Thus, this action turns $\mathrm{Aut}(\mathfrak{g}^L)$ into a $\mathrm{Gal}(L/K)$-group. Therefore we can talk about the associated first Galois cohomology set

$$H^1(L/K, \mathrm{Aut}(\mathfrak{g}^L)).$$

Next, we discuss the connection between this set and the set $E(L/K, \mathfrak{g}^L)$ from Section 5.2. To each cocycle $\rho : \mathrm{Gal}(L/K) \to \mathrm{Aut}(\mathfrak{n}^L)$, one can associate a subset $\mathfrak{g}_\rho \subset \mathfrak{g}^L$ by

$$\mathfrak{g}_\rho := \{v \in \mathfrak{g}^L \mid \forall \sigma \in \mathrm{Gal}(L/K) : \rho_\sigma({}^{\sigma}v) = v\}. \tag{5.4}$$

From properties (i), (ii), (iii) on page 113 and the fact that each ρ_σ is an automorphism of $\mathfrak{g}^L$, it follows that $\mathfrak{g}_\rho$ is closed under taking K-linear combinations and the Lie bracket. Therefore we have that $\mathfrak{g}_\rho$ is indeed a Lie algebra defined over K. We still need to check that $\mathfrak{g}_\rho \otimes L \cong \mathfrak{g}^L$. For this consider the map $\mathfrak{g}_\rho \otimes L \to \mathfrak{g}^L : v \otimes l \mapsto lv$. It is a standard result that this map is an L-vector space isomorphism as proven for example in [Ber10, Lemma III.8.21.]. It is then also straightforward to check this map preserves the Lie bracket. As it turns out, the K-forms constructed in this way are all the possible K-forms of $\mathfrak{g}^L$ up to K-isomorphism.

Theorem 5.4.1 (Galois descent for Lie algebras)**.** *Let L/K be a Galois extension and $\mathfrak{g}$ a Lie algebra defined over K. The map*

$$H^1\left(L/K, \mathrm{Aut}\left(\mathfrak{g}^L\right)\right) \to E(L/K, \mathfrak{g}^L) : [\rho] \mapsto [\mathfrak{g}_\rho]$$

is a bijection, which sends the trivial cocycle to $[\mathfrak{g}]$.

Proof. See [GS84, Theorem 1.3 and 1.4] or [Ber10, Proposition III.9.1., Remark III.9.2. and Remark III.9.8.] for a proof of this statement. □

To see how the inverse of this map works, let $\mathfrak{h}$ be a K-form of $\mathfrak{g}^L$. By definition, there exists an isomorphism $f : \mathfrak{h}^L \to \mathfrak{g}^L$. Then we can associate to $\mathfrak{h}$ a cocycle $\rho^{\mathfrak{h}} \in Z^1(L/K, \mathrm{Aut}(\mathfrak{g}^L))$ defined by

$$\rho^{\mathfrak{h}}_\sigma = f\left({}^{\sigma}f\right)^{-1}$$

for all $\sigma \in \mathrm{Gal}(L/K)$. Of course the cocycle $\rho^{\mathfrak{h}}$ depends on the choice of isomorphism f, but its class $[\rho^{\mathfrak{h}}]$ in $H^1\left(L/K, \mathrm{Aut}\left(\mathfrak{g}^L\right)\right)$ does not. In fact, this class only depends on the equivalence class of h in $E(L/K, \mathfrak{g}^L)$. The inverse of the map from Theorem 5.4.1 is then given by

$$E(L/K, \mathfrak{g}^L) \to H^1(L/K, \mathrm{Aut}(\mathfrak{g}^L)) : [\mathfrak{h}] \mapsto \left[\rho^{\mathfrak{h}}\right].$$

Next, we discuss how the inclusion of equivalence classes of forms from (5.1) works on the cohomology level in case of Galois extensions. Let $L/M/K$ be a tower of fields where both L/M and M/K are Galois and $\mathfrak{g}$ a Lie algebra defined over K. From Theorem 5.1.14 we know that there is a quotient morphism

$$\pi : \mathrm{Gal}(L/M) \to \mathrm{Gal}(M/K) : \sigma \mapsto \sigma|_M.$$

As one can check, this gives a well-defined inclusion on the associated Galois cohomology

$$H^1(M/K, \mathrm{Aut}(\mathfrak{g}^M)) \hookrightarrow H^1(L/K, \mathrm{Aut}(\mathfrak{g}^L)) : [\rho] \mapsto [\rho \circ \pi].$$

In fact, this map, the inclusion from 5.1 and the map from Theorem 5.4.1 fit in a commutative diagram as follows:

$$\begin{array}{ccc} H^1(M/K, \mathrm{Aut}(\mathfrak{g}^M)) & \hookrightarrow & H^1(L/K, \mathrm{Aut}(\mathfrak{g}^L)) \\ \downarrow & & \downarrow \\ E(M/K, \mathfrak{g}^M) & \hookrightarrow & E(L/K, \mathfrak{g}^L). \end{array}$$

Another map that one can consider on the level of Galois groups is the inclusion:

$$i : \mathrm{Gal}(L/M) \to \mathrm{Gal}(L/K) : \sigma \mapsto \sigma.$$

Using the canonical isomorphism $\left(\mathfrak{g}^M\right)^L \cong \mathfrak{g}^L$, we obtain an induced map on cohomology

$$H^1(L/K, \mathrm{Aut}(\mathfrak{g}^L)) \to H^1(L/M, \mathrm{Aut}(\mathfrak{g}^L)) : [\rho] \mapsto [\rho \circ i].$$

This map fits again in a commutative diagram

$$\begin{array}{ccc} H^1(L/K, \mathrm{Aut}(\mathfrak{g}^L)) & \longrightarrow & H^1(L/M, \mathrm{Aut}(\mathfrak{g}^L)) \\ \downarrow & & \downarrow \\ E(L/K, \mathfrak{g}^L) & \longrightarrow & E(L/M, \mathfrak{g}^L), \end{array}$$

where the lower horizontal map is defined by the assignment $[\mathfrak{h}] \mapsto [\mathfrak{h} \otimes M]$.

Remark 5.4.2. The horizontal maps in the diagram above are not surjective in general. This is equivalent to the following statement: given a tower of fields $L/M/K$ and a Lie algebra $\mathfrak{g}$ over L, in general, not every M-form of $\mathfrak{g}$ has itself a K-form.

Recall from Section 4.6.1 of Chapter 4, that for any Lie algebra $\mathfrak{g}$, we have a projection

$$\pi_{\mathrm{ab}} : \mathrm{Aut}(\mathfrak{g}) \to \mathrm{GL}(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])$$

of which we write the image as $\mathrm{Aut}_{\mathrm{ab}}(\mathfrak{g})$. As one can check, the kernel of π_{ab} is a unipotent normal subgroup of $\mathrm{Aut}(\mathfrak{g})$, defined over K. Therefore, Theorem 5.3.10 applies and one obtains the following result, which was also described in section 3 of [GS84].

Theorem 5.4.3. *Let L/K be a Galois extension with K of characteristic 0 and $\mathfrak{n}$ a nilpotent Lie algebra defined over K. The induced map*

$$\pi_{\mathrm{ab}*} : H^1\left(L/K, \mathrm{Aut}(\mathfrak{n}^L)\right) \to H^1\left(L/K, \mathrm{Aut}_{\mathrm{ab}}(\mathfrak{n}^L)\right)$$

is a bijection.

We end this section with a remark on the existence of certain cocycles.

Remark 5.4.4. Let L/K be a Galois extension and $\mathfrak{g}$ a Lie algebra over K. Note that there is a natural inclusion $\mathrm{Aut}(\mathfrak{g}) \subset \mathrm{Aut}(\mathfrak{g}^L)$ and that this subset can be characterized as

$$\mathrm{Aut}(\mathfrak{g}) = \{\varphi \in \mathrm{Aut}(\mathfrak{g}^L) \mid \forall \sigma \in \mathrm{Gal}(L/K) : {}^{\sigma}\varphi = \varphi\}.$$

It is not hard to check that any continuous group morphism

$$\rho : \mathrm{Gal}(L/K) \to \mathrm{Aut}(\mathfrak{g})$$

is a cocycle and thus gives rise to a K-form $\mathfrak{g}_\rho$ of $\mathfrak{g}^L$.

5.5 Galois cohomology of certain semi-direct products

In what follows, we prove a result on the Galois cohomology of a certain class of semi-direct products including wreath products. This section is based on section 3.2 from [DW23a].

Let L/K be a Galois extension and A a finite group with a left action on the set $\{1, \ldots, n\}$ which we write as $a \cdot i$ for any $a \in A$ and $i \in \{1, \ldots, n\}$. Let

$G_1, \dots, G_n$ be $\mathrm{Gal}(L/K)$-groups such that $G_i = G_j$ for any $i \in \{1, \dots, n\}$ and $j \in A \cdot i$. We define the semi-direct product $(\prod_{i=1}^n G_i) \rtimes A$ by letting A act on $\prod_{i=1}^n G_i$ according to the law

$$a \cdot (g_1, \dots, g_n) := (g_{a^{-1}\cdot 1}, \dots, g_{a^{-1}\cdot n})$$

for any $a \in A$ and $g_i \in G_i$. Note that if all groups G_i are equal, this group is just a *wreath product*. We can turn this group into a $\mathrm{Gal}(L/K)$-group as follows. We endow A with the trivial $\mathrm{Gal}(L/K)$ action and $\prod_{i=1}^n G_i$ with the induced component wise left $\mathrm{Gal}(L/K)$-action, i.e. ${}^{\sigma}(g_1, \dots, g_n) = ({}^{\sigma}g_1, \dots, {}^{\sigma}g_n)$ for all $g_i \in G_i$ and $\sigma \in \mathrm{Gal}(L/K)$. Clearly, these are actions by group automorphisms. Note that the actions of A and $\mathrm{Gal}(L/K)$ on $\prod_{i=1}^n G_i$ commute:

$$\begin{aligned}
{}^{\sigma}(a \cdot (g_1, \dots, g_n)) &= {}^{\sigma}(g_{a^{-1}\cdot 1}, \dots, g_{a^{-1}\cdot n}) \\
&= ({}^{\sigma}g_{a^{-1}\cdot 1}, \dots, {}^{\sigma}g_{a^{-1}\cdot n}) \\
&= a \cdot ({}^{\sigma}g_1, \dots, {}^{\sigma}g_n) \\
&= a \cdot ({}^{\sigma}(g_1, \dots, g_n)) .
\end{aligned}$$

At last we define a $\mathrm{Gal}(L/K)$-action on $(\prod_{i=1}^n G_i) \rtimes A$ by ${}^{\sigma}(g, a) = ({}^{\sigma}g, {}^{\sigma}a) = ({}^{\sigma}g, a)$ for all $g \in \prod_{i=1}^n G_i, a \in A$ and $\sigma \in \mathrm{Gal}(L/K)$. This is an action by automorphisms since

$$\begin{aligned}
{}^{\sigma}((g, a)(h, b)) &= {}^{\sigma}(ga \cdot h, ab) \\
&= ({}^{\sigma}(ga \cdot h), ab) \\
&= ({}^{\sigma}g\, {}^{\sigma}(a \cdot h), ab) \\
&= ({}^{\sigma}ga \cdot {}^{\sigma}h, ab) \\
&= ({}^{\sigma}g, a)\, ({}^{\sigma}h, b) \\
&= {}^{\sigma}(g, a)\, {}^{\sigma}(h, b)
\end{aligned}$$

for any $g, h \in \prod_{i=1}^n G_i$ and $a, b \in A$.

We can thus talk about the first Galois cohomology set $H^1(L/K, (\prod_{i=1}^n G_i) \rtimes A)$. Let $\pi : (\prod_{i=1}^n G_i) \rtimes A \to A : (g, a) \mapsto a$ denote the projection morphism and $\iota : A \to (\prod_{i=1}^n G_i) \rtimes A : a \mapsto ((1, \dots, 1), a)$ the natural injection. Note that both π and ι are $\mathrm{Gal}(L/K)$-equivariant. As a consequence we have the well-defined maps on cohomology:

$$\pi_* : H^1\left(L/K, \left(\prod_{i=1}^n G_i\right) \rtimes A\right) \to H^1(L/K, A) : [\alpha] \mapsto [\pi \circ \alpha]$$

$$\iota_* : H^1(L/K, A) \to H^1\left(L/K, \left(\prod_{i=1}^n G_i\right) \rtimes A\right) : [\alpha] \mapsto [\iota \circ \alpha].$$

The set $H^1(L/K, A)$ is relatively easy to understand since A is a group with trivial Galois action, so it is given by all continuous group morphisms from $\mathrm{Gal}(L/K)$ to A up to conjugation by an element of A.

Theorem 5.5.1. *Let $(\prod_{i=1}^n G_i) \rtimes A$ be the $\mathrm{Gal}(L/K)$-group as defined above. Assume that for any finite degree intermediate extension N/K and any $i \in \{1, \ldots, n\}$, the first Galois cohomology set $H^1(L/N, G_i)$ is trivial. Then we have that*

$$\pi_* : H^1\left(L/K, \left(\prod_{i=1}^n G_i\right) \rtimes A\right) \to H^1\left(L/K, A\right) : [\rho] \mapsto [\pi \circ \rho]$$

is a bijection with inverse ι_*.

Proof. Since $\pi \circ \iota = \mathrm{Id}_A$, it follows that $\pi_* \circ \iota_*$ is the identity on $H^1(L/K, A)$ as well. Let us prove that $\iota \circ \pi = \mathrm{Id}$ as well. Thus, we need to prove that $[\rho] = [\iota \circ \pi \circ \rho]$ for an arbitrary $[\rho] \in H^1(L/K, (\prod_{i=1}^n G_i) \rtimes A)$.

Take any cocycle $\rho : \mathrm{Gal}(L/K) \to (\prod_{i=1}^n G_i) \rtimes A$. Let us write $\rho_\sigma = (g_\sigma, a_\sigma)$ for all $\sigma \in \mathrm{Gal}(L/K)$. We then have that

$$\rho_{\sigma\tau} = \rho_\sigma \, {}^\sigma\rho_\tau = (g_\sigma \, a_\sigma \cdot {}^\sigma g_\tau, a_\sigma a_\tau).$$

Thus we have $g_{\sigma\tau} = g_\sigma \, a_\sigma \cdot {}^\sigma g_\tau$ and $a_{\sigma\tau} = a_\sigma a_\tau$. Let us denote for $h \in \prod_{i=1}^n G_i$ by $(h)_i$ the i-th entry of h. Then we have

$$(g_{\sigma\tau})_i = (g_\sigma \, a_\sigma \cdot {}^\sigma g_\tau)_i = (g_\sigma)_i \, {}^\sigma(g_\tau)_{a_\sigma^{-1} \cdot i}$$

Note that the group morphism $\pi \circ \rho : \mathrm{Gal}(L/K) \to A$ induces an action of the Galois group on the set $\{1, \ldots, n\}$. The stabilizers $\mathrm{stab}(i)$ for some $i \in \{1, \ldots, n\}$ are subgroups of $\mathrm{Gal}(L/K)$. Moreover they are open subgroups since they can be written as the inverse image of $\{a \in A \mid a \cdot i = i\}$ under the continuous map $\pi \circ \rho$ and A is endowed with the discrete topology. As a consequence we have $\mathrm{stab}(i) = \mathrm{Gal}(L/L^{\mathrm{stab}(i)})$.

Now note that for $\sigma, \tau \in \mathrm{stab}(i)$, we have

$$(g_{\sigma\tau})_i = (g_\sigma)_i \, {}^\sigma(g_\tau)_i.$$

This shows that the assignment $\sigma \mapsto (g_\sigma)_i$ is a cocycle from $\mathrm{Gal}\left(L/L^{\mathrm{stab}(i)}\right)$ to G_i. By assumption, $H^1(L/L^{\mathrm{stab}(i)}, G_i)$ is trivial and thus there exists a $h_i \in G_i$ such that $h_i (g_\sigma)_i \, {}^\sigma h_i^{-1} = 1$ for all $\sigma \in \mathrm{stab}(i)$. This gives an element $h = (h_1, \ldots, h_n) \in \prod_{i=1}^n G_i$. Then define a new cocycle $\tilde\rho : \mathrm{Gal}(L/K) \to (\prod_{i=1}^n G_i) \rtimes A : \sigma \mapsto (h,1)(g_\sigma, a_\sigma)\, {}^\sigma (h,1)^{-1}$. By the way this $\tilde\rho$ is defined, it is clear that $[\rho] = [\tilde\rho]$. We also have that

$$\tilde\rho_\sigma = (\underbrace{h g_\sigma \, a_\sigma \cdot {}^\sigma h^{-1}}_{:= \tilde g_\sigma}, a_\sigma).$$

Note that now, for $\sigma \in \mathrm{stab}(i)$ we have

$$(\tilde g_\sigma)_i = h_i (g_\sigma)_i \, {}^\sigma h^{-1}_{a_\sigma^{-1} \cdot i} = h_i (g_\sigma)_i \, {}^\sigma h_i^{-1} = 1.$$

Next, let us choose from each orbit of the action defined by $\pi \circ \rho$ $(= \pi \circ \tilde\rho)$ on $\{1, \ldots, n\}$, exactly one element m_i, giving a subset $\{m_1, \ldots, m_k\} \subset \{1, \ldots, n\}$. For $j \in \mathrm{orb}(m_i)$, we now define the element $r_j := (\tilde g_\sigma)_j^{-1}$ where $\sigma \in \mathrm{Gal}(L/K)$ is chosen such that $a_\sigma \cdot m_i = j$. This does not depend on the choice of σ. Indeed, if $\tau \in \mathrm{Gal}(L/K)$ also satisfies $a_\tau \cdot m_i = j$, then we get

$$(\tilde g_\tau)_j = (\tilde g_{\sigma\sigma^{-1}\tau})_j = (\tilde g_\sigma)_j \, {}^\sigma (\tilde g_{\sigma^{-1}\tau})_{a_\sigma^{-1} \cdot j} = (\tilde g_\sigma)_j \, {}^\sigma (\tilde g_{\sigma^{-1}\tau})_{m_i} = (\tilde g_\sigma)_j$$

where we used that $\sigma^{-1}\tau \in \mathrm{stab}(m_i)$ and thus $(\tilde g_{\sigma^{-1}\tau})_{m_i} = 1$. This gives an element $r = (r_1, \ldots, r_n) \in \prod_{i=1}^n G_i$.

We now have that

$$(r,1)\tilde\rho_\sigma \, {}^\sigma (r,1)^{-1} = (r,1)(\tilde g_\sigma, a_\sigma)\left({}^\sigma r^{-1}, 1\right) = (r \tilde g_\sigma \, a_\sigma \cdot {}^\sigma r^{-1}, a_\sigma).$$

At last we show that $r\tilde g_\sigma \, a_\sigma \cdot {}^\sigma r^{-1} = (1, \ldots, 1)$ for all $\sigma \in \mathrm{Gal}(L/K)$. Let $j \in \mathrm{orb}(m_i)$ and let $\tau \in \mathrm{Gal}(L/K)$ such that $\tau(m_i) = j$. Then we have for any $\sigma \in \mathrm{Gal}(L/K)$:

$$\begin{aligned}
\left(r\tilde g_\sigma \, a_\sigma \cdot {}^\sigma r^{-1}\right)_j &= r_j (\tilde g_\sigma)_j \, {}^\sigma r^{-1}_{a_\sigma^{-1} \cdot j} \\
&= (\tilde g_\tau)_j^{-1} (\tilde g_\sigma)_j \, {}^\sigma (\tilde g_{\sigma^{-1}\tau})_{a_\sigma^{-1} \cdot j} \\
&= (\tilde g_\tau)_j^{-1} (\tilde g_\sigma)_j \, {}^\sigma \left((\tilde g_{\sigma^{-1}})_{a_\sigma^{-1} \cdot j} \, {}^{\sigma^{-1}} (\tilde g_\tau)_j \right) \\
&= (\tilde g_\tau)_j^{-1} (\tilde g_\sigma)_j \, {}^\sigma (\tilde g_{\sigma^{-1}})_{a_\sigma^{-1} \cdot j} (\tilde g_\tau)_j \\
&= (\tilde g_\tau)_j^{-1} (\tilde g_{\sigma\sigma^{-1}})_j (\tilde g_\tau)_j \\
&= (\tilde g_\tau)_j^{-1} (\tilde g_\tau)_j \\
&= 1.
\end{aligned}$$

As a consequence we have that $(r,1)\tilde{\rho}_\sigma{}^\sigma(r,1)^{-1} = ((1,\ldots,1),a_\sigma)$ for any $\sigma \in \mathrm{Gal}(L/K)$. Note that $(\iota \circ \pi \circ \rho)_\sigma = ((1,\ldots,1),a_\sigma)$ and thus that we have shown that $[\iota \circ \pi \circ \rho] = [\tilde{\rho}] = [\rho]$. □

5.6 Forms of partially commutative Lie algebras

Recall from Section 4.2 of Chapter 4 that to any graph $\mathcal{G} = (V,E)$ and any field K one can associate a Lie algebra $\mathfrak{g}^K(\mathcal{G})$ and for any integer $c > 1$ a nilpotent Lie algebra $\mathfrak{n}^K(\mathcal{G},c)$, both defined over K. In this section we describe the forms of these Lie algebras for $K \subset \mathbb{C}$. This section is partially based on [DW23a].

5.6.1 The associated Galois cohomology

Let L/K be a Galois extension. Write $\mathfrak{f}^K(V)$ for the free Lie algebra on V over K. The natural inclusion $\mathfrak{f}^K(V) \hookrightarrow \mathfrak{f}^L(V)$ induces an inclusion $\mathfrak{n}^K(\mathcal{G},c) \hookrightarrow \mathfrak{n}^L(\mathcal{G},c)$. Using this inclusion we get a canonical isomorphism

$$\mathfrak{n}^K(\mathcal{G},c) \otimes L \cong \mathfrak{n}^L(\mathcal{G},c) : v \otimes l \mapsto lv$$

of Lie algebras over L. Thus $\mathfrak{n}^K(\mathcal{G},c)$ is a distinguished K-from of $\mathfrak{n}^L(\mathcal{G},c)$ which can canonically be identified with a subset of $\mathfrak{n}^L(\mathcal{G},c)$. We will also call this the *standard K-form of* $\mathfrak{n}^L(\mathcal{G},c)$. As discussed in Section 5.4, this gives an action of $\mathrm{Gal}(L/K)$ on $\mathfrak{n}^L(\mathcal{G},c)$ by semi-linear maps which fix the vectors in the K-form $\mathfrak{n}^K(\mathcal{G},c)$. In particular the action fixes the vertices $V \subset \mathfrak{n}^L(\mathcal{G},c)$. This action induces a $\mathrm{Gal}(L/K)$-action on $\mathrm{Aut}(\mathfrak{n}^L(\mathcal{G},c))$ and thus we can talk about the first Galois cohomology set

$$H^1\left(L/K, \mathrm{Aut}\left(\mathfrak{n}^L\left(\mathcal{G},c\right)\right)\right).$$

In what follows, we compute this set for the Galois extensions L/K with L a subfield of $\mathbb{C}$. The reason that we only consider subfields of $\mathbb{C}$ is that the description of the group

$$G^L(\mathcal{G}) = \mathrm{Aut}_{\mathrm{ab}}(\mathfrak{n}^L(\mathcal{G},c))$$

from Section 4.6.2 is also limited to L being a subfield of $\mathbb{C}$. In the next section, we then apply the descent theorem to relate this computation on Galois cohomology to the set of equivalence classes of K-forms of $\mathfrak{n}^L(\mathcal{G},c)$.

So let L/K be a Galois extension of subfields of $\mathbb{C}$. Recall from Section 4.1 that any graph $\mathcal{G}$ has an associated quotient graph $\overline{\mathcal{G}}$ which has a finite automorphism group $\mathrm{Aut}(\mathcal{G})$. Let us equip this group with the trivial $\mathrm{Gal}(L/K)$-action, thus turning it in a $\mathrm{Gal}(L/K)$-group. The following maps are then $\mathrm{Gal}(L/K)$-equivariant group morphisms

$$\mathrm{Aut}(\mathfrak{n}^L(\mathcal{G},c)) \xrightarrow{\pi_{\mathrm{ab}}} G^L(\mathcal{G}) \xrightarrow{p} \mathrm{Aut}(\overline{\mathcal{G}}),$$

where p is the morphism from (4.7) obtained by modding out the irreducible identity component of $G^L(\mathcal{G})$. Thus one gets an induced map on cohomology

$$(p \circ \pi_{\mathrm{ab}})_* : H^1(L/K, \mathrm{Aut}(\mathfrak{n}^L(\mathcal{G},c))) \longrightarrow H^1(L/K, \mathrm{Aut}(\overline{\mathcal{G}})).$$

In what follows, we will prove that this is in fact a bijection. Note that Theorem 5.4.3 already tells us that $\pi_{\mathrm{ab}*}$ is a bijection. Thus, we are only left to prove that p_* is a bijection.

Theorem 5.6.1. *Let L/K be a Galois extension of subfields of $\mathbb{C}$. The map $(p \circ \pi_{\mathrm{ab}})_*$ as described above is a bijection.*

Proof. As already mentioned, by Theorem 5.4.3, we only need to prove that the induced map p_* is a bijection. Note that p decomposes as the composition of two quotient maps

$$G^L(\mathcal{G}) \xrightarrow{p_1} G^L(\mathcal{G})\Big/U^L(\mathcal{G}) \xrightarrow{p_2} G^L(\mathcal{G})\Big/G_0^L(\mathcal{G}),$$

where $U^L(\mathcal{G})$ is the unipotent radical and $G_0^L(\mathcal{G})$ is the irreducible component at the identity of $G^L(\mathcal{G})$. By Theorem 5.3.10, we know that the induced map $(p_1)_*$ is bijective. Thus, we are left to prove that $(p_2)_*$ is bijective.

By Theorem 4.6.6, we have

$$G^L(\mathcal{G})\Big/U^L(\mathcal{G}) \cong P(\mathrm{Aut}(\mathcal{G})) \cdot \prod_{\lambda \in \Lambda_{\mathcal{G}}} \mathrm{GL}(\mathrm{span}_L(\lambda)).$$

In the same way as we did in Remark 4.1.9, choose an ordering of the vertices in each coherent component

$$\lambda = \{v_{\lambda,1}, v_{\lambda,2}, \ldots, v_{\lambda,|\lambda|}\}$$

and write $r : \mathrm{Aut}(\overline{\mathcal{G}}) \to \mathrm{Aut}(\mathcal{G})$ for the associated splitting morphism. We then have that

$$G^L(\mathcal{G})\Big/U^L(\mathcal{G}) \cong (P \circ r)(\mathrm{Aut}(\overline{\mathcal{G}})) \cdot \prod_{\lambda \in \Lambda_{\mathcal{G}}} \mathrm{GL}(\mathrm{span}_L(\lambda)). \tag{5.5}$$

By using coordinates with respect to the ordered basis $v_{\lambda,1}, v_{\lambda,2}, \dots, v_{\lambda,|\lambda|}$, we get for each $\lambda \in \Lambda_{\mathcal{G}}$ a group isomorphism

$$f_\lambda : \mathrm{GL}(\mathrm{span}_L(\lambda)) \to \mathrm{GL}_{|\lambda|}(L).$$

Moreover, these are isomorphisms of $\mathrm{Gal}(L/K)$-groups (with the standard coefficient-wise $\mathrm{Gal}(L/K)$-action on $\mathrm{GL}_{|\lambda|}(L)$). The action of $\mathrm{Aut}(\overline{\mathcal{G}})$ on $\Lambda_{\mathcal{G}}$ allows one to define a semi-direct product

$$\left(\prod_{\lambda\in\Lambda_{\mathcal{G}}} \mathrm{GL}_{|\lambda|}(L)\right) \rtimes \mathrm{Aut}(\overline{\mathcal{G}})$$

in the same way as the semi-direct products constructed in Section 5.5. We also endow this group with the same $\mathrm{Gal}(L/K)$-action as was done in Section 5.5. As one can check we get a commutative diagram

$$\begin{array}{ccc} G^L(\mathcal{G})\big/U^L(\mathcal{G}) & \xrightarrow{\ p_2\ } & \mathrm{Aut}(\overline{\mathcal{G}}) \\ \big\downarrow{\scriptstyle\cong} & \nearrow{\scriptstyle\pi} & \\ \left(\prod_{\lambda\in\Lambda_{\mathcal{G}}} \mathrm{GL}_{|\lambda|}(L)\right) \rtimes \mathrm{Aut}(\overline{\mathcal{G}}) & & \end{array}$$

where the isomorphism is obtained by first applying the isomorphism from (5.5) and then using the isomorphisms f_λ. As one can check it is an isomorphism of $\mathrm{Gal}(L/K)$-groups. By Theorem 5.3.8, the groups $\mathrm{GL}_{|\lambda|}(L)$ have trivial Galois cohomology for any field extension L/K. Thus, we can apply Theorem 5.5.1 and find that π induces a bijection on the associated Galois cohomology sets. Thus, by the diagram, this proves that $(p_2)_*$ is a bijection. ☐

Remark 5.6.2. Let $\mathcal{G} = (V, E)$ be a graph and L/K a field extension of subfields of $\mathbb{C}$. Note that any graph automorphism $\varphi \in \mathrm{Aut}(\mathcal{G})$ induces a unique Lie algebra automorphism $q(\varphi) \in \mathrm{Aut}(\mathfrak{n}^L(\mathcal{G}, c))$ such that $q(\varphi)$ restricts to φ on V. Note that the map q is a $\mathrm{Gal}(L/K)$-equivariant group morphism. Choose a splitting morphism $r : \mathrm{Aut}(\overline{\mathcal{G}}) \to \mathrm{Aut}(\mathcal{G})$ like in Remark 4.1.9. Then we get an induced map

$$i = (q \circ r) : \mathrm{Aut}(\overline{\mathcal{G}}) \to \mathrm{Aut}(\mathfrak{n}^L(\mathcal{G}, c)).$$

As one can check, the induced map i_* on cohomology gives an inverse to the map $(p \circ \pi_{\mathrm{ab}})_*$ from Theorem 5.6.1.

Note that the set $H^1(L/K, \mathrm{Aut}(\overline{\mathcal{G}}))$ is easier to understand than the set $H^1(L/K, \mathrm{Aut}(\mathfrak{n}^L(\mathcal{G}, c)))$. Indeed, since the $\mathrm{Gal}(L/K)$-action on $\mathrm{Aut}(\overline{\mathcal{G}})$ is trivial, its cocycles are just continuous group morphisms

$$\rho : \mathrm{Gal}(L/K) \to \mathrm{Aut}(\overline{\mathcal{G}}),$$

and two such group morphisms ρ, η are equivalent if and only if there exists a $\varphi \in \operatorname{Aut}(\overline{\mathcal{G}})$ such that $\rho_\sigma = \varphi \eta_\sigma \varphi^{-1}$ for all $\sigma \in \operatorname{Gal}(L/K)$.

5.6.2 Rational forms of the real and complex Lie algebra

In this section we give a description of the rational forms of $\mathfrak{n}^{\mathbb{R}}(\mathcal{G}, c)$ and $\mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c)$ and give some examples with concrete graphs.

Consider a Galois extension L/K of subfields of $\mathbb{C}$. As discussed in Remark 5.6.2, after choosing a splitting morphism $r : \operatorname{Aut}(\overline{\mathcal{G}}) \to \operatorname{Aut}(\mathcal{G})$, we obtain a $\operatorname{Gal}(L/K)$-equivariant group morphism

$$i : \operatorname{Aut}(\overline{\mathcal{G}}) \to \operatorname{Aut}(\mathfrak{n}^L(\mathcal{G}, c)).$$

Thus, for any continuous morphism $\rho : \operatorname{Gal}(L/K) \to \operatorname{Aut}(\overline{\mathcal{G}})$, we get a cocycle $i \circ \rho \in Z^1(L/K, \operatorname{Aut}(\mathfrak{n}^L(\mathcal{G}, c)))$ and thus, by (5.4), a K-form of $\mathfrak{n}^L(\mathcal{G}, c)$. For notational purposes, we will simply write this form as

$$\mathfrak{n}^K(\rho, c) = \{v \in \mathfrak{n}^L(\mathcal{G}, c) \mid \forall \sigma \in \operatorname{Gal}(L/K) : \, i(\rho_\sigma)({}^\sigma v) = v\},$$

Note that the cocycle $i \circ \rho$ that we use, takes values in $\operatorname{Aut}(\mathfrak{n}^K(\mathcal{G}, c)) \subset \operatorname{Aut}(\mathfrak{n}^L(\mathcal{G}, c))$ and is thus of the type that is described in Remark 5.4.4.

The following theorem now tells us that all rational forms of $\mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c)$ arise in this way, and which ones will be rational forms of the real Lie algebra $\mathfrak{n}^{\mathbb{R}}(\mathcal{G}, c)$.

Let us write $\tau \in \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ for the complex conjugation automorphism on $\overline{\mathbb{Q}}$.

Theorem 5.6.3. *Let $\mathcal{G}$ be a graph and $c > 1$ an integer. The assignment*

$$[\rho] \mapsto \left[\mathfrak{n}^{\mathbb{Q}}(\rho, c)\right]$$

gives the bijections

$$H^1\left(\overline{\mathbb{Q}}/\mathbb{Q}, \operatorname{Aut}(\overline{\mathcal{G}})\right) \longrightarrow E\left(\mathbb{C}/\mathbb{Q}, \mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c)\right) \tag{5.6}$$

and

$$\left\{[\rho] \in H^1\left(\overline{\mathbb{Q}}/\mathbb{Q}, \operatorname{Aut}(\overline{\mathcal{G}})\right) \,\middle|\, \tau \in \ker(\rho)\right\} \longrightarrow E\left(\mathbb{R}/\mathbb{Q}, \mathfrak{n}^{\mathbb{R}}(\mathcal{G}, c)\right). \tag{5.7}$$

Proof. The first bijection follows immediately from combining Theorem 5.6.1 for $K = \mathbb{Q}$, Theorem 5.4.1 and Theorem 5.2.6.

For the second bijection, note that if we view $\overline{\mathbb{Q}}$ as a subfield of $\mathbb{C}$, we get a continuous morphism $\nu : \operatorname{Gal}(\mathbb{C}/\mathbb{R}) \to \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) : \sigma \mapsto \sigma|_{\overline{\mathbb{Q}}}$. This allows us to define the maps ω_1 and ω_2 with domain and codomain as in diagram (5.8) and which send $[\rho]$ to $[\rho \circ \nu]$. Note that we also have two bijections i_* in the diagram, induced by the map $i : \mathrm{Aut}(\overline{\mathcal{G}}) \to \mathrm{Aut}(\mathfrak{n}^K(\mathcal{G}, c))$ from Remark 5.6.2.

$$\begin{array}{ccc} H^1(\overline{\mathbb{Q}}/\mathbb{Q}, \mathrm{Aut}(\overline{\mathcal{G}})) & \xrightarrow{\omega_1} & H^1(\mathbb{C}/\mathbb{R}, \mathrm{Aut}(\overline{\mathcal{G}})) \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} \\ H^1(\overline{\mathbb{Q}}/\mathbb{Q}, \mathrm{Aut}(\mathfrak{n}^{\overline{\mathbb{Q}}}(\mathcal{G}, c))) & \xrightarrow{\omega_2} & H^1(\mathbb{C}/\mathbb{R}, \mathrm{Aut}(\mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c))) \\ \updownarrow & & \updownarrow \\ E(\overline{\mathbb{Q}}/\mathbb{Q}, \mathfrak{n}^{\overline{\mathbb{Q}}}(\mathcal{G}, c)) & \longrightarrow & E(\mathbb{C}/\mathbb{R}, \mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c)). \end{array} \tag{5.8}$$

Note that the trivial cocycle in $H^1(\mathbb{C}/\mathbb{R}, \mathrm{Aut}(\mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c)))$ corresponds to the standard real form $\mathfrak{n}^{\mathbb{R}}(\mathcal{G}, c)$ of $\mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c)$. Thus by the bottom square in the commutative diagram (5.8), it follows that the cocycles in $H^1(\overline{\mathbb{Q}}/\mathbb{Q}, \mathrm{Aut}(\mathfrak{n}^{\overline{\mathbb{Q}}}_{\mathcal{G},c}))$ that give a rational form of $\mathfrak{n}^{\mathbb{R}}(\mathcal{G}, c)$ are exactly the classes in $\omega_2^{-1}([1])$. By the top square in diagram (5.8), we get that the classes $[\rho] \in \omega_1^{-1}([1])$ are then exactly the ones for which $[\mathfrak{n}^{\mathbb{Q}}(\rho, c)]$ lies in $E(\mathbb{R}/\mathbb{Q}, \mathfrak{n}^{\mathbb{R}}(\mathcal{G}, c))$.

Since we have that

$$\begin{aligned} [\rho] \in \omega_1^{-1}([1]) &\Leftrightarrow [\rho \circ \nu] = [1] \\ &\Leftrightarrow \exists \varphi \in \mathrm{Aut}(\overline{\mathcal{G}}) : \forall \sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{R}) : (\rho \circ \nu)_\sigma = \varphi \,\mathrm{Id}\, \varphi^{-1} \\ &\Leftrightarrow \forall \sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{R}) : (\rho \circ \nu)_\sigma = \mathrm{Id} \\ &\Leftrightarrow \rho_\tau = \mathrm{Id} \\ &\Leftrightarrow \tau \in \ker(\rho), \end{aligned}$$

this proves that the second map is a bijection. □

We are now ready to prove the 'injective version' of Theorem 5.6.3. It only uses finite degree Galois extensions of $\mathbb{Q}$ and thus the Krull topology does not play a role anymore.

Theorem 5.6.4 (Injective version)**.** *Let $\mathcal{G}$ be a graph. Any rational form of the complex Lie algebra $\mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c)$ is isomorphic to*

$$\mathfrak{n}^{\mathbb{Q}}(\rho, c) = \{v \in \mathfrak{n}^L(\mathcal{G}, c) \mid \forall \sigma \in \mathrm{Gal}(L/K) : \; i(\rho_\sigma)({}^\sigma v) = v\},$$

for some finite degree Galois extension $L/\mathbb{Q}$ with $L \subset \mathbb{C}$ and an injective group morphism $\rho : \mathrm{Gal}(L/\mathbb{Q}) \to \mathrm{Aut}(\overline{\mathcal{G}})$. If $K/\mathbb{Q}$ with $K \subset \mathbb{C}$ is another finite degree

Galois extension with an injective group morphism $\eta : \mathrm{Gal}(K/\mathbb{Q}) \to \mathrm{Aut}(\overline{\mathcal{G}})$*, then*

$$\mathfrak{n}^{\mathbb{Q}}(\rho, c) \cong \mathfrak{n}^{\mathbb{Q}}(\eta, c)$$

$$\Updownarrow$$

$$L = K \text{ and } \exists\, \varphi \in \mathrm{Aut}(\overline{\mathcal{G}}) : \forall \sigma \in \mathrm{Gal}(L/\mathbb{Q}) :\ \rho_\sigma = \varphi\, \eta_\sigma\, \varphi^{-1}.$$

The Lie algebra $\mathfrak{n}^{\mathbb{Q}}(\rho, c)$ *is a rational form of the real Lie algebra* $\mathfrak{n}^{\mathbb{R}}(\mathcal{G}, c)$ *if and only if* $L \subset \mathbb{R}$*.*

Proof. First, take any rational form of $\mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c)$. Then it follows from Theorem 5.6.3 that up to isomorphism, the form is given by $\mathfrak{n}^{\mathbb{Q}}(\rho, c)$ for some continuous morphism $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}(\overline{\mathcal{G}})$. Note that $\ker(\rho)$ is an open normal subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. By Theorem 5.1.14, we get that $L_\rho := \overline{\mathbb{Q}}^{\ker(\rho)}$ is a finite degree Galois extension of $\mathbb{Q}$ and we have a natural isomorphism of groups

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})/\ker(\rho) \to \mathrm{Gal}(L_\rho/\mathbb{Q}) : \sigma \ker(\rho) \mapsto \sigma|_\rho.$$

We therefore get an induced injective morphism of groups

$$\overline{\rho} : \mathrm{Gal}(L_\rho/\mathbb{Q}) \to \mathrm{Aut}(\overline{\mathcal{G}})$$

which gives a class $[\overline{\rho}] \in H^1(L_\rho/\mathbb{Q}, \mathrm{Aut}(\overline{\mathcal{G}}))$. Note that we have a natural injection $\mathfrak{n}^{L_\rho}(\mathcal{G}, c) \hookrightarrow \mathfrak{n}^{\overline{\mathbb{Q}}}(\mathcal{G}, c)$ and that this injection restricts to a $\mathbb{Q}$-Lie algebra isomorphism $\mathfrak{n}^{\mathbb{Q}}(\overline{\rho}, c) \cong \mathfrak{n}^{\mathbb{Q}}(\rho, c)$. Following Theorem 5.6.3, we also have that $\mathfrak{n}^{\mathbb{Q}}(\overline{\rho}, c)$ is a rational form of $\mathfrak{n}^{\mathbb{R}}(\mathcal{G}, c)$ if and only if $\tau \in \ker(\rho)$ and thus if and only if $L_\rho \subset \mathbb{R}$.

Second, if $\eta : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}(\overline{\mathcal{G}})$ is another continuous morphism, we have the equivalences

$$\begin{aligned}
\mathfrak{n}^{\mathbb{Q}}(\overline{\rho}, c) \cong \mathfrak{n}^{\mathbb{Q}}(\overline{\eta}, c) &\Leftrightarrow \mathfrak{n}^{\mathbb{Q}}(\rho, c) \cong \mathfrak{n}^{\mathbb{Q}}(\eta, c) \\
&\Leftrightarrow [\rho] = [\eta] \\
&\Leftrightarrow \ker(\rho) = \ker(\eta) \text{ and } [\overline{\rho}] = [\overline{\eta}] \\
&\Leftrightarrow L_\rho = L_\eta \text{ and } [\overline{\rho}] = [\overline{\eta}].
\end{aligned}$$

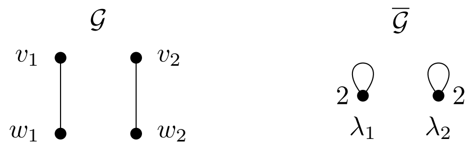
At last, note that conversely, if $L'/\mathbb{Q}$ is any finite degree Galois extension and $\rho' : \mathrm{Gal}(L'/\mathbb{Q}) \to \mathrm{Aut}(\overline{\mathcal{G}})$ is any injective group morphism, we can define the continuous morphism

$$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}(\overline{\mathcal{G}}) : \sigma \mapsto \rho'(\sigma|_{L'}),$$

which now satisfies $\overline{\rho} = \rho'$. This shows all that needed to be proven. □

The following example shows that Theorem 5.6.4 can be used to simplify certain classifications of Lie algebras, especially for quadratic extensions.

Example 5.6.5 (Direct sum of two Heisenberg Lie algebras)**.** Consider the graph $\mathcal{G} = (V, E)$ defined by $V = \{v_1, v_2, w_1, w_2\}$ and $E = \{\{v_1, w_1\}, \{v_2, w_2\}\}$. The set of coherent components is then given by $\Lambda_{\mathcal{G}} = \{\lambda_1 := \{v_1, w_1\}, \lambda_2 := \{v_2, w_2\}\}$. A figure of the graph and quotient graph are given below.



There are only two automorphisms of the quotient graph, namely the identity and $\varphi \in \mathrm{Aut}(\overline{\mathcal{G}})$ which is defined by $\varphi(\lambda_1) = \lambda_2$ and $\varphi(\lambda_2) = \lambda_1$. We can define the morphism $r : \mathrm{Aut}(\overline{\mathcal{G}}) \to \mathrm{Aut}(\mathcal{G})$ by letting $r(\varphi)(v_1) = v_2$, $r(\varphi)(v_2) = v_1$, $r(\varphi)(w_1) = w_2$ and $r(\varphi)(w_2) = w_1$. The associated 2-step nilpotent Lie algebra $\mathfrak{n}^L(\mathcal{G}, 2)$ is then isomorphic to a direct sum of two 3-dimensional Heisenberg Lie algebras with basis $\{v_1,\ v_2,\ w_1,\ w_2,\ u_1,\ u_2\}$ where $u_1 := [v_1, w_1]$ and $u_2 := [v_2, w_2]$.

It is clear that if $\mathrm{Gal}(L/\mathbb{Q}) \to \mathrm{Aut}(\overline{\mathcal{G}})$ is an injective group morphism, $L/\mathbb{Q}$ must have degree 2 or 1. All non-isomorphic degree 2 or 1 Galois extensions of $\mathbb{Q}$ are given by $\mathbb{Q}(\sqrt{d})$ for d a square free non-zero integer (see Example 5.1.2). Note that if $d = 1$, $\mathbb{Q}(\sqrt{d}) = \mathbb{Q}$. For all square free non-zero integers d, let ρ_d denote the uniquely determined injective group morphism $\rho_d : \mathrm{Gal}(\mathbb{Q}(\sqrt{d})/\mathbb{Q}) \to \mathrm{Aut}(\overline{\mathcal{G}})$. For simplicity, let us write $\mathfrak{n}_d^{\mathbb{Q}}$ for the associated rational form $\mathfrak{n}^{\mathbb{Q}}(\rho_d, 2)$ of $\mathfrak{n}^{\mathbb{C}}(\mathcal{G}, 2)$. From Theorem 5.6.4 we then get that the sets

$$\{\mathfrak{n}_d^{\mathbb{Q}} \mid d \neq 0 \text{ square free}\} \qquad \text{and} \qquad \{\mathfrak{n}_d^{\mathbb{Q}} \mid d \geq 1 \text{ square free}\}$$

give us a complete set of pairwise non-isomorphic rational forms of $\mathfrak{n}^{\mathbb{C}}(\mathcal{G}, 2)$ and $\mathfrak{n}^{\mathbb{R}}(\mathcal{G}, 2)$, respectively. Note that $\mathfrak{n}_1^{\mathbb{Q}} \cong \mathfrak{n}^{\mathbb{Q}}(\mathcal{G}, 2)$ is the standard rational form of $\mathfrak{n}^{\mathbb{C}}(\mathcal{G}, 2)$. We thus get an alternative proof of [Lau08, Proposition 3.2.] without using the Pfaffian form on 2-step nilpotent Lie algebras.

For a non-zero square-free integer d, a basis for $\mathfrak{n}_d^{\mathbb{Q}} \subset \mathfrak{n}^{\mathbb{C}}(\mathcal{G}, 2)$ can be given by

$$X_1 := v_1 + v_2 \qquad Y_1 := w_1 + w_2 \qquad Z_1 := u_1 + u_2$$

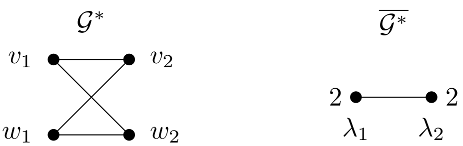
$$X_2 := \sqrt{d}(v_1 - v_2) \qquad Y_2 := \sqrt{d}(w_1 - w_2) \qquad Z_2 := \sqrt{d}(u_1 - u_2).$$

The bracket relations of the rational Lie algebra $\mathfrak{n}_d^{\mathbb{Q}}$ in this basis are then given by

$$[X_1, Y_1] = Z_1 \qquad [X_2, Y_1] = Z_2$$
$$[X_1, Y_2] = Z_2 \qquad [X_2, Y_2] = d\, Z_1.$$

We can also consider the complement graph.

Example 5.6.6. Let $\mathcal{G} = (V, E)$ be the graph from Example 5.6.5 and $\mathcal{G}^*$ its complement graph, i.e. $\mathcal{G}^* = (V, E^*)$ with $E^* = \{\{v, w\} \mid v, w \in V, v \neq w, \{v, w\} \notin E\}$. A figure of $\mathcal{G}^*$ and its quotient graph are given below.



Since $\mathrm{Aut}(\overline{\mathcal{G}^*}) = \mathrm{Aut}(\overline{\mathcal{G}})$, it follows that the only injective group morphisms $\mathrm{Gal}(L/\mathbb{Q}) \to \mathrm{Aut}(\overline{\mathcal{G}^*})$ are the morphisms $\rho_d : \mathrm{Gal}(\mathbb{Q}(\sqrt{d})/\mathbb{Q}) \to \mathrm{Aut}(\overline{\mathcal{G}})$ from Example 5.6.5 where d is any square free non-zero integer. Again, let us simply write $\mathfrak{n}_{d,*}^{\mathbb{Q}}$ for the associated rational form $\mathfrak{n}^{\mathbb{Q}}(\rho_d, 2)$ of $\mathfrak{n}^{\mathbb{C}}(\mathcal{G}^*, 2)$. From Theorem 5.6.4 it thus follows that the sets

$$\{\mathfrak{n}_{d,*}^{\mathbb{Q}} \mid d \neq 0 \text{ square free}\} \qquad \text{and} \qquad \{\mathfrak{n}_{d,*}^{\mathbb{Q}} \mid d \geq 1 \text{ square free}\}$$

give us a complete set of pairwise non-isomorphic rational forms of $\mathfrak{n}^{\mathbb{C}}(\mathcal{G}^*, 2)$ and $\mathfrak{n}^{\mathbb{R}}(\mathcal{G}^*, 2)$, respectively. A basis for $\mathfrak{n}^{\mathbb{C}}(\mathcal{G}^*, 2)$ can be given by $\{v_1, v_2, w_1, w_2, u_1, u_2, u_3, u_4\}$ where $u_1 := [v_1, w_2]$, $u_2 := [v_2, w_1]$, $u_3 = [v_1, v_2]$, $u_4 = [w_1, w_2]$. For non-zero square-free d, a basis for the form $\mathfrak{n}_{d,*}^{\mathbb{Q}}$ can then be given by:

$$X_1 := v_1 + v_2 \qquad Y_1 := w_1 + w_2$$
$$X_2 := \sqrt{d}(v_1 - v_2) \qquad Y_2 := \sqrt{d}(w_1 - w_2)$$
$$Z_1 := u_1 + u_2 \qquad Z_3 = -2\sqrt{d}\, u_3$$
$$Z_2 := \sqrt{d}(u_2 - u_1) \qquad Z_4 = -2\sqrt{d}\, u_4.$$

The bracket relations of the rational Lie algebra $\mathfrak{n}_{d,*}^{\mathbb{Q}}$ in this basis are then given by

$$\begin{aligned} [X_1, X_2] &= Z_3 & [X_2, Y_1] &= Z_2 \\ [X_1, Y_1] &= -Z_1 & [X_2, Y_2] &= dZ_1 \\ [X_1, Y_2] &= -Z_2 & [Y_1, Y_2] &= Z_4. \end{aligned}$$

We thus get an alternative proof of [Lau08, Proposition 4.5.].

5.6.3 Number of non-isomorphic rational forms

We apply Theorem 5.6.4 to show that the Lie algebras $\mathfrak{n}^{\mathbb{R}}(\mathcal{G}, c)$ and $\mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c)$ have either exactly one or infinitely many rational forms, up to isomorphism. The number of rational forms is always countable by Remark 5.2.3.

First, we need the following fundamental observation that ensures the existence of enough non-isomorphic cyclic Galois extensions of a certain degree.

Lemma 5.6.7. *For every positive integer $d > 1$, there exist infinitely many real Galois extensions $L_i/\mathbb{Q}$ with $i \in \mathbb{N}$ such that $\mathrm{Gal}(L_i/\mathbb{Q}) \cong \mathbb{Z}/d\mathbb{Z}$ and $L_i \cap L_j = \mathbb{Q}$ for any $i, j \in \mathbb{N}$ with $i \neq j$.*

Proof. By Dirichlet's theorem [Dir13], there are infinitely many different primes p_i for $i \in \mathbb{N}$ such that $p_i = 1 \mod 2d$ for all $i \in \mathbb{N}$. If we denote by $\zeta_k = e^{2\pi i/k}$ the primitive k-th root of unity, we can define the cyclotomic field extensions $K_i = \mathbb{Q}(\zeta_{p_i})$, for which the Galois group $\mathrm{Gal}(K_i/\mathbb{Q})$ is cyclic of order $p_i - 1$. Since $2d \mid p_i - 1$, there exists a (unique) cyclic subgroup $H_i \subset \mathrm{Gal}(K_i/\mathbb{Q})$ of index $2d$. Let $K_i^{H_i}$ denote the field which is fixed under H_i. Since $\mathrm{Gal}(K_i/\mathbb{Q})$ is abelian, H_i is a normal subgroup and thus we have $\mathrm{Gal}(K_i^{H_i}/\mathbb{Q}) \cong \frac{\mathrm{Gal}(K_i/\mathbb{Q})}{H_i} \cong \mathbb{Z}/2d\mathbb{Z}$.

Note that $\mathrm{Gal}(K_i^{H_i}/\mathbb{Q})$ has a unique element σ of order 2 and in case $K_i^{H_i}$ is not totally real, this must be the complex conjugation automorphism. Let L_i be the subfield of $K_i^{H_i}$ which is fixed by $\{1, \sigma\}$. As before we have that $\{1, \sigma\}$ is a normal subgroup of $\mathrm{Gal}(K_i^{H_i}/\mathbb{Q})$ and thus that $\mathrm{Gal}(L_i/\mathbb{Q}) \cong \frac{\mathrm{Gal}(K_i^{H_i}/\mathbb{Q})}{\{1,\sigma\}} \cong \mathbb{Z}/d\mathbb{Z}$. Note that, even if K_i was not totally real, the fields L_i must be real since they are fixed by complex conjugation. We have thus constructed infinitely many real Galois extensions $L_i/\mathbb{Q}$, $i \in \mathbb{N}$ such that $\mathrm{Gal}(L_i/\mathbb{Q}) \cong \mathbb{Z}/d\mathbb{Z}$. Moreover since for $i \neq j$, the primes p_i and p_j are different, we know that $K_i \cap K_j = \mathbb{Q}(\zeta_{p_i}) \cap \mathbb{Q}(\zeta_{p_j}) = \mathbb{Q}(\zeta_{\gcd(p_i,p_j)}) = \mathbb{Q}$. As a consequence also $L_i \cap L_j = \mathbb{Q}$. $\square$

Theorem 5.6.8. *The Lie algebras $\mathfrak{n}^{\mathbb{R}}(\mathcal{G}, c)$ and $\mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c)$ associated to a graph $\mathcal{G}$ for $c > 1$ have either exactly one or infinitely many rational forms up to isomorphism. The former is true if and only if $\operatorname{Aut}(\overline{\mathcal{G}})$ is trivial.*

Proof. If $\operatorname{Aut}(\overline{\mathcal{G}})$ is trivial, then clearly $H^1(\overline{\mathbb{Q}}/\mathbb{Q}, \operatorname{Aut}(\overline{\mathcal{G}}))$ is trivial as well which implies by Theorem 5.6.3 that both $E(\mathbb{R}/\mathbb{Q}, \mathfrak{n}^{\mathbb{R}}(\mathcal{G}, c))$ and $E(\mathbb{C}/\mathbb{Q}, \mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c))$ count only one element.

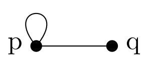
Conversely, assume that $\operatorname{Aut}(\overline{\mathcal{G}})$ is non-trivial. Then there exists an element $\varphi \in \operatorname{Aut}(\overline{\mathcal{G}})$ of prime order p. Let L_i with $i \in \mathbb{N}$ be the finite degree Galois extensions of $\mathbb{Q}$ with Galois group $\mathbb{Z}/p\mathbb{Z}$ as in Lemma 5.6.7. Choose for all $i \in \mathbb{N}$ a generator $\sigma_i \in \operatorname{Gal}(L_i/\mathbb{Q})$ and define the injective morphisms

$$\rho_i : \operatorname{Gal}(L_i/\mathbb{Q}) \to \operatorname{Aut}(\overline{\mathcal{G}}) : \sigma_i^k \mapsto \varphi^k.$$

The fields L_i are all different and hence the corresponding rational forms are non-isomorphic by Theorem 5.6.4. Moreover, since each L_i is a real field, the aforementioned theorem implies $\mathfrak{n}^{\mathbb{Q}}(\rho_i, c)$ is a rational form of the real Lie algebra $\mathfrak{n}^{\mathbb{R}}(\mathcal{G}, c)$ for all $i \in \mathbb{N}$. This proves that $E(\mathbb{R}/\mathbb{Q}, \mathfrak{n}^{\mathbb{R}}(\mathcal{G}, c))$ counts infinitely many elements and thus the same is true for $E(\mathbb{C}/\mathbb{Q}, \mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c))$. □

As a consequence, we present a family of graphs such that the corresponding real and complex Lie algebras have a unique rational form, up to isomorphism.

Example 5.6.9. Let p, q be two non-negative integers with $q > 1$. Take two disjoint sets V_1 and V_2 which have cardinalities p and q, respectively. We can define a graph $\mathcal{G} = (V_1 \sqcup V_2, E)$ where $E = \{\{v, w\} \mid v \in V_1, w \in W_2\} \cup \{\{v, w\} \mid v, w \in V_1, v \neq w\}$. The quotient graph of $\mathcal{G}$ can be drawn as:



From this it is clear that $\operatorname{Aut}(\overline{\mathcal{G}})$ is trivial and thus that the c-step nilpotent Lie algebras over $\mathbb{R}$ (or $\mathbb{C}$) which are associated to these graphs have only one rational form up to isomorphism. Note that these graphs were also considered in [DM05] in the study of Anosov diffeomorphisms.

Question 5.6.10. Does Theorem 5.6.8 hold for all real and complex nilpotent Lie algebras?

5.6.4 Number of non-isomorphic real forms

In this section we prove a characterization of real forms in complex partially commutative Lie algebras and apply it to some examples. If G is a group, we say an element $g \in G$ is an *involution* of G if $g^2 = 1$. Note that, in particular, the neutral element $e \in G$ is defined to be an involution. We write $\tau \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ for the complex conjugation automorphism.

Theorem 5.6.11. *Let $\mathcal{G}$ be a graph, $c > 1$ an integer and $\mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c)$ the associated complex Lie algebra. Every real form of $\mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c)$ is isomorphic to*

$$\mathfrak{n}^{\mathbb{R}}(\varphi, c) = \{v \in \mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c) \mid \varphi({}^{\tau}v) = v\}$$

for some involution $\varphi \in \mathrm{Aut}(\overline{\mathcal{G}})$. If $\phi \in \mathrm{Aut}(\overline{\mathcal{G}})$ is another involution, then the real forms $\mathfrak{n}^{\mathbb{R}}(\varphi, c)$ and $\mathfrak{n}^{\mathbb{R}}(\phi, c)$ are isomorphic if and only if φ and ϕ are conjugate in $\mathrm{Aut}(\overline{\mathcal{G}})$.

Proof. Combining Theorem 5.6.1 for $L = \mathbb{C}, K = \mathbb{R}$ and Theorem 5.4.1 we get a bijection

$$H^1(\mathbb{C}/\mathbb{R}, \mathrm{Aut}(\overline{\mathcal{G}})) \to E(\mathbb{C}/\mathbb{R}, \mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c)) : [\rho] \mapsto [\mathfrak{n}^{\mathbb{Q}}(\rho, c)].$$

It is clear that any $\rho \in Z^1(\mathbb{C}/\mathbb{R}, \mathrm{Aut}(\overline{\mathcal{G}}))$ is determined by the involution $\rho(\tau)$. Conversely, every involution $\varphi \in \mathrm{Aut}(\overline{\mathcal{G}})$ gives a unique morphism

$$\rho_{\varphi} : \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to \mathrm{Aut}(\overline{\mathcal{G}}) : 1 \mapsto \mathrm{Id},\ \tau \mapsto \varphi.$$

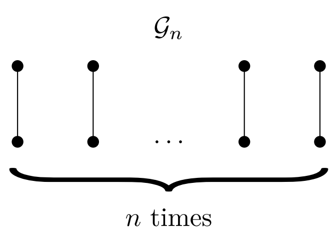
If we write $\mathfrak{n}^{\mathbb{R}}(\varphi, c)$ for the real form $\mathfrak{n}^{\mathbb{R}}(\rho_{\varphi}, c)$, then the statements in the theorem follow straightforward. $\square$

The following fact about symmetric groups is easily verified and thus we omit the proof.

Lemma 5.6.12. *For any $k \in \mathbb{Z}^{>0}$, the number of involutions up to conjugation in the symmetric group $\mathrm{Perm}(X)$ with $|X| = 2k$ or $|X| = 2k + 1$ is equal to $k + 1$.*

In what follows, we give examples of graphs for which we discuss the real forms of the associated complex Lie algebra.

Example 5.6.13 (n-fold direct sum of Heisenberg Lie algebras)**.** Let $n > 1$ be an integer and let $\mathcal{G}_n = (V_n, E_n)$ denote the graph defined by $V_n = \{1, \dots, 2n\}$ and $E_n = \{\{2i - 1, 2i\} \mid 1 \leq i \leq n\}$ as drawn below.



If we let $\mathfrak{h}_3^K$ denote the 3-dimensional Heisenberg Lie algebra over the field K, then it is not hard to see that for $c = 2$ we get a decomposition

$$\mathfrak{n}^{\mathbb{C}}(\mathcal{G}_n, 2) \cong \underbrace{\mathfrak{h}_3^{\mathbb{C}} \oplus \ldots \oplus \mathfrak{h}_3^{\mathbb{C}}}_{n \text{ times}}.$$

The set of coherent components of $\mathcal{G}_n$ is given by $\Lambda_n = \{\{2i-1, 2i\} \mid 1 \leq i \leq n\}$. It is straightforward to verify that the automorphism group $\operatorname{Aut}(\overline{\mathcal{G}_n})$ is isomorphic to the permutation group on a set with n elements. By lemma 5.6.12, we thus find for any integers $c > 1$ and $k \geq 1$ that the number of real forms of $\mathfrak{n}^{\mathbb{C}}(\mathcal{G}_n, c)$ is equal to $k+1$ both for $n = 2k$ and $n = 2k+1$. In particular, we can achieve every natural number ≥ 1 as the number of different real forms.

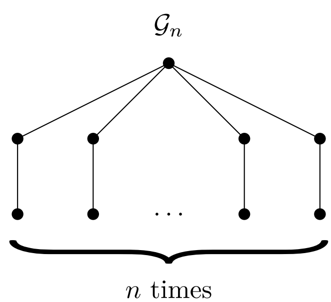
In the special case of $c = 2$, we can even describe the Lie algebras explicitly. Let $\mathfrak{n}^{\mathbb{Q}}_{-1,*}$ be the rational Lie algebra as defined in Example 5.6.6 and write $\mathfrak{n}^{\mathbb{R}}_{-1,*} = \mathfrak{n}^{\mathbb{Q}}_{-1,*} \otimes_{\mathbb{Q}} \mathbb{R}$. Note that $\mathfrak{n}^{\mathbb{R}}_{-1,*}$ is a 6-dimensional real Lie algebra that is also isomorphic to the real Lie algebra obtained by restricting the scalar multiplication on $\mathfrak{h}_3^{\mathbb{C}}$ to $\mathbb{R}$. Every real form of $\mathfrak{n}^{\mathbb{C}}(\mathcal{G}_n, 2)$ is isomorphic to a direct sum Lie algebra of the form

$$\underbrace{\mathfrak{h}_3^{\mathbb{R}} \oplus \ldots \oplus \mathfrak{h}_3^{\mathbb{R}}}_{k \text{ times}} \oplus \underbrace{\mathfrak{n}^{\mathbb{R}}_{-1,*} \oplus \ldots \oplus \mathfrak{n}^{\mathbb{R}}_{-1,*}}_{l \text{ times}}$$

for some non-negative integers k, l which satisfy $k + 2l = n$.

In Example 5.6.13, the real forms arise from permuting the summands in the direct sum decomposition. To show that any number of real forms can also be present in an indecomposable Lie algebra (see Section 5.6.5 for the definition), we give the following example.

Example 5.6.14. Let $n > 1$ be an integer and let $\mathcal{G}_n = (V_n, E_n)$ denote the graph defined by $V_n = \{1, \ldots, 2n+1\}$ and $E_n = \{\{1, 2i\} \mid 1 \leq i \leq n\} \cup \{\{2i, 2i+1\} \mid 1 \leq i \leq n\}$ as drawn below.



The set of coherent components is given by all singletons $\Lambda_n = \{\{i\} \mid 1 \leq i \leq 2n+1\}$. It is straightforward to verify that the automorphism group $\mathrm{Aut}(\overline{\mathcal{G}_n})$ is isomorphic to the permutation group on a set with n elements. By Lemma 5.6.12, we thus find for any integers $c > 1$ and $k \geq 1$ that the number of real forms of $\mathfrak{n}^{\mathbb{C}}(\mathcal{G}_n, c)$ is equal to $k+1$ both for $n = 2k$ and $n = 2k+1$. In particular, we can achieve every natural number ≥ 1 as the number of non-isomorphic real forms of an indecomposable complex Lie algebra.

Thus, the above example shows the following observation that the author did not find yet in the literature. Again, we refer to the following section for the definition of an indecomposable Lie algebra.

Corollary 5.6.15. *For any integer $k > 1$, there exist indecomposable nilpotent complex Lie algebras with exactly k real forms up to isomorphism.*

5.6.5 Indecomposable forms

Example 5.6.5 illustrates that the direct sum of two complex Heisenberg Lie algebras has rational forms that do not admit a direct sum of two non-trivial rational Lie algebras, in contrast to the original complex Lie algebra. We say those rational forms are indecomposable and in what follows we determine which forms of a partially commutative nilpotent Lie algebra have this property. First, let us give a rigorous definition of indecomposable Lie algebras.

Definition 5.6.16. A Lie algebra $\mathfrak{g}$ defined over a field K is said to be *decomposable* if there exist two non-trivial Lie ideals $\mathfrak{h}, \mathfrak{k} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$. We say a Lie algebra is *indecomposable* if it is not decomposable.

Note that if a Lie algebra $\mathfrak{g}^L$ is indecomposable, then any K-form $\mathfrak{g}^K$ for $K \subset L$ is indecomposable as well. As mentioned above, the converse does not hold.

If a Lie algebra is decomposable, one can decompose it into its indecomposable summands. Such a decomposition is not unique in general, but in [FGH13, Theorem 3.3.], uniqueness was proven in case the Lie algebra is centreless, i.e. in case

$$Z(\mathfrak{g}) = \{X \in \mathfrak{g} \mid \forall Y \in \mathfrak{g} : [X, Y] = 0\} = \{0\}.$$

The theorem was proven for real Lie algebras, but the proof works for any subfield of $\mathbb{C}$. We can restate the result that we need as follows:

Theorem 5.6.17 ([FGH13])**.** *Let $\mathfrak{g}$ be a Lie algebra over a field $K \subset \mathbb{C}$, s, t positive integers and $\mathfrak{h}_1, \ldots, \mathfrak{h}_s, \mathfrak{k}_1, \ldots, \mathfrak{k}_t \subset \mathfrak{g}$ indecomposable ideals such that*

$$\mathfrak{g} = \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_s \qquad \text{and} \qquad \mathfrak{g} = \mathfrak{k}_1 \oplus \ldots \oplus \mathfrak{k}_t,$$

then $s = t$ and up to reordering the summands $\mathfrak{h}_i$, we have $\mathfrak{h}_i \subset \mathfrak{k}_i + Z(\mathfrak{g})$ for all $i \in \{1, \ldots, s\}$.

We want to apply the above theorem to the partially commutative nilpotent Lie algebras $\mathfrak{n}^K(\mathcal{G}, c)$. In order to do so we first determine a canonical decomposition of $\mathfrak{n}^K(\mathcal{G}, c)$ into indecomposable ideals.

First, recall that for any two Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$, the *direct sum Lie algebra* is the vector space $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ endowed with a Lie bracket determined by $[X_1 + X_2, Y_1 + Y_2] := [X_1, Y_1] + [X_2, Y_2]$ for any $X_1, Y_1 \in \mathfrak{g}_1$ and $X_2, Y_2 \in \mathfrak{g}_2$.

Second, recall from Remark 4.2.3 that the Lie algebra $\mathfrak{n}^K(\mathcal{G}, c)$ satisfies a universal property. Using this property, it is not hard to show the following lemma.

Lemma 5.6.18. *Let $\mathcal{G} = (V, E)$ be a graph and $K \subset \mathbb{C}$ a field. Let $V_1, V_2 \subset V$ be disjoint subsets such that $V = V_1 \cup V_2$ and $\forall v \in V_1, w \in V_2 : \{v, w\} \notin E$. Write $\mathcal{G}_1$, $\mathcal{G}_2$ for the subgraphs of $\mathcal{G}$ spanned by V_1, V_2, respectively. There exists a unique Lie algebra isomorphism*

$$\mathfrak{n}^K(\mathcal{G}, c) \xrightarrow{\cong} \mathfrak{n}^K(\mathcal{G}_1, c) \oplus \mathfrak{n}^K(\mathcal{G}_2, c)$$

which restricts to the identity on the vertices V.

Recall the definition of a *connected graph* from Section 4.1.

Lemma 5.6.19. *Let $\mathcal{G} = (V, E)$ be a graph and $K \subset \mathbb{C}$ a field. The following are equivalent:*

(i) *$\mathcal{G}$ is connected,*

(ii) *$\mathfrak{n}^K(\mathcal{G}, 2)$ is indecomposable,*

(iii) *$\mathfrak{n}^K(\mathcal{G}, c)$ is indecomposable for any $c > 1$.*

Proof. (i) $\Rightarrow$ (ii). This is essentially [DM05, Lemma 6.6] which deals with $K = \mathbb{R}$ but works for an arbitrary field.

(ii) $\Rightarrow$ (iii). Take any $c > 1$ and assume that there exist ideals $\mathfrak{h}, \mathfrak{k} \subset \mathfrak{n}^K(\mathcal{G}, c)$ such that $\mathfrak{n}^K(\mathcal{G}, c) = \mathfrak{h} \oplus \mathfrak{k}$. Note that both $\mathfrak{h}$ and $\mathfrak{k}$ are also nilpotent Lie algebras. We have a sequence of isomorphisms:

$$\mathfrak{n}^K(\mathcal{G}, 2) \cong \mathfrak{n}^K(\mathcal{G}, c)/\gamma_3(\mathfrak{n}^K(\mathcal{G}, c)) \cong \mathfrak{h}/\gamma_3(\mathfrak{h}) \oplus \mathfrak{k}/\gamma_3(\mathfrak{k}).$$

Since $\mathfrak{n}^K(\mathcal{G}, 2)$ is assumed to be indecomposable, this implies that $\mathfrak{h}/\gamma_3(\mathfrak{h}) = \{0\}$ or $\mathfrak{k}/\gamma_3(\mathfrak{k}) = \{0\}$. By nilpotency of $\mathfrak{h}$ and $\mathfrak{k}$, we then get that $\mathfrak{h} = \{0\}$ or $\mathfrak{k} = \{0\}$.

(iii) $\Rightarrow$ (i). We prove this by contraposition. Assume $\mathcal{G}$ is not connected, then there exists a partition of the vertices $V = V_1 \sqcup V_2$ with V_1 and V_2 non-empty, such that for any $v \in V_1$ and any $w \in V_2$ it holds that $\{v, w\} \notin E$. It then readily follows from Lemma 5.6.18 that $\mathfrak{n}^K(\mathcal{G}, c)$ is decomposable and concludes the proof. $\square$

Let us write for the remainder of this section

$$W = \mathfrak{n}^K(\mathcal{G}, c)/[\mathfrak{n}^K(\mathcal{G}, c), \mathfrak{n}^K(\mathcal{G}, c)]$$

and note that we can see the vertices V as a subset of W, by identifying them with their images under the projection to the abelianization

$$\pi_{\mathrm{ab}} : \mathfrak{n}^K(\mathcal{G}, c) \to W.$$

Moreover, the vertices give a basis for W. The projection of the centre to the abelianization can be described by the following lemma. The *degree* of a vertex $v \in V$ is defined as the number of vertices adjacent to v.

Lemma 5.6.20. *For any graph $\mathcal{G} = (V, E)$ and field $K \subset \mathbb{C}$ we have*

$$\pi_{\mathrm{ab}}\left(Z\left(\mathfrak{n}^K(\mathcal{G}, c)\right)\right) = \mathrm{span}_K\left(\{v \in V \mid v \textit{ has degree } 0\}\right) \subset W.$$

Proof. Take any $v \in V$ of degree 0. It is clear that $[v, w] = 0$ in $\mathfrak{n}^K(\mathcal{G}, c)$ for any $w \in V$ and thus since V generates $\mathfrak{n}^K(\mathcal{G}, c)$ that $v \in Z(\mathfrak{n}^K(\mathcal{G}, c))$. As a consequence we get the inclusion

$$\mathrm{span}_K\left(\{v \in V \mid v \text{ has degree } 0\}\right) \subseteq \pi_{\mathrm{ab}}\left(Z\left(\mathfrak{n}^K(\mathcal{G}, c)\right)\right).$$

For the other inclusion, take any $X \in \pi_{\mathrm{ab}}\left(Z\left(\mathfrak{n}^K(\mathcal{G}, c)\right)\right)$. Then there exists a $Y \in \gamma_2(\mathfrak{n}^K(\mathcal{G}, c))$ such that $X + Y \in Z\left(\mathfrak{n}^K(\mathcal{G}, c)\right)$. As a consequence, for any $w \in V$ it must hold that $[w, X + Y] = [w, X] + [w, Y] = 0$. From the Carnot grading of $\mathfrak{n}^K(\mathcal{G}, c)$ as given in (4.3), it follows that both $[w, X]$ and $[w, Y]$ are zero, and this for any vertex $w \in V$. Let $f : V \to K$ be the unique function such that $X = \sum_{v \in V} f(v)v$. We then get that $\sum_{v \in V} f(v)[w, v] = 0$ for any $w \in V$. Using the relations in $\mathfrak{n}^K(\mathcal{G}, c)$, this reduces to $\sum_{v \in N(w)} f(v)[w, v] = 0$, where we remind the reader of the definition of the open neighbourhood $N(w)$ of a vertex w from Section 4.1. Since the set $\{[w, v] \mid v \in N(w)\}$ is linearly independent in $\mathfrak{n}^K(\mathcal{G}, c)$, it follows that $f(v) = 0$ for all $v \in N(w)$ and all $w \in V$. This exactly means that $X \in \mathrm{span}_K\left(\{v \in V \mid v \text{ has degree } 0\}\right)$, which proves the other inclusion. $\square$

Let $\mathcal{G} = (V, E)$ be a graph. Recall from Section 4.1 that a walk is a tuple of vertices $(v_1, \ldots, v_n)$ such that $\{v_i, v_{i+1}\} \in E$ for all $i \in \{1, \ldots, n-1\}$. In that case, we say that $(v_1, \ldots, v_n)$ is a walk between v_1 and v_n. Note that *the existence of a walk between vertices* defines an equivalence relation on V. The equivalence classes are called the *connected components of* $\mathcal{G}$. Note that every connected component is in fact connected (definition from Section 4.1). We let $\mathcal{C}(\mathcal{G})$ denote the set of all connected components of $\mathcal{G}$. We can now combine Theorem 5.6.17, Lemma 5.6.18, Lemma 5.6.19 and Lemma 5.6.20 to prove the following result for the decomposition of Lie algebras associated to graphs.

Proposition 5.6.21. *Let $\mathcal{G} = (V, E)$ be a graph with no vertices of degree 0, $K \subset \mathbb{C}$ a field k a positive integer and $\mathfrak{h}_1, \ldots, \mathfrak{h}_k \subset \mathfrak{n}^K(\mathcal{G}, c)$ non-trivial indecomposable ideals such that $\mathfrak{n}^K(\mathcal{G}, c) = \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_k$. Then $k = |\mathcal{C}(\mathcal{G})|$ and there exists an ordering of the connected components $\mathcal{C}(\mathcal{G}) = \{C_1, \ldots, C_k\}$ such that $\pi_{\mathrm{ab}}(\mathfrak{h}_i) = \mathrm{span}_K(C_i)$ for all $i \in \{1, \ldots, k\}$.*

Proof. Let $\{C_1, \ldots, C_l\}$ be some ordering of the connected components of $\mathcal{G}$ with $l = |\mathcal{C}(\mathcal{G})|$. Let $\mathfrak{n}_i \subset \mathfrak{n}^K(\mathcal{G}, c)$ be the Lie subalgebra in $\mathfrak{n}^K(\mathcal{G}, c)$ generated by C_i for all $i \in \{1, \ldots, l\}$. By Lemma 5.6.18, it is clear that the $\mathfrak{n}_i$'s are ideals, that $\mathfrak{n}^K(\mathcal{G}, c) = \mathfrak{n}_1 \oplus \ldots \oplus \mathfrak{n}_k$ and if $\mathcal{G}_i$ denotes the subgraph spanned by C_i, that $\mathfrak{n}_i \cong \mathfrak{n}^K(\mathcal{G}_i, c)$. Moreover, since each graph $\mathcal{G}_i$ is connected, Lemma 5.6.19 implies that $\mathfrak{n}_i \cong \mathfrak{n}^K(\mathcal{G}_i, c)$ is indecomposable for each $i \in \{1, \ldots, l\}$. By Theorem 5.6.17, we have $k = l = |\mathcal{C}(\mathcal{G})|$ and we can fix a reordering of the connected components of $\mathcal{G}$ such that $\mathfrak{h}_\mathfrak{i} \subset \mathfrak{n}_i + Z(\mathfrak{n}^K(\mathcal{G}, c))$ for all $i \in \{1, \ldots, k\}$. Since $\mathcal{G}$ has no vertices of degree 0, we find by Lemma 5.6.20 that $\pi_{\mathrm{ab}}(Z(\mathfrak{n}^K(\mathcal{G}, c))) = \{0\}$ and thus that $\pi_{\mathrm{ab}}(\mathfrak{h}_i) \subset \pi_{\mathrm{ab}}(\mathfrak{n}_i)$ for all $i \in \{1, \ldots, k\}$. Note that $\pi_{\mathrm{ab}}(\mathfrak{n}_i) = \mathrm{span}_K(C_i)$ and thus that $V = \pi_{\mathrm{ab}}(\mathfrak{n}_1) \oplus \ldots \oplus \pi_{\mathrm{ab}}(\mathfrak{n}_k)$. Since π_{ab} is surjective we must have $V = \pi_{\mathrm{ab}}(\mathfrak{h}_1 + \ldots + \mathfrak{h}_k) = \pi_{\mathrm{ab}}(\mathfrak{h}_1) + \ldots + \pi_{\mathrm{ab}}(\mathfrak{h}_k)$ which implies that $\pi_{\mathrm{ab}}(\mathfrak{h}_i) = \pi_{\mathrm{ab}}(\mathfrak{n}_i) = \mathrm{span}_K(C_i)$ for all $i \in \{1, \ldots, k\}$. This concludes the proof. $\square$

Let $\mathcal{G} = (V, E)$ be a graph and L/K a Galois extension of subfields of $\mathbb{C}$. Recall the natural action of $\mathrm{Gal}(L/K)$ on $\mathfrak{n}^L(\mathcal{G}, c)$. Note that $\mathrm{span}_L(V) = W \subset \mathfrak{n}^L(\mathcal{G}, c)$ is preserved under this action and thus that we have an induced action of $\mathrm{Gal}(L/K)$ on W. The vertices $V \subset W$ are fixed under this action. If $\rho : \mathrm{Gal}(L/K) \to \mathrm{Aut}(\mathcal{G})$ is a continuous morphism and $\mathfrak{n}^K(\rho, c)$ the associated K-form of $\mathfrak{n}^L(\mathcal{G}, c)$, one can check that

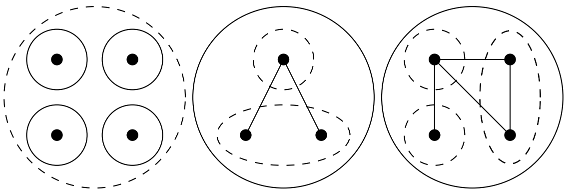
$$\pi_{\mathrm{ab}}(\mathfrak{n}^K(\rho, c)) = \{v \in W \mid \forall \sigma \in \mathrm{Gal}(L/K) : \overline{P}(\rho_\sigma)({}^\sigma v) = v\} \tag{5.9}$$

where

$$\overline{P} = (P \circ r)$$

with $r : \mathrm{Aut}(\overline{\mathcal{G}}) \to \mathrm{Aut}(\mathcal{G})$ is the splitting morphism as chosen in 5.6.2 and $P : \mathrm{Aut}(\mathcal{G}) \to \mathrm{GL}(W)$ is the permutation representation.

Let $\Lambda_{\mathcal{G}}$ be the set of coherent components of $\mathcal{G}$. As one can check, every connected component which counts at least two vertices is a disjoint union of coherent components. On the other hand, the union of all connected components which are singletons is equal to a coherent component. This is illustrated by the example drawn below. Dashed lines represent coherent components while full lines (apart from de edges) represent connected components.



For any $\varphi \in \mathrm{Aut}(\overline{\mathcal{G}})$, the images of two coherent components which are subsets of the same connected component are again subsets of the same connected component. Let $C \in \mathcal{C}(\mathcal{G})$ be a connected component. In case C counts at least two vertices, there exist coherent components $\lambda_1, \ldots, \lambda_k$ such that $C = \lambda_1 \sqcup \ldots \sqcup \lambda_k$ and we define $\chi(\varphi)(C) = \varphi(\lambda_1) \sqcup \ldots \sqcup \varphi(\lambda_k)$. In case C is a singleton, we define $\chi(\varphi)(C) = C$. This gives a morphism

$$\chi : \mathrm{Aut}(\overline{\mathcal{G}}) \to \mathrm{Perm}(\mathcal{C}(\mathcal{G})).$$

Recall that an action is called *transitive* if it has only one orbit.

Theorem 5.6.22. *Let $\mathcal{G} = (V, E)$ be a graph, L/K a Galois extension of subfields of $\mathbb{C}$ and $\rho : \mathrm{Gal}(L/K) \to \mathrm{Aut}(\overline{\mathcal{G}})$ a continuous morphism. The form $\mathfrak{n}^K(\rho, c)$ is indecomposable if and only if the $(\chi \circ \rho)$-action on the set of connected components $\mathcal{C}(\mathcal{G})$ is transitive.*

Proof. Assume $\mathfrak{n}^K(\rho, c)$ is indecomposable. Let $C \in \mathcal{C}(\mathcal{G})$ be a connected component. We write $\tilde{V}$ for the vertices which lie in a connected component which lies in the $(\chi \circ \rho)$-orbit of C, i.e. $\tilde{V} = \{v \in V \mid \exists \sigma \in \mathrm{Gal}(L/K) : v \in (\chi \circ \rho)_\sigma(C)\}$. Let $\mathfrak{n}_1$ and $\mathfrak{n}_2$ denote the subalgebras of $\mathfrak{n}^L(\mathcal{G}, c)$ generated by $\tilde{V}$ and $V \setminus \tilde{V}$, respectively. It follows that $r(\rho_\sigma)(\tilde{V}) = \tilde{V}$ and as a consequence that $i(\rho_\sigma)(\mathfrak{n}_1) = \mathfrak{n}_1$ and $i(\rho_\sigma)(\mathfrak{n}_2) = \mathfrak{n}_2$ for any $\sigma \in \mathrm{Gal}(L/K)$. Moreover, since $\mathfrak{n}_1$ and $\mathfrak{n}_2$ are generated by sets of vertices, we have that $i(\rho_\sigma)({}^\sigma\mathfrak{n}_1) = \mathfrak{n}_1$ and $i(\rho_\sigma)({}^\sigma\mathfrak{n}_2) = \mathfrak{n}_2$ for any $\sigma \in \mathrm{Gal}(L/K)$. By Lemma 5.6.18, $\mathfrak{n}_1$ and $\mathfrak{n}_2$ are ideals and we have a direct sum $\mathfrak{n}^L(\mathcal{G}, c) = \mathfrak{n}_1 \oplus \mathfrak{n}_2$. Thus, for an arbitrary $v \in \mathfrak{n}^L(\mathcal{G}, c)$

there exist unique vectors $v_1 \in \mathfrak{n}_1$ and $v_2 \in \mathfrak{n}_2$ such that $v = v_1 + v_2$. For any $\sigma \in \mathrm{Gal}(L/K)$, we then have the equivalences

$$\begin{aligned} i(\rho_\sigma)({}^\sigma v) = v &\Leftrightarrow i(\rho_\sigma)({}^\sigma(v_1 + v_2)) = v_1 + v_2 \\ &\Leftrightarrow i(\rho_\sigma)({}^\sigma v_1) + i(\rho_\sigma)({}^\sigma v_2) = v_1 + v_2 \\ &\Leftrightarrow i(\rho_\sigma)({}^\sigma v_1) = v_1 \quad \text{and} \quad i(\rho_\sigma)({}^\sigma v_2) = v_2. \end{aligned}$$

where the last equivalence uses $i(\rho_\sigma)({}^\sigma \mathfrak{n}_j) = \mathfrak{n}_j$ for $j = 1, 2$. Since $\mathfrak{n}_1$, $\mathfrak{n}_2$ are themselves Lie algebras associated to a graph, we know that

$$\mathfrak{m}_j = \{v \in \mathfrak{n}_j \mid \forall \sigma \in \mathrm{Gal}(L/K) : i(\rho_\sigma)(v) = v\}$$

defines a K-form of $\mathfrak{n}_j$. The above equivalences then imply that $\mathfrak{n}^K(\rho, c) = \mathfrak{m}_1 \oplus \mathfrak{m}_2$. Moreover, since $\mathfrak{n}_1$ and $\mathfrak{n}_2$ are ideals, the same holds for $\mathfrak{m}_1$ and $\mathfrak{m}_2$. Since we assumed $\mathfrak{n}^K(\rho, c)$ to be indecomposable and $\mathfrak{m}_1 \neq \{0\}$ by construction, we must have that $\mathfrak{m}_2 = \{0\}$ and thus that $V \setminus \tilde{V} = \emptyset$. This proves that the $(\chi \circ \rho)$-action on $\mathcal{C}(\mathcal{G})$ is transitive.

Conversely, assume that the $(\chi \circ \rho)$-action on $\mathcal{C}(\mathcal{G})$ is transitive. Since this action fixes connected components which are singletons, transitivity implies that either there are no connected components which are singletons or that V itself is a singleton. In the latter case, the Lie algebra $\mathfrak{n}^L(\mathcal{G}, c)$ is itself indecomposable and as a consequence so are all of its forms. Thus we can assume that there are no connected components which are singletons. This is equivalent to saying there are no vertices of degree 0. Assume there are ideals $\mathfrak{m}_1$, $\mathfrak{m}_2$ of the Lie algebra $\mathfrak{n}^K(\rho, c)$ such that $\mathfrak{n}^K(\rho, c) = \mathfrak{m}_1 \oplus \mathfrak{m}_2$. Define $\mathfrak{n}_j = \mathrm{span}_L(\mathfrak{m}_j)$ for $j = 1, 2$. Since $\mathfrak{n}^K(\mathcal{G}, c)$ is a form of $\mathfrak{n}^L(\mathcal{G}, c)$, it follows that $\mathfrak{n}_1$, $\mathfrak{n}_2$ are ideals of $\mathfrak{n}^L(\mathcal{G}, c)$ and that $\mathfrak{n}^L(\mathcal{G}, c) = \mathfrak{n}_1 \oplus \mathfrak{n}_2$. By Proposition 5.6.21, we must have subsets of vertices $V_1, V_2 \subset V$, each a union of connected components, such that $V = V_1 \sqcup V_2$ and $\pi_{\mathrm{ab}}(\mathfrak{n}_j) = \mathrm{span}_L(V_j)$ for $j = 1, 2$. Take $\sigma \in \mathrm{Gal}(L/K)$ arbitrarily. Note that ${}^\sigma(\mathrm{span}_L(V_j)) = \mathrm{span}_L(V_j)$ for $j = 1, 2$, since these subspaces are spanned by vertices. Using that $\pi_{\mathrm{ab}}(\mathfrak{m}_j) \subset \mathrm{span}_L(V_j)$, we thus find that ${}^\sigma(\pi_{\mathrm{ab}}(\mathfrak{m}_j)) \subset \mathrm{span}_L(V_j)$ for $j = 1, 2$. From equation (5.9) it follows that $\overline{P}(\rho_\sigma)^{-1}(\pi_{\mathrm{ab}}(\mathfrak{m}_j)) = {}^\sigma(\pi_{\mathrm{ab}}(\mathfrak{m}_j)) \subset \mathrm{span}_L(V_j)$. But since $\mathrm{span}_L(\mathfrak{m}_j) = \mathrm{span}_L(V_j)$, we also have $\overline{P}(\rho_\sigma)^{-1}(\mathrm{span}_L(V_j)) = \mathrm{span}_L(V_j)$. Since σ was chosen arbitrarily, this implies that $r(\rho_\sigma)(V_j) = V_j$ for all $\sigma \in \mathrm{Gal}(L/K)$. Since we assumed that the $(\chi \circ \rho)$-action on $\mathcal{C}(\mathcal{G})$ is transitive, it follows that either V_1 or V_2 is the empty set and thus that either $\mathfrak{m}_1 = \{0\}$ or $\mathfrak{m}_2 = \{0\}$. □

Chapter 6

Anosov Lie algebras

In this chapter we will deal with Question 3.4.23 from Chapter 3 that reads: which rational Lie algebras admit an integer-like hyperbolic automorphism? Such an automorphism is called an Anosov automorphism and a rational Lie algebra admitting one is called an Anosov Lie algebra. In Section 6.4 we prove a result on the indecomposable factors of Anosov Lie algebras (see Theorem 6.4.5). In Section 6.5 we give the first example of a 12-dimensional Anosov Lie algebra without a positive grading and we prove that this is the smallest dimension for such an example (see Theorem 6.5.17). In Section 6.6, we prove a characterization of Anosov rational forms in partially commutative Lie algebras (see Theorem 6.6.13). This uses Theorem 5.6.3 from Chapter 5 on the rational forms of partially commutative Lie algebras.

6.1 Definitions and known results

Definition 6.1.1. Let $\mathfrak{n}$ be a nilpotent Lie algebra defined over a subfield of $\mathbb{C}$. We say an automorphism $A \in \operatorname{Aut}(\mathfrak{g})$ is an *Anosov automorphism* if it is hyperbolic and integer-like. A rational Lie algebra which admits an Anosov automorphism will be called an *Anosov Lie algebra.*

The following observations are elementary, yet important for the results in this section.

Lemma 6.1.2. *The semi-simple part of an Anosov automorphism is an Anosov automorphism. Any non-zero power of an Anosov automorphism is an Anosov automorphism.*

Proof. Let A be an Anosov automorphism on the Lie algebra $\mathfrak{g}$ defined over a subfield $K \subset \mathbb{C}$.

For the first part, recall from sections 4.5 and 4.6 that the automorphism group $\mathrm{Aut}(\mathfrak{g})$ is a linear algebraic group defined over K. Let $A = A_s A_u$ be the multiplicative Jordan-Chevalley decomposition of A. By Theorem 4.5.8, it follows that also $A_s \in \mathrm{Aut}(\mathfrak{g})$ and thus that A_s is also a Lie algebra automorphism. Moreover, the characteristic polynomial of A_s is the same as the one of A, implying that A is also hyperbolic and integer-like. We conclude that A_s is an Anosov automorphism.

For the second part, let k be a non-zero integer. The eigenvalues of A^k are the k-powers of the eigenvalues of A. Clearly this does not alter the fact that they have modulus different from 1 as $k \neq 0$. By Remark 3.4.19, there exists a basis with respect to which A is represented by an integral matrix with determinant equal to 1 or -1. The k-th power of this matrix is still integral with determinant equal to 1 or -1. Therefore, we conclude that A^k is also integer-like and thus Anosov. □

Let $K/\mathbb{Q}$ be an algebraic field extension. We recall that an *algebraic integer* in K is an element $\alpha \in K$ whose minimal polynomial has integer coefficients. It is a classical result that the set of all algebraic integers in K form a ring. The units of this ring, which are the algebraic integers $\alpha \in K$ such that α^{-1} is also an algebraic integer, are called the *algebraic units* in K. This is a group for the multiplication and we write this group as $\mathcal{U}_K$. Another standard result states that $\alpha \in K$ is an algebraic unit if and only if its minimal polynomial has integer coefficients and has a constant term equal to $\pm$ 1. As a consequence, the eigenvalues of an Anosov automorphism are algebraic units.

The proof of the following two lemmas is based on [DW23b, Lemma 4.3.].

Lemma 6.1.3. *Let $f, f_1, \ldots, f_k \in \mathbb{Q}[X]$ be monic polynomials with rational coefficients such that $f = f_1 \cdot \ldots \cdot f_k$. Then the following are equivalent:*

(i) $f \in \mathbb{Z}[X]$ *and* $f(0) = \pm 1$,

(ii) $f_i \in \mathbb{Z}[X]$ *and* $f_i(0) = \pm 1$ *for all* i.

Proof. The implication $(ii) \implies (i)$ is trivial.

For the other implication, note that the roots of f are algebraic units and therefore the roots of each polynomial f_i are algebraic units as well. Using that the algebraic integers form a ring, we find that the coefficients of the polynomials f_i, which can be expressed as products and sums of the roots of

f_i, are algebraic integers. Since these coefficients are also rational (and since the algebraic integers in $\mathbb{Q}$ are exactly the integers $\mathbb{Z}$) we must conclude that $f_i \in \mathbb{Z}[X]$ for all i. For each f_i, its constant term is a product of its roots. Since these roots are algebraic units, this product is again an algebraic unit. Thus, the constant term of f_i is an algebraic unit which also has to lie in $\mathbb{Z}$. We conclude (since the only algebraic units in $\mathbb{Q}$ are 1 and -1), that $f(0) = \pm 1$ for all i. □

Lemma 6.1.4. *Let V be a vector space defined over $\mathbb{Q}$ which decomposes into a direct sum $V = V_1 \oplus \ldots \oplus V_k$. Let $A \in \mathrm{GL}(V)$ satisfy $A(V_i) = V_i$ and write A_i for the restriction of A to V_i. Then the following are true*

(i) A is integer-like if and only if A_i is integer-like for all $i \in \{1, \ldots, k\}$,

(ii) A is hyperbolic if and only if A_i is hyperbolic for all $i \in \{1, \ldots, k\}$.

Proof. Denote by $f, f_1, \ldots, f_k$ the characteristic polynomials of $A, A_1, \ldots, A_k$, respectively. Clearly we have that $f, f_1, \ldots, f_k \in \mathbb{Q}[X]$ and $f = f_1 \cdot \ldots \cdot f_k$. Lemma 6.1.3 then immediately proves part (i). Part (ii) follows immediately from $f = f_1 \cdot \ldots \cdot f_k$. □

Another observation is the following.

Lemma 6.1.5. *Let V be a non-trivial vector space defined over a subfield of $\mathbb{C}$. If there exists an $A \in \mathrm{GL}(V)$ which is integer-like hyperbolic, then* $\dim V \geq 2$.

Proof. Assume that $\dim(V) = 1$ and that $A \in \mathrm{GL}(V)$ is integer-like hyperbolic. Note that A has one eigenvalue which is equal to its determinant. Since this determinant needs to be equal to 1 or -1, we get a contradiction with A being hyperbolic. □

Definition 6.1.6. Let $\mathfrak{n}$ be a nilpotent Lie algebra of class c. The *type* of $\mathfrak{n}$ is the c-tuple $(n_1, \ldots, n_c)$ where $n_i = \dim(\gamma_i(\mathfrak{n})) - \dim(\gamma_{i+1}(\mathfrak{n}))$.

Let $\mathfrak{n}$ be a rational nilpotent Lie algebra with an Anosov automorphism A. By Lemma 6.1.2 its semi-simple part A_s is also an Anosov automorphism. Hence, for all $i \in \{1, \ldots, c\}$, one can choose a complement $\mathfrak{n}_i$ to $\gamma_{i+1}(\mathfrak{n})$ inside $\gamma_i(\mathfrak{n})$ for which $A_s(\mathfrak{n}_i) = \mathfrak{n}_i$. Note that if $(n_1, \ldots, n_c)$ is the type of $\mathfrak{n}$, then $\dim(\mathfrak{n}_i) = n_i$. Using Lemma 6.1.4 and Lemma 6.1.5, we see that if a rational nilpotent Lie algebra $\mathfrak{n}$ admits an Anosov automorphism, its type $(n_1, \ldots, n_c)$ must satisfy $n_i \geq 2$ for all $i \in \{1, \ldots, c\}$.

More necessary conditions on the type can be derived and we list them in the following theorem. The proof can be found in [LW08] and [Pay09].

Theorem 6.1.7. *Let* $\mathfrak{n}$ *be a non-abelian rational nilpotent Lie algebra of type* $(n_1, \ldots, n_c)$*. If* $\mathfrak{n}$ *is Anosov, then the following are true:*

(i) $n_1 \geq 3$ *and* $n_i \geq 2$ *for all* i*,*

(ii) if $n_1 = 3$*, then* $n_2 = 3$ *and* $3 \mid n_i$ *for all* i*,*

(iii) if $n_1 = 4$*, then* $2 \mid n_i$ *for all* i*.*

Definition 6.1.8. Let $\mathfrak{g}$ be a Lie algebra. An *abelian factor* of $\mathfrak{g}$ is a Lie ideal $\mathfrak{h} \subset \mathfrak{g}$ such that $\mathfrak{h}$ is abelian and there exists another ideal $\mathfrak{k} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$. An abelian factor is called *maximal* if it is not properly included in another abelian factor of $\mathfrak{g}$.

Let $\mathfrak{g}$ be a Lie algebra over a field K. It is immediate that a subspace $\mathfrak{a} \subset \mathfrak{g}$ is an abelian factor if and only if it is contained in the center $Z(\mathfrak{g})$ with $\mathfrak{a} \cap [\mathfrak{g}, \mathfrak{g}] = 0$. A maximal abelian factor is then exactly any complement to $Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]$ inside $Z(\mathfrak{g})$. Therefore, we define the number

$$m(\mathfrak{g}) = \dim\left(Z(\mathfrak{g})\right) - \dim\left(Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]\right),$$

which is exactly the dimension of any maximal abelian factor. Note that this number does not change over field extensions. If L/K is an extension of fields, we have that $m(\mathfrak{g}) = m(\mathfrak{g}^L = \mathfrak{g} \otimes_K L)$.

The following theorem was proven in [LW08, Theorem 3.1] and relates maximal abelian factors to the existence of an Anosov automorphism on nilpotent rational Lie algebras.

Theorem 6.1.9. *Let* $\mathfrak{n}$ *be a rational nilpotent Lie algebra with a vector space decomposition* $\mathfrak{n} = \mathfrak{a} \oplus \tilde{\mathfrak{n}}$ *where* $\mathfrak{a}$ *is a maximal abelian factor and* $\tilde{\mathfrak{n}}$ *a Lie ideal. Then* $\mathfrak{n}$ *is Anosov if and only if* $\dim(\mathfrak{a}) \neq 1$ *and* $\tilde{\mathfrak{n}}$ *is Anosov.*

In low dimensions, there is a classification of the real Lie algebras that admit an Anosov rational form. It is enough to list those Lie algebras $\mathfrak{n}$ with $m(\mathfrak{n}) = 0$ (i.e. those Lie algebras without a non-trivial abelian factor) due to Theorem 6.1.9. In Table 6.1 we define the Lie algebras that occur in the classification. Note that $\mathfrak{h}_3$ and $\mathfrak{l}_4$ are the Heisenberg Lie algebra of dimension 3 and the standard filiform Lie algebra of dimension 4, respectively, as introduced in Example 2.3.19 of Chapter 2. The Lie algebra $\mathfrak{f}_{3,2}$ is the free 2-step nilpotent Lie algebra on 3 generators.

	Type	Bracket relations
$\mathfrak{h}_3$	$(2,1)$	$[X,Y]=Z$
$\mathfrak{f}_{3,2}$	$(3,3)$	$[X_1,X_2]=Z_1$, $[X_2,X_3]=Z_2$, $[X_3,X_1]=Z_3$
$\mathfrak{g}_{6,2}$	$(6,2)$	$[X_1,X_2]=Z_1$, $[X_1,X_3]=Z_2$, $[Y_1,Y_2]=Z_1$, $[Y_1,Y_3]=Z_2$
$\mathfrak{g}_{4,4}$	$(4,4)$	$[X_1,X_2]=Z_1$, $[X_1,X_4]=Z_2$, $[X_2,X_3]=Z_3$, $[X_2,X_4]=Z_4$
$\mathfrak{l}_4$	$(2,1,1)$	$[X,Y]=Z$, $[X,Z]=W$

Table 6.1: Definitions of Lie algebras occurring in the classification of real Lie algebras with an Anosov rational form in low dimensions

The classification was proven up to dimension 8 by [LW09].

Theorem 6.1.10. *Any real Lie algebra of dimension ≤ 8 that admits an Anosov rational form and does not have a non-trivial abelian factor is isomorpic to one of the following Lie algebras:*

$$\mathfrak{h}_3 \oplus \mathfrak{h}_3, \quad \mathfrak{f}_{3,2}, \quad \mathfrak{g}_{6,2}, \quad \mathfrak{g}_{4,4}, \quad \mathfrak{l}_4 \oplus \mathfrak{l}_4.$$

The following existence result was proven in [LW08].

Theorem 6.1.11. *Let $\mathfrak{n}$ be a rational nilpotent Lie algebra with a positive grading. The real Lie algebra $(\mathfrak{n} \oplus \ldots \oplus \mathfrak{n})^{\mathbb{R}}$ (where the direct sum counts at least 2 summands) has an Anosov rational form.*

6.2 Dirichlet's unit theorem

Recall that a *number field* is a finite degree extension of $\mathbb{Q}$. We consider any number field as a subfield of $\mathbb{C}$. An (injective) field morphism $\sigma : K \to \mathbb{C}$ will be called an *embedding*. Every number field K has exactly $[K : \mathbb{Q}]$ different embeddings $K \to \mathbb{C}$. If $\tau : \mathbb{C} \to \mathbb{C}$ denotes the complex conjugation automorphism, then an embedding $\sigma : K \to \mathbb{C}$ satisfies $\tau \circ \sigma = \sigma$ if and only if

$\sigma(K) \subset \mathbb{R}$. We call σ a *real embedding* if $\sigma(K) \subset \mathbb{R}$ and $\sigma, \tau \circ \sigma$ a *conjugated pair of complex embeddings* if $\sigma(K) \not\subseteq \mathbb{R}$. If s is the number of real embeddings and t the number of conjugated pairs of complex embeddings, then $s + 2t = [K : \mathbb{Q}]$. We say a number field is *totally real* if all its embeddings in $\mathbb{C}$ are real and *totally imaginary* if none of its embeddings are real.

Note that for any embedding $\sigma : K \to \mathbb{C}$ of a number field K, we have that $\sigma(K) \subset \overline{\mathbb{Q}}$. As a consequence we also have an embedding $\sigma : K \to \overline{\mathbb{Q}}$. Conversely, every embedding of K into $\overline{\mathbb{Q}}$ gives an embedding of K into $\mathbb{C}$ by composing it with the inclusion of $\overline{\mathbb{Q}}$ in $\mathbb{C}$. A number field K is a Galois extension of $\mathbb{Q}$ if and only if $\sigma(K) = K$ for any embedding $\sigma : K \to \mathbb{C}$. Note that $\mathbb{C}/\mathbb{Q}$ is not a Galois extension, but $\overline{\mathbb{Q}}/\mathbb{Q}$ is. Therefore, in some cases, it will be more convenient to work with $\overline{\mathbb{Q}}$ instead of $\mathbb{C}$.

Recall that the *algebraic units* of K are all elements in K which have a minimal polynomial with integer coefficients and constant term equal to ± 1. We write the set of algebraic units in K as $\mathcal{U}_K$. This set is a group for multiplication in K. If a number field K has s real embeddings: $\sigma_1, \ldots, \sigma_s$ and t conjugated pairs of complex embeddings: $\sigma_{s+1}, \tau \circ \sigma_{s+1}, \ldots, \sigma_{s+t}, \tau \circ \sigma_{s+t}$, then one can define a map

$$l : \mathcal{U}_K \to \mathbb{R}^{s+t} \tag{6.1}$$

by

$$l(\xi) = (\log|\sigma_1(\xi)|, \ldots, \log|\sigma_s(\xi)|, 2\log|\sigma_{s+1}(\xi)|, \ldots, 2\log|\sigma_{s+t}(\xi)|).$$

Clearly, l defines a group homomorphism from $\mathcal{U}_K$ (with multiplication in K) to $\mathbb{R}^{s+t}$ (with addition). Moreover, it is not hard to check that the image of l lies in the vector subspace

$$W_{s+t} := \{(x_1, \ldots, x_{s+t}) \in \mathbb{R}^{s+t} \mid x_1 + \ldots + x_{s+t} = 0\}.$$

Dirichlet's unit theorem describes the structure of the group of algebraic units of K by use of the map l above (see [ST87] for more details).

Theorem 6.2.1 (Dirichlet's unit theorem)**.** *For any number field K, the kernel of the map $l : \mathcal{U}_K \to \mathbb{R}^{s+t}$ is finite and equal to the set of roots of unity in K. The image $l(\mathcal{U}_K)$ is a cocompact lattice in the subspace W_{s+t}. As a consequence, the group $\mathcal{U}_K$ is isomorphic to the direct product $F \times \mathbb{Z}^{s+t-1}$ with F a finite abelian group.*

Using this description of $\mathcal{U}_K$, we can prove the following lemma on the existence of algebraic units with hyperbolic properties. This lemma is a generalization of [DD14, Proposition 3.6] and the proof is very similar.

Lemma 6.2.2. *Let $K/\mathbb{Q}$ be a number field with s real embeddings $\sigma_1, \ldots, \sigma_s$ and t conjugated pairs of complex embeddings, where we list one of each pair as $\sigma_{s+1}, \ldots, \sigma_{s+t}$. For any $c \in \mathbb{N}$, there exists an algebraic unit $\xi \in K$ such that for any $e_1, \ldots, e_{s+t} \in \mathbb{N}$ with $e_1 + \ldots + e_{s+t} \leq c$ it holds that*

$$\begin{gathered} |\sigma_1(\xi)^{e_1} \cdot \ldots \cdot \sigma_{s+t}(\xi)^{e_{s+t}}| = 1 \\ \Downarrow \\ 2e_1 = \ldots = 2e_s = e_{s+1} = \ldots = e_{s+t} \end{gathered} \tag{6.2}$$

Proof. Let $l : \mathcal{U}_K \to \mathbb{R}^{s+t}$ be the map as defined in (6.1) for the number field K. From Dirichlet's Unit Theorem, we know that $l(\mathcal{U}_K)$ is a cocompact lattice in the vector subspace $W_{s+t} \subset \mathbb{R}^{s+t}$. For any $e = (e_1, \ldots, e_{s+t}) \in \mathbb{N}^{s+t}$ define the hyperplane

$$H_e \leftrightarrow \sum_{i=1}^{s} 2e_i x_i + \sum_{i=s+1}^{s+t} e_i x_i = 0$$

in $\mathbb{R}^{s+t}$. Then define the collection of hyperplanes

$$\left\{ H_e \;\middle|\; \begin{array}{c} e \in \mathbb{N}^{s+t}, \sum_{i=1}^{s+t} e_i \leq c \text{ and} \\ \neg(2e_1 = \ldots = 2e_s = e_{s+1} = \ldots = e_{s+t}) \end{array} \right\}$$

Note that $\mathcal{H}$ is finite and that for any $H \in \mathcal{H}$ we have that $H \neq W_{s+t}$ and thus that $H \cap W_{s+t}$ is a $s + t - 2$ dimensional vector subspace of W_{s+t}. Since a cocompact lattice in a real vector space can never be contained in a finite union of proper vector subspaces, it follows that there exists a $\xi \in \mathcal{U}_K$ such that

$$l(\xi) \in W_{s+t} \setminus \left(\bigcup_{H \in \mathcal{H}} H \right).$$

As one can check, ξ satisfies the required property. ☐

Remark 6.2.3. Note that in the above lemma, if $s + t \geq 2$ and $c \geq 1$, then the algebraic unit ξ will also satisfy $|\xi| \neq 1$.

6.3 Constructing Anosov rational forms

Let L/K be a Galois extension and $\mathfrak{h}$ a K-form of $\mathfrak{g}$, a Lie algebra over L. The following Lemma tells us when an automorphism of $\mathfrak{g}$ restricts to an automorphism of $\mathfrak{h}$. We use the same notation as introduced in section 5.4. In particular, recall that for a Galois extension L/K, a Lie algebra $\mathfrak{g}$ over K and any cocycle $\rho : \mathrm{Gal}(L/K) \to \mathrm{Aut}(\mathfrak{g}^L)$, we have a K-form of $\mathfrak{g}^L$ defined by

$$\mathfrak{g}_\rho = \{v \in \mathfrak{g}^L \mid \forall \sigma \in \mathrm{Gal}(L/K) : \rho_\sigma({}^\sigma v) = v\}.$$

Lemma 6.3.1. *Let L/K be a Galois extension, $\mathfrak{g}$ a finite dimensional Lie algebra over K, $\rho : \mathrm{Gal}(L/K) \to \mathrm{Aut}(\mathfrak{g}^L)$ a cocycle and $\mathfrak{g}_\rho \subset \mathfrak{g}^L$ the associated K-form. For any automorphism $f \in \mathrm{Aut}(\mathfrak{g}^L)$ it holds that*

$$f(\mathfrak{g}_\rho) = \mathfrak{g}_\rho \quad \Leftrightarrow \quad \forall \sigma \in \mathrm{Gal}(L/K) : f\rho_\sigma = \rho_\sigma {}^\sigma f.$$

Proof. Note that the condition $f(\mathfrak{g}_\rho) = \mathfrak{g}_\rho$ is equivalent to the condition $f(\mathfrak{g}_\rho) \subseteq \mathfrak{g}_\rho$, since f is also a K-linear map and thus preserves the dimension of $\mathfrak{g}_\rho$ as a K-vector space. We then see that $f(\mathfrak{g}_\rho) \subset \mathfrak{g}_\rho$ if and only if for all $v \in \mathfrak{g}_\rho$ and $\sigma \in \mathrm{Gal}(L/K)$ it holds that

$$\begin{aligned} & \rho_\sigma\left({}^\sigma(f(v))\right) = f(v) \\ \Leftrightarrow \quad & \rho_\sigma\left({}^\sigma\left(f\left({}^{\sigma^{-1}}({}^\sigma v)\right)\right)\right) = f(v) \\ \Leftrightarrow \quad & (\rho_\sigma {}^\sigma f)({}^\sigma v) = f(v) \\ \Leftrightarrow \quad & (\rho_\sigma {}^\sigma f)({}^\sigma v) = f(\rho_\sigma({}^\sigma v)) \\ \Leftrightarrow \quad & (\rho_\sigma {}^\sigma f)({}^\sigma v) = (f\rho_\sigma)({}^\sigma v). \end{aligned}$$

Since this must hold for all $v \in \mathfrak{g}_\rho$ and $\mathfrak{g}_\rho$ contains a basis for $\mathfrak{g}^L$, it follows that this is equivalent with $\rho_\sigma {}^\sigma f = f\rho_\sigma$ for all $\sigma \in \mathrm{Gal}(L/K)$, which concludes the proof. □

Corollary 6.3.2. *Let $K/\mathbb{Q}$ be a Galois extension, $\mathfrak{n}$ a finite dimensional nilpotent rational Lie algebra, $\rho : \mathrm{Gal}(K/\mathbb{Q}) \to \mathrm{Aut}(\mathfrak{n}^K)$ a cocycle and $\mathfrak{n}_\rho \subset \mathfrak{n}^K$ the associated rational form. Then $\mathfrak{n}_\rho$ is Anosov if and only if there exists an Anosov automorphism $f \in \mathrm{Aut}(\mathfrak{n}^K)$ such that $\forall \sigma \in \mathrm{Gal}(L/K) : f\rho_\sigma = \rho_\sigma {}^\sigma f$.*

If we apply this corollary to the cocycles as described in Remark 5.4.4 for finite degree Galois extensions, then we obtain the following corollary, which is exactly the main result of [Der16a].

Corollary 6.3.3. *Let $K/\mathbb{Q}$ be a finite degree Galois extension, $\mathfrak{n}$ a finite dimensional nilpotent rational Lie algebra, $\rho : \mathrm{Gal}(K/\mathbb{Q}) \to \mathrm{Aut}(\mathfrak{n})$ a group morphism and $\mathfrak{n}_\rho \subset \mathfrak{n}^K$ the associated rational form. Then $\mathfrak{n}_\rho$ is Anosov if and only if there exists an Anosov automorphism $f \in \mathrm{Aut}(\mathfrak{n}^K)$ such that $\forall \sigma \in \mathrm{Gal}(L/K) : f\rho_\sigma = \rho_\sigma {}^\sigma f$.*

Let us illustrate how this corollary in combination with Lemma 6.2.2 can be used to prove a result like Theorem 6.1.11.

Alternative proof of Theorem 6.1.11. Let $\mathfrak{n}$ be a rational Lie algebra which is positively graded and write this grading as $\mathfrak{n} = \mathfrak{n}_1 \oplus \ldots \oplus \mathfrak{n}_k$. Let us take $n \geq 2$ copies of the Lie algebra $\mathfrak{n}$ and write them as $\mathfrak{m}_i = \mathfrak{n}$ for $i \in \{1, \ldots, n\}$. We thus also get for any $i \in \{1, \ldots, n\}$ the positive grading $\mathfrak{m}_i = \mathfrak{m}_{i1} \oplus \ldots \oplus \mathfrak{m}_{ik}$ with $\mathfrak{m}_{ij} = \mathfrak{n}_j$. Let $\mathfrak{m}$ be the direct sum Lie algebra $\mathfrak{m} = \mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_n$ in which we can represent an element as a tuple $(X_1, \ldots, X_n)$ with $X_i \in \mathfrak{m}_i$. Let $K/\mathbb{Q}$ be a real Galois extension with Galois group isomorphic to $\mathbb{Z}/n\mathbb{Z}$ which exists by Lemma 5.6.7. Let $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ be a generator for this Galois group. Define a group morphism $\rho : \mathrm{Gal}(K/\mathbb{Q}) \to \mathrm{Aut}(\mathfrak{m})$ by

$$\rho_\sigma(X_1, \ldots, X_n) = (X_n, X_1, \ldots, X_{n-1})$$

for any $X_i \in \mathfrak{m}_i$.

Next, for any matrix $A = (a_{ij}) \in K^{n \times k}$, define the linear self-map f_A on $\mathfrak{m}^K$ by

$$f_A(X) = a_{ij} X \quad \text{if} \quad X \in \mathfrak{m}_{ij}.$$

For such a map, we have that ${}^\sigma f_A = f_B$ with $B = (b_{ij})$ satisfying $b_{ij} = \sigma(a_{ij})$. On the other hand we also have that $\rho_\sigma^{-1} f_A \rho_\sigma = f_B$ with $B = (b_{ij})$ satisfying $b_{ij} = a_{i+1\,j}$, where we reduce the first index modulo n. Using Lemma 6.2.2, there exists an algebraic unit ξ in K such that

$$\left| \left(\sigma^{i-1}(\xi)\right)^j \right| \neq 1$$

for any $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, k\}$. Note that we used that $n \geq 2$. Let us now fix $A = (a_{ij})$ with

$$a_{ij} = \left(\sigma^{i-1}(\xi)\right)^j X$$

and consider the invertible linear map f_A on $\mathfrak{m}^K$. By the discussion above, it is not hard to check that $\rho_\eta^{-1} f_A \rho_\eta = {}^\eta f_A$ for all $\eta \in \mathrm{Gal}(K/\mathbb{Q})$. Thus f_A restricts to an invertible linear map on the rational form $\mathfrak{m}_\rho \subset \mathfrak{m}^K$. Since every $\mathfrak{m}_i$ is positively graded, it follows from the way that f_A is defined, that f_A restricts to a Lie algebra automorphism on every $\mathfrak{m}_i$ and thus that f_A is a Lie algebra automorphism on $\mathfrak{m}$. At last, note that all eigenvalues of f_A are algebraic units of absolute value different from one (by the choice of ξ) and thus that f_A is an Anosov automorphism on $\mathfrak{m}_\rho$. We conclude that $\mathfrak{m}^K$ has an Anosov rational form $\mathfrak{m}_\rho$. $\square$

In the specific case of the rational forms of the partially commutative Lie algebras (see section 5.6.2), we can formulate Lemma 6.3.1 as follows.

Lemma 6.3.4. *Let L/K be a Galois extension and $\mathcal{G} = (V, E)$ a graph. Let $\rho : \mathrm{Gal}(L/K) \to \mathrm{Aut}(\overline{\mathcal{G}})$ be a continuous group morphism and $\mathfrak{n}^K(\rho, c) \subset \mathfrak{n}^L(\mathcal{G}, c)$ the associated K-form. Then for any $f \in \mathrm{Aut}(\mathfrak{n}^L(\mathcal{G}, c))$ we have that $f(\mathfrak{n}^K(\rho, c)) = \mathfrak{n}^K(\rho, c)$ if and only if*

$$i(\rho_\sigma^{-1})\, f\, i(\rho_\sigma) = {}^\sigma f$$

for all $\sigma \in \mathrm{Gal}(L/K)$.

6.4 Indecomposable factors of Anosov Lie algebras

This section is based on section 4 of [DW23b].

Every Lie algebra has an essentially unique decomposition into indecomposable factors. In this section, we relate the existence of an Anosov automorphism on the whole Lie algebra to the existence of an Anosov automorphism on its indecomposable factors, and do the same for the existence of a positive grading. This corresponds to studying the existence of Anosov diffeomorphisms and expanding maps on products of nilmanifolds. The special case of abelian factors and Anosov automorphisms was considered in [LW08]. In particular, the results of this section will be used in the next section to construct families of Anosov Lie algebras with no expanding maps in every dimension ≥ 14.

Recall from Definition 5.6.16 that a Lie algebra $\mathfrak{g}$ is called *indecomposable* if it is not equal to the direct sum of two non-trivial ideals. Since we only consider finite-dimensional Lie algebras in this section, every Lie algebra $\mathfrak{g}$ has a vector space decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$, with each $\mathfrak{g}_i$ a non-trivial indecomposable ideal. In this case we call $\mathfrak{g}_i$ a *factor of* $\mathfrak{g}$. By the work in [FGH13], it follows that up to reordering, the isomorphism type of the factors is unique. Note that being indecomposable is not preserved under extending the scalars to a bigger field (see section 5.6.5).

The first step we need is a description of the automorphisms on a direct sum of indecomposable Lie algebras. The following result was proven in [FGH13, Theorem 3.4] for real Lie algebras, but the proof also works for any field of characteristic zero.

Theorem 6.4.1. *Let $\mathfrak{g}$ be a Lie algebra over a field of characteristic zero with a decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$ into indecomposable factors. An invertible linear map $\phi : \mathfrak{g} \to \mathfrak{g}$ is an automorphism of $\mathfrak{g}$ if and only if it has the form $\theta + \eta$, where θ is an automorphism of $\mathfrak{g}$ which maps each $\mathfrak{g}_i$ to itself or to an isomorphic summand $\mathfrak{g}_j$, and where η is a linear endomorphism of $\mathfrak{g}$ such that $\eta(\mathfrak{g}) \subset Z(\mathfrak{g})$ and $\eta([\mathfrak{g}, \mathfrak{g}]) = \{0\}$.*

To prove the main theorems, we will apply this result to an Anosov automorphism or an expanding automorphism. Since the properties we consider are invariant under taking non-zero powers, the following lemma will be useful.

Lemma 6.4.2. *Let $\mathfrak{g}$ be a Lie algebra over a field of characteristic zero with a decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$ into indecomposable factors and ϕ an automorphism of $\mathfrak{g}$. There exists an integer $l > 0$ such that ϕ^l has a decomposition $\phi^l = \tilde{\theta} + \tilde{\eta}$ satisfying the properties of Theorem 6.4.1 and with $\tilde{\theta}(\mathfrak{g}_i) = \mathfrak{g}_i$ for all $i \in \{1, \ldots, k\}$.*

Proof. Given the decomposition $\phi = \theta + \eta$ from Theorem 6.4.1, we give a decomposition for the map ϕ^m. Let us first define for all positive integers $m > 0$ the linear maps $\eta_m = \phi^m - \theta^m$.

We start by proving inductively that $\eta_m(\mathfrak{g}) \subset Z(\mathfrak{g})$ and $\eta_m([\mathfrak{g}, \mathfrak{g}]) = \{0\}$ for any $m > 0$. For $m = 1$ we have $\eta_1 = \eta$ and thus by the assumption it holds that $\eta_1(\mathfrak{g}) \subset Z(\mathfrak{g})$ and $\eta_1([\mathfrak{g}, \mathfrak{g}]) = \{0\}$. Now assume that $\eta_m(\mathfrak{g}) \subset Z(\mathfrak{g})$ and $\eta_m([\mathfrak{g}, \mathfrak{g}]) = \{0\}$ for a fixed $m > 0$. We can rewrite η_{m+1} as

$$\eta_{m+1} = \phi^{m+1} - \theta^{m+1} = (\theta + \eta)\phi^m - \theta\theta^m = \theta\eta_m + \eta\phi^m.$$

Since θ and ϕ^m are automorphisms, they preserve both the center $Z(\mathfrak{g})$ and the derived algebra $[\mathfrak{g}, \mathfrak{g}]$. This fact together with the induction hypothesis clearly gives $\eta_{m+1}(\mathfrak{g}) \subset Z(\mathfrak{g})$ and $\eta_{m+1}([\mathfrak{g}, \mathfrak{g}]) = \{0\}$.

Write $S_k = \mathrm{Perm}(\{1, \ldots, k\})$ for the symmetric group on the set $\{1, \ldots, k\}$. Note that the automorphism θ induces a permutation $\sigma \in S_k$ such that $\theta(\mathfrak{g}_i) = \mathfrak{g}_{\sigma(i)}$ for all $1 \leq i \leq k$. Let l be the order of this permutation σ. It follows that $\theta^l(\mathfrak{g}_i) = \mathfrak{g}_i$ for all $1 \leq i \leq k$. If we set $\tilde{\theta} = \theta^l$ and $\tilde{\eta} = \eta_l$, we get the desired decomposition $\phi^l = \theta^l + (\phi^l - \theta^l) = \tilde{\theta} + \tilde{\eta}$. □

Note that if an indecomposable Lie algebra is abelian, it has dimension ≤ 1. if $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$ is a decomposition of a Lie algebra $\mathfrak{g}$ into indecomposable factors, then up to reordering the summands, there is an $1 \leq m \leq k$ (which is equal to $m(\mathfrak{g})$ from the previous section) such that $\mathfrak{a} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_m$ is a maximal abelian factor of $\mathfrak{g}$. Thus we get the decomposition

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{g}_{m+1} \oplus \ldots \oplus \mathfrak{g}_k.$$

As we will show below, one needs to make this distinction between abelian indecomposable factors and non-abelian indecomposable factors.

The following lemma (who's proof follows the idea from [LW08]) helps us study automorphisms with respect to maximal abelian factors.

Lemma 6.4.3. *Let* $\mathfrak{g}$ *be a Lie algebra with a decomposition* $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{h}$ *where* $\mathfrak{a}$ *is a maximal abelian factor and* $\mathfrak{h}$ *is a Lie ideal. Then for any semi-simple automorphism* $\phi \in \mathrm{Aut}(\mathfrak{g})$ *there exists an automorphism* $\varphi \in \mathrm{Aut}(\mathfrak{g})$ *such that* $(\varphi\phi\varphi^{-1})(\mathfrak{a}) = \mathfrak{a}$ *and* $(\varphi\phi\varphi^{-1})(\mathfrak{h}) = \mathfrak{h}$.

Proof. Let us write $\mathfrak{b} = Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]$. Since ϕ is an automorphism, it must preserve the center and the derived algebra and therefore also $\mathfrak{b}$. Using that ϕ is semi-simple, there exists a complementary subspace $\tilde{\mathfrak{a}}$ to $\mathfrak{b}$ in $Z(\mathfrak{g})$ for which $\phi(\tilde{\mathfrak{a}}) = \tilde{\mathfrak{a}}$. Clearly $\tilde{\mathfrak{a}}$ is a maximal abelian factor of $\mathfrak{g}$. Since $m(\mathfrak{h}) = 0$, it follows that $\tilde{\mathfrak{a}} \cap \mathfrak{h} = \{0\}$ and thus that $\mathfrak{g} = \tilde{\mathfrak{a}} \oplus \mathfrak{h}$. Note that $\tilde{\mathfrak{a}} \oplus [\mathfrak{g}, \mathfrak{g}]$ is ϕ-invariant. Using again that ϕ is semi-simple, we get a subspace $\mathfrak{k}$ with $\phi(\mathfrak{k}) = \mathfrak{k}$ and a vector space decomposition

$$\mathfrak{g} = \tilde{\mathfrak{a}} \oplus \underbrace{[\mathfrak{g}, \mathfrak{g}] \oplus \mathfrak{k}}_{:=\tilde{\mathfrak{h}}}.$$

An easy check shows that the subspace $\tilde{\mathfrak{h}}$ is an ideal of $\mathfrak{g}$. We now clearly get the isomorphisms of Lie algebras $\tilde{\mathfrak{a}} \approx \mathfrak{a}$ and $\tilde{\mathfrak{h}} \approx (\tilde{\mathfrak{a}} \oplus \tilde{\mathfrak{h}})/\tilde{\mathfrak{a}} \approx (\tilde{\mathfrak{a}} \oplus \mathfrak{h})/\tilde{\mathfrak{a}} \approx \mathfrak{h}$. By applying these isomorphisms component-wise we get a map $\varphi : \tilde{\mathfrak{a}} \oplus \tilde{\mathfrak{h}} \to \mathfrak{a} \oplus \mathfrak{h}$ which is the automorphism on $\mathfrak{g}$ that we want. ☐

Now we are ready to prove the main theorems of this section.

Theorem 6.4.4. *Let* $\mathfrak{g}$ *be a Lie algebra with decomposition*

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$$

into non-abelian indecomposable ideals $\mathfrak{g}_i$ *and an abelian ideal* $\mathfrak{a}$. *Then for any automorphism* $\phi \in \mathrm{Aut}(\mathfrak{g})$ *there exists an integer* $l > 0$ *and an automorphism* $\tilde{\phi} \in \mathrm{Aut}(\mathfrak{g})$ *which has the same characteristic polynomial as* ϕ^l *and such that* $\tilde{\phi}(\mathfrak{a}) = \mathfrak{a}$ *and* $\tilde{\phi}(\mathfrak{g}_i) = \mathfrak{g}_i$ *for all* $i \in \{1, \ldots, k\}$.

Proof. By Lemma 6.1.2, the semi-simple part of ϕ is also an automorphism of $\mathfrak{g}$. It has the same characteristic polynomial as ϕ and thus we can just assume that ϕ is semi-simple.

Write $\mathfrak{h} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$. It is clear that $\mathfrak{a}$ is a maximal abelian factor of $\mathfrak{g}$. From Lemma 6.4.3 we get an automorphism $\varphi \in \mathrm{Aut}(\mathfrak{g})$ such that $(\varphi\phi\varphi^{-1})(\mathfrak{a}) = \mathfrak{a}$ and $(\varphi\phi\varphi^{-1})(\mathfrak{h}) = \mathfrak{h}$. Therefore, we get the well-defined restrictions $\phi_1 := (\varphi\phi\varphi^{-1})|_{\mathfrak{a}}$ and $\phi_2 := (\varphi\phi\varphi^{-1})|_{\mathfrak{h}}$. Lemma 6.4.2 then gives us an integer $l > 0$ such that $\phi_2^l = \theta + \eta$ with θ an automorphism on $\mathfrak{h}$ such that $\theta(\mathfrak{g}_i) = \mathfrak{g}_i$ for all $1 \leq i \leq k$ and η a linear endomorphism on $\mathfrak{h}$ such that $\eta(\mathfrak{h}) \subset Z(\mathfrak{h})$ and $\eta([\mathfrak{h}, \mathfrak{h}]) = \{0\}$. Since $\mathfrak{a}$ is a maximal abelian factor of $\mathfrak{g}$ we must have $m(\mathfrak{h}) = 0$. As a consequence

$Z(\mathfrak{h}) \subset [\mathfrak{h}, \mathfrak{h}]$ and the map η now also satisfies $\eta(\mathfrak{h}) \subset [\mathfrak{h}, \mathfrak{h}]$. Write $\mathfrak{k}$ for some complement of $[\mathfrak{h}, \mathfrak{h}]$ in $\mathfrak{h}$, the matrix representation of the maps θ and η with respect to the direct sum $\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}] \oplus \mathfrak{k}$ takes the form

$$\theta = \begin{pmatrix} \theta_{11} & \theta_{12} \\ 0 & \theta_{22} \end{pmatrix}, \qquad \eta = \begin{pmatrix} 0 & \eta_{12} \\ 0 & 0 \end{pmatrix}.$$

From this it follows that θ has the same characteristic polynomial as ϕ_2^l. As a consequence the automorphism $\tilde{\phi}$ defined by $\tilde{\phi}|_{\mathfrak{a}} = \phi_1^l$ and $\tilde{\phi}|_{\mathfrak{h}} = \theta$ has the same characteristic polynomial as ϕ^l and preserves every factor of the decomposition $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$. This proves the theorem. □

Combining Theorem 6.4.4 with Lemma 6.1.4 gives us the result on the existence of Anosov automorphisms on a direct sum of indecomposable factors.

Theorem 6.4.5. *Let $\mathfrak{n}$ be a rational nilpotent Lie algebra with a decomposition*

$$\mathfrak{n} = \mathfrak{a} \oplus \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$$

into non-abelian indecomposable ideals $\mathfrak{g}_i$ and an abelian ideal $\mathfrak{a}$. The Lie algebra $\mathfrak{n}$ is Anosov if and only if $\dim(\mathfrak{a}) \neq 1$ and $\mathfrak{g}_i$ is Anosov for all $i \in \{1, \ldots, k\}$.

Proof. By Lemma 6.1.5, we have that $\mathfrak{a}$ is Anosov if and only if $\dim(\mathfrak{a}) \neq 1$. Let us write $\mathfrak{g}_0 = \mathfrak{a}$.

If all Lie algebras $\mathfrak{g}_0, \ldots, \mathfrak{g}_k$ are Anosov with Anosov automorphisms $A_0, \ldots, A_k$, respectively, then the automorphism A on $\mathfrak{n}$ defined by $A|_{\mathfrak{g}_i} = A_i$ is Anosov on $\mathfrak{n}$ (see Lemma 6.1.4).

Conversely, assume that $\mathfrak{n}$ is Anosov with Anosov automorphism A. Let $\tilde{A}$ be the automorphism we get from Theorem 6.4.4. It is clear that $\tilde{A}$ is also integer-like hyperbolic and thus an Anosov automorphism on $\mathfrak{n}$ with $\tilde{A}(\mathfrak{g}_i) = \mathfrak{g}_i$ for all $0 \leq i \leq k$. Using Lemma 6.1.4 we get that $\tilde{A}|_{\mathfrak{g}_i}$ defines an Anosov automorphism on $\mathfrak{g}_i$ for all $0 \leq i \leq k$. In particular we get that $\dim(\mathfrak{a}) = \dim(\mathfrak{g}_0) \neq 1$. □

We can now state this result on the level of nilmanifolds by using the algebraic characterization of the existence of Anosov diffeomorphisms on nilmanifolds (see Theorem 3.4.21). First, we relate direct products of nilmanifolds to direct products of rational nilpotent Lie algebras.

Lemma 6.4.6. *Let M, M_1, M_2 be nilmanifolds with associated rational nilpotent Lie algebras $\mathfrak{n}, \mathfrak{n}_1, \mathfrak{n}_2$. If M is homeomorphic to the product $M_1 \times M_2$, then $\mathfrak{n}$ is isomorphic to the direct sum Lie algebra $\mathfrak{n}_1 \oplus \mathfrak{n}_2$.*

Proof. Let $M = \Gamma \backslash N$, $M_i = \Gamma_i \backslash N_i$ where Γ, Γ_i are cocompact lattices in the simply connected nilpotent Lie groups N, N_i. Since M is homeomorphic to $M_1 \times M_2$, we get on the level of fundamental groups that Γ is isomorphic to the direct product $\Gamma_1 \times \Gamma_2$. Then the same holds for their Mal'cev completions $\Gamma^{\mathbb{Q}} \cong \Gamma_1^{\mathbb{Q}} \times \Gamma_2^{\mathbb{Q}}$. Thus the associated rational nilpotent Lie algebras are isomorphic: $\mathfrak{n} \cong \mathfrak{n}_1 \oplus \mathfrak{n}_2$. □

Corollary 6.4.7. *Let M be a nilmanifold with rational Lie algebra without non-trivial abelian factor. If M is homeomorphic to the product $M_1 \times M_2$ of two nilmanifolds M_1, M_2, then M admits an Anosov diffeomorphism if and only if both M_1 and M_2 admit an Anosov diffeomorphism.*

Proof. Let $\mathfrak{n}, \mathfrak{n}_1, \mathfrak{n}_2$ be the rational nilpotent Lie algebras associated to M, M_1, M_2, respectively. Since, by Lemma 6.4.6 we have $\mathfrak{n} \cong \mathfrak{n}_1 \oplus \mathfrak{n}_2$. Note that $\mathfrak{n}_1$ can be decomposed into non-abelian indecomposable ideals $\mathfrak{n}_1 = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$ and the same for $\mathfrak{n}_2 = \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_l$. By Theorem 6.4.4 it follows that all $\mathfrak{g}_i$, $\mathfrak{h}_i$ are Anosov and thus also $\mathfrak{n}_1$ and $\mathfrak{n}_2$ are Anosov. □

A similar result is obtained for expanding automorphisms.

Theorem 6.4.8. *Let $\mathfrak{n}$ be a nilpotent Lie algebra over a subfield of $\mathbb{C}$ with a decomposition*

$$\mathfrak{n} = \mathfrak{a} \oplus \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$$

into non-abelian indecomposable ideals $\mathfrak{g}_i$ and an abelian ideal $\mathfrak{a}$. The Lie algebra $\mathfrak{n}$ has a positive grading if and only if $\mathfrak{g}_i$ has a positive grading for all $i \in \{1, \ldots, k\}$.

Proof. Let us write $\mathfrak{g}_0 = \mathfrak{a}$. Note that, by Theorem 3.5.6, the existence of a positive grading on $\mathfrak{n}$ is equivalent to the existence of an expanding automorphism on $\mathfrak{n}$ and that an abelian Lie algebra always admits a positive grading.

Assume that for all $i \in \{1, \ldots, k\}$, the factor $\mathfrak{g}_i$ admits an expanding automorphism ϕ_i. and let ϕ_0 be an expanding automorphism on $\mathfrak{g}_0$, which always exists. The automorphism ϕ on $\mathfrak{n}$ defined by $\phi|_{\mathfrak{g}_i} = \phi_i$ for all $i \in \{0, \ldots, k\}$ is an expanding automorphism on $\mathfrak{n}$.

Conversely, assume that $\mathfrak{n}$ admits an expanding automorphism ϕ. Theorem 6.4.4 gives us an integer $l > 0$ and an automorphism $\tilde{\phi}$ with the same characteristic polynomial as ϕ^l and such that $\tilde{\phi}(\mathfrak{g}_i) = \mathfrak{g}_i$ for all $i \in \{0, \ldots, k\}$. It is clear that each $\tilde{\phi}|_{\mathfrak{g}_i}$ is an expanding automorphism as well and thus that each $\mathfrak{g}_i$ has a positive grading. □

Similarly as with Anosov automorphisms, we can state this result on the level of nilmanifolds as follows.

Corollary 6.4.9. *If a nilmanifold M is homeomorphic to a product of two nilmanifolds $M_1 \times M_2$, then M admits an expanding map if and only if both M_1 and M_2 admit an expanding map.*

6.5 Anosov Lie algebras without a positive grading

The discussion throughout this section is partially based on [DW23b].

Something we can observe from the classification (over $\mathbb{R}$) of Anosov Lie algebras up to dimension 8 (see Theorem 6.1.10) is that they all have a positive grading. Here we implicitly use a result proven in [Der17], which states that $\mathfrak{g}$ has a positive grading if and only if $\mathfrak{g}^L$ has a positive grading for $\mathfrak{g}$ a Lie algebra over K and L/K a field extension of subfields of $\mathbb{C}$. We can thus ask the following question.

Question 6.5.1. Does every Anosov Lie algebra have a positive grading?

This was answered negatively in [Der17], where an example of an Anosov Lie algebra without positive grading was given. Unfortunately, this example is not explicit and its dimension is unknown. This is due to the nature of the construction, since the Lie algebra is constructed from a quotient of a free nilpotent Lie algebra by an ideal defined from a generating set with specific elements, making it hard to compute its dimension. This raises the following question.

Question 6.5.2. What is the minimal dimension of an Anosov Lie algebra with no positive grading?

In this section, we will answer this question. First, in section 6.5.1, we introduce some notation and methodology that will be used. Second, in section 6.5.2, we give a concrete family of examples of Anosov Lie algebras without positive grading in dimension 12. Next, in section 6.5.3, we introduce the notion of rank of an Anosov automorphism and Anosov automorphisms of full rank. At last, in section 6.5.4, we use these notions to prove that 12 is the minimal dimension for an Anosov Lie algebra without positive grading.

6.5.1 Notation and methodology

Let $\mathfrak{n}$ be a rational nilpotent Lie algebra. Assume that $\mathfrak{n}$ is Anosov. By Lemma 6.1.2, we can assume that there exists a semi-simple Anosov automorphism $A : \mathfrak{n} \to \mathfrak{n}$. As discussed in section 6.1, one can choose subspaces $\mathfrak{n}_i \subset \mathfrak{n}$ such that $A(\mathfrak{n}_i) = \mathfrak{n}_i$,

$$\mathfrak{n} = \mathfrak{n}_1 \oplus \ldots \oplus \mathfrak{n}_c \quad \text{and} \quad \gamma_i(\mathfrak{n}) = \mathfrak{n}_i \oplus \ldots \oplus \mathfrak{n}_c,$$

for all $i \in \{1, \ldots, c\}$. Consequently, the type $(n_1, \ldots, n_c)$ satisfies $\dim(\mathfrak{n}_i) = n_i$. Let us write A_i for the restriction of A to $\mathfrak{n}_i$ and write $f, f_1, \ldots, f_c$ for the characteristic polynomials of $A, A_1, \ldots, A_c$, respectively. Note that $f = f_1 \cdot \ldots \cdot f_c$ and thus, by Lemma 6.1.3, all these polynomials have integer coefficients and constant term equal to ± 1.

Using the Lie algebra structure of $\mathfrak{n}$ we can deduce relations among the eigenvalues of A. In particular, the eigenvalues of A_i are i-fold products of the eigenvalues of A_1.

Lemma 6.5.3. *For all $i \in \{1, \ldots, c\}$, the eigenvalues of A_i (or equivalently, the roots of f_i) are contained in the set*

$$\left\{ \prod_{j=1}^{i} \alpha_j \;\middle|\; \alpha_j \in \mathbb{C},\, f_1(\alpha_j) = 0 \right\}.$$

Proof. Recall that we write $\mathfrak{n}^{\mathbb{C}} = \mathfrak{n} \otimes_{\mathbb{Q}} \mathbb{C}$ and that A extends to an automorphism of $\mathfrak{n}^{\mathbb{C}}$ which is diagonalizable since it is semi-simple. Since the subspaces $\mathfrak{n}_i^{\mathbb{C}}$ are invariant under A, a basis of eigenvectors for A can be chosen such that each eigenvector lies in a single $\mathfrak{n}_i^{\mathbb{C}}$.

Fix any $i \in \{1, \ldots, c\}$. Note that the multilinear map

$$\phi : \prod_{j=1}^{i} \mathfrak{n}_1^{\mathbb{C}} \to \gamma_i(\mathfrak{n}^{\mathbb{C}})/\gamma_{i+1}(\mathfrak{n}^{\mathbb{C}})$$

defined by the assignment

$$(X_1, \ldots, X_i) \mapsto [\ldots[[[X_1, X_2], X_3], \ldots], X_i]$$

is surjective. Note that A also induces a linear map $\overline{A}_i$ on $\gamma_i(\mathfrak{n}^{\mathbb{C}})/\gamma_{i+1}(\mathfrak{n}^{\mathbb{C}})$ and that it is conjugated to A_i under the natural identification $\mathfrak{n}_i^{\mathbb{C}} \cong \gamma_i(\mathfrak{n}^{\mathbb{C}})/\gamma_{i+1}(\mathfrak{n}^{\mathbb{C}})$. In particular, $\overline{A}_i$ and A_i have the same eigenvalues. Let $B = \{Y_1, \ldots, Y_k\}$ be a basis of eigenvectors for A_1 in $\mathfrak{n}_1^{\mathbb{C}}$ with $k = \dim(\mathfrak{n}_1^{\mathbb{C}})$. From surjectivity of ϕ, it follows that the set

$$C = \{\phi(Y) \mid Y \in B^i\}$$

spans $\gamma_i(\mathfrak{n}^{\mathbb{C}})/\gamma_{i+1}(\mathfrak{n}^{\mathbb{C}})$. Moreover, one can check that by Lemma 4.6.13, all elements in C are eigenvectors for $\overline{A}_i$ and the eigenvalues are contained in the set

$$\left\{ \prod_{j=1}^{i} \alpha_j \;\middle|\; \alpha_j \in \mathbb{C},\, f_1(\alpha_j) = 0 \right\}.$$

By thinning out the set C to a basis for $\gamma_i(\mathfrak{n}^{\mathbb{C}})/\gamma_{i+1}(\mathfrak{n}^{\mathbb{C}})$, the claim follows. $\square$

Following this lemma, we thus have for the splitting fields of the characteristic polynomials

$$\mathbb{Q}(f_i) \subset \mathbb{Q}(f) = \mathbb{Q}(f_1)$$

where every field is viewed as a subfield of $\mathbb{C}$. Moreover, $\mathbb{Q}(f_1)/\mathbb{Q}(f_i)$ is a Galois extension and thus the Galois group of $\mathbb{Q}(f_1)$ acts on $\mathbb{Q}(f_i)$ by restriction. In particular, the Galois group of $\mathbb{Q}(f_1)$ acts on the eigenvalues of each A_i. It is this action that can often be used to find obstructions to the existence of an Anosov automorphism and this will be exploited in the proof in section 6.5.4. In what follows, we introduce some notation regarding this action.

Galois groups acting on roots. In general, if g is a polynomial over $\mathbb{Q}$ and $K/\mathbb{Q}(g)$ is a Galois extension, then $\mathrm{Gal}(K/\mathbb{Q})$ acts on $\mathbb{Q}(g)$ by restriction. Moreover, for any root α of g and automorphism $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$, the element $\sigma(\alpha)$ is again a root of g. This gives an action of $\mathrm{Gal}(K/\mathbb{Q})$ on the set of roots of g. In particular, if g is irreducible of degree n all its roots $\alpha_1, \ldots, \alpha_n$ are distinct and the action of the Galois group is transitive, i.e. for any two roots α_i, α_j, there exists an automorphism $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ such that $\sigma(\alpha_i) = \alpha_j$. Write $S_n = \mathrm{Perm}(\{1, \ldots, n\})$ for the symmetric group on the set $\{1, \ldots, n\}$. The action of $\mathrm{Gal}(K/\mathbb{Q})$ on the roots is determined by the morphism

$$\iota_g : \mathrm{Gal}(K/\mathbb{Q}) \to S_n : \sigma \mapsto \iota_g(\sigma)$$

with

$$\iota_g(\sigma)(i) = j \Leftrightarrow \sigma(\alpha_i) = \alpha_j.$$

Note that ι_g depends on the chosen ordering of the roots of g. In general g is not irreducible, but it can be written as a product of its irreducible factors over $\mathbb{Q}$, say $g = g_1 \cdot \ldots \cdot g_m$ with $\deg(g_i) = r_i$. For each factor g_i we have that $\mathbb{Q}(g_i) \subset \mathbb{Q}(g)$. Note that since $r = r_1 + \ldots + r_m$, we get the natural inclusion

$S_{r_1} \oplus \ldots \oplus S_{r_m} \hookrightarrow S_r$. After ordering the roots of each polynomial g_i, this gives a map

$$\iota : \mathrm{Gal}(\mathbb{Q}(g)/\mathbb{Q}) \xrightarrow{\iota_{g_1} \oplus \ldots \oplus \iota_{g_m}} S_1 \oplus \ldots \oplus S_{r_m} \hookrightarrow S_r.$$

This map is injective as a field automorphism of $\mathbb{Q}(g)$ is completely determined by its restriction to the roots of g. In this section (section 6.5) we will, after ordering of the roots, omit the 'ι' and just write σ for both the field automorphism and the corresponding permutation in S_r, as there is no confusion possible. We will write elements of S_r with standard cycle notation, i.e. if $\sigma = (a_1\, a_2\, \ldots\, a_s)$, then $\sigma(a_1) = a_2$, $\sigma(a_2) = a_3$, $\ldots$,$\sigma(a_s) = a_1$ and σ fixes all other elements.

It is natural to apply the orbit-stabilizer theorem to the action of the Galois group on the roots.

Theorem 6.5.4 (Orbit-stabilizer)**.** *Let G be a group which has an action on a set X. For any $x \in X$, we have that*

$$|\operatorname{Orb}_G(x)| \cdot |\operatorname{Stab}_G(x)| = |G|.$$

where $\operatorname{Orb}_G(x) = \{g \cdot x \mid g \in G\}$ is the orbit of x under the action of G and $\operatorname{Stab}_G(x) = \{g \in G \mid g \cdot x = x\}$ is the stabilizer subgroup of x in G.

A consequence that we will use often is stated below.

Lemma 6.5.5. *Let f be an irreducible polynomial over $\mathbb{Q}$ of prime degree p with splitting field E over $\mathbb{Q}$. There exists an element $\sigma \in \mathrm{Gal}(E, \mathbb{Q})$ of order p and an ordering of the roots such that $\sigma = (1\,2\, \ldots\, p)$.*

Proof. Since f is irreducible, the action of $G := \mathrm{Gal}(E, \mathbb{Q})$ on the roots of f is transitive. Therefore we get for any root α of f that $|\operatorname{Orb}_G(\alpha)| = p$. The orbit-stabilizer thus tells us that p divides the order of G. Following Cauchy's theorem there must exist an element $\sigma \in G$ of order p. Next we apply the orbit-stabilizer theorem to the group generated by σ, which we denote by $\langle\sigma\rangle$. This gives for any root α of f that $|\operatorname{Orb}_{\langle\sigma\rangle}(\alpha)| \cdot |\operatorname{Stab}_{\langle\sigma\rangle}(\alpha)| = p$ and thus that $|\operatorname{Orb}_{\langle\sigma\rangle}(\alpha)| = 1$ or $|\operatorname{Orb}_{\langle\sigma\rangle}(\alpha)| = p$. If the orbit under σ of any root (and hence every root) is equal to one, then σ is the identity map which is a contradiction because it is of order p. Therefore there exists a root β of which the orbit under $\langle\sigma\rangle$ counts p elements and thus contains all roots of f. Then define $\alpha_i := \sigma^i(\beta)$ for $1 \leq i \leq p$. It follows that with this ordering of the roots, σ is equal to the permutation $(1\,2\, \ldots\, p)$. □

6.5.2 Example in minimal dimension

In this section we exhibit a family of 12-dimensional Anosov Lie algebras that do not have a positive grading. The family consists of rational nilpotent Lie algebras $\mathfrak{m}_k$ for $k > 0$ of type $(4, 2, 2, 2, 2)$ with $\mathfrak{m}_k$ and $\mathfrak{m}_l$ isomorphic if and only if $\frac{k}{l} = m^2$ for some integer $m \in \mathbb{Z}$. In order to show that the Lie algebras are Anosov, we construct these from Corollary 6.3.3 by using a fixed rational nilpotent Lie algebra $\mathfrak{n}$ and different field extensions K of $\mathbb{Q}$. To check that the Lie algebras do not have a positive grading we apply the methods of [Der17], whereas the isomorphisms between the Lie algebras $\mathfrak{m}_k$ follow from a classification given in [Lau08]. In section 6.5.4 we show that 12 is indeed the minimal dimension for such an example.

The rational Lie algebra $\mathfrak{n}$ is defined as the vector space over $\mathbb{Q}$ with basis

$$\{X_1, X_2, X_3, X_4, Y_1, Y_2, Z_1, Z_2, V_1, V_2, W_1, W_2\}$$

and Lie bracket given by the relations

$$[X_1, X_3] = Y_1 \qquad [X_1, Y_1] = Z_1 \qquad [X_1, Z_1] = V_1 \qquad [X_3, V_1] = W_1$$

$$[X_2, X_4] = Y_2 \qquad [X_2, Y_2] = Z_2 \qquad [X_2, Z_2] = V_2 \qquad [X_4, V_2] = W_2$$

$$[Z_1, Y_1] = W_1 \qquad [X_1, X_4] = W_1$$

$$[Z_2, Y_2] = W_2 \qquad [X_2, X_3] = W_2.$$

The last two brackets ensure that $\mathfrak{n}$ is not isomorphic to the direct sum of two filiform Lie algebras, where a filiform Lie algebra is a nilpotent Lie algebra of nilpotency class $c > 1$ and dimension $c+1$, being the smallest possible dimension of such a Lie algebra.

In order to apply Corollary 6.3.3, we take K the field $\mathbb{Q}(\sqrt{k}) \subset \mathbb{R}$ where k is a positive integer which is not a square. This is a quadratic extension of $\mathbb{Q}$ and its Galois group is given by $\mathrm{Gal}(K, \mathbb{Q}) = \{\mathrm{Id}, \sigma\}$ with $\sigma(a + b\sqrt{k}) = a - b\sqrt{k}$ for all $a, b \in \mathbb{Q}$. Let $\xi \in K$ be any algebraic unit with minimal polynomial of degree 2, which exists by Dirichlet's unit theorem (Theorem 6.2.1). It has only one conjugate which is also its inverse $\xi^{-1} = \sigma(\xi)$, moreover it holds that $|\xi| \neq 1 \neq |\xi^{-1}|$. We define the linear map $A : \mathfrak{n}^K \to \mathfrak{n}^K$ and the representation

$\rho : \{1, \sigma\} \to \mathrm{Aut}(\mathfrak{n})$ by

$$A = \begin{pmatrix} B^3 & & & & & \\ & B^{-2} & & & & \\ & & B & & & \\ & & & B^4 & & \\ & & & & B^7 & \\ & & & & & B^5 \end{pmatrix}$$

and

$$\rho_\sigma = \begin{pmatrix} C & & & & & \\ & C & & & & \\ & & C & & & \\ & & & C & & \\ & & & & C & \\ & & & & & C \end{pmatrix}$$

with respect to the basis $\{X_1, X_2, X_3, X_4, Y_1, Y_2, Z_1, Z_2, V_1, V_2, W_1, W_2\}$ and where

$$B = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

One can check that A and ρ_σ are automorphisms of the Lie algebras $\mathfrak{n}^K$ and $\mathfrak{n}$, respectively. Since

$$CBC^{-1} = \begin{pmatrix} \xi^{-1} & 0 \\ 0 & \xi \end{pmatrix} = \begin{pmatrix} \sigma(\xi) & 0 \\ 0 & \sigma(\xi^{-1}) \end{pmatrix} = {}^{\sigma}B$$

it follows that $\rho_\sigma A \rho_{\sigma^{-1}} = {}^{\sigma}A$. Therefore the hypothesis of Corollary 6.3.3 is satisfied and A induces an Anosov automorphism on the rational form

$$\mathfrak{m}_k := \mathfrak{n}_\rho \subset \mathfrak{n}^K$$

We give $\mathfrak{m}_k$ the subscript k since its structure depends on the choice of field extension $K = \mathbb{Q}(\sqrt{k})$.

To see that the Lie algebra $\mathfrak{n}$ has no positive grading, or equivalently it has no expanding automorphism, we use [Der17, Theorem 4.4.]. Assume by contradiction that there would be an expanding automorphism, then this result implies that there exists an expanding automorphism $\varphi \in \mathrm{Aut}(\mathfrak{n}^K)$ which commutes with the map A above. Since all the eigenvalues of A are distinct, this implies the expanding automorphism is diagonal in the basis $X_1, \dots, W_2$, in particular we have $\varphi(X_i) = \alpha_i X_i$ with $\alpha_i > 1$ for $1 \leq i \leq 4$. The relations on $\mathfrak{n}$ imply that

$$\varphi(Y_1) = \alpha_1\alpha_3 Y_1, \qquad \varphi(Z_1) = \alpha_1^2\alpha_3 Z_1,$$
$$\varphi(Y_2) = \alpha_2\alpha_4 Y_2, \qquad \varphi(Z_2) = \alpha_2^2\alpha_4 Z_2,$$
$$\varphi(V_1) = \alpha_1^3\alpha_3 V_1, \qquad \varphi(W_1) = \alpha_1^3\alpha_3^2 W_1,$$
$$\varphi(V_2) = \alpha_2^3\alpha_4 V_2, \qquad \varphi(W_2) = \alpha_2^3\alpha_4^2 W_2.$$

The last relations moreover imply that $\alpha_1\alpha_4 = \alpha_1^3\alpha_3^2$ and $\alpha_2\alpha_3 = \alpha_2^3\alpha_4^2$. Combining these equations, we get that $\alpha_4 = (\alpha_1\alpha_3)^2 = \alpha_1^2\alpha_2^4\alpha_4^4$ or thus $1 = \alpha_1^2\alpha_2^4\alpha_4^3$ which is a contradiction since φ is expanding.

By using the explicit form of the rational Lie algebra $\mathfrak{m}_k$, it is possible to compute the bracket relations. We first need a basis for which we can take

$$\overline{X_1} = X_1 + X_2 \qquad \overline{Y_1} = Y_1 + Y_2 \qquad \overline{V_1} = V_1 + V_2$$
$$\overline{X_2} = \sqrt{k}(X_1 - X_2) \qquad \overline{Y_2} = \sqrt{k}(Y_1 - Y_2) \qquad \overline{V_2} = \sqrt{k}(V_1 - V_2)$$
$$\overline{X_3} = X_3 + X_4 \qquad \overline{Z_1} = Z_1 + Z_2 \qquad \overline{W_1} = W_1 + W_2$$
$$\overline{X_4} = \sqrt{k}(X_3 - X_4) \qquad \overline{Z_2} = \sqrt{k}(Z_1 - Z_2) \qquad \overline{W_2} = \sqrt{k}(W_1 - W_2).$$

For example we have

$$\begin{aligned}\rho_\sigma\left({}^{\sigma}\overline{X_2}\right) &= \rho_\sigma\left(-\sqrt{k}(X_1 - X_2)\right)\\ &= -\sqrt{k}(X_2 - X_1)\\ &= \sqrt{k}(X_1 - X_2)\\ &= \overline{X_2},\end{aligned}$$

which shows that indeed $\overline{X_2}$ lies in the rational form $\mathfrak{m}_k = \mathfrak{n}_\rho$. As one can now calculate, the Lie bracket on $\mathfrak{m}_k$ is given by the relations

$$
\begin{array}{lll}
[\overline{X_1}, \overline{X_3}] = \overline{Y_1} + \overline{W_1} & [\overline{X_1}, \overline{Y_1}] = \overline{Z_1} & [\overline{X_1}, \overline{Z_1}] = \overline{V_1} \\
[\overline{X_2}, \overline{X_4}] = k(\overline{Y_1} - \overline{W_1}) & [\overline{X_2}, \overline{Y_2}] = k\overline{Z_1} & [\overline{X_2}, \overline{Z_2}] = k\overline{V_1} \\
[\overline{X_1}, \overline{X_4}] = \overline{Y_2} - \overline{W_2} & [\overline{X_1}, \overline{Y_2}] = \overline{Z_2} & [\overline{X_1}, \overline{Z_2}] = \overline{V_2} \\
[\overline{X_2}, \overline{X_3}] = \overline{Y_2} + \overline{W_2} & [\overline{X_2}, \overline{Y_1}] = \overline{Z_2} & [\overline{X_2}, \overline{Z_1}] = \overline{V_2}
\end{array}
$$

$$
\begin{array}{ll}
[\overline{X_3}, \overline{V_1}] = \overline{W_1} & [\overline{Z_1}, \overline{Y_1}] = \overline{W_1} \\
[\overline{X_4}, \overline{V_2}] = k\overline{W_1} & [\overline{Z_2}, \overline{Y_2}] = k\overline{W_1} \\
[\overline{X_3}, \overline{V_2}] = \overline{W_2} & [\overline{Z_1}, \overline{Y_2}] = \overline{W_2} \\
[\overline{X_4}, \overline{V_1}] = \overline{W_2} & [\overline{Z_2}, \overline{Y_1}] = \overline{W_2}.
\end{array}
$$

Clearly we have that $\mathfrak{m}_k / [\mathfrak{m}_k, [\mathfrak{m}_k, \mathfrak{m}_k]]$ is a rational Lie algebra of type $(4,2)$. In [Lau08] a complete list of rational Lie algebras of type $(4,2)$ up to isomorphism is given. From this it follows that if k/l is not the square of an integer, then the rational Lie algebras $\mathfrak{m}_k^{\mathbb{Q}}$ and $\mathfrak{m}_l^{\mathbb{Q}}$ are not isomorphic. On the other hand, if $k = lm^2$ for some integer $m \in \mathbb{Z}$, then the fields $\mathbb{Q}(\sqrt{k}) = \mathbb{Q}(\sqrt{l})$ are equal and hence the Lie algebras $\mathfrak{m}_k = \mathfrak{m}_l$ by the construction above.

We conclude that the Lie algebras $\mathfrak{m}_k$ have the claimed properties which proves the following theorem.

Theorem 6.5.6. *There exists a family of Anosov Lie algebras of dimension 12 with no positive grading.*

From the above example, we can construct other families for most dimensions by using the results of section 6.4.

Corollary 6.5.7. *For every dimension $n \geq 14$, there exist a family of Anosov Lie algebras of dimension n with no positive grading.*

Proof. This follows immediately by considering the Lie algebra $\mathfrak{m}^{\mathbb{Q}} \oplus \mathbb{Q}^{n-12}$, which satisfies the hypotheses by Theorem 6.4.5 and 6.4.8. $\square$

Note that these examples are decomposable, contrary to the example in dimension 12. This raises the following question.

Question 6.5.8. Does there exist for any $n \geq 14$ an indecomposable Anosov Lie algebra of dimension n with no positive grading?

As another consequence of the results of section 6.4, an example of minimal dimension must be indecomposable, reducing the possibilities to check in the proof of minimality in section 6.5.4.

6.5.3 Rank of an Anosov automorphism

By definition, the eigenvalues of an Anosov automorphism are algebraic units, with certain relations induced by the fact that they are eigenvalues of an automorphisms of the Lie algebra. Therefore it is important to study the multiplicative group generated by these eigenvalues. We define the rank of an Anosov automorphism as the rank of this abelian group, i.e. the maximal number of $\mathbb{Z}$-independent elements. Using the rank will prove useful for constructing positive gradings on Lie algebras. We start by introducing some terminology, partially coming from [Pay09].

Let us call a monic polynomial $f \in \mathbb{Z}[X]$ an *Anosov polynomial* if it has only roots of absolute value different from 1 and constant term equal to ± 1. The Anosov polynomials are exactly the characteristic polynomials of Anosov automorphisms. Note that there are no Anosov polynomials of degree one and that by Lemma 6.1.3 the irreducible factors of an Anosov polynomial over $\mathbb{Q}$ are Anosov as well. Let us fix an Anosov polynomial f of degree k with roots $\alpha_1, \ldots, \alpha_k$. We define the rank of f to be the rank of the abelian multiplicative group generated by the roots $\alpha_1, \ldots, \alpha_k$. This group can also be seen as the image of the following map

$$\phi_f : \mathbb{Z}^k \to K^* : (z_1, \ldots, z_k) \mapsto \alpha_1^{z_1} \cdot \ldots \cdot \alpha_k^{z_n} \tag{6.3}$$

where K is the splitting field of f over $\mathbb{Q}$. As a consequence the rank of f can be expressed as $k - \operatorname{rank}(\ker \phi_f)$ and thus lies between 0 and k.

Since we assumed the constant term of f to be ± 1, the product of all its roots is equal to ± 1. This implies that $\mathbb{Z}(2, \ldots, 2) \in \ker \phi_f$ and thus that the rank of an Anosov polynomial f is at most $k - 1$. When there is equality, we say that f satisfies the *full rank condition* or that its roots have *full rank*. Note that every polynomial that satisfies the full rank condition must be irreducible. This notion was first introduced in [Pay09]. Combining [Pay09, Proposition 3.6.(2)] with Lemma 6.5.5 we obtain the following result.

Proposition 6.5.9. *Let f be an irreducible Anosov polynomial of prime degree, then the roots of f satisfy the full rank condition.*

Note that there is also a lower bound on the rank of f, namely it has to be at least 1. If the rank of f would be 0, then the group generated by the roots would be finite which implies that each α_i is a root of unity. This contradicts the fact that their absolute value has to be different from 1.

We can naturally extend the notion of rank to Anosov automorphisms by using their characteristic polynomial.

Definition 6.5.10. Let A be an Anosov automorphism on a rational Lie algebra $\mathfrak{n}$. We define the *rank* of A to be the rank of the multiplicative group generated by its eigenvalues.

Note that the rank of an Anosov automorphism does not change when taking the semi-simple part. Let A be a semi-simple Anosov automorphism on a rational Lie algebra $\mathfrak{n}$ with characteristic polynomial f. Let $\mathfrak{n} = \mathfrak{n}_1 \oplus \ldots \oplus \mathfrak{n}_c$ be the decomposition as introduced in section 6.5.1, together with the restrictions A_i and characteristic polynomials f_i. Write $(n_1, \ldots, n_c)$ for the type of $\mathfrak{n}$. Since $\mathbb{Q}(f) = \mathbb{Q}(f_1)$, the rank of A is actually equal to the rank of f_1. For the Anosov automorphism A we will write $\phi_A := \phi_{f_1}$ as defined by equation (6.3). The Anosov automorphism A is said to be of *full rank* if f_1 satisfies the full rank condition.

Lemma 6.5.11. *The rank of A is equal to the rank of A^k for every $k > 0$.*

Proof. Consider the two morphisms $\phi_A : \mathbb{Z}^{n_1} \to K$ and $\phi_{A^k} : \mathbb{Z}^{n_1} \to K'$ as defined above where K and K' are the splitting fields of the characteristic polynomials of A and A^k, respectively. It is clear that $K' \subset K$ since the eigenvalues of A^k are k-powers of the eigenvalues of A. Let θ_k be the morphism from $\mathbb{Z}^{n_1}$ to itself given by multiplication by k, then we have $\phi_{A^k} = \phi_A \circ \theta_k$. Since θ_k is an injective morphism, it preserves $\mathbb{Z}$-linear independence. Therefore it follows that $\operatorname{rank}(\ker \phi_{A^k}) \leq \operatorname{rank}(\ker \phi_A)$. On the other hand we also have that $\ker \phi_A \subset \ker \phi_{A^k}$ which implies that $\operatorname{rank}(\ker \phi_A) = \operatorname{rank}(\ker \phi_{A^k})$. We conclude that the rank of an Anosov automorphism is invariant under taking non-zero powers. □

During the remainder of Section 6.5, we will often take powers of Anosov automorphisms to achieve stronger assumptions. For example, we will from now on always assume that the constant term of f_1 is equal to 1, by squaring A if necessary.

Consider the following morphism between abelian groups

$$\psi : \mathbb{Z}^{n_1} \to \mathbb{Z} : (z_1, \ldots, z_{n_1}) \mapsto \sum_{i=1}^{n_1} z_i.$$

For any Anosov automorphism A, the image of $\ker \phi_A$ under this map is a subgroup of $\mathbb{Z}$. Therefore there exists a unique positive integer d_A such that $\psi(\ker \phi_A) = d_A \cdot \mathbb{Z}$. In the special case where A has full rank, d_A equals n_1. The following theorem gives a restriction on the Lie algebra structure using this integer d_A.

Proposition 6.5.12. *Let A be an Anosov automorphism on the rational nilpotent Lie algebra $\mathfrak{n}$ with corresponding decomposition $\mathfrak{n} = \mathfrak{n}_1 \oplus \ldots \oplus \mathfrak{n}_c$. Then for all $1 \leq i, j \leq c$, we have that*

$$[\mathfrak{n}_i, \mathfrak{n}_j] \subset \bigoplus_{k \in \mathbb{N}} \mathfrak{n}_{i+j+k \cdot d_A}$$

where we set $\mathfrak{n}_i = 0$ if $i > c$. As a consequence we have that if $c \leq d_A + 1$, the Lie algebra $\mathfrak{n}$ is positively graded.

Proof. Let $A : \mathfrak{n}^{\mathbb{Q}} \to \mathfrak{n}^{\mathbb{Q}}$ be a semi-simple Anosov automorphism for which $A(\mathfrak{n}_i) = \mathfrak{n}_i$. We denote the splitting field of the characteristic polynomial of A_1 by K and the roots by $\alpha_1, \ldots, \alpha_{n_1}$. Then A has a basis of eigenvectors in $\mathfrak{n}^K = K \otimes \mathfrak{n}$ and since the subspaces $\mathfrak{n}_i^K := K \otimes \mathfrak{n}_i$ are invariant under A, it follows that we can choose a basis of eigenvectors that respects the direct sum $\mathfrak{n}^K = \mathfrak{n}_1^K \oplus \ldots \oplus \mathfrak{n}_c^K$.

Now take any two of these eigenvectors X, Y with $X \in \mathfrak{n}_k^K$ and $Y \in \mathfrak{n}_l^K$. If their bracket $[X, Y]$ is non-zero, it is again an eigenvector of A. Define $\mathcal{I}$ as the set of all integers m with $1 \leq m \leq c$ such that the eigenvalue of $[X, Y]$ is an eigenvalue of A_m. Now take an $m \in \mathcal{I}$. Since the eigenvalues of A_i are i-fold products of the α_j's, there exist non-negative integers $e_1, \ldots, e_{n_1}, f_1, \ldots, f_{n_1}, g_1, \ldots, g_{n_1}$ with $\sum_{i=1}^{n_1} e_i = k, \sum_{i=1}^{n_1} f_i = l$ and $\sum_{i=1}^{n_1} g_i = m$ such that the eigenvalues of X, Y and $[X, Y]$ are given by $\prod_{i=1}^{n_1} \alpha_i^{e_i}, \prod_{i=1}^{n_1} \alpha_i^{f_i}$ and $\prod_{i=1}^{n_1} \alpha_i^{g_i}$, respectively. Since A is an automorphism of $\mathfrak{n}$, we must have that the product of the eigenvalues of X and Y is equal to the eigenvalue of $[X, Y]$. This implies that

$$\left(\prod_{i=1}^{n_1} \alpha_i^{e_i} \right) \cdot \left(\prod_{i=1}^{n_1} \alpha_i^{f_i} \right) = \prod_{i=1}^{n_1} \alpha_i^{g_i} \qquad \Rightarrow \qquad \prod_{i=1}^{n_1} \alpha_i^{e_i + f_i - g_i} = 1.$$

As a consequence $(e_1 + f_1 - g_1, \ldots, e_{n_1} + f_{n_1} - g_{n_1}) \in \ker \phi_A$ and so we know that its image under ψ is a multiple of d_A. This gives that $m = k + l + r \cdot d_A$ for some $r \in \mathbb{Z}$. Since m was chosen arbitrarily in $\mathcal{I}$, we know that $\mathcal{I} \subset \{k + l + r \cdot d_A \mid r \in \mathbb{Z}\}$. Therefore we must have that $[X, Y]$ lies in the direct sum $\bigoplus_{r \in \mathbb{Z}} \mathfrak{n}_{k+l+r \cdot d_A}^K$, where we set $\mathfrak{n}_i^K = 0$ for $i < 1$ or $i > c$. As the eigenvectors $X \in \mathfrak{n}_k^K$ and $Y \in \mathfrak{n}_l^K$ were arbitrary, it follows that $\left[\mathfrak{n}_k^K, \mathfrak{n}_l^K\right] \subset \bigoplus_{r\in\mathbb{Z}} \mathfrak{n}_{k+l+r\cdot d_A}^K$. From this we get that the same holds for the rational spaces $\mathfrak{n}_i$ and because $[\mathfrak{n}_k, \mathfrak{n}_l] \subset \gamma_{k+l}(\mathfrak{n})$, the direct sum only needs to run over non-negative integers r. This gives

$$[\mathfrak{n}_k, \mathfrak{n}_l] \subset \bigoplus_{r\in\mathbb{N}} \mathfrak{n}_{k+l+r\cdot d_A}. \tag{6.4}$$

Now, if we assume in addition that $c \leq d_A + 1$, then we have for any $1 \leq k, l \leq c$ that $\mathfrak{n}_{k+l+r\cdot d_A} = 0$ for $r > 0$. Therefore Equation (6.4) becomes $[\mathfrak{n}_k, \mathfrak{n}_l] \subset \mathfrak{n}_{k+l}$ which shows that in this case $\mathfrak{n} = \mathfrak{n}_1 \oplus \ldots \oplus \mathfrak{n}_c$ is a positive grading for $\mathfrak{n}$. ☐

Using tools of Galois theory, we find more information about the rank of an Anosov automorphism. After ordering the roots of an irreducible polynomial of degree n, the Galois group of its splitting field K can be seen as a subgroup of S_n as explained in Section 6.5.1. For any finite set X we write $\operatorname{Perm}(X)$ for the permutation group on X. It is clear that if $X \subset \{1, \ldots, n\}$ there is a natural inclusion of $\operatorname{Perm}(X)$ in S_n by extending a permutation of X by the identity on the complement. The following lemma from [Pay09, Lemma 3.7.] gives more information about the subgroup of $\operatorname{Gal}(K/\mathbb{Q})$ that fixes a certain element of K. We recall the proof for completeness.

Lemma 6.5.13. *Let f be an irreducible Anosov polynomial which satisfies the full rank condition. Let K denote the splitting field of f and $\alpha_1, \ldots, \alpha_n$ its roots. Consider an element $\beta = \alpha_1^{e_1} \cdot \ldots \cdot \alpha_n^{e_n}$ with $e_i \in \mathbb{N}$ and a corresponding equivalence relation $\sim$ on $I := \{1, \ldots, n\}$ defined by $i \sim j \Leftrightarrow e_i = e_j$. Then we have*

$$\operatorname{Aut}(K, \mathbb{Q}(\beta)) \subseteq \bigoplus_{[i]\in I/\sim} \operatorname{Perm}([i]).$$

As a consequence if $\beta \neq \pm 1$, $\operatorname{Aut}(K/\mathbb{Q}(\beta))$ does not act transitively on the roots $\alpha_1, \ldots, \alpha_n$.

Proof. Take any $\sigma \in \operatorname{Aut}(K/\mathbb{Q}(\beta))$, so σ fixes β and we have

$$\begin{aligned} & \sigma(\alpha_1^{e_1} \cdot \ldots \cdot \alpha_n^{e_n}) = \alpha_1^{e_1} \cdot \ldots \cdot \alpha_n^{e_n} \\ \Rightarrow\quad & \alpha_{\sigma(1)}^{e_1} \cdot \ldots \cdot \alpha_{\sigma(n)}^{e_n} = \alpha_1^{e_1} \cdot \ldots \cdot \alpha_n^{e_n} \\ \Rightarrow\quad & \alpha_1^{e_{\sigma^{-1}(1)} - e_1} \cdot \ldots \cdot \alpha_n^{e_{\sigma^{-1}(n)} - e_n} = 1. \end{aligned}$$

Since the α_i have full rank, this implies that

$$e_{\sigma^{-1}(1)} - e_1 = \ldots = e_{\sigma^{-1}(n)} - e_n. \tag{6.5}$$

Then let $(i_1 \cdots i_k)$ be any of the disjoint cycles of σ seen as an element of S_n. Then it follows from equation (6.5) that $e_{i_1} = \ldots = e_{i_k}$. So $i_1, \ldots, i_k \in [i_1]$ and as a consequence $(i_1 \cdots i_k) \in \mathrm{Perm}([i_1])$. We thus see that σ is a composition of elements which each lie in some $\mathrm{Perm}([i])$ and thus this proves the claim.

For the last statement, we know that if $\mathrm{Aut}(K/\mathbb{Q}(\beta))$ acts transitively, it must hold that $i \sim j$ for all i and j and thus that $e_i = e_j$ for all i and j. In particular, we have that $\beta = (\alpha_1 \cdot \ldots \cdot \alpha_n)^{e_1} = (\pm 1)^{e_1} = \pm 1$. □

We can now use this lemma to prove the following proposition from [Pay09, Corollary 3.9.], which gives us more information about the type of an Anosov Lie algebra.

Proposition 6.5.14. *Let $\mathfrak{n}$ be a rational nilpotent Lie algebra with n_1 prime and $A : \mathfrak{n} \to \mathfrak{n}$ an Anosov automorphism of full rank. Then n_1 divides n_i for all $2 \leq i \leq c$.*

Proof. We can assume that A is semi-simple. Write f_i for the characteristic polynomial of A_i. Since f_1 satisfies the full rank condition, it is irreducible as well. By Lemma 6.5.5 there exists an ordering of the roots of f, say $\alpha_1, \alpha_2, \ldots, \alpha_{n_1}$ and an element σ of order n_1 which corresponds to the permutation $(1\,2\,\ldots\,n_1)$ on these roots. Each f_i can be written as a product of its irreducible factors $f_i = g_{i1} \cdot g_{i2} \cdot \ldots \cdot g_{ik}$. Let M_{ij} denote the set of roots of the polynomial g_{ij}. It is clear that $\mathrm{Gal}(K/\mathbb{Q})$ has an action on this set and that $|M_{ij}| = \deg g_{ij}$. Now take an element $\beta \in g_{ij}$. By Lemma 6.5.13 we know that σ can not be an element of $\mathrm{Aut}(K/\mathbb{Q}(\beta))$, meaning that $\sigma(\beta) \neq \beta$. The orbit of β under the subgroup of $\mathrm{Gal}(K/\mathbb{Q})$ generated by σ, written $\langle \sigma \rangle$ must therefore count at least two elements. Since σ has prime order, the orbit stabilizer theorem (Theorem 6.5.4) then tells us that this orbit counts exactly n_1 elements. This proves that all the orbits under $\langle \sigma \rangle$ in M_{ij} count n_1 elements. Since these orbits form a partition of M_{ij}, this proves that n_1 divides $|M_{ij}| = \deg g_{ij}$. As a consequence $n_1 \mid \deg f_i = n_i$ for all $2 \leq i \leq c$. □

The final lemma shows that we can assume that the irreducible factors of Anosov polynomials of rank 1 have degree 2 by taking a finite power.

Lemma 6.5.15. *Let A be an Anosov automorphism of rank one. Up to taking a power of A its characteristic polynomial is a product of Anosov polynomials of degree 2.*

Proof. Let g be an irreducible factor of f. It follows that g is Anosov of rank 1 as well. Let $\alpha_1, \ldots, \alpha_k$ be the roots of g. Since they generate an abelian group of rank 1, there exist non-zero integers e_i such that

$$\alpha_1^{e_1} = \alpha_2^{e_2} = \ldots = \alpha_n^{e_n}.$$

Since g is irreducible, there exists for each index $2 \leq i \leq n$ a field automorphism σ_i in the Galois group of f over $\mathbb{Q}$ such that $\sigma_i(\alpha_1) = \alpha_i$. This implies that $\sigma_i(\alpha_1^{e_i}) = \alpha_i^{e_i} = \alpha_1^{e_1}$. Write s_i for the order of σ_i, then it follows that

$$\alpha_1^{\left(e_i^{s_i}\right)} = \sigma_i^{s_i}\left(\alpha_1^{\left(e_i^{s_i}\right)}\right) = \alpha_1^{\left(e_1^{s_i}\right)}.$$

Since α_1 has absolute value different from 1 we must have that $e_1^{s_i} = e_i^{s_i}$ and thus that $e_1 = \pm e_i$. Note that we can assume $e_1 = 1$ after taking the e_1-th power of A if necessary. This can be done for every irreducible factor of f since there are only finitely many of them. As a consequence we get that the roots of g all lie in the set $\{\alpha_1, \alpha_1^{-1}\}$. This shows that the degree of g is 2 since it is irreducible and thus must have distinct roots. We conclude that f is a product of degree 2 Anosov polynomials. □

Example 6.5.16. To illustrate that it is necessary to consider A up to a power in Lemma 6.5.15, consider the polynomial $f(X) = X^4 - X^2 - 1$. This is an irreducible Anosov polynomial with roots

$$\sqrt{\frac{1+\sqrt{5}}{2}},\ -\sqrt{\frac{1+\sqrt{5}}{2}},\ \sqrt{\frac{1-\sqrt{5}}{2}} \text{ and } -\sqrt{\frac{1-\sqrt{5}}{2}}.$$

Alternatively, they can be listed as $\alpha, -\alpha, \alpha^{-1}, -\alpha^{-1}$ for $\alpha = \sqrt{\frac{1+\sqrt{5}}{2}}$. As a consequence, we see that f has rank one, but f does not split as a product of Anosov polynomials of degree 2 as it is irreducible. By squaring all the roots (which corresponds to squaring the Anosov automorphism), we get the collection of roots $\alpha, \alpha, \alpha^{-1}, \alpha^{-1}$ which are exactly the roots of the polynomial $(X^2 - X - 1)(X^2 - X - 1)$. This polynomial is indeed a product of degree 2 Anosov polynomials.

6.5.4 Proof of minimality

In this section, we use the notation and the results of the preceding sections to prove that every nilpotent Anosov Lie algebra $\mathfrak{n}$ of dimension strictly less than 12 admits a positive grading. On the level of nilmanifolds, this implies that a nilmanifold which admits an Anosov diffeomorphism but no expanding map must have dimension at least 12. Since an Anosov diffeomorphism on an infra-nilmanifold lifts to one on the covering nilmanifold and the existence of expanding maps on infra-nilmanifolds only depends on the covering nilpotent Lie group (see Theorem 3.5.6), we find that this generalizes to infra-nilmanifolds as well. At the end of this section we will thus have proved the following.

Theorem 6.5.17. *Every infra-nilmanifold of dimension* < 12 *which admits an Anosov diffeomorphism also admits an expanding map.*

Using a result from [Mai12] and our results from section 6.4, we can immediately prove the following consequence.

Corollary 6.5.18. *Every infra-nilmanifold of dimension* 13 *which admits an Anosov diffeomorphism also admits an expanding map.*

Proof. As argued above it suffices to prove that every 13-dimensional Anosov Lie algebra has a positive grading. Let $\mathfrak{n}$ be a rational nilpotent Lie algebra which is Anosov. If it has a non-trivial abelian factor, Theorem 6.1.9 gives a decomposition into factors $\mathfrak{n} = \mathfrak{a} \oplus \mathfrak{m}$ with $\mathfrak{a}$ abelian of dimension ≥ 2 and $\mathfrak{m}$ Anosov. Therefore $\mathfrak{m}$ is Anosov of dimension at most 11 and thus has a positive grading by Theorem 6.5.17. Using Theorem 6.4.8 we find that $\mathfrak{n}$ must have a positive grading as well. If, on the other hand, $\mathfrak{n}$ has no non-trivial abelian factor, then [Mai12, Theorem 1] implies that $\mathfrak{n}^{\mathbb{C}}$ is 2-step nilpotent and thus positively graded. Using [Der17, Theorem 3.1], we find that $\mathfrak{n}$ has a positive grading as well. This concludes the proof. ☐

The proof of Theorem 6.5.17 is divided into separate cases, depending on the value of n_1 of the type $(n_1, \ldots, n_c)$ of the Lie algebra $\mathfrak{n}$. Since every 2-step nilpotent Lie algebra is positively graded, we only need to check Lie algebras of nilpotency class at least 3. Since by Theorem 6.1.7, we also know that $n_1 \geq 3$, $n_i \geq 2$ for all $2 \leq i \leq c$ and that $\dim \mathfrak{n} < 12$, this leaves us only with the cases $n_1 = 3, 4, 5, 6, 7$. We are only interested in constructing a positive grading on the Lie algebras, therefore we do not have to determine the full Lie bracket on the Lie algebra. As discussed in section 6.5.1, we can and will always assume that our Anosov automorphism $A : \mathfrak{n} \to \mathfrak{n}$ is semi-simple with characteristic polynomial f satisfying $f(0) = 1$. Furthermore, we always write $\mathfrak{n} = \mathfrak{n}_1 \oplus \ldots \oplus \mathfrak{n}_c$ for the associated decomposition, A_i for the restriction to $\mathfrak{n}_i$ and f_i for the characteristic polynomial of A_i. At last, we always let K denote the splitting field of f (or equivalently of f_1).

Case $n_1 = 3$

Since in this case, every Anosov automorphism is of full rank, we immediately have the following result.

Proposition 6.5.19. *Let $\mathfrak{n}$ be a Anosov rational Lie algebra with $n_1 = 3$ and $\dim \mathfrak{n} \leq 12$, then $\mathfrak{n}$ has a positive grading.*

Proof. It is clear that f_1 is irreducible since linear factors give an eigenvalue equal to ± 1. Therefore Proposition 6.5.9 implies that f_1 satisfies the full rank condition. Using Proposition 6.5.14, we know that $3 \mid n_i$ for all $2 \leq i \leq c$ and thus that $\dim \mathfrak{n} \geq 3 \cdot c$. By our assumption on the dimension of $\mathfrak{n}$ we thus have that $c \leq 4$. At last we use Proposition 6.5.12 to conclude that $\mathfrak{n}$ has a positive grading. □

Case $n_1 = 4$

For this case, we will make a distinction according to the rank of the Anosov automorphism. This gives us more information about the eigenvalues by the following proposition.

Proposition 6.5.20. *Let $\mathfrak{n}$ be a Anosov rational Lie algebra of type $(4, n_2, \ldots, n_c)$ and let α_1, α_2, α_3, α_4 be the eigenvalues of A_1. Then, up to replacing A by some power of A and reordering the roots, we either have:*

- *(i) A has full rank and $d_A = 4$,*
- *(ii) A has rank 2, the eigenvalues satisfy $\alpha_1 = \alpha_2^{-1}$, $\alpha_3 = \alpha_4^{-1}$ and $d_A = 2$ or*
- *(iii) A has rank 1 and the eigenvalues satisfy $\alpha_1 = \alpha_2^{-1}$, $\alpha_3 = \alpha_4^{-1}, \alpha_1^k = \alpha_3^l$ with l, k non-zero integers which are coprime. If $k + l$ is even then $d_A = 2$, otherwise $d_A = 1$.*

Proof. As per usual, let f_1 denote the characteristic polynomial of A_1. If A has full rank, then it is clear that $d_A = 4$ since $\ker \phi_A = \mathbb{Z} \cdot (1, 1, 1, 1)$.

If A has rank 2, we get that $\operatorname{rank}(\ker \phi_A) = 2$. We first consider the case when f_1 is reducible. Under this assumption, f_1 factors into two irreducible polynomials of degree 2 since it does not have linear factors and thus $\alpha_1 = \alpha_2^{-1}$ and $\alpha_3 = \alpha_4^{-1}$. Now consider the case when f_1 is irreducible. Since $\operatorname{rank}(\ker \phi_A) = 2$, there exists an element $(z_1, z_2, z_3, z_4) \in \ker \phi_A$ linearly independent from $(1, 1, 1, 1)$. This implies that not all z_i are equal. Up to some permutation of the indices and possibly adding an integer multiple of $(1, 1, 1, 1)$, we may also assume that

$z_1 \geq z_2 \geq z_3 \geq z_4 = 0$. Since f_1 is irreducible, there exists an element σ in its Galois group such that $\sigma(\alpha_3) = \alpha_4$. It follows that $(z_{\sigma(1)}, z_{\sigma(2)}, 0, z_{\sigma(4)})$ is also an element of $\ker \phi_A$. Since $\operatorname{rank}(\ker \phi_A) = 2$, there exist integers r, s, t such that

$$r(z_{\sigma(1)}, z_{\sigma(2)}, 0, z_{\sigma(4)}) = s(1,1,1,1) + t(z_1, z_2, z_3, 0).$$

Note that this implies that either $z_{\sigma(1)} \geq z_{\sigma(2)} \geq 0 \geq z_{\sigma(4)}$ or $z_{\sigma(1)} \leq z_{\sigma(2)} \leq 0 \leq z_{\sigma(4)}$, depending on the sign of the integer $r \cdot t$. But since we also had $z_i \geq 0$ for all i, this gives that at least two of the z_i are zero. This shows that there is, again up to permutation of the indices, an element in the kernel of the form $(u, -v, 0, 0)$ with $u > 0, v < 0$ integers and therefore $\alpha_1^u = \alpha_2^v$. Since f_1 was assumed to be irreducible, there is an element τ in the Galois group of f_1 such that $\tau(\alpha_1) = \alpha_2$. Therefore we get that $\tau(\alpha_1^v) = \tau(\alpha_1)^v = \alpha_2^v = \alpha_1^u$. Let $|\tau|$ denote the order of τ. We get that

$$\alpha_1^{v^{|\tau|}} = \tau^{|\tau|}(\alpha_1^{v^{|\tau|}}) = \alpha_1^{u^{|\tau|}}.$$

Since $|\alpha_1| \neq 1$ this implies that $u = -v$ and thus that $\alpha_1^u = \alpha_2^{-u}$. We can then assume that $u = 1$ after taking the u-th power of the automorphism A if necessary. So in both the reducible as the irreducible case, we get that $\alpha_1 = \alpha_2^{-1}$ and $\alpha_3 = \alpha_4^{-1}$, up to taking a power of A. Combined with the assumption that $\operatorname{rank}(\ker \phi_A) = 2$, this implies that $\ker \phi_A$ is generated by the elements $(1,1,0,0)$ and $(0,0,1,1)$, showing that $d_A = 2$.

If A has rank 1, Lemma 6.5.15 tells us that, by taking a power of A if necessary, we can assume that f_1 is the product of two Anosov polynomials of degree 2. This shows that up to changing the indices of the roots, $\alpha_1 = \alpha_2^{-1}$ and $\alpha_3 = \alpha_4^{-1}$. Since A has rank 1, there must also be integers l' and k' such that $\alpha_1^{k'} = \alpha_3^{l'}$. If m denotes the greatest common divisor of k' and l' it is clear that after taking the m-th power of A, the roots satisfy $\alpha_1 = \alpha_2^{-1}$, $\alpha_3 = \alpha_4^{-1}$ and $\alpha_1^k = \alpha_3^l$ with $k = k'/m$ and $l = l'/m$ coprime. Since $\operatorname{rank}(\ker \phi_A) = 3$ it follows that every element $(z_1, \ldots, z_4)$ in $\ker \phi_A$ must satisfy $lz_1 - lz_2 - kz_3 + kz_4 = 0$. This implies that $l(z_1 + z_2 + z_3 + z_4) = 2lz_2 + (k+l)z_3 + (l-k)z_4$. If $k + l$ is even, then $l - k$ is even and l is odd. As a consequence $z_1 + z_2 + z_3 + z_4$ is even and $d_A = 2$. If $k + l$ is odd, then d_A can not be even since $k + l \in d_A \cdot \mathbb{Z}$. We also have that $\psi(1,1,0,0) = 2 \in d_A \cdot \mathbb{Z}$ and thus it follows that $d_A = 1$. □

This result tells us more about the possible Galois groups of an Anosov polynomial f of degree 4. Let $\alpha_1, \ldots, \alpha_4$ denote the roots of f. If f is reducible, it factors as two irreducible Anosov polynomials of degree 2. Since the Galois group can only permute the roots of each factor separately, we get that it is either isomorphic to $\mathbb{Z}_2$ or to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. If f is irreducible, but does not satisfy the full rank condition, Proposition 6.5.20 tells us the relations $\alpha_1^k = \alpha_2^{-k}$ and $\alpha_3^k = \alpha_4^{-k}$ must hold for some integer $k > 0$. If the Galois group of f contains an element σ of order 3, then without loss of generality its action on the roots of f is given by $\sigma(\alpha_1) = \alpha_2$, $\sigma(\alpha_2) = \alpha_3$, $\sigma(\alpha_3) = \alpha_1$ and $\sigma(\alpha_4) = \alpha_4$. In particular, $\sigma^2(\alpha_4^{-k}) = \left(\sigma^2(\alpha_4)\right)^{-k} = \alpha_4^{-k} = (\sigma(\alpha_4))^{-k} = \sigma(\alpha_4^{-k})$. This gives that

$$\alpha_1^k = \sigma(\alpha_3^k) = \sigma(\alpha_4^{-k}) = \sigma^2(\alpha_4^{-k}) = \sigma^2(\alpha_3^k) = \alpha_2^k = \alpha_1^{-k}.$$

This contradicts the fact that α_1 has absolute value different from 1. As a consequence there are no elements of order three in the Galois group of f, thus implying that it is isomorphic to either the cyclic group $\mathbb{Z}_4$, the Klein-four group K_4 or the dihedral group D_4 of order eight. We summarized this in Table 6.2 below. We included the Anosov polynomials of degree two and three as well for completeness.

deg	full rank	irreducible	Galois group	order
2	yes	yes	$\mathbb{Z}_2$	2
3	yes	yes	$\mathbb{Z}_3, S_3$	3, 6
4	yes	yes	$\mathbb{Z}_4, K_4, D_8, A_4, S_4$	4, 4, 8, 12, 24
4	no	yes	$\mathbb{Z}_4, K_4, D_8$	4, 4, 8
4	no	no	$\mathbb{Z}_2, \mathbb{Z}_2 \oplus \mathbb{Z}_2$	2, 4

Table 6.2: Possible Galois groups of Anosov polynomials up to degree 4.

This now gives the following information about the type.

Proposition 6.5.21. *Let f be an Anosov polynomial of degree 4 with roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and let $e_1, e_2, e_3, e_4 \in \mathbb{Z}$ such that $\beta = \alpha_1^{e_1}\alpha_2^{e_2}\alpha_3^{e_3}\alpha_4^{e_4}$ is not a rational number. Then the minimal polynomial of β over $\mathbb{Q}$ has even degree.*

Proof. Let G denote the Galois group of $K/\mathbb{Q}$, with K the splitting field of f. If the order of G is a power of 2, the orbit stabilizer theorem (Theorem 6.5.4) tells us that the orbit of β under the action of G must have either 1 element or an even number of elements. The former is not possible since it implies β is rational. Therefore, we conclude that the minimal polynomial of β has an even number of roots and thus has even degree. Hence we only need to prove the statement in the case where the order of G is not a power of 2.

From Table 6.2 it follows that this only occurs when f_1 satisfies the full rank condition and G is isomorphic to either A_4 or S_4. Again, arguing as before, we find by the orbit stabilizer theorem that the orbit of β under the action of G either has 3 or an even number of elements. In the latter case we are done so assume the orbit counts 3 elements. This implies that the minimal polynomial of β, which we denote by g, has degree 3. The Galois group of $K/\mathbb{Q}(g)$, denoted as $\mathrm{Gal}(K/\mathbb{Q}(g))$, is a strictly smaller normal subgroup of G. For S_4 these subgroups are up to automorphism given by $A_4, K_4, \{1\}$ and for A_4 they are given by $K_4, \{1\}$. Note that $\mathrm{Gal}(K/\mathbb{Q}(g)) \subset \mathrm{Aut}(K/\mathbb{Q}(\beta))$ can not be a transitive subgroup of S_4 by Lemma 6.5.13. This leaves us only with the case $\mathrm{Gal}(K/\mathbb{Q}(g)) = \{1\}$, which in turn implies $\mathbb{Q}(g) = K$. This is impossible since by our assumption g had degree 3 and thus $[\mathbb{Q}(g) : \mathbb{Q}] \leq 6$ whereas $[K : \mathbb{Q}] \geq 12$. □

By using our techniques, we find another proof of [Pay09, Theorem 1.3.] which was also incorporated in Theorem 6.1.7.

Corollary 6.5.22. *If $\mathfrak{n}$ is a Anosov rational Lie algebra of type $(4, n_2, \dots, n_c)$, then $2|n_i$ for all $2 \leq i \leq c$.*

Proof. As per usual, let f_i denote the characteristic polynomial of A_i. It is clear that f_1 is an Anosov polynomial of degree 4. The roots of f_i are i-fold products of the roots of f_1. They are also not rational since the only rational algebraic units are 1 and -1 which have absolute value equal to 1 contradicting the hyperbolicity of A. Using Proposition 6.5.21 we get that the irreducible factors of each f_i must be of even degree and thus each f_i has even degree itself. This shows that n_i is even for all $2 \leq i \leq c$. □

This leads to the main result about Anosov Lie algebras of type $(4, \dots, n_c)$.

Proposition 6.5.23. *Let $\mathfrak{n}$ be a Anosov rational Lie algebra with $n_1 = 4$ and $\dim \mathfrak{n} < 12$, then $\mathfrak{n}$ has a positive grading.*

Proof. Since 2-step nilpotent Lie algebras always have a positive grading, we will only consider Lie algebras of nilpotency class at least 3. Let f_1 be the characteristic polynomial of A_1 and K its splitting field over $\mathbb{Q}$. Let $X_1, \dots, X_4 \in \mathfrak{n}_1^K$ be a basis of eigenvectors of A_1 with eigenvalues $\alpha_1, \dots, \alpha_4$, respectively. If these roots have full rank, then $\mathfrak{n}$ is positively graded by Proposition 6.5.12 where we keep in mind that $\dim \mathfrak{n} < 12$, so we focus on the case when $\alpha_1, \dots, \alpha_4$ do not have full rank. We can assume by Proposition 6.5.20 that $\alpha_1 = \alpha_2^{-1}$ and $\alpha_3 = \alpha_4^{-1}$. Let us write for simplicity $\alpha_1 = \alpha$ and $\alpha_3 = \beta$, then the eigenvalues of A_1 take the form $\alpha, \alpha^{-1}, \beta, \beta^{-1}$. From here on we prove the statement separately for each possible type. Recall that by Corollary 6.5.22 all n_i are even, leading to four possibilities for the type.

- Type $(4,4,2)$: We have that $\mathfrak{n}_2^K$ is spanned by the vectors $\{[X_i, X_j] \mid 1 \leq i < j \leq 4\}$. The brackets $[X_1, X_2]$ and $[X_3, X_4]$ have to be zero, since if this is not the case they are eigenvectors of A with eigenvalues $\alpha\alpha^{-1} = 1$ and $\beta\beta^{-1} = 1$, respectively. This would contradict the hyperbolicity of A. Because $\dim \mathfrak{n}_2 = 4$ it follows that a basis for $\mathfrak{n}_2^K$ can be given by the vectors $[X_1, X_3]$, $[X_2, X_4]$, $[X_1, X_4]$ and $[X_2, X_3]$. Therefore we have that $[\mathfrak{n}_1, \mathfrak{n}_1] \subset \mathfrak{n}_2$, which implies that $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \mathfrak{n}_3$ is a positive grading for $\mathfrak{n}$.

- Type $(4,2,4)$: The eigenvalues of A_2 must be 2-fold products of the eigenvalues of A_1. Without loss of generality we can assume that one eigenvalue of A_2 is given by $\alpha\beta$. It follows that the other eigenvalue of A_2 is the inverse namely $\alpha^{-1}\beta^{-1}$. The brackets $[X_2, X_3]$ and $[X_1, X_4]$ must lie in $\gamma_3(\mathfrak{n}^K)$, since otherwise their eigenvalues given by $\alpha^{-1}\beta$ and $\alpha\beta^{-1}$, respectively, must equal one of the two eigenvalues of A_2 which gives either that $\beta^2 = 1$ or that $\alpha^2 = 1$, a contradiction. We also know that $[X_1, X_2] = 0$ and $[X_3, X_4] = 0$ since otherwise these would be eigenvectors with eigenvalue 1. Therefore it is clear that a basis of eigenvectors of A_2 is given by $Y_1 := [X_1, X_3]$ and $Y_2 := [X_2, X_4]$.

  The eigenvectors of A_3 can be written as a linear combination of the elements $[X_i, Y_j]$. Using the Jacobi identity we get that

$$[X_2, Y_1] = [X_1, \underbrace{[X_2, X_3]}_{\in \gamma_3(\mathfrak{n}^K)}] + [X_3, \cancel{[X_1, X_2]}] = 0,$$

$$[X_4, Y_1] = [X_1, \cancel{[X_4, X_3]}] + [X_3, \underbrace{[X_4, X_2]}_{\in \gamma_3(\mathfrak{n}^K)}] = 0,$$

$$[X_1, Y_2] = [X_2, \underbrace{[X_1, X_4]}_{\in \gamma_3(\mathfrak{n}^K)}] + [X_4, \cancel{[X_2, X_1]}] = 0,$$

$$[X_3, Y_2] = [X_2, \cancel{[X_3, X_4]}] + [X_4, \underbrace{[X_2, X_3]}_{\in \gamma_3(\mathfrak{n}^K)}] = 0.$$

  This gives that $[X_1, Y_1]$, $[X_3, Y_1]$, $[X_2, Y_2]$ and $[X_4, Y_2]$ form a basis of eigenvectors for A_3. Its eigenvalues are therefore given by $\alpha^2\beta$, $\alpha\beta^2$, $\alpha^{-2}\beta^{-1}$ and $\alpha^{-1}\beta^{-2}$, respectively.

We know that $[X_2, X_3], [X_1, X_4] \in \mathfrak{n}_3^K$. If they are non-zero then they must be eigenvectors of A_3 and thus this gives one of the following equalities:

$$\alpha^{-1}\beta = \begin{cases} \alpha^2\beta & \Rightarrow \alpha^3 = 1 \\ \alpha\beta^2 & \Rightarrow \alpha^2\beta = 1 \\ \alpha^{-2}\beta^{-1} & \Rightarrow \alpha\beta^2 = 1 \\ \alpha^{-1}\beta^{-2} & \Rightarrow \beta^3 = 1. \end{cases}, \quad \alpha\beta^{-1} = \begin{cases} \alpha^2\beta & \Rightarrow \alpha\beta^2 = 1 \\ \alpha\beta^2 & \Rightarrow \beta^3 = 1 \\ \alpha^{-2}\beta^{-1} & \Rightarrow \alpha^3 = 1 \\ \alpha^{-1}\beta^{-2} & \Rightarrow \alpha^2\beta = 1. \end{cases}$$

which gives in each case a contradiction, either because α or β would be a root of unity or because an eigenvalue of A_3 would be equal to 1. So we must conclude that $[X_2, X_3] = [X_1, X_4] = 0$. As a consequence $[\mathfrak{n}_1, \mathfrak{n}_1] \subset \mathfrak{n}_2$ and thus the decomposition $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \mathfrak{n}_3$ gives a positive grading.

- Type $(4, 2, 2)$: Just as in the previous case, we find that $Y_1 := [X_1, X_3]$ and $Y_2 := [X_2, X_4]$ form a basis of eigenvectors for A_2, with eigenvalues $\alpha\beta$ and $\alpha^{-1}\beta^{-1}$, respectively. Moreover, an identical argument as before shows that $[X_2, Y_1] = [X_4, Y_1] = [X_1, Y_2] = [X_3, Y_2] = 0$, implying that the eigenvalues on $\mathfrak{n}_3^K$ are, by interchanging α and β if necessary, of the form $\alpha^2\beta$ and $\alpha^{-2}\beta^{-1}$, with corresponding eigenvectors $Z_1 := [X_1, Y_1]$ and $Z_2 := [X_2, Y_2]$.

  If $[X_1, X_4]$ is non-zero, it is an eigenvector for eigenvalue $\alpha\beta^{-1}$, which is different from $\alpha^{-2}\beta^{-1}$ because otherwise α would have absolute value 1. In particular, $[X_1, X_4] = a[X_1, Y_1]$ for some $a \in K$. By replacing the vector X_4 by $X_4 - aY_1$, which is again an eigenvector for the same eigenvalue, we get that $[X_1, X_4] = 0$, but with all the other relations identical. Similarly, we can replace X_3 to achieve $[X_2, X_3] = 0$, showing that $[\mathfrak{n}_1^K, \mathfrak{n}_1^K] = \mathfrak{n}_2^K$ and thus realizing a positive grading $\mathfrak{n}_1^K \oplus \mathfrak{n}_2^K \oplus \mathfrak{n}_3^K$ for $\mathfrak{n}^K$, which in turn implies $\mathfrak{n}$ admits a positive grading. In fact, this argument shows that $\mathfrak{n}^K$ is a direct sum of two filiform Lie algebras, one generated by the elements X_1, X_3 and the other by X_2, X_4. Note that if $[X_3, Y_1]$ is non-zero, it is an eigenvector of eigenvalue $\alpha\beta^2$, which must be different from $\alpha^{-2}\beta^{-1}$ and thus $[X_3, Y_1] = b[X_1, Y_1]$ for $b \in K$. By replacing X_3 by $X_3 - bX_1$, we can thus assume that $[X_3, Y_1] = 0$. Similarly, we realize the assumption $[X_4, Y_2] = 0$.

- Type $(4, 2, 2, 2)$: We use the same notations and basis for the spaces $\mathfrak{n}_1^K \oplus \mathfrak{n}_2^K \oplus \mathfrak{n}_3^K$ as for type $(4, 2, 2)$, in particular with the property that $[X_1, X_4] \in \mathfrak{n}_4^K$ and $[X_2, X_3] \in \mathfrak{n}_4^K$. By using the Jacobi identity, we see that $[X_2, Z_1] = [X_3, Z_1] = [X_4, Z_1] = [X_1, Z_2] = [X_3, Z_2] = [X_4, Z_2] = 0$. In particular, a basis of eigenvectors for A_4 is given by $[X_1, Z_1]$ and $[X_2, Z_2]$ with eigenvalues $\alpha^3\beta$ and $\alpha^{-3}\beta^{-1}$. Now, if $[X_1, X_4]$ is non-zero, it lies in $\mathfrak{n}_4$ and has eigenvalue $\alpha\beta^{-1}$ which is different from $\alpha^{-3}\beta^{-1}$. In particular, $[X_1, X_4] = a[X_1, Z_1]$ for some $a \in K$, and thus by replacing X_4 by $X_4 - aZ_1$ we get that $[X_1, X_4] = 0$. In a similar fashion, we can assume $[X_2, X_3] = 0$ and thus $[\mathfrak{n}_1, \mathfrak{n}_1] = \mathfrak{n}_2$, showing that the decomposition $\mathfrak{n}_1^K \oplus \mathfrak{n}_2^K \oplus \mathfrak{n}_3^K \oplus \mathfrak{n}_4^K$ gives a positive grading for $\mathfrak{n}^K$. In fact, we have shown that $\mathfrak{n}^K$ is a direct sum of two filiform Lie algebras.

□

Due to the proof for Anosov Lie algebras of type $(4, 2, 2)$ and $(4, 2, 2, 2)$, one might conjecture that for every Anosov Lie algebra $\mathfrak{n}$ of type $(4, 2, \ldots, 2)$ it holds that $\mathfrak{n}^{\mathbb{C}}$ is isomorphic to the direct sum of two filiform Lie algebras. The family of examples in Section 6.5.2 shows that this is not the case, since they have no positive grading.

Case $n_1 = 5$

We first consider the possibilities if the polynomial f_1 is reducible.

Proposition 6.5.24. *Let $\mathfrak{n}$ be an Anosov rational Lie algebra with $n_1 = 5$ and $c \leq 4$. If the characteristic polynomial of A_1 is reducible and $\mathfrak{n}$ has no non-trivial abelian factor, then $n_2 \geq 6$.*

Proof. Denote the characteristic polynomial of A_1 by f_1. Because f_1 is reducible it is equal to a product $f_1 = g_1 g_2$ with g_1 and g_2 irreducible of degree 3 and 2, respectively. After squaring the Anosov automorphism A if necessary we can assume that the constant terms of g_1 and g_2 are equal to 1. Let K, K_1 and K_2 denote the splitting fields of f, g_1 and g_2, respectively. Because $K_1 \subset K$ and $K_2 \subset K$ are subfields, we get that both 3 and 2 divide the order of $\mathrm{Gal}(K/\mathbb{Q})$. Let $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ be an element of order 3. Let $\alpha_1, \alpha_2, \alpha_3$ denote the roots of g_1 with corresponding eigenvectors $X_1, X_2, X_3 \in \mathfrak{n}_1^K$ and β, β^{-1} the roots of g_2 with corresponding eigenvectors $Y_1, Y_2 \in \mathfrak{n}_1^K$. We must have, up to reordering the α_i, that

$$\sigma(\alpha_1) = \alpha_2, \quad \sigma(\alpha_2) = \alpha_3, \quad \sigma(\alpha_3) = \alpha_1$$

$$\sigma(\beta) = \beta, \quad \sigma(\beta^{-1}) = \beta^{-1}.$$

This follows from the fact that σ must permute the roots of each irreducible polynomial separately and from the orbit stabilizer theorem (Theorem 6.5.4) which implies that the orbit of an element under σ must have either 1 or 3 elements.

We first show that $[X_1, Y_1] \in \mathfrak{n}_2^K$. Indeed, if the vector $[X_1, Y_1]$ is non-zero, it is an eigenvector of A with eigenvalue $\alpha_1\beta$. All the eigenvalues of A_k are k-fold products of eigenvalues of A_1. So if the eigenvalue $\alpha_1\beta$ occurs on $\mathfrak{n}_k^K$, we must have $\alpha_1\beta = \alpha_1^{e_1}\alpha_2^{e_2}\alpha_3^{e_3}\beta^s$ for some positive integers $e_i, s \in \mathbb{N}$ with $e_1 + e_2 + e_3 + s = k$. This implies that $\beta^{1-s} = \alpha_1^{e_1-1}\alpha_2^{e_2}\alpha_3^{e_3}$. Clearly, the left hand side is invariant under σ so the same holds for the right hand side. By Proposition 6.5.9 the roots $\alpha_1, \alpha_2, \alpha_3$ satisfy the full rank condition. Using that $\alpha_1^{e_1-1}\alpha_2^{e_2}\alpha_3^{e_3}$ is invariant under σ, Lemma 6.5.13 implies that $e_1 - 1 = e_2 = e_3$. From this we get that $\beta^{1-s} = 1$ and thus since β is not a root of unity that $s = 1$. As a consequence we also have that $k = e_1 + e_2 + e_3 + s = e_1 + (e_1 - 1) + (e_1 - 1) + 1 = 3e_1 - 1$. Since c is assumed to be less or equal than 4, this proves that $k = 2$ and thus that $[X_1, Y_1] \in \mathfrak{n}_2^K$. Analogously, it can be proven that $[X_i, Y_j] \in \mathfrak{n}_2^K$ for all $1 \leq i \leq 3$ and $1 \leq j \leq 2$.

Therefore, we know that either all the brackets $[X_i, Y_j]$ are zero or that A_2 has an eigenvalue of the form $\alpha_i\beta^{\pm 1}$. In the first case $\text{span}_K\{Y_1, Y_2\}$ is an abelian factor of $\mathfrak{n}^K$. As a consequence the rational Lie algebra $\mathfrak{n}$ has a non-trivial abelian factor as well, which is in contradiction with the assumption. We thus have without loss of generality, that $\alpha_1\beta$ is an eigenvalue of A_2. By letting σ act on this we get that $\alpha_1\beta, \alpha_2\beta, \alpha_3\beta$ are distinct eigenvalues of A_2. Their product is equal to $\alpha_1\alpha_2\alpha_3\beta^3 = \beta^3$ which can not be equal to 1 since β is not a root of unity. Therefore A_2 must have at least one other eigenvalue. This can be either one of the form $\alpha_i\alpha_j$ or of the form $\alpha_i\beta^{-1}$. In either case, by letting σ act on these roots we see they each have at least 2 other conjugates. As a consequence n_2 must be greater or equal than 6. □

As a consequence, we can study positive gradings on Anosov Lie algebras of type $(5, n_2, \ldots, n_c)$.

Corollary 6.5.25. *Let $\mathfrak{n}$ be an Anosov rational Lie algebra with $n_1 = 5$ and $\dim \mathfrak{n} < 12$, then $\mathfrak{n}$ has a positive grading.*

Proof. Let f_1 be the characteristic polynomial of A_1. By Proposition 6.5.12 we know that if the roots of f_1 have full rank, then the Lie algebra $\mathfrak{n}$ has a positive grading. In the other case, if the roots of f_1 are not of full rank, then Proposition 6.5.9 implies that f_1 is reducible. We can also assume that $\mathfrak{n}$ has no non-trivial abelian factor since otherwise our work in the previous sections and Theorem 6.4.5 and 6.4.8 assure that $\mathfrak{n}$ has a positive grading. As a consequence we can apply Proposition 6.5.24 and thus $n_2 \geq 6$. This implies $c = 2$ because of the assumption on our dimension. In particular, $\mathfrak{n}$ has a positive grading. □

Case $n_1 = 6$

Since for any $2 \leq i \leq c$ we have that $n_i \geq 2$, the only possible types of dimension < 12 we need to check are $(6, 2, 2)$, $(6, 2, 3)$ and $(6, 3, 2)$.

For types $(6, 2, 2)$ and $(6, 2, 3)$, we will use the fact that for these types the quotient by the third ideal in the upper central series gives an Anosov Lie algebra of type $(6, 2)$ which have been classified in [LW09]. Let $\mathfrak{h}_3$ denote the Heisenberg Lie algebra of dimension 3. From Theorem 6.1.10 we know that if $\mathfrak{n}$ is an Anosov rational Lie algebra of type (6, 2), there are, up to isomorphism, only two possibilities for the Lie algebra $\mathfrak{n}^{\mathbb{R}}$. The first possibility is $\mathfrak{h}_3 \oplus \mathfrak{h}_3 \oplus \mathbb{R}^2$ with $\mathfrak{h}_3$ the 3-dimensional real Heisenberg Lie algebra, whereas the second possibility, denoted as $\mathfrak{g}_{6,2}$, is the real Lie algebra spanned by $X_1, X_2, X_3, X_4, X_5, X_6, Z_1, Z_2$ with the Lie bracket defined by

$$[X_1, X_2] = Z_1, \quad [X_1, X_3] = Z_2, \quad [X_4, X_5] = Z_1, \quad [X_4, X_6] = Z_2. \tag{6.6}$$

We immediately state a result about possible extensions of the Lie algebra $\mathfrak{g}_{6,2}^{\mathbb{C}}$.

Lemma 6.5.26. *Let* $\mathfrak{n}$ *be a complex Lie algebra of nilpotency class at least 3. Then* $\mathfrak{g}_{6,2}^{\mathbb{C}}$ *can not be isomorphic to* $\mathfrak{n}/\gamma_3(\mathfrak{n})$.

Proof. Let us write for simplicity $\mathfrak{g} = \mathfrak{g}_{6,2}^{\mathbb{C}}$. Suppose that there is an isomorphism $\mathfrak{g} \cong \mathfrak{n}/\gamma_3(\mathfrak{n})$. Using this isomorphism we can identify $\mathfrak{g}$ with a complementary subspace of $\gamma_3(\mathfrak{n})$ in $\mathfrak{n}$. In this way we get $\mathfrak{n} = \text{span}_{\mathbb{C}}\{X_1, \ldots, X_6, Z_1, Z_2\} \oplus \gamma_3(\mathfrak{n})$. The relations from (6.6) then still hold modulo $\gamma_3(\mathfrak{n})$.

Now take any three-fold bracket $[X_i, [X_j, X_k]]$ with $1 \leq i, j, k \leq 6$, then we claim it must lie in $\gamma_4(\mathfrak{n})$. If $j \in \{1, 2, 3\}$ and $k \in \{4, 5, 6\}$ or the other way around, we have that $[X_j, X_k] \in \gamma_3(\mathfrak{n})$. As a consequence $[X_i, [X_j, X_k]] \in \gamma_4(\mathfrak{n})$. So we can assume without loss of generality that both $j, k \in \{1, 2, 3\}$. If $i \in \{1, 2, 3\}$ as well, there exists a $Y \in \gamma_3(\mathfrak{n})$ such that $[X_j, X_k] = [X_{j+3}, X_{k+3}] + Y$. Thus we have $[X_i, [X_j, X_k]] = [X_i, Y] + [X_i, [X_{j+3}, X_{k+3}]]$. Since $[X_i, Y]$ lies in $\gamma_4(\mathfrak{n})$ it now suffices to show that $[X_i, [X_{j+3}, X_{k+3}]] \in \gamma_4(\mathfrak{n})$. So it suffices to show that $[X_i, [X_j, X_k]] \in \gamma_4(\mathfrak{n})$ for $i \in \{4, 5, 6\}$ and $j, k \in \{1, 2, 3\}$. From the Jacobi identity we then get

$$[X_i, [X_j, X_k]] = [X_j, \underbrace{[X_i, X_k]}_{\in \gamma_3(\mathfrak{n})}] + [X_k, \underbrace{[X_j, X_i]}_{\in \gamma_3(\mathfrak{n})}] \in \gamma_4(\mathfrak{n}).$$

This proves that any bracket of the form $[X_i, [X_j, X_k]]$ with $1 \leq i, j, k \leq 6$ lies in $\gamma_4(\mathfrak{n})$ and thus that $\gamma_3(\mathfrak{n}) = \gamma_4(\mathfrak{n}) = [\gamma_3(\mathfrak{n}), \mathfrak{n}]$. Inductively, we thus find that $\gamma_3(\mathfrak{n}) = \gamma_i(\mathfrak{n})$ for any $i \geq 3$, in particular, $\gamma_3(\mathfrak{n}) = \gamma_{c+1}(\mathfrak{n}) = \{0\}$, where c is the nilpotency class of $\mathfrak{n}$. Since we assumed $\mathfrak{n}$ to be at least 3-step nilpotent, this gives a contradiction. ◻

We apply this result to show the existence of positive gradings.

Proposition 6.5.27. *There are no Anosov Lie algebras of type* $(6, 2, 3)$ *and an Anosov Lie algebra of type* $(6, 2, 2)$ *admits a positive grading.*

Proof. Let $\mathfrak{n}$ be an Anosov rational Lie algebra of type $(6, 2, n_3)$ and let $A : \mathfrak{n} \to \mathfrak{n}$ be a semi-simple Anosov automorphism. As usual we get the decomposition $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \mathfrak{n}_3$ such that $A(\mathfrak{n}_i) = \mathfrak{n}_i$ and we write $A_i = A|_{\mathfrak{n}_i}$. The Anosov automorphism A induces an Anosov automorphism on $\mathfrak{n}/\gamma_3(\mathfrak{n})$ which we call $\tilde{A}$. The Lie algebra $\mathfrak{n}/\gamma_3(\mathfrak{n})$ is thus an Anosov Lie algebra of type $(6, 2)$. We use the natural identification of vector spaces $\mathfrak{n}_1 \oplus \mathfrak{n}_2 \cong \mathfrak{n}/\gamma_3(\mathfrak{n})$ and see that under this identification $\tilde{A}$ on $\mathfrak{n}/\gamma_3(\mathfrak{n})$ corresponds to $A_1 \oplus A_2$ on $\mathfrak{n}_1 \oplus \mathfrak{n}_2$. From Theorem 6.1.10, we know that the Lie algebra $\mathfrak{n}^{\mathbb{C}}/\gamma_3(\mathfrak{n}^{\mathbb{C}})$ is isomorphic to either $\mathfrak{g}_{6,2}^{\mathbb{C}}$ or $\mathfrak{h}_3^{\mathbb{C}} \oplus \mathfrak{h}_3^{\mathbb{C}} \oplus \mathbb{C}^2$. From Lemma 6.5.26 we get that $\mathfrak{n}^{\mathbb{C}}/\gamma_3(\mathfrak{n}^{\mathbb{C}})$ must be isomorphic to $\mathfrak{h}_3^{\mathbb{C}} \oplus \mathfrak{h}_3^{\mathbb{C}} \oplus \mathbb{C}^2$, moreover by following the proof of [LW09, Theorem 3.8.], we get a basis of eigenvectors $X_1, \dots, X_4, Y_1, Y_2$ of A_1 and a basis of eigenvectors Z_1, Z_2 of A_2 such that the following relations hold in $\mathfrak{n}^{\mathbb{C}}$:

$$[X_1, X_2] = Z_1 \mod \gamma_3(\mathfrak{n}^{\mathbb{C}}) \tag{6.7}$$

$$[X_3, X_4] = Z_2 \mod \gamma_3(\mathfrak{n}^{\mathbb{C}}) \tag{6.8}$$

$$[X_1, X_3], [X_1, X_4], [X_2, X_3], [X_2, X_4] \in \gamma_3(\mathfrak{n}^{\mathbb{C}}) \tag{6.9}$$

$$[\mathbb{C}Y_i, \mathfrak{n}_1^{\mathbb{C}} \oplus \mathfrak{n}_2^{\mathbb{C}}] \subset \gamma_3(\mathfrak{n}^{\mathbb{C}}) \qquad \forall \quad i \in \{1, 2\}. \tag{6.10}$$

Let $\alpha_1, \dots, \alpha_4, \beta_1, \beta_2$ be the eigenvalues of $X_1, \dots, X_4, Y_1, Y_2$, respectively. We know by the proof in [LW09] as well that the characteristic polynomial f_1 of A_1 factors as $f_1 = gh$ with $\alpha_1, \dots, \alpha_4$ the roots of g and β_1, β_2 the roots of h.

Now, we use the fact that $[\mathfrak{n}^{\mathbb{C}}, \gamma_3(\mathfrak{n}^{\mathbb{C}})] = 0$ and the Jacobi identity to find

$$\begin{aligned}
\forall\, i,j \in \{1,2\}: \quad [Y_i, Z_j] &= [Y_i, [X_{2j-1}, X_{2j}]] \\
&= [X_{2j-1}, \underbrace{[Y_i, X_{2j}]}_{\in\gamma_3(n^{\mathbb{C}})}] + [X_{2j}, \underbrace{[X_{2j-1}, Y_i]}_{\in\gamma_3(n^{\mathbb{C}})}] = 0 \\
\forall\, i \in \{1,2\}: \quad [X_i, Z_2] &= [X_i, [X_3, X_4]] \\
&= [X_3, \underbrace{[X_i, X_4]}_{\in\gamma_3(n^{\mathbb{C}})}] + [X_4, \underbrace{[X_3, X_i]}_{\in\gamma_3(n^{\mathbb{C}})}] = 0 \\
\forall\, i \in \{3,4\}: \quad [X_i, Z_1] &= [X_i, [X_1, X_2]] \\
&= [X_1, \underbrace{[X_i, X_2]}_{\in\gamma_3(n^{\mathbb{C}})}] + [X_2, \underbrace{[X_1, X_i]}_{\in\gamma_3(n^{\mathbb{C}})}] = 0.
\end{aligned}$$

Since the brackets above vanish, it follows that $\mathfrak{n}_3$ is spanned by the vectors $[X_1, Z_1]$, $[X_2, Z_1]$, $[X_3, Z_2]$ and $[X_4, Z_2]$. This implies that the eigenvalues of A_3 lie in the set $\{\alpha_1^2\alpha_2, \alpha_1\alpha_2^2, \alpha_3^2\alpha_4, \alpha_3\alpha_4^2\}$. Applying Proposition 6.5.21 on the Anosov polynomial g and the eigenvalues of A_3 we see that n_3 must be even. As a consequence a Lie algebra of type $(6, 2, 3)$ can not be Anosov. We can thus from here assume that $n_3 = 2$.

From (6.7) and (6.8) it follows that Z_1 has eigenvalue $\alpha_1\alpha_2$ and Z_2 has eigenvalue $\alpha_3\alpha_4$. If $\{\alpha_1^2\alpha_2,\ \alpha_1\alpha_2^2\}$ or $\{\alpha_3^2\alpha_4,\ \alpha_3\alpha_4^2\}$ are the eigenvalues of A_3 we get that $(\alpha_1\alpha_2)^3 = 1$ or $(\alpha_3\alpha_4)^3 = 1$, respectively. This contradicts the fact that the eigenvalues of A_2 can not have absolute value equal to 1. As a consequence we have without loss of generality that the eigenvalues of A_3 are $\alpha_1^2\alpha_2$ and $\alpha_3^2\alpha_4$. This shows that $\alpha_1\alpha_3 = 1$ and thus also that $\alpha_2\alpha_4 = 1$. From here on we can argue similarly as in the proof of type $(4, 2, 2)$ in Proposition 6.5.23 and get that the subalgebra $\operatorname{span}_{\mathbb{C}}\{X_1, \ldots, X_4\} \oplus \mathfrak{n}_2^{\mathbb{C}} \oplus \mathfrak{n}_3^{\mathbb{C}} \subset \mathfrak{n}^{\mathbb{C}}$ has a positive grading given by $V_1 \oplus \mathfrak{n}_2^{\mathbb{C}} \oplus \mathfrak{n}_3^{\mathbb{C}}$ for some vector subspace $V_1 \subset \operatorname{span}_E\{X_1, \ldots, X_4\} \oplus \mathfrak{n}_2^{\mathbb{C}}$. It is then straightforward to check that $W_1 = V_1$, $W_2 = \mathfrak{n}_2^{\mathbb{C}} \oplus \operatorname{span}_E\{Y_1, Y_2\}$, $W_3 = \mathfrak{n}_3^{\mathbb{C}}$ defines a positive grading for $\mathfrak{n}^{\mathbb{C}}$. Hence also $\mathfrak{n}$ is positively graded. $\square$

This leaves only one possible type, which we check via the action of the Galois group on the eigenvalues.

Proposition 6.5.28. *Let* $\mathfrak{n}$ *be an Anosov Lie algebra of type* $(6, 3, 2)$*, then* $\mathfrak{n}$ *admits a positive grading.*

Proof. Let $\alpha_1, \ldots, \alpha_6$ denote the eigenvalues of A_1 with corresponding eigenvectors $X_1, \ldots, X_6$ and $\beta_1, \beta_2, \beta_3$ the eigenvalues of A_2 with corresponding eigenvectors Y_1, Y_2, Y_3. As usual, we write K for the splitting field of the characteristic polynomial f_1 of A_1. Note that f_2, the characteristic polynomial of A_2, is irreducible of degree 3 since otherwise it would have a rational root. By applying the orbit stabilizer theorem to the action of $\mathrm{Gal}(K/\mathbb{Q})$ on β_1, it follows that 3 divides the order of $\mathrm{Gal}(K/\mathbb{Q})$. As a consequence there is an element $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ of order 3. The field automorphism σ is completely determined by its action on the roots of f_1. Without loss of generality, there are two possibilities for σ, where we use the notation introduced in section 6.5.1.

- $\sigma = (1\,2\,3)(4\,5\,6)$ We first show that A_3 has no eigenvalue of the form $\alpha_i\alpha_j$ with $i \neq j$ and either $i, j \in \{1, 2, 3\}$ or $i, j \in \{4, 5, 6\}$. Without loss of generality we can take $i = 1$, $j = 2$. We must have that $\alpha_1\alpha_2$ is fixed under σ since otherwise A_3 has 3 distinct eigenvalues which is in contradiction with $n_3 = 2$. Thus it follows that $\alpha_1\alpha_2 = \alpha_2\alpha_3 = \alpha_3\alpha_1$ or with other words $\alpha_1 = \alpha_2 = \alpha_3$. This implies that f is a product of 3 irreducible polynomials of degree 2 which is in contradiction with the way σ acts on $\{\alpha_i\}_i$. This shows A_3 can not have an eigenvalue of this form.

  Next we show that A_3 has no eigenvalue of the form $\alpha_i\alpha_j$ with $i \in \{1, 2, 3\}$ and $j \in \{4, 5, 6\}$. Without loss of generality we can assume $i = 1$ and $j = 4$. Again we must have that $\alpha_1\alpha_4$ is fixed under σ since $n_3 = 2$. Thus we get $\alpha_1\alpha_4 = \alpha_2\alpha_5 = \alpha_3\alpha_6$. Therefore $1 = \alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6 = (\alpha_1\alpha_4)^3$ and thus $\alpha_1\alpha_4$ has norm 1. It can therefore not be an eigenvalue of A_3.

  We have shown above that A_3 can not have any eigenvalue of the form $\alpha_i\alpha_j$ with $1 \leq i < j \leq 6$. As a consequence we have $[\mathfrak{n}_1^K, \mathfrak{n}_1^K] \subset \mathfrak{n}_2^K$ and thus $\mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \mathfrak{n}_3$ is a positive grading for $\mathfrak{n}$.

- $\sigma = (1\,2\,3)(4)(5)(6)$. By a similar argument as in the previous case, it holds that $\alpha_i\alpha_j$ with $i \neq j$, $i \in \{1, 2, 3\}$ and $j \in \{1, 2, 3, 4, 5, 6\}$, is not an eigenvalue of A_3. We thus know that A_2 must have an eigenvalue of this form, otherwise $\{X_1, X_2, X_3\}$ spans an abelian factor of $\mathfrak{n}^K$ and thus $\mathfrak{n}$ would be a direct sum of the 3-dimensional abelian Lie algebra and an Anosov Lie algebra of type $(3, 3, 2)$ which does not exist. So A_2 has an eigenvalue of the form $\alpha_i\alpha_j$ with eigenvector $[X_i, X_j]$ where $i \in \{1, 2, 3\}$. Without loss of generality we can assume this eigenvalue is $\alpha_1\alpha_2$ or $\alpha_1\alpha_4$. We treat each case separately.

  In the first case, as we argued before, σ can not fix $\alpha_1\alpha_2$ and thus the other eigenvalues of A_2 are given by $\alpha_2\alpha_3$ and $\alpha_3\alpha_1$. This shows as well that $(\alpha_1\alpha_2\alpha_3)^2 = 1$. Without loss of generality we can assume A_3 has an eigenvalue of the form $\alpha_1\alpha_2\alpha_i$. If $i \in \{4, 5, 6\}$, then σ does not fix $\alpha_1\alpha_2\alpha_i$ which is in contradiction with $n_3 = 2$. So we must have $i \in \{1, 2, 3\}$. If $i = 1$ or $i = 2$ we get, since σ must fix $\alpha_1\alpha_2\alpha_i$, that $\alpha_1^2\alpha_2 = \alpha_2^2\alpha_3 = \alpha_3^2\alpha_1$

or that $\alpha_1\alpha_2^2 = \alpha_2\alpha_3^2 = \alpha_3\alpha_1^2$, respectively. In either case we can derive that $\alpha_1^2 = \alpha_2\alpha_3$, $\alpha_2^2 = \alpha_1\alpha_3$ and $\alpha_3^2 = \alpha_1\alpha_2$. Therefore we have

$$\alpha_1^8 = \alpha_2^4\alpha_3^4 = (\alpha_1\alpha_3)^2(\alpha_1\alpha_2)^2 = (\alpha_1\alpha_2\alpha_3)^2\alpha_1^2 = \alpha_1^2.$$

As a consequence $\alpha_1^6 = 1$, which is in contradiction with the fact that α_6 has absolute value different from 1. We thus conclude that only $\alpha_1\alpha_2\alpha_3$ can be an eigenvalue of A_3 which contradicts the fact that $n_3 = 2$.

Now consider the second case where $\alpha_1\alpha_4$ is an eigenvalue of A_2. The other eigenvalues of A_2 are then $\alpha_2\alpha_4$ and $\alpha_3\alpha_4$. Without loss of generality A_3 must have an eigenvalue of the form $\alpha_1\alpha_4\alpha_i$. If $i \in \{4, 5, 6\}$ it follows that its orbit under σ has three distinct elements: $\alpha_1\alpha_4\alpha_i, \alpha_2\alpha_4\alpha_i$ and $\alpha_3\alpha_4\alpha_i$, which can not happen since $n_3 = 2$. So we must have that $i \in \{1, 2, 3\}$, but then again its orbit under sigma counts three distinct elements contradicting $n_3 = 2$.

Since both cases lead to contradictions, we conclude that the case $\sigma = (1\,2\,3)(4)(5)(6)$ does not occur at all, which finishes the proof.

□

Case $n_1 = 7$

As a first lemma, we show how reducibility of the polynomial f_1 sometimes gives us information about the type.

Lemma 6.5.29. *Let $\mathfrak{n}$ be a rational nilpotent Lie algebra with semi-simple Anosov automorphism $A : \mathfrak{n} \to \mathfrak{n}$ and corresponding decomposition $\mathfrak{n} = \mathfrak{n}_1 \oplus \ldots \oplus \mathfrak{n}_c$. Let f_1 be the characteristic polynomial of A_1 and assume its factorisation in irreducible polynomials is given by $f_1 = g \cdot h_1, \ldots, h_k$ where $\deg g = p$ is prime. We write K_i for the splitting field of the polynomial h_i over $\mathbb{Q}$. If $\mathfrak{n}$ has no non-trivial abelian factors and p does not divide the order of $\mathrm{Gal}(K_i/\mathbb{Q})$ for all $1 \leq i \leq k$, then there exists an index $2 \leq j \leq c$ such that $n_j \geq p$.*

Proof. We write K for the splitting field of f_1. Let $\alpha_1, \ldots, \alpha_p$ be the roots of g with corresponding eigenvectors $X_1, \ldots, X_p \in \mathfrak{n}_1^K$ and $\beta_1, \ldots, \beta_{n_1-p}$ the roots of $h_1 \cdot \ldots \cdot h_k$ with eigenvectors $Y_1, \ldots, Y_{n_1-p}$. Since the Galois group $\mathrm{Gal}(K/\mathbb{Q})$ acts transitively on the roots of g, it follows by the orbit stabilizer theorem that p divides the order of $\mathrm{Gal}(K/\mathbb{Q})$. As a consequence, there exists an element $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ of order p. By restricting σ to the subfield K_i, we get an automorphism $\sigma|_{K_i} \in \mathrm{Gal}(K_i/\mathbb{Q})$ which must have order 1 or p. By our assumption p does not divide the order of $\mathrm{Gal}(K_i/\mathbb{Q})$ and thus we must have that $\sigma|_{K_i} = \mathrm{Id}$. So σ acts trivially on the roots β_i. By following a similar argument as in the proof of Lemma 6.5.5, we find that σ acts, without loss of generality, by the cyclic permutation $(1\,2\,\ldots\,p)$ on the roots of g.

Since the Lie algebra $\mathfrak{n}$ has no abelian factor, there exists either X_m such that $[X_1, X_m] \neq 0$ or there exits Y_m such that $[X_1, Y_m] \neq 0$. In the first case, the eigenvalue of $[X_1, X_m]$ is $\alpha_1\alpha_m$ and in the other case, the eigenvalue of $[X_1, Y_m]$ is $\alpha_1\beta_m$. Since p is prime, both of these eigenvalues must be either fixed by σ or have p conjugates under the action of σ. By Proposition 6.5.9 we know g satisfies the full rank condition and thus if $\alpha_1\alpha_m$ would be fixed under σ, Lemma 6.5.13 gives a contradiction. If $\alpha_1\beta_m$ is fixed by σ, we get that $\alpha_1 = \ldots = \alpha_p$ since β_m is fixed by σ. This contradicts the fact that g is irreducible and thus the minimal polynomial of $\alpha_1\alpha_m$ or $\alpha_1\beta_m$ has degree at least p. In particular the statement of the lemma follows for the j for which $\alpha_1\alpha_m$ or $\alpha_1\beta_m$ is an eigenvalue of A_j. □

Before dealing with the final case, we first prove this technical lemma.

Lemma 6.5.30. *If $f = g \cdot h$ is a polynomial with g, h irreducible polynomials of degree p and n, respectively with p prime and p not a divisor of n, then for any root α of h there exists an element of the Galois group of f that fixes α but does not fix any root of g.*

Proof. Let K, K_1 and K_2 denote the splitting fields of f, g and h, respectively. For any field automorphism $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ we thus have that $\sigma(K_1) = K_1$ and $\sigma(K_2) = K_2$. Therefore we have the natural injection

$$\mathrm{Gal}(K/\mathbb{Q}) \hookrightarrow \mathrm{Gal}(K_1/\mathbb{Q}) \times \mathrm{Gal}(K_2/\mathbb{Q}) : \sigma \mapsto (\sigma|_{K_1}, \sigma|_{K_2})$$

To see this map is injective, note that K_1 and K_2 generate K as a field and thus $\sigma|_{K_1}$ and $\sigma|_{K_2}$ completely determine σ. From here we use this identification to write elements of the Galois group of f as an ordered pair of an element of the Galois group of g and one of the Galois group of h.

Let α be one of the roots of h. Take an element σ of order p in the Galois group of g. Since $K/\mathbb{Q}$ is normal, we can extend $\sigma : K_1 \to K_1$ to an element (σ, τ) of the Galois group of f (see [Ste89, Theorem 10.1]), where τ lies in the Galois group of h. The order of τ can be written as $|\tau| = p^l m$ where p and m are coprime and $l \in \mathbb{N}$.

Now consider the field automorphism $(\sigma, \tau)^m = (\sigma^m, \tau^m)$. Since p and m are coprime and σ has order p, we get that σ^m has order p as well. The order of τ^m is clearly equal to p^l. Let H denote the set of roots of h, then the action of $\langle \tau^m \rangle$ on H gives a partition $H = H_1 \sqcup \ldots \sqcup H_k$ where H_i are the orbits under this action. By the orbit stabilizer theorem we get that the size of each orbit H_i, must divide p^l. If p divides $|H_i|$ for all $1 \leq i \leq k$, then it follows that p divides $\sum_i |H_i| = |H| = n$, which is in contradiction with the assumption. Therefore there must be at least one orbit which counts only one element and thus τ^m fixes a root of h, call it $\beta \in H$. Since $\mathrm{Gal}(K_2/\mathbb{Q})$ acts transitively on H, there exists an element $\pi \in \mathrm{Gal}(K_2/\mathbb{Q})$ such that $\pi(\beta) = \alpha$. Again, extend this to a field automorphism (π', π) of K where $\pi' \in \mathrm{Gal}(K_1/\mathbb{Q})$. Then $(\pi', \pi)(\sigma^m, \tau^m)(\pi', \pi)^{-1}$ fixes the root α and is still of order p when restricted to K_1 which means it does not fix any root of g. □

Combining the previous results leads to the final case for Theorem 6.5.17.

Proposition 6.5.31. *Let $\mathfrak{n}$ be an Anosov rational Lie algebra with $n_1 = 7$ and $\dim \mathfrak{n} < 12$, then $\mathfrak{n}$ has a positive grading.*

Proof. Let $A : \mathfrak{n} \to \mathfrak{n}$ be a semi-simple Anosov automorphism with corresponding decomposition $\mathfrak{n} = \mathfrak{n}_1 \oplus \ldots \oplus \mathfrak{n}_c$. Let f_1 denote the characteristic polynomial of A_1, the restriction of A to $\mathfrak{n}_1$. We prove the statement separately for the possible factorisations of f in irreducible polynomials.

- f_1 is irreducible: Combine Proposition 6.5.9 and Proposition 6.5.14 to find that 7 divides n_i for all $1 \leq i \leq c$. Together with the assumption $\dim \mathfrak{n} < 12$ we must have $c = 1$. In particular $\mathfrak{n}$ has a positive grading.

- $f_1 = g \cdot h_1 \cdot h_2$ with $\deg g = 3$ and $\deg h_1 = \deg h_2 = 2$: Since the degree of g is prime and strictly bigger than the degrees of h_1 and h_2, we have that the degree of g does not divide the order of the Galois groups of h_1 and h_2. We can assume $\mathfrak{n}$ has no non-trivial abelian factor since in this case our work in the previous sections for smaller n_1 and Theorems 6.4.5 and 6.4.8 assure $\mathfrak{n}$ has a positive grading. Lemma 6.5.29 then gives the existence of an index $1 \leq j \leq c$ such that $n_j \geq 3$. Together with the restriction $\dim \mathfrak{n} < 12$, we get that $c \leq 2$ and thus $\mathfrak{n}$ admits a positive grading.

- $f_1 = g \cdot h$ with $\deg g = 5$ and $\deg h = 2$: Since g has prime degree and is strictly greater than the degree of h, we have that the degree of g does not divide the order of the Galois group of h. We can assume $\mathfrak{n}$ has no non-trivial abelian factor since in this case our work in the previous sections for smaller n_1 and Theorems 6.4.5 and 6.4.8 assure $\mathfrak{n}$ has a positive grading. Lemma 6.5.29 then gives an index $1 \leq j \leq c$ such that $n_j \geq 5$. Together with the restriction $\dim \mathfrak{n} < 12$, we get that $c < 2$ and thus $\mathfrak{n}$ admits a positive grading.

- $f_1 = g \cdot h$ with $\deg g = 3$ and $\deg h = 4$: We can assume that 3 divides the order of the Galois group of h, otherwise we can use Lemma 6.5.29 in a similar fashion as with the previous two cases and find that $\mathfrak{n}$ must be positively graded. It follows that the Galois group of h is either the alternating group A_4 or the whole permutation group S_4. Let $\alpha_1, \alpha_2, \alpha_3$ be the roots of g and $\beta_1, \beta_2, \beta_3, \beta_4$ the roots of h. Since $\mathfrak{n}$ is not abelian there must be without loss of generality an eigenvector in $\mathfrak{n}_2$ with one of following three eigenvalues: $\alpha_1\alpha_2$, $\alpha_1\beta_1$ or $\beta_1\beta_2$.

  In case the eigenvalue is $\alpha_1\alpha_2$, take the element in the Galois group of g corresponding with the permutation $(1\,2\,3)$ and extend it to an element σ of the Galois group of f. Then it is clear that $\sigma(\alpha_1\alpha_2) = \alpha_2\alpha_3$ and $\sigma^2(\alpha_1\alpha_2) = \alpha_3\alpha_1$ are also eigenvalues of A_2. These are three different eigenvalues and thus we get $n_2 \geq 3$.

  In case the eigenvalue is equal to $\alpha_1\beta_1$, we use Lemma 6.5.30 and find an element σ in the Galois group of f such that σ fixes β_1 and acts by the permutation $(1\,2\,3)$ on the roots of g. Then A_2 must also have eigenvalues $\sigma(\alpha_1\beta_1) = \alpha_2\beta_1$ and $\sigma^2(\alpha_1\beta_1) = \alpha_3\beta_1$. This gives three different eigenvalues of A_2 and thus $n_2 \geq 3$.

  In case the eigenvalue equals $\beta_1\beta_2$, take the element in the Galois group of h corresponding to the permutation $(1)(2\,3\,4)$ and extend it to an element σ of the Galois group of f. Then A_2 must also have the eigenvalues $\sigma(\beta_1\beta_2) = \beta_1\beta_3$ and $\sigma^2(\beta_1, \beta_2) = \beta_1\beta_4$. This gives three different eigenvalues of A_2 and thus $n_2 \geq 3$.

  In all cases we see that $n_2 \geq 3$. Therefore by the restriction on the dimension of $\mathfrak{n}$ we find that $c = 2$ and thus that $\mathfrak{n}$ admits a positive grading.

□

6.6 Anosov rational forms of partially commutative Lie algebras

This section is based on [DW24].

Recall from Chapter 4 that for any non-empty graph $\mathcal{G} = (V, E)$, we defined a c-step nilpotent partially commutative Lie algebra $\mathfrak{n}^K(\mathcal{G}, c)$ over a field K. In section 5.6.2 of Chapter 5, we classified the rational forms of the complex Lie algebra $\mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c)$ by use of continuous morphisms $\rho : \mathrm{Gal}(K/\mathbb{Q}) \to \mathrm{Aut}(\overline{\mathcal{G}})$ where $K/\mathbb{Q}$ is a Galois extension. The rational form associated to ρ is written as $\mathfrak{n}^{\mathbb{Q}}(\rho, c)$. In this section, we will answer the following question.

Question 6.6.1. For which continuous morphisms $\rho : \mathrm{Gal}(K/\mathbb{Q}) \to \mathrm{Aut}(\overline{\mathcal{G}})$ is the rational Lie algebra $\mathfrak{n}^{\mathbb{Q}}(\rho, c)$ Anosov?

The answer to this question is formulated in Theorems 6.6.13 and 6.6.14. The former deals with continuous morphisms from the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ to $\mathrm{Aut}(\overline{\mathcal{G}})$ while the latter deals with injective morphisms from finite Galois groups to $\mathrm{Aut}(\overline{\mathcal{G}})$. Several corollaries are then deduced from this general result. If we only look at the rational forms of the real Lie algebra we obtain Corollary 6.6.15. The special case of the standard rational form is formulated in Corollary 6.6.16.

In section 6.6.3 we give some examples to illustrate the effects of the main theorems. In particular, we apply the theorem to certain classes of graphs, a first of which are trees (see Definition 4.1.4). If we write $N(\mathcal{G}, c)$ for the real simply connected nilpotent Lie group associated to the real Lie algebra $\mathfrak{n}^{\mathbb{R}}(\mathcal{G}, c)$, then we can formulate the obtained result as follows.

Theorem 6.6.2. *Let $\mathcal{G}$ be a non-empty tree, $c > 1$ and $N(\mathcal{G}, c)$ the associated c-step nilpotent simply connected Lie group. For any cocompact lattice $\Gamma \leq N(\mathcal{G}, c)$, we have that the nilmanifold $\Gamma\backslash N(\mathcal{G}, c)$ does not admit an Anosov diffeomorphism.*

Note that this theorem is an improvement of the existing theorem in [Mai06] which considers only one specific cocompact lattice in $N(\mathcal{G}, c)$ (namely the one corresponding to the standard rational form $\mathfrak{n}^{\mathbb{Q}}(\mathcal{G}, c)$ of $\mathfrak{n}^{\mathbb{R}}(\mathcal{G}, c)$).

The next class to consider are the cycle graphs (see Example 4.1.2).

Theorem 6.6.3. *Let $\mathcal{G}$ be the cycle graph on $n \geq 5$ vertices, $c > 1$ and $N(\mathcal{G}, c)$ the associated c-step nilpotent simply connected Lie group. There exists a cocompact lattice $\Gamma \leq N(\mathcal{G}, c)$ with the nilmanifold $\Gamma\backslash N(\mathcal{G}, c)$ admitting an Anosov diffeomorphism if and only if $n > c$.*

Before we prove the main theorem in section 6.6.2, we first prove a result in the following section which relates the existence of an Anosov automorphism to the existence of hyperbolic algebraic units for each vertex of the graph, compatible with the action on the quotient graph.

6.6.1 Reduction to algebraic units on the vertices

First we introduce some notation that will be used in this and the following sections.

For any group morphism $\rho : G \to \mathrm{Perm}(X)$ from a group G to a the permutation group on a set X, we refer to the induced action by G on X as the *ρ-action*.

The orbit and stabilizer of an element $x \in X$ under this action will be written as $\mathrm{Orb}_\rho(x)$ and $\mathrm{Stab}_\rho(x)$, respectively. A subset $Y \subset X$ is said to be *ρ-invariant* if for any $y \in Y$ and any $g \in G$ we have $\rho_g(y) \in Y$.

Let X be a finite set and $\mathcal{P}$ a *partition* of X, i.e. a collection $\mathcal{P}$ of non-empty subsets of X such that any two different sets in $\mathcal{P}$ have empty intersection and the union of all sets in $\mathcal{P}$ is equal to X. An automorphism of the partition $\mathcal{P}$ is defined as a permutation $\varphi \in \mathrm{Perm}(\mathcal{P})$ such that for any $Y \in \mathcal{P}$ we have $|\varphi(Y)| = |Y|$. Clearly, when $\mathcal{G} = (V, E)$ is a graph, its set of coherent components $\Lambda_\mathcal{G}$ is a partition of the vertices V. With the above introduced notation, we then have that the group of automorphisms of the quotient graph $\mathrm{Aut}(\overline{\mathcal{G}})$ is a subgroup of $\mathrm{Aut}(\Lambda_\mathcal{G})$.

Let X be a finite set, $\mathcal{P}$ a partition of X and K a field. Write K^X for the set of maps from X to K. We can naturally identify the group $\prod_{Y\in\mathcal{P}} \mathrm{Perm}(Y)$ with a subgroup of $\mathrm{Perm}(X)$. Thus, we get a right action of $\prod_{Y\in\mathcal{P}} \mathrm{Perm}(Y)$ on K^X by precomposition, namely

$$\Psi \cdot \theta := \Psi \circ \theta$$

for all $\theta \in \prod_{Y\in\mathcal{P}} \mathrm{Perm}(Y)$ and $\Psi \in K^X$. Let us write $\mathcal{H}^K_\mathcal{P}$ for the corresponding orbit space, so

$$\mathcal{H}^K_\mathcal{P} := K^X \Big/ \prod_{Y\in\mathcal{P}} \mathrm{Perm}(Y).$$

Thus elements of $\mathcal{H}^K_\mathcal{P}$ are determined by maps from X to K, but without caring, inside each set $Y \in \mathcal{P}$, about which element of Y is assigned to which value.

Note that for any element $\varphi \in \mathrm{Aut}(\mathcal{P})$ there exists a (not necessarily unique) permutation $\psi \in \mathrm{Perm}(X)$ such that $\varphi(Y) = \psi(Y)$ for all $Y \in \mathcal{P}$. This is easily seen using the same arguments as in Remark 4.1.9 for the partition $\Lambda_\mathcal{G}$ of the vertices V of a graph $\mathcal{G} = (V, E)$. Moreover, as shown in that same remark, one can choose a group morphism $r : \mathrm{Aut}(\mathcal{P}) \to \mathrm{Perm}(X)$ such that $r(\varphi)(Y) = \varphi(Y)$ for all $Y \in \mathcal{P}$. The groups $\mathrm{Aut}(K)$ and $\mathrm{Aut}(\mathcal{P})$ then have a well-defined left, respectively right action on $\mathcal{H}^K_\mathcal{P}$ by

$$\sigma \cdot [\Psi] := [\sigma \circ \Psi], \qquad [\Psi] \cdot \varphi := [\Psi \circ r(\varphi)]$$

for all $\sigma \in \mathrm{Aut}(K)$, $\varphi \in \mathrm{Aut}(\mathcal{P})$ and $\Psi \in K^X$. Although we use the morphism r to define the action of $\mathrm{Aut}(\mathcal{P})$ on $\mathcal{H}^K_\mathcal{P}$, one can check that this action is independent of the choice of such a morphism r.

Note that for a graph $\mathcal{G} = (V, E)$, we did however fix a morphism $r : \mathrm{Aut}(\overline{\mathcal{G}}) \to \mathrm{Aut}(\mathcal{G})$ which we used to define the morphism $i : \mathrm{Aut}(\overline{\mathcal{G}}) \to \mathrm{Aut}(\mathfrak{n}^{\overline{\mathbb{Q}}}(\mathcal{G}, c))$ which in turn is used to describe the rational form

$$\mathfrak{n}^{\mathbb{Q}}(\rho, c) = \{v \in \mathfrak{n}^{\overline{\mathbb{Q}}}(\mathcal{G}, c) \mid \forall \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) : \; i(\rho_\sigma)({}^\sigma v) = v\},$$

for any continuous group morphism $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}(\overline{\mathcal{G}})$ (see section 5.6.2). The following theorem characterizes the existence of an Anosov automorphism on the rational form $\mathfrak{n}^{\mathbb{Q}}(\rho, c)$ by the existence of a function from the vertices V to $\overline{\mathbb{Q}}$ and makes use of the above introduced notation for the partition of V into coherent components.

Theorem 6.6.4. *Let $\mathcal{G} = (V, E)$ be a simple undirected graph and $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}(\overline{\mathcal{G}})$ a continuous morphism. The rational form $\mathfrak{n}^{\mathbb{Q}}(\rho, c)$ of $\mathfrak{n}^{\overline{\mathbb{Q}}}(\mathcal{G}, c)$ is Anosov if and only if there exists a map $\Psi : V \to \overline{\mathbb{Q}}$ such that*

(i) for any vertex $v \in V$, $\Psi(v)$ is an algebraic unit,

(ii) for any $k \in \{1, \ldots, c\}$ and any (not necessarily distinct) vertices $v_1, \ldots, v_k \in V$ such that $\{v_1, \ldots, v_k\}$ spans a connected subgraph of $\mathcal{G}$, we have $|\Psi(v_1) \cdot \ldots \cdot \Psi(v_n)| \neq 1$,

(iii) and for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ it holds that $\sigma \cdot [\Psi] = [\Psi] \cdot \rho_\sigma$ in $\mathcal{H}^{\overline{\mathbb{Q}}}_{\Lambda_{\mathcal{G}}}$.

Note that the conditions in this theorem do not depend on the choice of representative Ψ in its equivalence class $[\Psi] \in \mathcal{H}^{\overline{\mathbb{Q}}}_{\Lambda_{\mathcal{G}}}$.

Proof. First assume that $\mathfrak{n}^{\mathbb{Q}}(\rho, c) \subset \mathfrak{n}^{\overline{\mathbb{Q}}}(\mathcal{G}, c)$ is Anosov with Anosov automorphism $f \in \mathrm{Aut}\left(\mathfrak{n}^{\mathbb{Q}}(\rho, c)\right)$. Recall that we have the projection onto the abelianization π_{ab} and that, by Theorem 4.6.6, the image of $\mathrm{Aut}\left(\mathfrak{n}^{\overline{\mathbb{Q}}}(\mathcal{G}, c)\right)$ under π_{ab} is equal to

$$G^{\overline{\mathbb{Q}}}(\mathcal{G}) = \mathrm{Aut}_{\mathrm{ab}}\left(\mathfrak{n}^{\overline{\mathbb{Q}}}(\mathcal{G}, c)\right) = P(\mathrm{Aut}(\mathcal{G})) \cdot \left(\prod_{\lambda \in \Lambda_{\mathcal{G}}} \mathrm{GL}(\mathrm{span}_{\overline{\mathbb{Q}}}(\lambda))\right) \cdot U^{\overline{\mathbb{Q}}}(\mathcal{G}).$$

We also have morphisms

$$p : G^{\overline{\mathbb{Q}}}(\mathcal{G}) \to \mathrm{Aut}(\overline{\mathcal{G}}) \quad \text{and} \quad i : \mathrm{Aut}(\overline{\mathcal{G}}) \to \mathrm{Aut}\left(\mathfrak{n}^{\overline{\mathbb{Q}}}(\mathcal{G}, c)\right)$$

where p is defined in equation (4.7) and i was defined in Remark 5.6.2. Recall that i depends on the chosen morphism $r : \mathrm{Aut}(\mathcal{G}) \to \mathrm{Aut}(\overline{\mathcal{G}})$. At last, let us write

$$\pi = p \circ \pi_{\mathrm{ab}} \quad \text{and} \quad \overline{P} = P \circ r$$

and recall from equation (4.8) that we have an internal semi-direct product

$$G^{\overline{\mathbb{Q}}}(\mathcal{G}) = \overline{P}(\mathrm{Aut}(\mathcal{G})) \cdot \left(\prod_{\lambda \in \Lambda_{\mathcal{G}}} \mathrm{GL}(\mathrm{span}_{\overline{\mathbb{Q}}}(\lambda)) \right) \cdot U^{\overline{\mathbb{Q}}}(\mathcal{G})$$

$$\cong \overline{P}(\mathrm{Aut}(\mathcal{G})) \ltimes \left(\prod_{\lambda \in \Lambda_{\mathcal{G}}} \mathrm{GL}(\mathrm{span}_{\overline{\mathbb{Q}}}(\lambda)) \right) \ltimes U^{\overline{\mathbb{Q}}}(\mathcal{G}).$$

Note that since $\mathfrak{n}^{\mathbb{Q}}(\rho, c)$ is a rational form of $\mathfrak{n}^{\overline{\mathbb{Q}}}(\mathcal{G}, c)$, we can naturally extend f to an automorphism of $\mathfrak{n}^{\overline{\mathbb{Q}}}(\mathcal{G}, c)$. The semisimple part of f is again an Anosov automorphism, as $\mathrm{Aut}(\mathfrak{n}^{\mathbb{Q}}(\rho, c))$ is a linear algebraic group. Moreover, any positive power f^k for $k > 0$ of f is an Anosov automorphism as well, so without loss of generality we can assume that f is semi-simple and lies in the Zariski-connected component of the identity of $\mathrm{Aut}\left(\mathfrak{n}^{\overline{\mathbb{Q}}}(\mathcal{G}, c)\right)$. Hence, under these assumptions, f lies in some maximal torus of $\mathrm{Aut}\left(\mathfrak{n}^{\overline{\mathbb{Q}}}(\mathcal{G}, c)\right)$. By Lemma 4.6.15, we know that the subgroup of vertex diagonal automorphisms $D^{\overline{\mathbb{Q}}}(\mathcal{G}, c)$ is a maximal torus of $\mathrm{Aut}\left(\mathfrak{n}^{\overline{\mathbb{Q}}}(\mathcal{G}, c)\right)$. Since $\mathrm{Aut}\left(\mathfrak{n}^{\overline{\mathbb{Q}}}(\mathcal{G}, c)\right)$ is a linear algebraic group over an algebraically closed field, all its maximal tori are conjugate and thus there exists an $h \in \mathrm{Aut}\left(\mathfrak{n}^{\overline{\mathbb{Q}}}(\mathcal{G}, c)\right)$ and an $\tilde{f} \in D^{\overline{\mathbb{Q}}}(\mathcal{G}, c)$ such that $h\tilde{f}h^{-1} = f$. Moreover, since $i\left(\mathrm{Aut}\left(\overline{\mathcal{G}}\right)\right)$ normalizes $D^{\overline{\mathbb{Q}}}(\mathcal{G}, c)$, we can assume that $\pi(h) = 1$.

Let us define $\Psi : V \to \overline{\mathbb{Q}}$ by assigning to a vertex $v \in V$ its corresponding eigenvalue under $\tilde{f}$. Since $\tilde{f}$ is an Anosov automorphism, it follows that $\Psi(v)$ is a hyperbolic algebraic unit for all $v \in V$. As a consequence we also have $|\Psi(v)^k| \neq 0$ for any $v \in V$ and $k \in \mathbb{N}$. By Proposition 4.6.14 from Chapter 4, we know that for any vertices $v_1, \ldots, v_n \in V$ with $|\{v_1, \ldots, v_n\}| \geq 2$ and such that $\{v_1, \ldots, v_n\}$ spans a connected subgraph of $\mathcal{G}$, the product $\Psi(v_1) \cdot \ldots \cdot \Psi(v_n)$ is an eigenvalue of $\tilde{f}$ and thus $|\Psi(v_1) \cdot \ldots \cdot \Psi(v_n)| \neq 1$. All together this proves Ψ satisfies conditions (i) and (ii).

Let us now show Ψ also satisfies condition (iii). Since $f\left(\mathfrak{n}^{\mathbb{Q}}(\rho, c)\right) = \mathfrak{n}^{\mathbb{Q}}(\rho, c)$, it follows from Lemma 6.3.4 that $f\, i(\rho_\sigma) = i(\rho_\sigma)\, {}^{\sigma}f$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Substituting $h\tilde{f}h^{-1}$ for f, we find the equality

$$\tilde{f} a_\sigma = a_\sigma\, {}^{\sigma}\tilde{f} \qquad \text{where} \quad a_\sigma := h^{-1} i(\rho_\sigma)\, {}^{\sigma}h. \tag{6.11}$$

Note that $\pi(a_\sigma) = \pi(h^{-1})\pi(i(\rho_\sigma))\pi(h) = \pi(i(\rho_\sigma)) = \rho_\sigma$, and hence $\pi_{\mathrm{ab}}(a_\sigma) = \overline{P}(\rho_\sigma) A_\sigma m_\sigma$ for unique $m_\sigma \in U^{\overline{\mathbb{Q}}}(\mathcal{G})$ and $A_\sigma \in \prod_{\lambda \in \Lambda_{\mathcal{G}}} \mathrm{GL}(\mathrm{span}_{\overline{\mathbb{Q}}}(\lambda))$. Applying the morphism π_{ab} to equation (6.11), we get that

$$\pi_{\mathrm{ab}}(\tilde{f})\overline{P}(\rho_\sigma)A_\sigma m_\sigma = \overline{P}(\rho_\sigma)A_\sigma m_\sigma \,{}^\sigma\pi_{\mathrm{ab}}(\tilde{f}).$$

Note that $\pi_{\mathrm{ab}}(\tilde{f})$ and ${}^\sigma\pi_{\mathrm{ab}}(\tilde{f})$ are elements in $\prod_{\lambda\in\Lambda_{\mathcal{G}}} \mathrm{GL}(\mathrm{span}_{\overline{\mathbb{Q}}}(\lambda))$, since $\tilde{f}$ is diagonal on the vertices V. Rearranging the equation above according to the semi-direct product decomposition of $G^{\overline{\mathbb{Q}}}(\mathcal{G})$, we find

$$\begin{aligned}\overline{P}(\rho_\sigma)\cdot\Big(\overline{P}(\rho_\sigma)^{-1}\,\pi_{\mathrm{ab}}(\tilde{f})\,\overline{P}(\rho_\sigma)A_\sigma\Big)\cdot m_\sigma \\ = \overline{P}(\rho_\sigma)\cdot\Big(A_\sigma\,{}^\sigma\pi_{\mathrm{ab}}(\tilde{f})\Big)\cdot\Big({}^\sigma\pi_{\mathrm{ab}}(\tilde{f})^{-1}\,m_\sigma\,{}^\sigma\pi_{\mathrm{ab}}(\tilde{f})\Big).\end{aligned}$$

Thus, the equality on $\prod_{\lambda\in\Lambda_{\mathcal{G}}} \mathrm{GL}(\mathrm{span}_{\overline{\mathbb{Q}}}(\lambda))$ is given by

$$\overline{P}(\rho_\sigma)^{-1}\pi_{\mathrm{ab}}(\tilde{f})\overline{P}(\rho_\sigma)A_\sigma = A_\sigma\,{}^\sigma\pi_{\mathrm{ab}}(\tilde{f}),$$

or equivalently

$$\overline{P}(\rho_\sigma)^{-1}\pi_{\mathrm{ab}}(\tilde{f})\overline{P}(\rho_\sigma) = A_\sigma\,{}^\sigma\pi_{\mathrm{ab}}(\tilde{f})A_\sigma^{-1},$$

which holds for any $\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Since both sides are elements of $\prod\limits_{\lambda\in\Lambda_{\mathcal{G}}} \mathrm{GL}(\mathrm{span}_{\overline{\mathbb{Q}}}(\lambda))$, we can look at their projection onto $\mathrm{GL}(\mathrm{span}_{\overline{\mathbb{Q}}}(\lambda))$ for any $\lambda\in\Lambda_{\mathcal{G}}$. We then find that

$$f|_{\mathrm{span}_{\overline{\mathbb{Q}}}(\rho_\sigma(\lambda))} \quad\text{and}\quad {}^\sigma\Big(f|_{\mathrm{span}_{\overline{\mathbb{Q}}}(\lambda)}\Big)$$

are conjugate linear maps. Thus, their eigenvalues, counted with multiplicities, coincide. This shows exactly that $[\Psi\circ r(\rho_\sigma)] = [\sigma\circ\Psi]$ for any $\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and thus that Ψ satisfies all the required properties.

Conversely, assume that a map $\Psi : V\to\overline{\mathbb{Q}}$ satisfying conditions (i), (ii) and (iii) exists. Let m be the number of orbits for the ρ-action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\Lambda_{\mathcal{G}}$ and choose coherent components $\lambda_1,\dots,\lambda_m\in\Lambda_{\mathcal{G}}$ such that $\Lambda_{\mathcal{G}} = \bigsqcup_{j=1}^m \mathrm{Orb}_\rho(\lambda_i)$. For any $j\in\{1,\dots,m\}$, define the polynomial $g_j(X)\in\overline{\mathbb{Q}}[X]$ by

$$g_j(X) = \prod_{v\in\lambda_j}(X-\Psi(v)).$$

Let us fix an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\overline{\mathbb{Q}}[X]$ by acting on the coefficients of the polynomials. As one can check this is an action by ring automorphisms. Take an arbitrary $\sigma\in\mathrm{Stab}_\rho(\lambda_j)$. By the assumption, we have that $[\sigma\circ\Psi] =$

$[\Psi \circ r(\rho_\sigma)]$. As a consequence, there exists a $\theta \in \prod_{\lambda \in \Lambda_{\mathcal{G}}} \mathrm{Perm}(\lambda)$ such that $\sigma \circ \Psi = \Psi \circ r(\rho_\sigma) \circ \theta$. We now have that

$$\begin{aligned} {}^{\sigma}g_j(X) &= \prod_{v \in \lambda_j} {}^{\sigma}(X - \Psi(v)) \\ &= \prod_{v \in \lambda_j} (X - (\sigma \circ \Psi)(v)) \\ &= \prod_{v \in \lambda_j} (X - (\Psi \circ r(\rho_\sigma) \circ \theta)(v)) \\ &= \prod_{v \in \lambda_j} (X - (\Psi \circ r(\rho_\sigma))(v)) \\ &= \prod_{v \in \rho_\sigma(\lambda_i)} (X - \Psi(v)) \\ &= \prod_{v \in \lambda_j} (X - \Psi(v)) = g_j(X). \end{aligned} \tag{6.12}$$

So the coefficients of $g_j(X)$ lie in the field extension $\overline{\mathbb{Q}}^{\mathrm{Stab}_\rho(\lambda_j)}/\mathbb{Q}$. Note that since ρ is continuous, the stabilizer $\mathrm{Stab}_\rho(\lambda_j)$ is an open subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (for the Krull topology) and thus that $\overline{\mathbb{Q}}^{\mathrm{Stab}_\rho(\lambda_j)}/\mathbb{Q}$ is a finite degree extension.

Next, for any $j \in \{1, \ldots, m\}$, let $B_j \in \mathrm{GL}(\mathrm{span}_{\overline{\mathbb{Q}}}(\lambda_j))$ be the linear map given by the companion matrix of $g_j(X)$ in a basis of vertices of $\mathrm{span}_{\overline{\mathbb{Q}}}(\lambda_j)$, where the order of the basis does not matter. Clearly, we have that ${}^{\sigma}B_j = B_j$ for any $\sigma \in \mathrm{Stab}_\rho(\lambda_j)$. Now define the linear map $A \in \mathrm{GL}(\mathrm{span}_{\overline{\mathbb{Q}}}(V))$ by setting for any $\mu \in \mathrm{Orb}_\rho(\lambda_j)$ and $v \in \mathrm{span}_{\overline{\mathbb{Q}}}(\mu)$:

$$A(v) = i(\rho_\sigma)\, {}^{\sigma}B_j\, i(\rho_\sigma)^{-1} v$$

where $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is chosen such that $\sigma(\lambda_j) = \mu$. Let us first show this is well-defined and independent of the choice of $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Say $\nu \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is another element which also satisfies $\nu(\lambda_j) = \mu$, then $\sigma^{-1}\nu \in \mathrm{stab}_\rho(\lambda_j)$ and we get

$$\begin{aligned} i(\rho_\nu)\, {}^{\nu}B_j\, i(\rho_\nu)^{-1} v &= i(\rho_\sigma) i(\rho_{\sigma^{-1}\nu})\, {}^{\sigma}\left({}^{\sigma^{-1}\nu}B_j\right) i(\rho_{\sigma^{-1}\nu})^{-1} i(\rho_\sigma)^{-1} v \\ &= i(\rho_\sigma)\, {}^{\sigma}B_j i(\rho_\sigma)^{-1} v. \end{aligned}$$

We thus have a well-defined linear map A and by construction, we have $A \in \prod_{\lambda \in \Lambda_{\mathcal{G}}} \mathrm{GL}(\mathrm{span}_{\overline{\mathbb{Q}}}(\lambda))$. This gives a unique graded automorphism $f \in \mathrm{Aut}_g\left(\mathfrak{n}^{\overline{\mathbb{Q}}}(\mathcal{G}, c)\right)$ with $\pi_{\mathrm{ab}}(f) = A$ (see section 4.6.1 of Chapter 4).

We claim that f induces an Anosov automorphism on $\mathfrak{n}^{\mathbb{Q}}(\rho, c)$. To check that $f(\mathfrak{n}^{\mathbb{Q}}(\rho, c)) = \mathfrak{n}^{\mathbb{Q}}(\rho, c)$, we need to check that $i(\rho_\sigma)^{-1} f\, i(\rho_\sigma) = {}^{\sigma} f$ for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (Lemma 6.3.4). Since both f and $i(\rho_\sigma)$ are graded automorphisms of $\mathfrak{n}^{\overline{\mathbb{Q}}}(\mathcal{G}, c)$, it suffices to check this on $\mathrm{span}_{\overline{\mathbb{Q}}}(V)$. Take any $j \in \{1, \ldots, m\}$, $\mu \in \mathrm{Orb}_\rho(\lambda_j)$ and $v \in \mathrm{span}_{\overline{\mathbb{Q}}}(\mu)$. Let $\nu \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be an element such that $\rho_\nu(\lambda_j) = \rho_\sigma(\mu)$ or equivalently $\rho_{\sigma^{-1}\nu}(\lambda_j) = \mu$. Then we have

$$\begin{aligned} i(\rho_\sigma)^{-1} f\, i(\rho_\sigma) v &= i(\rho_\sigma)^{-1} i(\rho_\nu)\ {}^{\nu} B_j\, i(\rho_\nu)^{-1} i(\rho_\sigma) v \\ &= i(\rho_{\sigma^{-1}\nu})\ {}^{\sigma}\left({}^{\sigma^{-1}\nu} B_j\right) i(\rho_{\nu^{-1}\sigma}) v \\ &= {}^{\sigma}\left(i(\rho_{\sigma^{-1}\nu})\ {}^{\sigma^{-1}\nu} B_j\, i(\rho_{\sigma^{-1}\nu})^{-1}\right) v \\ &= ({}^{\sigma} f) v. \end{aligned}$$

The set of eigenvalues of f on $\mathrm{span}_{\overline{\mathbb{Q}}}(V)$ is equal to the image of Ψ which are algebraic units by assumption. The other eigenvalues of f are products of the eigenvalues on $\mathrm{span}_{\overline{\mathbb{Q}}}(V)$ and are thus algebraic units as well. This shows that f is integer-like, since its characteristic polynomial has coefficients in $\mathbb{Q}$ from the fact that $f(\mathfrak{n}^{\mathbb{Q}}(\rho, c)) = \mathfrak{n}^{\mathbb{Q}}(\rho, c)$ as proven above. To see that f is hyperbolic, note that by the way we constructed f, it follows that f is conjugated to the vertex diagonal automorphism f_Ψ. We thus only need to check that f_Ψ is hyperbolic. By the assumption on Ψ, we know that the n-fold products of the form $\Psi(v_1) \cdot \ldots \cdot \Psi(v_n)$ with $\{v_1, \ldots, v_n\}$ connected and $1 \leq n \leq c$, have absolute value different from 1. By Proposition 4.6.14, only these products can occur as eigenvalues of f_Ψ. This proves that f_Ψ is hyperbolic. □

6.6.2 A characterization on the graph and the action

In this section, we give a condition for a rational form $\mathfrak{n}^{\mathbb{Q}}(\rho, c)$ to be Anosov which is easier to check than the condition in Theorem 6.6.4 and which solely depends on how the orbits of the action on $\overline{\mathcal{G}}$ induced by ρ look like and on which coherent components are fixed under the action of the complex conjugation automorphism. Before we prove this characterization, we first prove several lemmas. At the end of the section we list some corollaries, one of which is a correction to a result of [Mai06].

Given a number field, the following lemma shows the existence of a Galois extension of that field which satisfies a condition on the number of real and complex embeddings.

Lemma 6.6.5. *Let K be a number field with s real embeddings and t conjugated pairs of complex embeddings in $\mathbb{C}$. For any positive integer $d \in \mathbb{N}$, there exists a Galois extension F/K of degree $[F : K] = d$ such that F has $d \cdot s$ real embeddings and $d \cdot t$ conjugated pairs of complex embeddings in $\mathbb{C}$.*

Proof. We can assume that all fields are subfields of $\mathbb{C}$. From Lemma 5.6.7, it is clear that there exists a totally real Galois extension $L/\mathbb{Q}$ of degree d such that $K \cap L = \mathbb{Q}$. Now take $F := KL$. Using Proposition 3.19 and Corollary 3.20 from [Mil22], we find that F/K is Galois and that

$$[F : \mathbb{Q}] = \frac{[L : \mathbb{Q}][K : \mathbb{Q}]}{[L \cap K : \mathbb{Q}]} = [L : \mathbb{Q}][K : \mathbb{Q}] = d \cdot (s + 2t).$$

Let us write in general $\mathcal{M}_K$ for the set of embeddings from the number field K to $\mathbb{C}$. Consider the map

$$g : \mathcal{M}_F \to \mathcal{M}_K \times \mathcal{M}_L : \sigma \mapsto (\sigma|_K, \sigma|_L).$$

It is clear this map is injective since any embedding of F in $\mathbb{C}$ is completely determined by its values on K and L. Since the number of embeddings equals the degree of the number field, it follows from the above that $\mathcal{M}_F$ and $\mathcal{M}_K \times \mathcal{M}_L$ have the same cardinality. We thus have that g is a bijection. If we let τ denote the complex conjugation automorphism on $\mathbb{C}$, we have for any $\sigma \in \mathcal{M}_F$ the equivalences

$$\begin{aligned}
\tau \circ \sigma = \sigma &\Leftrightarrow g(\tau \circ \sigma) = g(\sigma) \\
&\Leftrightarrow (\tau \circ \sigma|_K, \tau \circ \sigma|_L) = (\sigma|_K, \sigma|_L) \\
&\Leftrightarrow (\tau \circ \sigma|_K, \sigma|_L) = (\sigma|_K, \sigma|_L) \\
&\Leftrightarrow \tau \circ \sigma|_K = \sigma|_K.
\end{aligned}$$

From this it follows that the number of real embeddings of F into $\mathbb{C}$ is exactly equal to $d \cdot s$. As a consequence, the number of conjugated pairs of complex embeddings of F into $\mathbb{C}$ must be equal to $d \cdot t$. □

Let X be a finite set. For any Galois extension $L/\mathbb{Q}$ and any continuous morphism $\rho : \mathrm{Gal}(L/\mathbb{Q}) \to \mathrm{Perm}(X)$, inducing an action on X, we define a function

$$z_\rho : X \to \left\{1, \frac{1}{2}\right\}$$

by

$$z_\rho(x) = \begin{cases} 1 & \text{if } \exists \sigma \in \mathrm{Gal}(L/\mathbb{Q}) : \rho_{\tau\sigma}(x) = \rho_\sigma(x) \\ \frac{1}{2} & \text{else} \end{cases}, \tag{6.13}$$

where $\tau \in \mathrm{Gal}(L/\mathbb{Q})$ denotes the complex conjugation automorphism (with the convention that it is the identity automorphism if $L \subset \mathbb{R}$). Note that z_ρ is constant on ρ-orbits and that it takes the value 1 on an orbit if and only if ρ_τ fixes an element in that orbit. The reason why the function z_ρ takes the values 1 and 1/2 will become clear in the formulation of the main theorem of this section. The following lemma shows the relation between this function and totally imaginary number fields. Recall from section 6.2 that a number field is said to be *totally imaginary* if it has no real embeddings.

Lemma 6.6.6. *Let X be a set and $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Perm}(X)$ a continuous morphism. For any $x \in X$, the field $\overline{\mathbb{Q}}^H$ with $H = \mathrm{Stab}_\rho(x)$ is totally imaginary if and only if $z_\rho(x) = 1/2$.*

Proof. Let $\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be the complex conjugation automorphism. Note that $\overline{\mathbb{Q}}^H$ is totally imaginary if and only if for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ we have

$$\tau \notin \mathrm{Gal}\left(\overline{\mathbb{Q}}\middle/\sigma\left(\overline{\mathbb{Q}}^H\right)\right).$$

We have the series of equivalences

$$\begin{aligned} \tau \in \mathrm{Gal}\left(\overline{\mathbb{Q}}\middle/\sigma\left(\overline{\mathbb{Q}}^H\right)\right) &\Leftrightarrow \tau \in \mathrm{Gal}\left(\overline{\mathbb{Q}}\middle/\overline{\mathbb{Q}}^{\sigma H \sigma^{-1}}\right) \\ &\Leftrightarrow \tau \in \sigma H \sigma^{-1} \\ &\Leftrightarrow \tau \in \mathrm{Stab}_\rho(\rho_\sigma(x)) \\ &\Leftrightarrow \rho_{\tau\sigma}(x) = \rho_\sigma(x). \end{aligned}$$

Thus, $\overline{\mathbb{Q}}^H$ is totally imaginary if and only if for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ we have $\rho_{\tau\sigma}(x) \neq \rho_\sigma(x)$, which is by the definition of z_ρ equivalent with $z_\rho(x) = 1/2$. □

The following definition will also be used in the statement of the main theorem.

Definition 6.6.7. Let $\mathcal{G} = (V, E)$ be a graph and $\Lambda_\mathcal{G}$ its set of coherent components. A subset of coherent components $A \subset \Lambda_\mathcal{G}$ will be called *connected* if the underlying set of vertices

$$\bigcup_{\lambda \in A} \lambda$$

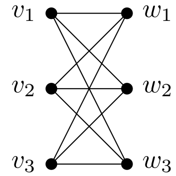
is connected in the graph $\mathcal{G}$ (according to Definition 4.1.3).

The next lemma is a result on the existence of connected subsets of coherent components satisfying a property with respect to a given involution of the quotient graph. This lemma will later be applied to the involution determined by the action of the complex conjugation automorphism on $\overline{\mathcal{G}}$.

Lemma 6.6.8. *Let $\mathcal{G} = (V, E)$ be a graph, $\Lambda_{\mathcal{G}}$ its set of coherent components and $\iota \in \mathrm{Aut}(\overline{\mathcal{G}})$ an involution, i.e. ι satisfies $\iota^2 = \mathrm{Id}$. Take any connected subset $A \subset \Lambda_{\mathcal{G}}$ such that for all $\lambda \in \Lambda_{\mathcal{G}}$ with $\iota(\lambda) \neq \lambda$ it holds that $A \neq \{\lambda, \iota(\lambda)\}$. Then there exists a connected subset $B \subset \Lambda_{\mathcal{G}}$ such that $A \cup \iota(A) = B \cup \iota(B)$ and for any $\lambda \in B$ with $\lambda \neq \iota(\lambda)$ we have $\iota(\lambda) \notin B$.*

Before proving it, we first illustrate it with 2 different examples.

Example 6.6.9. The above lemma requires the set $A \subset \Lambda_{\mathcal{G}}$ to satisfy $A \neq \{\lambda, \iota(\lambda)\}$ for any $\lambda \in \Lambda_{\mathcal{G}}$ with $\lambda \neq \iota(\lambda)$. This condition is indeed necessary by considering the following example. Take any positive integer $n > 1$ and let $V = \{v_1, \ldots, v_n, w_1, \ldots, w_n\}$. Define $E = \{\{v_k, w_l\} \mid 1 \leq k, l \leq n\}$. The resulting graph $\mathcal{G} = (V, E)$ is the *complete biparte graph on $n + n$ vertices* and is drawn below for $n = 3$.

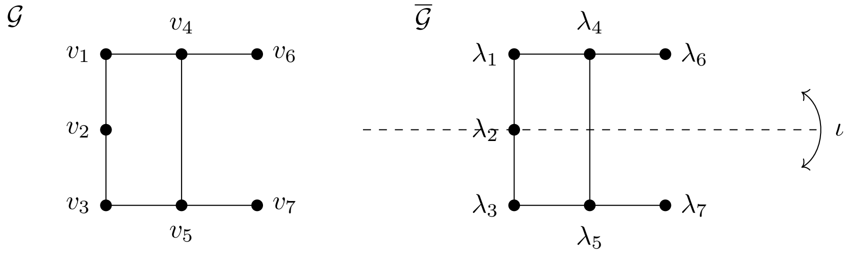


The coherent components are given by $\Lambda_{\mathcal{G}} = \{\lambda_1 = \{v_1, v_2, v_3\}, \lambda_2 = \{w_1, w_2, w_3\}\}$. Note that $A := \{\lambda_1, \lambda_2\}$ is connected since $\lambda_1 \cup \lambda_2 = S$ is a connected set of vertices. Let $\iota \in \mathrm{Aut}(\overline{\mathcal{G}})$ denote the involution defined by $\iota(\lambda_1) = \lambda_2$. It is clear that the above lemma can not be valid for this choice of set A since λ_1 and λ_2 are each not connected subsets in $\mathcal{G}$ and thus $\{\lambda_1\}$ and $\{\lambda_2\}$ are each not a connected subset of coherent components.

Example 6.6.10. To illustrate the conclusion of Lemma 6.6.8, consider the graph $\mathcal{G} = (V, E)$ with $V = \{v_1, \ldots, v_7\}$ and

$$E = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_1, v_4\}, \{v_4, v_6\}, \{v_3, v_5\}, \{v_5, v_7\}, \{v_4, v_5\}\}.$$

The coherent components are simply all the singletons $\Lambda_{\mathcal{G}} = \{\lambda_i = \{v_i\} \mid 1 \leq i \leq 7\}$. Let $\iota \in \mathrm{Aut}(\overline{\mathcal{G}})$ be the involution defined by $\iota(\lambda_2) = \lambda_2$, $\iota(\lambda_1) = \lambda_3$, $\iota(\lambda_4) = \lambda_5$ and $\iota(\lambda_6) = \lambda_7$. The graph and its quotient graph are drawn below together with the involution ι.



Define the subset of coherent components $A = \{\lambda_2, \lambda_3, \lambda_5, \lambda_4, \lambda_6\} \subset \Lambda_{\mathcal{G}}$, which is easily seen to be connected. This set is drawn in Figure 6.1.

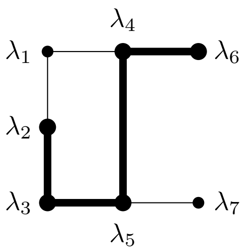


Figure 6.1: The subset A drawn in bold.

Lemma 6.6.8 gives a set B such that $B \cup \iota(B) = A \cup \iota(A)$ and for any $\lambda \in B$ with $\lambda \neq \iota(\lambda)$ it holds that $\iota(\lambda) \notin B$. In this example, such a set B can be given by either $B = \{\lambda_2, \lambda_3, \lambda_5, \lambda_7\}$ or $B = \{\lambda_2, \lambda_1, \lambda_4, \lambda_6\}$. These sets are drawn in Figure 6.2.

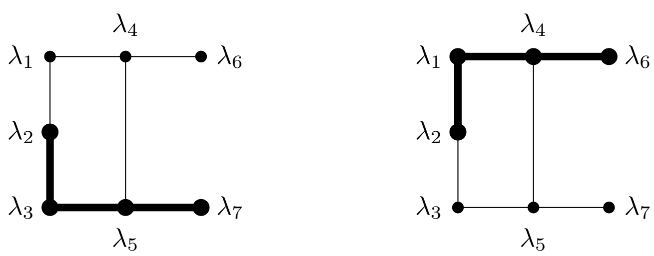


Figure 6.2: The two options for the subset B drawn in bold.

Proof of Lemma 6.6.8: We first prove the cases where A counts 1, 2 or 3 elements and then proceed by induction on $|A|$. If $|A| = 1$, we can just take $B = A$. If $|A| = 2$, the assumptions on A imply that $\iota(\lambda) \neq \lambda$ for any $\lambda \in A$. Thus we can again take $B = A$. If $|A| = 3$, we can assume that there is a coherent component $\lambda \in A$ such that $\iota(\lambda) \neq \lambda$ and $\iota(\lambda) \in A$, otherwise we can just take $B = A$. Without loss of generality we can then write $A = \{\lambda_1, \lambda_2, \lambda_3\}$ with $\iota(\lambda_2) = \lambda_3$. Since A is connected, we must have that either $\{\lambda_1, \lambda_2\} \in \overline{E}$, in which case we take $B = \{\lambda_1, \lambda_2\}$ or that $\{\lambda_1, \lambda_3\} \in \overline{E}$, in which case we take $B = \{\lambda_1, \lambda_3\}$.

Next, assume that $|A| > 3$ and that the theorem holds for lower cardinalities of A. As a basic property of connected subsets of graphs (which also lifts to connected sets of coherent components containing at least 3 coherent components), there exists a coherent component $\lambda \in A$ such that $A' = A \setminus \{\lambda\}$ is still connected. In addition, there must exist an element $\mu \in A'$ such that $\{\lambda, \mu\} \in \overline{E}$ since A was assumed connected. We can apply the induction hypothesis on A' and get a connected set $B' \subset \Lambda$ such that $A' \cup \iota(A') = B' \cup \iota(B')$ and for any $\lambda \in B'$ with $\lambda \neq \iota(\lambda)$ we have $\iota(\lambda) \notin B'$. The following three cases are to be considered:

- $\iota(\lambda) \in A'$. In this case, we have that $A \cup \iota(A) = A' \cup \iota(A') = B' \cup \iota(B')$. As a consequence we find that $B := B'$ satisfies the required properties.
- $\iota(\lambda) \notin A'$ and $\mu \in B'$. Then it follows from $\{\lambda, \mu\} \in \overline{E}$, that $B := B' \cup \{\lambda\}$ is connected. We also have

$$\begin{aligned} B \cup \iota(B) &= B' \cup \{\lambda\} \cup \iota(B' \cup \{\lambda\}) \\ &= B' \cup \iota(B') \cup \{\lambda\} \cup \{\iota(\lambda)\} \\ &= A' \cup \iota(A') \cup \{\lambda\} \cup \{\iota(\lambda)\} \\ &= A' \cup \{\lambda\} \cup \iota(A' \cup \{\lambda\}) \\ &= A \cup \iota(A) \end{aligned}$$

  and $\iota(\lambda) \notin B$ for any $\lambda \in B$ with $\lambda \neq \iota(\lambda)$. Thus B satisfies all required properties.
- $\iota(\lambda) \notin A'$ and $\mu \notin B'$. Then because $\mu \in A' \cup \iota(A') = B' \cup \iota(B')$, we must have $\iota(\mu) \in B'$. Since $\iota \in \mathrm{Aut}(\overline{\mathcal{G}})$ and $\{\lambda, \mu\} \in \overline{E}$, we must have that $\{\iota(\lambda), \iota(\mu)\} \in \overline{E}$. As a consequence we have that $B := B' \cup \{\iota(\lambda)\}$ is

connected. We also have

$$\begin{aligned} B \cup \iota(B) &= B' \cup \{\iota(\lambda)\} \cup \iota(B' \cup \{\iota(\lambda)\}) \\ &= B' \cup \iota(B') \cup \{\lambda\} \cup \{\iota(\lambda)\} \\ &= A' \cup \iota(A') \cup \{\lambda\} \cup \{\iota(\lambda)\} \\ &= A' \cup \{\lambda\} \cup \iota(A' \cup \{\lambda\}) \\ &= A \cup \iota(A) \end{aligned}$$

and $\iota(\lambda) \notin B$ for any $\lambda \in B$ with $\lambda \neq \iota(\lambda)$. Thus B satisfies all required properties.

This concludes the proof. □

Next, we prove a lemma which will help us find eigenvalues equal to ± 1 in vertex-diagonal automorphisms defined by $\Psi : V \to \overline{\mathbb{Q}}$ which satisfies conditions (i) and (iii) of Theorem 6.6.4.

Recall that for any finite set X, group G and morphism $\rho : G \to \mathrm{Perm}(X)$, we say a subset $A \subset X$ is a *ρ-invariant subset* if it is invariant under the ρ-action, i.e. for any $x \in A$ and $g \in G$, we have $\rho_g(x) \in A$. Similarly, a function $f : X \to Y$ (to any set Y) is said to be a *ρ-invariant function* if for any $g \in G$ we have $f = f \circ \rho_g$.

Lemma 6.6.11. *Let X be a finite set equipped with a partition $\mathcal{P}$. Consider a continuous morphism $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}(\mathcal{P})$ satisfying that*

(i) for any $x \in X$, $\Psi(x)$ is an algebraic unit and

(ii) for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ it holds that $\sigma \cdot [\Psi] = [\Psi] \cdot \rho_\sigma$ in $\mathcal{H}_{\mathcal{P}}^{\overline{\mathbb{Q}}}$.

If $A \subset \mathcal{P}$ is a ρ-invariant subset and $f : \mathcal{P} \to \mathbb{N}$ a ρ-invariant function, then

$$\prod_{Y \in A} \prod_{x \in Y} \Psi(x)^{f(Y)} = \pm 1.$$

Proof. Take any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Since Ψ satisfies condition (ii), there exists a $\theta \in \prod_{Y \in \mathcal{P}} \mathrm{Perm}(Y)$ such that $\sigma \circ \Psi = \Psi \circ r(\rho_\sigma) \circ \theta$. We thus have

$$\begin{aligned}
\sigma\left(\prod_{Y\in A}\prod_{x\in Y}\Psi(x)^{f(Y)}\right) &= \prod_{Y\in A}\prod_{x\in Y}(\sigma\circ\Psi)(x)^{f(Y)} \\
&= \prod_{Y\in A}\prod_{x\in Y}(\Psi\circ r(\rho_\sigma)\circ\theta)(x)^{f(Y)} \\
&= \prod_{Y\in A}\prod_{x\in Y}(\Psi\circ r(\rho_\sigma))(x)^{(f\circ\rho_\sigma)(Y)} \\
&= \prod_{Y\in A}\prod_{x\in \rho_\sigma(Y)}\Psi(x)^{f(Y)} \\
&= \prod_{Y\in A}\prod_{x\in Y}\Psi(x)^{f(Y)}.
\end{aligned}$$

This proves that $\left(\prod_{Y\in A}\prod_{x\in Y}\Psi(x)^{f(Y)}\right) \in \mathbb{Q}$. Since Ψ satisfies condition (i), every factor in this product is an algebraic unit and thus the product itself is an algebraic unit. The only algebraic units in $\mathbb{Q}$ are 1 and -1, which proves the claim. $\square$

At last, we prove a lemma which for a graph $\mathcal{G} = (V, E)$ and a continuous morphism $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}(\overline{\mathcal{G}})$ helps us construct a map Ψ on the vertices underneath a single ρ-orbit such that the map satisfies conditions (i) and (iii) of Theorem 6.6.4 on that ρ-orbit and enjoys an additional 'hyperbolicity' property. In the proof of the main theorem we will combine these maps for each orbit to construct a map on the whole vertex set which then also satisfies (ii) of Theorem 6.6.4.

Recall that an action is said to be *transitive* if is has only one orbit. For a transitive ρ-action, we have that the function z_ρ defined in (6.13) is constant. In this case we also refer to this constant value as z_ρ.

Lemma 6.6.12. *Let X be a finite set equipped with a partition $\mathcal{P}$ and $c > 1$ an integer. Consider a continuous morphism $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}(\mathcal{P})$ inducing a transitive ρ-action on $\mathcal{P}$. If $z_\rho \cdot |X| \geq 2$, then there exists a map $\Psi : X \to \overline{\mathbb{Q}}$ such that*

- *(i) for any $x \in X$, $\Psi(x)$ is an algebraic unit with $|\Psi(x)| \neq 1$,*
- *(ii) for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ we have $\sigma \cdot [\Psi] = [\Psi] \cdot \rho_\sigma$ in $\mathcal{H}_{\mathcal{P}}^{\overline{\mathbb{Q}}}$ and*

(iii) for any weight $e : X \to \mathbb{N}$ with $\sum_{x\in X} e(x) \leq c$ it holds that if

$$\left|\prod_{x\in X} \Psi(x)^{e(x)}\right| = 1,$$

then the assignment

$$x \mapsto e(x) + e(r(\rho_\tau)(x))$$

takes a constant value on X. Moreover, if this value is non-zero, it is greater or equal than $2 \cdot z_\rho$.

Proof. Fix some $Y \in \mathcal{P}$. Write $H = \mathrm{Stab}_\rho(Y)$. Note that

$$\left[\overline{\mathbb{Q}}^H : \mathbb{Q}\right] = \left[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) : H\right] = |\mathrm{Orb}_\rho(Y)| = |\mathcal{P}| \tag{6.14}$$

where we used the Galois correspondence for the first equality and the orbit-stabilizer theorem for the second one. The field $\overline{\mathbb{Q}}^H$ is a number field and let s be the number of real embeddings and t the number of conjugated pairs of complex embeddings of $\overline{\mathbb{Q}}^H$ in $\overline{\mathbb{Q}}$. Lemma 6.6.5 now gives us a Galois extension $F\big/\overline{\mathbb{Q}}^H$ of degree $m := |Y|$ such that F has $s \cdot m$ real embeddings and $t \cdot m$ conjugated pairs of complex embeddings. Applying Lemma 6.2.2 on the field F, we get an algebraic unit $\xi \in F$ which satisfies (6.2) for the given $c > 1$. We list the cosets of H in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ as $\sigma_1 H, \dots, \sigma_n H$ where $n := |\mathrm{Orb}_\rho(Y)| = s + 2t$. We also order the cosets of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ in H as $\gamma_1 \mathrm{Gal}(\overline{\mathbb{Q}}/F), \dots, \gamma_m \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ and order the elements of Y as $x_1, \dots, x_m$. Note that $|X| = n \cdot m$. By setting

$$x_{ij} := r(\rho_{\sigma_i})(x_j),$$

we have thus parametrized the elements in X as $X = \{x_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq m\}$. We define the map Ψ as

$$\Psi : X \to \overline{\mathbb{Q}} : x_{ij} \mapsto \sigma_i \gamma_j(\xi).$$

First, we show that Ψ satisfies condition (ii). Take any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Let us prove that there exists a $\theta \in \prod_{Y\in\mathcal{P}} \mathrm{Perm}(Y)$ such that for all $1 \leq i \leq n$, $1 \leq j \leq m$ it holds that $(\sigma \circ \Psi)(x_{ij}) = (\Psi \circ r(\rho_\sigma) \circ \theta)(x_{ij})$. By our listing of the cosets in the Galois group above, it is not hard to see that there exist permutations $a \in \mathrm{Perm}(\{1, \dots, n\})$, $b_i \in \mathrm{Perm}(\{1, \dots, m\})$ and elements $\tilde{\sigma} \in H, \overline{\sigma} \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ such that

$$\sigma\sigma_i = \sigma_{a(i)}\tilde{\sigma} \quad \text{and} \quad \tilde{\sigma}\gamma_j = \gamma_{b_i(j)}\overline{\sigma}$$

for all $1 \leq i \leq n$, $1 \leq j \leq m$. The permutations b_i allow us to define a permutation $\theta \in \prod_{Y\in\mathcal{P}} \mathrm{Perm}(Y)$ by setting $\theta(x_{ij}) = x_{ib_i(j)}$. Note that

$$r(\rho_\sigma)(x_{ij}) = r(\rho_{\sigma\sigma_i})(x_j) = r(\rho_{\sigma_{a(i)}\tilde{\sigma}})(x_j) = x_{a(i)j}$$

since $\tilde{\sigma} \in H = \mathrm{Stab}_\rho(Y)$. Thus, we get that

$$\begin{aligned}(\sigma \circ \Psi)(x_{ij}) &= \sigma\sigma_i\gamma_j(\xi) \\ &= \sigma_{a(i)}\gamma_{b_i(j)}\overline{\sigma}(\xi) \\ &= \sigma_{a(i)}\gamma_{b_i(j)}(\xi) \\ &= \Psi(x_{a(i)b_i(j)}) \\ &= (\Psi \circ r(\rho_\sigma))(x_{ib_i(j)}) \\ &= (\Psi \circ r(\rho_\sigma) \circ \theta)(x_{ij})\end{aligned}$$

where we used that $\overline{\sigma}(\xi) = \xi$ since $\overline{\sigma} \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$. This proves that Ψ satisfies condition (ii).

Next, we show that Ψ satisfies condition (i). Clearly, since ξ is an algebraic unit, so are all its conjugates $\sigma_i\gamma_j(\xi)$, which are exactly all the images of Ψ.

At last, we show Ψ satisfies condition (iii). Take a map $e : X \to \mathbb{N}$ with $\sum_{x\in X} e(x) \leq c$ and assume that

$$\left| \prod_{x\in X} \Psi(x)^{e(x)} \right| = 1.$$

By the construction of Ψ this gives

$$\left| \prod_{i=1}^{n} \prod_{j=1}^{m} \sigma_i\gamma_j(\xi)^{e(x_{ij})} \right| = 1. \tag{6.15}$$

Up to reordering in the index i, all real embeddings of F in $\overline{\mathbb{Q}}$ are given by the restrictions

$$\sigma_i\gamma_j : F \to \overline{\mathbb{Q}} \qquad 1 \leq i \leq s,\ 1 \leq j \leq m,$$

and all conjugated pairs of complex embeddings of F in $\overline{\mathbb{Q}}$ are given by the restrictions

$$\sigma_i\gamma_j : F \to \overline{\mathbb{Q}}, \quad \sigma_{i+t}\gamma_j = \tau\sigma_i\gamma_j : F \to \overline{\mathbb{Q}}, \quad s+1 \leq i \leq s+t,\ 1 \leq j \leq m$$

where $\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is the complex conjugation automorphism. Given this new ordering of the σ_i's, it is not hard to check that

$$\rho_\tau(x_{ij}) = x_{ij} \quad \Leftrightarrow \quad 1 \leq i \leq s$$

and that for any $s+1 \leq i \leq s+t$ we have:

$$x_{i+t\,j} = r(\rho_\tau)(x_{ij}).$$

Equation (6.15) can thus be rewritten as

$$1 = \left| \prod_{j=1}^{m} \left(\prod_{i=1}^{s} \sigma_i \gamma_j(\xi)^{e(x_{ij})} \cdot \prod_{i=s+1}^{s+t} \sigma_i \gamma_j(\xi)^{e(x_{ij})} \cdot \tau\sigma_i \gamma_j(\xi)^{e(r(\rho_\tau)(x_{ij}))} \right) \right|$$

$$= \left| \prod_{j=1}^{m} \left(\prod_{i=1}^{s} \sigma_i \gamma_j(\xi)^{e(x_{ij})} \cdot \prod_{i=s+1}^{s+t} \sigma_i \gamma_j(\xi)^{e(x_{ij})+e(r(\rho_\tau)(x_{ij}))} \right) \right|$$

where we use in the second equality that complex conjugation does not affect the absolute value of a complex number. After observing that $2e(x_{ij}) = e(x_{ij}) + e(r(\rho_\tau)(x_{ij}))$ for all $1 \leq i \leq s$, we now find immediately from the way ξ was constructed using Lemma 6.2.2, that the assignment

$$x \mapsto e(x) + e(r(\rho_\tau)(x))$$

is constant on X. Let us write k for this constant value and assume that $k > 0$. Using Lemma 6.6.6, we find that

$$z_\rho(Y) = \frac{1}{2} \quad \Leftrightarrow \quad \overline{\mathbb{Q}}^H \text{ is totally imaginary} \quad \Leftrightarrow \quad s = 0.$$

In case $z_\rho(Y) = \frac{1}{2}$, there is really nothing to prove since evidently $k \geq 1 = 2z_\rho(Y)$. If $z_\rho(Y) = 1$, then $s > 0$ and thus there exists an $x \in X$ with $r(\rho_\tau)(x) = x$, implying that $k = e(x) + e(r(\rho_\tau)(x)) = 2e(x) \geq 2z_\rho(Y)$. This concludes the proof.

□

We are now ready to prove the main theorem of section 6.6.

Theorem 6.6.13. *Let $\mathcal{G} = (V, E)$ be a simple undirected graph, $c > 1$ an integer and $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}(\overline{\mathcal{G}})$ a continuous morphism. The associated rational form $\mathfrak{n}^{\mathbb{Q}}(\rho, c)$ of $\mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c)$ is Anosov if and only if for each non-empty connected set of coherent components $A \subset \Lambda_{\mathcal{G}}$ such that $A \cup \rho_\tau(A)$ is ρ-invariant, it holds that*

$$c < \sum_{\lambda \in A \cup \rho_\tau(A)} z_\rho(\lambda) \cdot |\lambda|. \tag{6.16}$$

Proof. $\boxed{\Rightarrow}$ Assume that $\mathfrak{n}^{\mathbb{Q}}(\rho, c)$ is Anosov. By Theorem 6.6.4, there exists a map $\Psi : V \to \overline{\mathbb{Q}}$ which satisfies conditions (i), (ii) and (iii) from that theorem.

Take any non-empty connected subset of coherent components $A \subset \Lambda_{\mathcal{G}}$ such that $A \cup \rho_\tau(A)$ is ρ-invariant. We will prove that the inequality (6.16) holds.

First assume that A is not of the form $\{\lambda, \rho_\tau(\lambda)\}$ for some $\lambda \in \Lambda_{\mathcal{G}}$ with $\rho_\tau(\lambda) \neq \lambda$. By Lemma 6.6.8, there exists a connected set of coherent components $B \subset \Lambda_{\mathcal{G}}$ such that $A \cup \rho_\tau(A) = B \cup \rho_\tau(B)$ and for any $\lambda \in B$ with $\lambda \neq \rho_\tau(\lambda)$ we have $\rho_\tau(\lambda) \notin B$. Define the function

$$g : B \to \{1, 2\} : \lambda \mapsto \begin{cases} 1 & \text{if } z_\rho(\lambda) = 1/2 \\ 2 & \text{if } z_\rho(\lambda) = 1 \text{ and } \rho_\tau(\lambda) \neq \lambda \\ 1 & \text{if } z_\rho(\lambda) = 1 \text{ and } \rho_\tau(\lambda) = \lambda. \end{cases}$$

and note that it satisfies the equality

$$\sum_{\lambda \in B} g(\lambda) = \sum_{\lambda \in A \cup \rho_\tau(A)} z_\rho(\lambda).$$

Consider the algebraic unit

$$\zeta = \prod_{\lambda \in B} \prod_{v \in \lambda} \Psi(v)^{g(\lambda)}. \tag{6.17}$$

We now prove that $|\zeta| = 1$. Write $X = z_\rho^{-1}(1)$ and $Y = z_\rho^{-1}(1/2)$, then $X \sqcup Y = \Lambda_{\mathcal{G}}$ and X is exactly equal to the set of coherent components for which their ρ-orbit contains a fixed point of ρ_τ. Since Ψ satisfies (iii) of Theorem 6.6.4, there exists a $\theta \in \prod_{\lambda \in \Lambda_{\mathcal{G}}} \operatorname{Perm}(\lambda)$ such that $\tau \circ \Psi = \Psi \circ r(\rho_\tau) \circ \theta$. We then have

$$\begin{aligned}
|\zeta|^2 = \zeta\bar{\zeta} &= \prod_{\lambda \in B} \prod_{v \in \lambda} \Psi(v)^{g(\lambda)} \overline{\Psi(v)}^{g(\lambda)} \\
&= \prod_{\lambda \in B} \prod_{v \in \lambda} \Psi(v)^{g(\lambda)} (\Psi \circ r(\rho_\tau) \circ \theta)(v)^{g(\lambda)} \\
&= \prod_{\lambda \in B} \prod_{v \in \lambda} \Psi(v)^{g(\lambda)} (\Psi \circ r(\rho_\tau))(v)^{g(\lambda)} \\
&= \left(\prod_{\lambda \in B \cap X} \prod_{v \in \lambda} \Psi(v)^{g(\lambda)} (\Psi \circ r(\rho_\tau))(v)^{g(\lambda)} \right) \\
&\qquad \cdot \left(\prod_{\lambda \in B \cap Y} \prod_{v \in \lambda} \Psi(v)^{g(\lambda)} (\Psi \circ r(\rho_\tau))(v)^{g(\lambda)} \right) \\
&= \left(\prod_{\substack{\lambda \in B \cup \rho_\tau(B) \\ \lambda \in X}} \prod_{v \in \lambda} \Psi(v)^2 \right) \cdot \left(\prod_{\substack{\lambda \in B \cup \rho_\tau(B) \\ \lambda \in Y}} \prod_{v \in \lambda} \Psi(v) \right) \\
&= \prod_{\lambda \in B \cup \rho_\tau(B)} \prod_{v \in \lambda} \Psi(v)^{2 z_\rho(\lambda)} \\
&= \prod_{\lambda \in A \cup \rho_\tau(A)} \prod_{v \in \lambda} \Psi(v)^{2 z_\rho(\lambda)}.
\end{aligned}$$

This last product satisfies all requirements to apply Lemma 6.6.11 for the partition $\mathcal{P} = \Lambda_{\mathcal{G}}$ and thus we find that $|\zeta|^2 = \pm 1$ which implies $|\zeta| = 1$. Using that B is connected and that Ψ satisfies condition (ii) of Theorem 6.6.4, we must thus have that the number of factors in the product in (6.17) is strictly greater than c. The number of factors can be calculated as:

$$\sum_{\lambda \in B} \sum_{v \in \lambda} g(\lambda) = \sum_{\lambda \in B \cup \rho_\tau(B)} z_\rho(\lambda) \cdot |\lambda| = \sum_{\lambda \in A \cup \rho_\tau(A)} z_\rho(\lambda) \cdot |\lambda|$$

which proves that $c < \displaystyle\sum_{\lambda \in A \cup \rho_\tau(A)} z_\rho(\lambda) \cdot |\lambda|$.

Now assume that there exists a $\mu \in \Lambda_{\mathcal{G}}$ such that $A = \{\mu, \rho_\tau(\mu)\}$ and $\mu \neq \rho_\tau(\mu)$. Note that this implies that A is ρ-invariant. Let $\theta \in \prod_{\lambda \in \Lambda_{\mathcal{G}}} \operatorname{Perm}(\lambda)$ be the permutation such that $\tau \circ \Psi = \Psi \circ r(\rho_\tau) \circ \theta$. Choose a vertex $u \in \mu$ and define $u' = r(\rho_\tau)(\theta(u))$. Since A is connected, we have that $\{\mu, \rho_\tau(\mu)\} \in \overline{E}$ and thus that $\{u', v\} \in E$ for any $v \in \mu$. This implies that the set of vertices $\{u'\} \cup (\mu \setminus \{u\})$ is connected in $\mathcal{G}$. Note that

$$\begin{aligned}
\left| \Psi(u') \cdot \prod_{v \in \mu \setminus \{u\}} \Psi(v) \right|^2 &= \left| (\Psi \circ r(\rho_\tau) \circ \theta)(u) \cdot \prod_{v \in \mu \setminus \{u\}} \Psi(v) \right|^2 \\
&= \left| \overline{\Psi(u)} \cdot \prod_{v \in \mu \setminus \{u\}} \Psi(v) \right|^2 \\
&= \prod_{v \in \mu} \Psi(v) \overline{\Psi(v)} \\
&= \prod_{v \in \mu} \Psi(v) \prod_{w \in \rho_\tau(\mu)} \Psi(w) \\
&= 1
\end{aligned}$$

where the last equation follows from applying Lemma 6.6.11. Since $z_\rho(\mu) = z_\rho(\rho_\tau(\mu)) = 1/2$, it follows from condition (ii) of Theorem 6.6.4 that

$$c < |\{u'\} \cup (\mu \setminus \{u\})| = |\mu| = \sum_{\lambda \in A} z_\rho(\lambda) \cdot |\lambda|,$$

which completes the proof of the 'only if' direction.

$\boxed{\Leftarrow}$ Conversely, assume that for any non-empty connected subset of coherent components $A \subset \Lambda_{\mathcal{G}}$ such that $A \cup \rho_\tau(A)$ is ρ-invariant, inequality (6.16) holds. In what follows, we will construct a map $\Psi : V \to \overline{\mathbb{Q}}$ which satisfies conditions (i), (ii) and (iii) from Theorem 6.6.4, proving that $\mathfrak{n}^{\mathbb{Q}}(\rho, c)$ is Anosov.

Let us list all the ρ-orbits as $\mathrm{Orb}_\rho(\lambda_1), \ldots, \mathrm{Orb}_\rho(\lambda_l)$ such that $\Lambda_{\mathcal{G}} = \bigsqcup_{i=1}^{l} \mathrm{Orb}_\rho(\lambda_i)$ and define the underlying sets of vertices

$$V_i := \bigcup_{\mu \in \mathrm{Orb}_\rho(\lambda_i)} \mu$$

such that also $V = \sqcup_{i=1}^{l} V_i$.

We claim that $z_\rho(\lambda_i)|V_i| \geq 2$ for all $1 \leq i \leq l$. Indeed, if $z_\rho(\lambda_i) = 1$ and $|V_i| < 2$, then $\mathrm{Orb}_\rho(\lambda_i)$ consists of a single coherent component which counts a single vertex and thus is a connected subset of coherent components. The assumption then tells us that

$$\sum_{\lambda \in \mathrm{Orb}_\rho(\lambda_i)} |\lambda| = 1 > c,$$

but since c is assumed > 1, we get a contradiction. If $z_\rho(\lambda_i) = \frac{1}{2}$ and $|V_i| < 4$, then per definition of z_ρ we have that ρ_τ had no fixed points on $\mathrm{Orb}_\rho(\lambda_i)$. As a consequence, we must have that $|V_i| = 2$ and $|\lambda_i| = |\rho_\tau(\lambda_i)| = 1$. Since λ_i is a singleton, $\{\lambda_i\}$ is a connected subset of coherent components with $\{\lambda_i\} \cup \rho_\tau(\{\lambda_i\}) = \mathrm{Orb}_\rho(\lambda_i)$ being ρ-invariant. Thus, the assumption tells us that

$$\sum_{\lambda \in \mathrm{Orb}_\rho(\lambda_i)} \frac{1}{2}|\lambda| = \frac{1}{2} + \frac{1}{2} > c,$$

but since c is assumed > 1, we get again a contradiction. Thus, we must conclude that $z_\rho(\lambda_i)|V_i| \geq 2$ for all $1 \leq i \leq l$.

As a consequence, for each $1 \leq i \leq l$, we can apply Lemma 6.6.12 to the ρ-action restricted to the partition $\mathrm{Orb}_\rho(\lambda_i)$ of V_i. Thus, we obtain for each $1 \leq i \leq l$ a map $\Psi_i : V_i \to \overline{\mathbb{Q}}$ which satisfies the properties as described in the statement of that lemma. Next, we combine these maps and define for any tuple of integers $\vec{N} = (N_1, \ldots, N_l) \in \mathbb{Z}^l$ the map

$$\Psi_{\vec{N}} : V \to \overline{\mathbb{Q}} : v \mapsto \Psi_i(v)^{N_i} \quad \text{for} \quad v \in V_i.$$

From the properties that every Ψ_i possesses ((i) and (ii) of Lemma 6.6.12), it is straightforward to check that the map $\Psi_{\vec{N}}$ satisfies conditions (i) and (iii) of Theorem 6.6.4 for any $\vec{N} \in \mathbb{Z}^l$. To finish the proof we will choose an $\vec{N} \in \mathbb{Z}^l$ such that condition (ii) of Theorem 6.6.4 holds as well.

For any weight $e : V \to \mathbb{N}$ define the additive group morphism

$$\varphi_e : \mathbb{Z}^l \to \mathbb{R} : \vec{N} \mapsto \log \left| \prod_{v \in V} \Psi_{\vec{N}}(v)^{e(v)} \right|$$

and define the set of weights

$$\mathcal{E}(\mathcal{G}, c) = \{e : V \to \mathbb{N} \mid \mathrm{supp}(e) \text{ is connected in } \mathcal{G} \text{ and } 0 < \sum_{v \in V} e(v) \leq c\}.$$

We will want to choose $\vec{N}$ outside of the set $\bigcup_{e \in \mathcal{E}(\mathcal{G},c)} \ker(\varphi_e)$. For this to be possible, we need that the morphisms φ_e are non-trivial for any $e \in \mathcal{E}(\mathcal{G}, c)$. Let us prove this by contradiction.

Take any $e \in \mathcal{E}(\mathcal{G}, c)$ and assume that φ_e is the trivial morphism. First, note that $\operatorname{supp}(e)$ then has to count at least 2 elements. Indeed, if $\operatorname{supp}(e)$ were a singleton, say $\operatorname{supp}(e) = \{v\}$ with $v \in V_i$, then the image of φ_e would be equal to

$$\left\{\log \left|\Psi_i(v)^{n \cdot e(v)}\right| \;\middle|\; n \in \mathbb{Z}\right\} = \left\{n \cdot e(v) \cdot \log |\Psi_i(v)| \;\middle|\; n \in \mathbb{Z}\right\}.$$

This set can clearly not be equal to $\{0\}$ since $|\Psi_i(v)| \neq 1$ (see (i) of Lemma 6.6.12). Next, choose any integer $i \in \{1, \ldots, l\}$. Let $\vec{N_i} = (0, \ldots, 0, 1, 0, \ldots, 0)$ be the l-tuple with a one on the i-th entry and 0's elsewhere. Since φ_e is assumed to be the trivial morphism, we get $\varphi_e(\vec{N_i}) = 0$. This exactly means that

$$\left|\prod_{v \in V_i} \Psi_i(v)^{e(v)}\right| = 1,$$

and thus, since Ψ_i satisfies property (iii) of Lemma 6.6.12, the assignment $v \mapsto e(v) + e(r(\rho_\tau)(v))$ is constant on V_i. Moreover, if we write this constant value as k_i, we have $k_i/2 \geq z_\rho(\lambda_i)$ whenever $k_i > 0$. Doing this for any $i \in \{1, \ldots, l\}$, we get that for the set

$$A := \{\lambda \in \Lambda_{\mathcal{G}} \mid \lambda \cap \operatorname{supp}(e) \neq \emptyset\}$$

it holds that $A \cup \rho_\tau(A)$ is ρ-invariant. Since $\operatorname{supp}(e)$ counts at least 2 elements and is connected in $\mathcal{G}$, it is clear that $A \subset \Lambda_{\mathcal{G}}$ is non-empty and connected. By the hypothesis of the theorem we must thus have that

$$c < \sum_{\lambda \in A \cup \rho_\tau(A)} z_\rho(\lambda) \cdot |\lambda|.$$

Since $A \cup \rho_\tau(A)$ is ρ-invariant, there exists a subset $I \subset \{1, \ldots, l\}$ such that $A \cup \rho_\tau(A) = \bigcup_{i \in I} \operatorname{Orb}_\rho(\lambda_i)$. Note that the set I consists exactly of those indices i for which $k_i > 0$. We then have that

$$c < \sum_{\lambda \in A \cup \rho_\tau(A)} z_\rho(\lambda) \cdot |\lambda| \leq \sum_{i \in I} \sum_{\mu \in \operatorname{Orb}_\rho(\lambda_i)} \frac{k_i}{2} \cdot |\mu| = \sum_{v \in V} e(v)$$

which is in contradiction with $\sum_{v \in V} e(v) \leq c$. This proves that φ_e is not the trivial morphism for any $e \in \mathcal{E}(\mathcal{G}, c)$ and thus that $\ker(\varphi_e) \neq \mathbb{Z}^l$ for any $e \in \mathcal{E}(\mathcal{G}, c)$.

Note that since φ_e maps into the torsion-free group $(\mathbb{R}, +)$, the quotient $\mathbb{Z}^l / \ker(\varphi_e)$ is torsion-free as well. Thus for any $e \in \mathcal{E}(\mathcal{G}, c)$ we have that the free abelian group $\ker(\varphi_e)$ has rank at most $l-1$. From this observation, it follows that

$$B := \bigcup_{e \in \mathcal{E}(\mathcal{G}, c)} \ker(\varphi_e)$$

is not equal to $\mathbb{Z}^l$ since $\mathcal{E}(\mathcal{G}, c)$ is a finite set. Thus, there exists an element $\vec{M} \in \mathbb{Z}^l \setminus B$. As one can check $\Psi := \Psi_{\vec{M}}$ now satisfies also condition (ii) of Theorem 6.6.4. This concludes the proof. $\square$

Analogously as we did with Theorem 5.6.4 from the classification of rational forms of $\mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c)$, we can also reformulate Theorem 6.6.13 by considering finite degree Galois extensions $K/\mathbb{Q}$ and injective group morphisms $\rho : \mathrm{Gal}(K/\mathbb{Q}) \to \mathrm{Aut}(\overline{\mathcal{G}})$. Note that any such morphism corresponds uniquely to an extended morphism $\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}(\overline{\mathcal{G}})$. Since the orbit structure of ρ and $\overline{\rho}$ coincide, we obtain the following statement.

Theorem 6.6.14 (Injective version)**.** *Let $\mathcal{G} = (V, E)$ be a simple undirected graph, $K/\mathbb{Q}$ a finite degree Galois extension and $\rho : \mathrm{Gal}(K/\mathbb{Q}) \to \mathrm{Aut}(\overline{\mathcal{G}})$ an injective group morphism. The associated rational form $\mathfrak{n}^{\mathbb{Q}}(\rho, c)$ of $\mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c)$ is Anosov if and only if for each non-empty connected set of coherent components $A \subset \Lambda_{\mathcal{G}}$ such that $A \cup \rho_{\tau}(A)$ is ρ-invariant, it holds that*

$$c < \sum_{\lambda \in A \cup \rho_{\tau}(A)} z_{\rho}(\lambda) \cdot |\lambda|.$$

If we only consider rational forms of the real Lie algebra $\mathfrak{n}^{\mathbb{R}}(\mathcal{G}, c)$, then ρ_{τ} is always the identity and thus the above theorem simplifies to the following statement.

Corollary 6.6.15 (Real version)**.** *Let $\rho : \mathrm{Gal}(K/\mathbb{Q}) \to \mathrm{Aut}(\overline{\mathcal{G}})$ be an injective group morphism, where $K/\mathbb{Q}$ is a real finite degree Galois extension. The associated rational form $\mathfrak{n}^{\mathbb{Q}}(\mathcal{G}, \rho)$ of the real Lie algebra $\mathfrak{n}^{\mathbb{R}}(\mathcal{G}, c)$ is Anosov if and only if for any non-empty connected ρ-invariant subset $A \subset \Lambda_{\mathcal{G}}$, it holds that*

$$c < \sum_{\lambda \in A} |\lambda|.$$

We can also solely look at the standard rational form $\mathfrak{n}^{\mathbb{Q}}(\mathcal{G}, c)$ which corresponds to the trivial morphism $\rho : \mathrm{Gal}(\mathbb{Q}/\mathbb{Q}) \to \mathrm{Aut}(\overline{\mathcal{G}}) : \mathrm{Id} \mapsto \mathrm{Id}$. A condition for $\mathfrak{n}^{\mathbb{Q}}(\mathcal{G}, c)$ to be Anosov can then be formulated as follows. Recall from Section 4.1 that the edges of the quotient graph where defined as the set

$$\overline{E} = \{\{\lambda, \mu\} \mid \lambda, \mu \in \Lambda_{\mathcal{G}} : \exists v \in \lambda : \exists w \in \mu : \{v, w\} \in E\}.$$

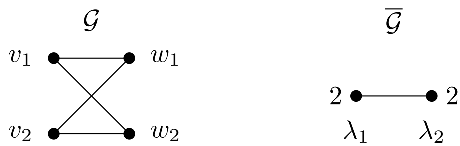
In particular, the quotient graph can contain loops which are elements in $\overline{E}$ of the form $\{\lambda\}$ for a $\lambda \in \Lambda_{\mathcal{G}}$.

Corollary 6.6.16 (Standard rational form)**.** *Let $\mathcal{G} = (V, E)$ be a non-empty graph and $\mathfrak{n}^{\mathbb{Q}}(\mathcal{G}, c)$ the associated rational c-step nilpotent Lie algebra. Then*

$\mathfrak{n}^{\mathbb{Q}}(\mathcal{G}, c)$ is Anosov if and only if $|\lambda| > 1$ for any $\lambda \in \Lambda_{\mathcal{G}}$ and for any edge in the quotient graph $e \in \overline{E}$ (possibly a loop) we have $\sum_{\lambda \in e} |\lambda| > c$.

Remark 6.6.17 (Correction to a result of [Mai06])**.** We like to note that the above corollary is a correction to the characterization stated in [Mai06, Theorem 4.3.], which only requires the condition of Corollary 6.6.16 to hold on loops in the quotient graph. We present an example to show that [Mai06, Theorem 4.3.] is false.

Let $\mathcal{G}$ be the graph from Example 6.6.9 for $n = 2$. The graph and its reduced graph are drawn below.



The false result in [Mai06] claims that $\mathfrak{n}^{\mathbb{Q}}(\mathcal{G}, c)$ is Anosov for any integer $c > 1$, while in fact this Lie algebra is only Anosov for $c < 4$. The problem in the proof of [Mai06, Theorem 4.3.] lies with the eigenvectors arising from Lie brackets of vertices lying in different coherent components. In this example, the Lie bracket $[v_1, [w_1, [v_2, w_2]]]$ is non-zero in $\mathfrak{n}^{\mathbb{Q}}(\mathcal{G}, c)$ with $c \geq 4$ and will be an eigenvector with eigenvalue ± 1 for any vertex-diagonal integer-like automorphism of $\mathfrak{n}^{\mathbb{Q}}(\mathcal{G}, c)$. This essentially proves $\mathfrak{n}^{\mathbb{Q}}(\mathcal{G}, c)$ is not Anosov for $c \geq 4$, but a detailed proof is provided by the main results above, in particular Corollary 6.6.16.

The above two corollaries show that from all rational forms in $\mathfrak{n}^{\mathbb{R}}(\mathcal{G}, c)$ that can be Anosov, the standard one $\mathfrak{n}^{\mathbb{Q}}(\mathcal{G}, c)$ leads to the strongest condition.

Corollary 6.6.18. *Let $\mathcal{G}$ be a simple undirected graph and $\mathfrak{n}^{\mathbb{Q}}(\mathcal{G}, c)$ the associated c-step nilpotent rational Lie algebra. If $\mathfrak{n}^{\mathbb{Q}}(\mathcal{G}, c)$ admits an Anosov automorphism, then so does any other rational form of the real Lie algebra $\mathfrak{n}^{\mathbb{R}}(\mathcal{G}, c)$.*

Remark 6.6.19. Note that Corollary 6.6.18 does not generalize to the rational forms of the complex Lie algebra $\mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c)$. Example 6.6.27 will illustrate this.

6.6.3 Examples and applications

In this section we illustrate how easy it is to apply our main result for some families of graphs. In particular, certain classifications in low dimension, as given in [LW09] are an immediate consequence of Theorem 6.6.14. First, let us give a concrete example to make the reader more familiar with checking the condition of Theorem 6.6.14.

Example 6.6.20. Consider the graph $\mathcal{G}$ as drawn together with its quotient graph in Figure 6.3. It has 4 coherent components which are singletons and 2 coherent components which count 2 elements. The sizes of the coherent components are also depicted in the figure.

Figure 6.3: The graph $\mathcal{G}$ and its quotient graph $\overline{\mathcal{G}}$ with the size of each coherent coherent component depicted.

Let us consider the Galois group of the extension $\mathbb{Q}(\sqrt{2}, i)/\mathbb{Q}$. It is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ and is generated by σ and τ which are defined by $\sqrt{2} \mapsto -\sqrt{2}, i \mapsto i$ and $\sqrt{2} \mapsto \sqrt{2}, i \mapsto -i$, respectively. Note that τ is the complex conjugation automorphism. Let us consider 2 injective group morphisms from this Galois group to $\mathrm{Aut}(\overline{\mathcal{G}})$, resulting in the two different actions as drawn in Figure 6.4. Both actions give rise to different functions z_ρ of which the values are also depicted in the figure.

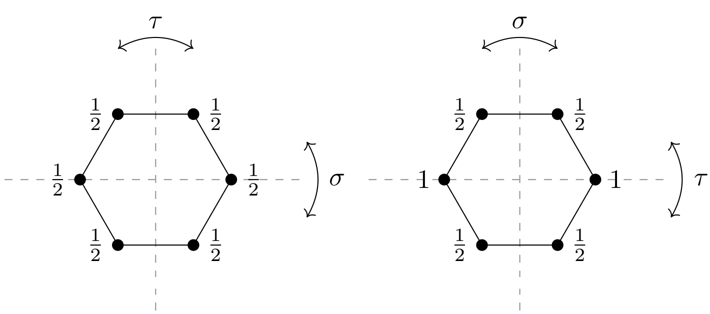


Figure 6.4: Two actions of $\mathrm{Gal}(\mathbb{Q}(\sqrt{2}, i)/\mathbb{Q})$ on $\overline{\mathcal{G}}$ with the associated values of z_ρ at each coherent component. Both σ and τ act as reflections on the graph.

For both actions, there is a choice of set A of coherent components which will result in the strongest condition on the nilpotency class c, following Theorem 6.6.14. For both actions such a set A is drawn in Figure 6.5 below. For the first action, it results in the condition

$$c < 2 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 4.$$

For the second action, we get the condition

$$c < \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 2.$$

Thus, for the first action, the associated rational form will be Anosov for $c = 2, 3$, while for the second action the associated rational form will not be Anosov for any $c > 1$.

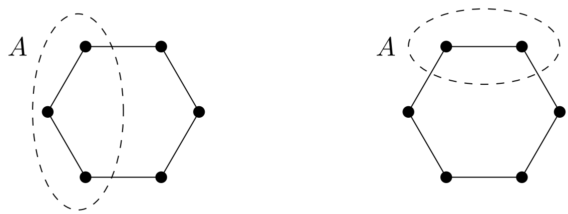


Figure 6.5: Choice of set $A \subset \Lambda$ for the two actions from Figure 6.4.

Second, recall from Theorem 4.1.11 that almost all unlabelled graphs have a trivial automorphism group. In particular, these graphs have coherent components of size 1 and also trivial automorphism group of the quotient graph. Using Theorem 5.6.4 and Theorem 6.6.14 we thus immediately get the following result.

Theorem 6.6.21. *Fixing an integer $c > 1$, the Lie algebra $\mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c)$ has an (up to $\mathbb{Q}$-isomorphism) unique rational form which is not Anosov for almost all unlabelled graphs $\mathcal{G}$.*

This shows that having an Anosov rational form is a rare condition for Lie algebras associated to graphs and raises the question whether a similar statement holds for nilpotent Lie algebras in general.

Next, let us apply Theorem 6.6.13 to certain classes of simple undirected graphs, a first of which is *trees*. Recall that a tree is a connected graph with no cycles (see Definition 4.1.4). For two vertices v, w in a connected graph $\mathcal{G} = (V, E)$, let

$d(v, w)$ denote the distance between v and w, given by the minimal number of edges needed to go from v to w in the graph. The *eccentricity* $e(v)$ of a vertex v is defined as $e(v) = \max\{d(v, w) \mid w \in V\}$. The *center* of $\mathcal{G}$ is then defined as the set of vertices of $\mathcal{G}$ which have minimal eccentricity. It is a standard result that a tree has a center consisting of either one vertex or two adjacent vertices. To illustrate this, two trees with their center are draw in Figure 6.6 and 6.7.

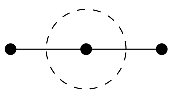


Figure 6.6: A tree with a center consisting of one vertex. The center is represented by the vertices inside the dashed loop.

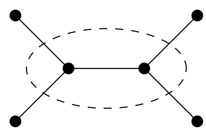


Figure 6.7: A tree with a center consisting of two adjacent vertices. The center is represented by the vertices inside the dashed loop.

Proposition 6.6.22. *If $\mathcal{G}$ is a tree, then $\mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c)$ has no Anosov rational forms for any $c > 1$.*

Proof. Take an arbitrary continuous morphism $\rho : \mathrm{Gal}(L/\mathbb{Q}) \to \mathrm{Aut}(\overline{\mathcal{G}})$. Let $C \subset V$ be the center of $\mathcal{G} = (V, E)$. Note that vertices in the same coherent component can be mapped onto each other by an automorphism of $\mathcal{G}$ and must thus have the same eccentricity. As a consequence, the center is a union of coherent components invariant under $\mathrm{Aut}(\mathcal{G})$. Since $\mathcal{G}$ is a tree, we are left with three cases:

- $|C| = 1$. Note that C is itself a coherent component and must be preserved under any automorphism of $\overline{\mathcal{G}}$, implying that $\{C\}$ is ρ-invariant non-empty and connected. Since $|C| = 1 < c$, we get by Theorem 6.6.14 that $\mathfrak{n}^{\mathbb{Q}}(\rho, c)$ is not Anosov.
- $|C| = 2$ and C is a coherent component. Since the coherent component C is preserved under any automorphism of $\overline{\mathcal{G}}$, we get that $\{C\}$ is ρ-invariant and non-empty. If the center of a tree contains two vertices, then they are adjacent. As a consequence $\{C\}$ is connected as well. Since $|C| = 2 \leq c$, Theorem 6.6.14 tells us that $\mathfrak{n}^{\mathbb{Q}}(\rho, c)$ is not Anosov.
- $|C| = 2$ and C is a union of two disjoint coherent components. Let us write $C = \lambda \cup \mu$ with λ and μ disjoint coherent components. Note that $\{\lambda, \mu\} \in \overline{E}$ and thus that $\{\lambda, \mu\}$ is a non-empty connected set of coherent components. Since the center is preserved under any automorphism of $\overline{\mathcal{G}}$, we get that $\{\lambda, \mu\}$ is ρ-invariant as well. Depending on whether ρ_τ fixes

λ or not, we get that $z_\rho(\lambda)|\lambda| + z_\rho(\mu)|\mu|$ is equal to 1 or 2, respectively. In either case, Theorem 6.6.14 tells us that $\mathfrak{n}^{\mathbb{Q}}(\rho, c)$ is not Anosov.

This concludes the proof. □

As a second class, let us consider the *cycle graphs*. These graphs can be considered as the simplest graphs which are not trees and therefore the natural class to consider next. The cycle graph of size n is given by vertices $V = \{1, \ldots, n\}$ and edges $E = \{\{1,2\}, \{2,3\}, \ldots, \{n-1, n\}, \{n, 1\}\}$. If $n \geq 5$, then the coherent components are all singletons as illustrated below in Figure 6.8 for $n = 6$. It follows that for $n \geq 5$, the automorphism group of the quotient graph is isomorphic to the dihedral group of order $2n$. Let a be a generator of the rotation subgroup of $\mathrm{Aut}(\overline{\mathcal{G}})$ and b a reflection of $\mathrm{Aut}(\overline{\mathcal{G}})$. Then $\mathrm{Aut}(\overline{\mathcal{G}}) = \{\mathrm{Id}, a, \ldots, a^{n-1}, b, ab, \ldots a^{n-1}b\}$. Let us call a rational form of $\mathfrak{n}^{\mathbb{R}}(\mathcal{G}, c)$ of *reflection type* if the corresponding representation $\mathrm{Gal}(L/\mathbb{Q}) \to \mathrm{Aut}(\overline{\mathcal{G}})$ has image $\{\mathrm{Id}, a^i b\}$ for some $1 \leq i \leq n$. Then using Corollary 6.6.15, it is not hard to prove following statement.

Figure 6.8: The cycle graph on 6 vertices and its quotient graph.

Proposition 6.6.23. *Let $\mathcal{G}$ be a cycle graph of size $n \geq 5$ and take any $c > 1$. The standard rational form $\mathfrak{n}^{\mathbb{Q}}(\mathcal{G}, c)$ and all reflection-type rational forms of $\mathfrak{n}^{\mathbb{R}}(\mathcal{G}, c)$ are not Anosov. The other rational forms of $\mathfrak{n}^{\mathbb{R}}(\mathcal{G}, c)$ are Anosov if and only if $n > c$.*

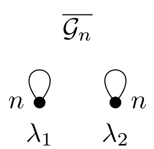
Proof. Since the coherent components of $\mathcal{G}$ are all singletons, it follows by Corollary 6.6.16 that $\mathfrak{n}^{\mathbb{Q}}(\mathcal{G}, c)$ is not Anosov. If $\mathfrak{n}^{\mathbb{Q}}(\rho, c)$ is a reflection-type rational form of $\mathfrak{n}^{\mathbb{R}}(\mathcal{G}, c)$, then after possibly conjugating the ρ-action by an automorphism of the quotient graph, either $\{\{1\}\}$ or $\{\{1\}, \{2\}\}$ is ρ-invariant. In any case Corollary 6.6.15 tells us that $\mathfrak{n}^{\mathbb{Q}}(\rho, c)$ is not Anosov. If $\mathfrak{n}^{\mathbb{Q}}(\rho, c)$ is any other rational form of $\mathfrak{n}^{\mathbb{R}}(\mathcal{G}, c)$, then the image of ρ must contain a non-trivial rotation. Now suppose $A \subset \Lambda_{\mathcal{G}}$ is a non-empty connected ρ-invariant subset of coherent components. Since A must be connected, we get that up to possibly conjugating the action of ρ by an automorphism of the quotient graph, that $A = \{\{1\}, \ldots, \{k\}\}$ for some $1 \leq k \leq n$. Using that A is also ρ-invariant, we know it must be preserved under the non-trivial rotation. This is only possible if $k = n$ implying that $A = \Lambda_{\mathcal{G}} = \{\{1\}, \ldots, \{n\}\}$ and $\sum_{\lambda \in A} |\lambda| = n$. By Corollary 6.6.15 we get that $\mathfrak{n}^{\mathbb{R}}(\rho, c)$ is Anosov if and only if $n > c$. □

Proposition 6.6.24. *If $\mathcal{G}$ is a graph for which there is a non-negative integer $k \geq 0$ such that $\mathcal{G}$ has a unique vertex of degree k, then $\mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c)$ has no Anosov rational forms for any $c > 1$.*

Proof. Note that all vertices in a coherent component have the same degree. As a consequence, if $\mathcal{G}$ has a vertex v for which there are no other vertices of the same degree, then $\{v\}$ is a coherent component of $\mathcal{G}$. Since graph automorphisms must preserve the degree of a vertex, it follows that v is fixed under any graph automorphism and as a consequence the coherent component $\{v\}$ is fixed under any automorphism of the quotient graph. It follows that for an arbitrary morphism $\rho : \mathrm{Gal}(L/\mathbb{Q}) \to \mathrm{Aut}(\overline{\mathcal{G}})$, the set $\{\{v\}\}$ is a non-empty connected ρ-invariant set. Since $|\{v\}| = 1 < c$, Theorem 6.6.13 tell us that $\mathfrak{n}^{\mathbb{Q}}(\rho, c)$ is not Anosov. □

The following examples show that our methods can be used to simplify certain classifications of Anosov Lie algebras and to extend them.

Example 6.6.25 (Direct sum of two free nilpotent Lie algebras of same rank and nilpotency class)**.** Consider the graph $\mathcal{G}_n = (V_n, E_n)$ defined by $V_n = \{v_1, \ldots, v_n, w_1, \ldots, w_n\}$ and $E_n = \{\{v_i, v_j\} \mid 1 \leq i < j \leq n\} \cup \{\{w_i, w_j\} \mid 1 \leq i < j \leq n\}$. The set of coherent components is then given by $\Lambda_{\mathcal{G}} = \{\lambda_1, \lambda_2\}$ with $\lambda_1 = \{v_1, \ldots, v_n\}$ and $\lambda_2 = \{w_1, \ldots, w_n\}$. A figure of the quotient graph is given below.



The Lie algebra $\mathfrak{n}^{\mathbb{C}}(\mathcal{G}_n, c)$ is isomorphic to the Lie algebra direct sum of two free c-step nilpotent Lie algebras of rank n. Let $\varphi \in \mathrm{Aut}(\overline{\mathcal{G}_n})$ denote the only non-trivial automorphism of $\overline{\mathcal{G}_n}$. For any non-zero square-free integer $d \neq 1$, let σ_d denote the only non-trivial automorphism in $\mathrm{Gal}(\mathbb{Q}(\sqrt{d})/\mathbb{Q})$. If $d = 1$, then $\mathrm{Gal}(\mathbb{Q}(\sqrt{d})/\mathbb{Q})$ is trivial and we let σ_1 denote the trivial automorphism. Note that up to isomorphism, the extensions $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$, with d non-zero square-free, are all possible Galois extensions of $\mathbb{Q}$ of degree 1 or 2. Consequently, all injective group morphisms from a Galois group over $\mathbb{Q}$ to $\operatorname{Aut}(\overline{\mathcal{G}_n})$ are given by

$$\rho_d : \operatorname{Gal}(\mathbb{Q}(\sqrt{d})/\mathbb{Q}) \to \operatorname{Aut}(\overline{\mathcal{G}_n}) : \sigma_d \mapsto \varphi$$

for some non-zero square-free integer d. Following Theorem 5.6.4, all rational forms of $\mathfrak{n}^{\mathbb{C}}(\mathcal{G}_n, c)$ can thus be written as

$$\mathfrak{n}(d, n, c) := \mathfrak{n}^{\mathbb{Q}}(\rho_d, c) \ \subset \mathfrak{n}^{\mathbb{Q}(\sqrt{d})}(\mathcal{G}_n, c)$$

for a non-zero square-free integer d. Moreover, for non-zero square-free integers d_1, d_2 it holds that $\mathfrak{n}(d_1, n, c) \cong \mathfrak{n}(d_2, n, c)$ if and only if $d_1 = d_2$. For $d = 1$, ρ_d is the trivial morphism and we retrieve the standard rational form $\mathfrak{n}(1, n, c) \cong \mathfrak{n}^{\mathbb{Q}}(\mathcal{G}_n, c)$. From Theorem 6.6.13, we can now easily see that for a square-free non-zero integer d:

$$\boxed{\mathfrak{n}(d, n, c) \text{ is Anosov } \Leftrightarrow d > 1 \vee (d \leq 1 \wedge c < n).}$$

This result was already known for $c = 2$ and $n = 2$ from the classification of Anosov Lie algebras of dimension 6, where the Lie algebra $\mathfrak{n}(k, 2, 2)$ was denoted by $\mathfrak{n}_k^{\mathbb{Q}}$ (see [LW09, Example 2.7. and Theorem 4.2.]).

Definition 6.6.26. Let $\mathcal{G} = (V, E)$ be a simple undirected graph. The graph $\mathcal{G}^* := (V, E^*)$ with

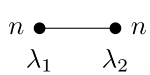
$$E^* = \{\{v, w\} \mid v, w \in V, v \neq w, \{v, w\} \notin E\}$$

is called the *complement graph of* $\mathcal{G}$.

Note that the coherent components of a simple undirected graph and its complement graph coincide. Moreover, we have that $\operatorname{Aut}(\overline{\mathcal{G}}) = \operatorname{Aut}(\overline{\mathcal{G}^*})$. This being said, let us look at the Lie algebra associated with the complement graph of the one from Example 6.6.25.

Example 6.6.27 (Free nilpotent sum of two abelian Lie algebras of same dimension)**.** Let $\mathcal{G}_n = (V_n, E_n)$ be the graph from Example 6.6.25 and $\mathcal{G}_n^*$ its complement graph. Note that $\mathcal{G}_n^*$ is also the cycle graph of size 4. The quotient graph $\overline{\mathcal{G}_n^*}$ is drawn below.

$\overline{\mathcal{G}_n^*}$



The Lie algebra $\mathfrak{n}^{\mathbb{C}}(\mathcal{G}_n, c)$ is isomorphic to the free c-step nilpotent sum of two abelian Lie algebras of dimension n. Let $\varphi \in \operatorname{Aut}(\overline{\mathcal{G}_n})$ denote the only non-trivial automorphism of $\overline{\mathcal{G}_n}$. Since $\operatorname{Aut}(\overline{\mathcal{G}_n}) = \operatorname{Aut}(\overline{\mathcal{G}_n^*})$, it follows that all injective group morphisms from finite Galois groups over $\mathbb{Q}$ into $\operatorname{Aut}(\overline{\mathcal{G}_n^*})$ are given by the same morphisms ρ_d for d a non-zero square-free integer as defined in Example 6.6.25. As a consequence, following Theorem 5.6.4, all rational forms of $\mathfrak{n}^{\mathbb{C}}(\mathcal{G}_n^*, c)$ are given by

$$\mathfrak{n}^*(d, n, c) := \mathfrak{n}^{\mathbb{Q}}(\rho_d, c) \ \subset \mathfrak{n}^{\mathbb{Q}(\sqrt{d})}(\mathcal{G}_n^*, c)$$

for a non-zero square-free integer d. Moreover, for non-zero square-free integers d_1, d_2 it holds that $\mathfrak{n}^*(d_1, n, c) \cong \mathfrak{n}^*(d_2, n, c)$ if and only if $d_1 = d_2$. For $d = 1$ we retrieve the standard rational form $\mathfrak{n}^*(1, n, c) \cong \mathfrak{n}^{\mathbb{Q}}(\mathcal{G}_n^*, c)$. From Theorem 6.6.14, we can now easily see that for a square-free non-zero integer d:

$$\boxed{\mathfrak{n}^*(d, n, c) \text{ is Anosov } \Leftrightarrow (d \geq 1 \wedge c < 2n) \vee (d < 1 \wedge c < n).}$$

This result was already known for $c = 2$, $n = 2$ and $d \geq 1$ from the classification of real Anosov Lie algebras of dimension 8, where the Lie algebra $\mathfrak{n}^*(k, 2, 2)$ was denoted by $\mathfrak{h}_k^{\mathbb{Q}}$ (see [LW09, Theorem 4.2.]).

When studying Anosov automorphisms on rational Lie algebras in low dimensions, one observes that in all known cases an Anosov Lie algebra always has an Anosov automorphism with only real eigenvalues. The results in Section 6.6 allow us to present an example where this is no longer the case, so where every Anosov automorphism has non-real eigenvalues.

Example 6.6.28 (Anosov rational form which does not admit Anosov automorphism with all eigenvalues real)**.** Let $\mathcal{G} = (V, E)$ be the cycle graph on 6 vertices as drawn in Figure 6.8. Let us write the vertices and edges as $V = \{v_1, \ldots, v_6\}$ and $E = \{\{v_1, v_2\}, \ldots \{v_5, v_6\}, \{v_6, v_1\}\}$. The coherent components of $\mathcal{G}$ are then simply all the singletons $\Lambda_{\mathcal{G}} = \{\lambda_i := \{v_i\} \mid 1 \leq i \leq 6\}$. As a consequence there is a natural bijection $h : V \to \Lambda_{\mathcal{G}} : v \mapsto \{v\}$ which gives us a splitting morphism $r : \operatorname{Aut}(\overline{\mathcal{G}}) \to \operatorname{Aut}(\mathcal{G}) : \varphi \mapsto h^{-1} \circ \varphi \circ h$. Let us write $\operatorname{Aut}(\overline{\mathcal{G}}) = \{1, a, \ldots a^5, b, ab, \ldots, a^5 b\}$ where a and b are defined by $a(\lambda_1) = \lambda_2$, $a(\lambda_2) = \lambda_3$, $b(\lambda_1) = \lambda_1$ and $b(\lambda_2) = \lambda_6$. Thus a is a generator for the rotations and b is a reflection, like in our general discussion of cycle graphs.

Now let L be the splitting field of the polynomial $X^3 - 2$ over $\mathbb{Q}$. The roots of this polynomial are given by $\sqrt[3]{2}$, $\omega \sqrt[3]{2}$ and $\overline{\omega} \sqrt[3]{2}$ where $\omega = e^{i \frac{2\pi}{3}}$. The Galois group $\operatorname{Gal}(L/\mathbb{Q})$ is generated by the elements σ and τ, defined by

$$\sigma(\sqrt[3]{2}) = \omega \sqrt[3]{2}, \quad \sigma(\omega \sqrt[3]{2}) = \overline{\omega} \sqrt[3]{2}, \quad \sigma(\overline{\omega} \sqrt[3]{2}) = \sqrt[3]{2},$$

$$\tau(\sqrt[3]{2}) = \sqrt[3]{2}, \quad \tau(\omega \sqrt[3]{2}) = \overline{\omega} \sqrt[3]{2}, \quad \tau(\overline{\omega} \sqrt[3]{2}) = \omega \sqrt[3]{2}.$$

Note that τ is just the complex conjugation automorphism on L and that $\mathrm{Gal}(L/\mathbb{Q})$ is isomorphic to the dihedral group of order 6. It follows that we have an injective group morphism

$$\rho : \mathrm{Gal}(L/\mathbb{Q}) \to \mathrm{Aut}(\overline{\mathcal{G}}) : \sigma \mapsto a^2, \tau \mapsto b$$

with corresponding rational form $\mathfrak{n}^{\mathbb{Q}}(\rho, 2)$ of the two-step nilpotent Lie algebra $\mathfrak{n}^{\mathbb{C}}(\mathcal{G}, 2)$. Using Theorem 6.6.14, it is straightforward to verify that $\mathfrak{n}^{\mathbb{Q}}(\rho, 2)$ is Anosov. Indeed, z_ρ takes only the value 1 on $\Lambda_{\mathcal{G}}$ and for any non-empty connected $A \subset \Lambda_{\mathcal{G}}$ for which $A \cup \rho_\tau(A)$ is ρ-invariant, we have that $A \cup \rho_\tau(A) = \Lambda_{\mathcal{G}}$.

For the convenience of the reader, we construct an explicit basis for the rational Lie algebra $\mathfrak{n}^{\mathbb{Q}}(\rho, 2)$ and compute the structure constants. We define the vectors $w_i = [v_i, v_{i+1}]$ for $1 \leq i \leq 5$ and $w_6 = [v_6, v_1]$. The following elements form a basis for $\mathfrak{n}^{\mathbb{Q}}(\rho, 2)$:

$$\begin{aligned}
X_i &= \left(\sqrt[3]{2}\right)^i v_1 + \left(\omega\sqrt[3]{2}\right)^i v_3 + \left(\overline{\omega}\sqrt[3]{2}\right)^i v_5 \\
Y_i &= \left(\overline{\omega}\sqrt[3]{2}\right)^i v_2 + \left(\sqrt[3]{2}\right)^i v_4 + \left(\omega\sqrt[3]{2}\right)^i v_6 \\
Z_i &= \left(\left(\sqrt[3]{2}\right)^i w_1 + \left(\omega\sqrt[3]{2}\right)^i w_3 + \left(\overline{\omega}\sqrt[3]{2}\right)^i w_5\right) \\
&\qquad - \left(\left(\omega\sqrt[3]{2}\right)^i w_2 + \left(\overline{\omega}\sqrt[3]{2}\right)^i w_4 + \left(\sqrt[3]{2}\right)^i w_6\right) \\
W_i &= \omega\left(\left(\sqrt[3]{2}\right)^i w_1 + \left(\omega\sqrt[3]{2}\right)^i w_3 + \left(\overline{\omega}\sqrt[3]{2}\right)^i w_5\right) \\
&\qquad - \overline{\omega}\left(\left(\omega\sqrt[3]{2}\right)^i w_2 + \left(\overline{\omega}\sqrt[3]{2}\right)^i w_4 + \left(\sqrt[3]{2}\right)^i w_6\right)
\end{aligned}$$

where $0 \leq i \leq 2$. In this basis, the Lie bracket of $\mathfrak{n}^{\mathbb{Q}}(\rho, 2)$ is given by the following relations

$$\begin{array}{lll}
[X_0, Y_0] = Z_0 & [X_1, Y_0] = Z_1 & [X_2, Y_0] = Z_2 \\
[X_0, Y_1] = -Z_1 - W_1 & [X_1, Y_1] = -Z_2 - W_2 & [X_2, Y_1] = -2Z_0 - 2W_0 \\
[X_0, Y_2] = W_2 & [X_1, Y_2] = 2W_0 & [X_2, Y_2] = 2W_1.
\end{array}$$

Let us prove by contradiction that $\mathfrak{n}^{\mathbb{Q}}(\rho, 2)$ does not admit an Anosov automorphism with real eigenvalues. So, assume $f : \mathfrak{n}^{\mathbb{Q}}(\rho, 2) \to \mathfrak{n}^{\mathbb{Q}}(\rho, 2)$ is an Anosov automorphism with real eigenvalues. From Theorem 6.6.4 and its proof, we know that there exists a map $\Psi : V \to \overline{\mathbb{Q}}$, which satisfies the properties formulated in that theorem and such that the algebraic units in the image of Ψ are eigenvalues of f. By the assumption, these eigenvalues are all real. Let $\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}(\overline{\mathcal{G}}) : \gamma \mapsto \rho(\gamma|_L)$ be the extended morphism. Note that since all coherent components are singletons, the group $\prod_{\lambda \in \Lambda_{\mathcal{G}}} \mathrm{Perm}(\lambda)$ is trivial and thus condition (iii) on Ψ becomes

$$\forall\, \gamma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) : \gamma \circ \Psi = \Psi \circ r(\overline{\rho}_\gamma).$$

Write $\xi := \Psi(v_1)$. Now let γ be any element in $\mathrm{Stab}_{\overline{\rho}}(\lambda_1)$. It follows that

$$\gamma(\xi) = (\gamma \circ \Psi)(v_1) = (\Psi \circ r(\overline{\rho}_\gamma))(v_1) = \Psi(v_1) = \xi.$$

As a consequence ξ is fixed under $\mathrm{Stab}_{\overline{\rho}}(\lambda_1)$ and in particular under $\ker(\overline{\rho})$. Note that $\overline{\mathbb{Q}}^{\ker(\overline{\rho})}$ is exactly equal to L and thus that $\xi \in L$. Now let $\overline{\sigma}$ be an extension of the field automorphism σ to $\overline{\mathbb{Q}}$ and note that

$$\sigma(\xi) = \overline{\sigma}(\xi) = (\overline{\sigma} \circ \Psi)(v_1) = (\Psi \circ r(\overline{\rho}_{\overline{\sigma}}))(v_1) = \Psi(v_3)$$

is also an eigenvalue of f. Since ξ and $\sigma(\xi)$ are both real, we have $\tau(\xi) = \xi$ and $\tau\sigma(\xi) = \sigma(\xi)$. This implies $\sigma(\xi) = \tau\sigma(\xi) = \sigma^2\tau(\xi) = \sigma^2(\xi)$ and thus that $\sigma(\xi) = \xi$. As a consequence ξ is an element of L, fixed by $\mathrm{Gal}(L/\mathbb{Q})$, which in turn tells us that $\xi \in \mathbb{Q}$. The only algebraic units in $\mathbb{Q}$ are 1 and -1 which are not hyperbolic. This gives us the contradiction.

A classification for small graphs

Using Corollary 6.6.15, we can check for a graph $\mathcal{G}$ whether $\mathfrak{n}^{\mathbb{R}}(\mathcal{G}, c)$ has an Anosov rational form or not. This corresponds to whether the associated simply connected Lie group $N(\mathcal{G}, c)$ admits a cocompact lattice Γ for which the nilmanifold $\Gamma \backslash N(\mathcal{G}, c)$ admits an Anosov diffeomorphism. In the following tables, we give a list of all connected graphs for which $\mathfrak{n}^{\mathbb{R}}(\mathcal{G}, c)$ has an Anosov rational form for some $c > 1$. We give this list for all graphs up to 6 vertices. Note that one can use Proposition 6.6.22 and Proposition 6.6.24 to narrow down the list of possible graphs. We refer to [Ber22] for a more exhaustive discussion on the computation of this list.

For each graph $\mathcal{G}$ in the list, we also give its quotient graph $\overline{\mathcal{G}}$, the number of rational forms of $\mathfrak{n}^{\mathbb{R}}(\mathcal{G}, c)$, the values of $c > 1$ for which the standard rational form $\mathfrak{n}^{\mathbb{Q}}(\mathcal{G}, c)$ is Anosov and the values of $c > 1$ for which $\mathfrak{n}^{\mathbb{R}}(\mathcal{G}, c)$ has an Anosov rational form. In the visual representation of the quotient graph, we omitted the weight 1 on the vertices.

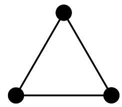
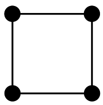
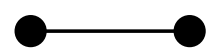
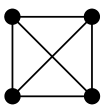
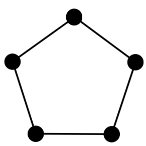
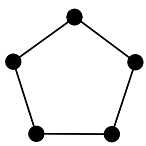
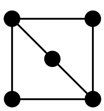
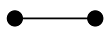
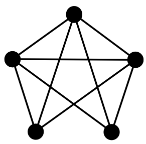
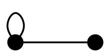
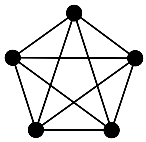
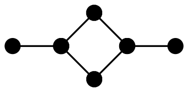
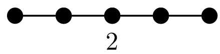
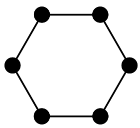
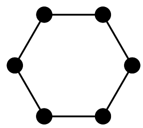
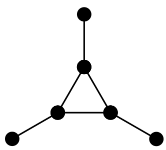
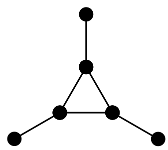
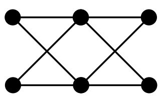
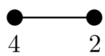
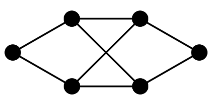
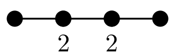
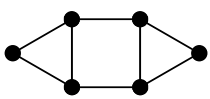
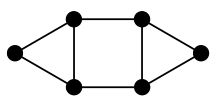
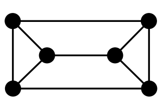
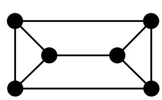
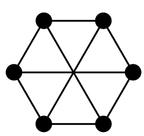
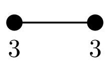
$\mathcal{G}$	$\overline{\mathcal{G}}$	Number of $\mathbb{Q}$-forms	$\mathfrak{n}^{\mathbb{Q}}(\mathcal{G}, c)$ Anosov?	$\mathfrak{n}^{\mathbb{R}}(\mathcal{G}, c)$ has Anosov $\mathbb{Q}$-form?
	 3	1	if $c \leq 2$	if $c \leq 2$
	 2 2	∞	if $c \leq 3$	if $c \leq 3$
	 4	1	if $c \leq 3$	if $c \leq 3$
		∞	no	if $c \leq 4$
	 3 2	1	if $c \leq 4$	if $c \leq 4$
	 3 2	1	if $c \leq 2$	if $c \leq 2$
	 5	1	if $c \leq 4$	if $c \leq 4$

Table 6.3: List of connected graphs $\mathcal{G}$ up to 5 vertices such that $\mathfrak{n}^{\mathbb{R}}(\mathcal{G}, c)$ has an Anosov rational form for some $c > 1$.

$\mathcal{G}$	$\overline{\mathcal{G}}$	Number of $\mathbb{Q}$-forms	$\mathfrak{n}^{\mathbb{Q}}(\mathcal{G}, c)$ Anosov?	$\mathfrak{n}^{\mathbb{R}}(\mathcal{G}, c)$ has Anosov $\mathbb{Q}$-form?
		∞	no	if $c \leq 3$
		∞	no	if $c \leq 5$
		∞	no	if $c \leq 2$
		1	if $c \leq 5$	if $c \leq 5$
		∞	no	if $c \leq 3$
		∞	no	if $c \leq 3$
		∞	no	if $c \leq 5$
		∞	if $c \leq 5$	if $c \leq 5$

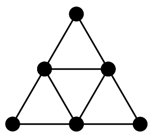
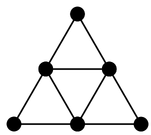
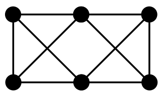
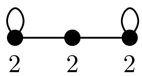
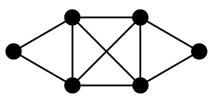
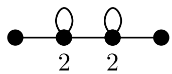
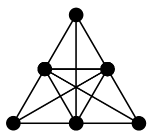
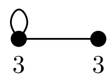
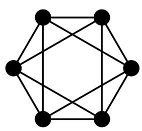
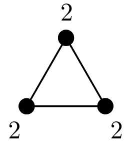
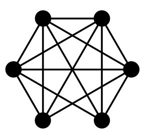
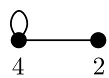
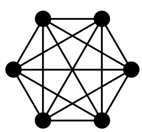
$\mathcal{G}$	$\overline{\mathcal{G}}$	Number of $\mathbb{Q}$-forms	$\mathfrak{n}^{\mathbb{Q}}(\mathcal{G}, c)$ Anosov?	$\mathfrak{n}^{\mathbb{R}}(\mathcal{G}, c)$ has Anosov $\mathbb{Q}$-form?
		∞	no	if $c \leq 2$
		∞	no	if $c \leq 5$
		∞	no	if $c \leq 3$
		1	if $c \leq 2$	if $c \leq 2$
		∞	if $c \leq 3$	if $c \leq 5$
		1	if $c \leq 3$	if $c \leq 3$
		1	if $c \leq 5$	if $c \leq 5$

Table 6.4: List of connected graphs $\mathcal{G}$ on 6 vertices such that $\mathfrak{n}^{\mathbb{R}}(\mathcal{G}, c)$ has an Anosov rational form for some $c > 1$.

Chapter 7

The R_∞-property

In Chapter 6, we investigated the existence of Anosov automorphisms on rational forms of nilpotent partially commutative Lie algebras. In Section 7.2 of this chapter, we study the related problem of determining which (nilpotent) partially commutative groups have the R_∞-property, a property which finds its origin in Nielsen fixed point theory. We prove that every non-abelian right-angled Artin group has the R_∞ property (see Theorem 7.2.4). For transposition-free graphs, we prove that the two-step nilpotent quotient of the associated right-angled Artin group has the R_∞-property (see Theorem 7.2.2). In Section 7.3, we explore the relation between the R_∞-property and abstract commensurability within the class of finitely generated torsion-free nilpotent groups which is joint work with Maarten Lathouwers. In particular we construct groups from edge-weighted graphs. Using this construction, we give the first known example of a pair of abstractly commensurable torsion-free finitely generated nilpotent groups of which one has the R_∞-property and the other does not (see Example 7.3.29). We begin this chapter with an introductory section on the R_∞-property and on how this property behaves with respect to nilpotent quotients of groups.

7.1 Definitions and preliminaries

Let G be a group. For any automorphism φ of G, one says $x, y \in G$ are *φ-conjugate* if and only if there exists an element $z \in G$ such that $x = zy\varphi(z)^{-1}$ and this is written as $x \sim_\varphi y$. Being φ-conjugate defines an equivalence relation on G. The equivalence classes are also called the *twisted conjugacy classes* of φ

and we write

$$[x]_\varphi = \{y \in G \mid y \sim_\varphi x\}$$

for the twisted conjugacy class containing x. Note that if φ is the identity automorphism, the twisted conjugacy classes are the ordinary conjugacy classes in the group G.

The number of twisted conjugacy classes of an automorphism φ is called the *Reidemeister number of* φ and is written as

$$R(\varphi) = \left|G/\sim_\varphi\right| = \left|\left\{[x]_\varphi \mid x \in G\right\}\right|.$$

Note that the Reidemeister number takes values in $\mathbb{N}_0 \cup \{\infty\}$. The *Reidemeister spectrum* of a group G is defined as the set

$$\mathrm{Spec}_R(G) = \{R(\varphi) \mid \varphi \in \mathrm{Aut}(G)\},$$

consisting of all Reidemeister numbers of automorphisms of G.

Definition 7.1.1. A group G is said to have the *R_∞-property* if for any automorphism φ of G it holds that $R(\varphi) = \infty$.

The Reidemeister number finds its origin in Nielsen fixed point theory. Under certain assumptions, the Reidemeister spectrum of the fundamental group of a topological space X can give information about the number of fixed points of homeomorphisms on X. For example, if the topological space is a nilmanifold $\Gamma\backslash N$ and its fundamental group (which is isomorphic to Γ) has the R_∞-property, then every self-homeomorphism of $\Gamma\backslash N$ is homotopic to a map which has no fixed points [Jia83] [Ano85].

The fundamental groups of nilmanifolds are exactly the finitely generated torsion-free nilpotent groups (see Theorem 2.3.21). Nilpotent partially commutative groups (see Section 4.2) fall under this class and are, among others, groups that will be considered in sections 7.2 and 7.3. A natural question for these groups is: how does the R_∞-property behave with respect to the nilpotency class? The following Lemma 7.1.2 and Corollary 7.1.3 are key observations in answering this question. These observations can be found in several papers on twisted conjugacy, for example in [GW09].

Lemma 7.1.2. *Let G be a group and H a characteristic subgroup. If G/H has the R_∞-property, then so does G.*

Proof. Assume G/H has the R_∞-property. Let φ be any automorphism of G. Since H is characteristic, we have that $\varphi(H) = H$ and thus we obtain an induced automorphism $\overline{\varphi} : G/H \to G/H : gH \mapsto \varphi(g)H$. Moreover, this automorphism fits into the commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & G \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ G/H & \xrightarrow{\overline{\varphi}} & G/H \end{array}$$

where $\pi : G \to G/H : g \mapsto gH$ is the projection. Since G/H is assumed to have the R_∞-property, we know that $R(\overline{\varphi}) = \infty$.

Assume that $x, y \in G$ are φ-conjugate. Then there exists a third element $z \in G$ such that $x = zy\varphi(z)^{-1}$. Applying π to both sides of this equation, we find that

$$\pi(x) = \pi\left(zy\varphi(z)^{-1}\right) = \pi(z)\pi(y)\pi\left(\varphi(z)\right)^{-1} = \pi(z)\pi(y)\overline{\varphi}\left(\pi(z)\right)^{-1}.$$

This implies that $\pi(x)$ and $\pi(y)$ are $\overline{\varphi}$-conjugate in G/H. As a consequence, π induces a surjective map:

$$\hat{\pi} : \left(G\big/ \sim_\varphi\right) \to \left((G/H)\big/ \sim_{\overline{\varphi}}\right) : [x]_\varphi \mapsto [x]_{\overline{\varphi}}.$$

Hence, since $R(\overline{\varphi}) = \infty$, we find that $R(\varphi) = \infty$. As φ is an arbitrary automorphism of G, this shows that G has the R_∞-property. □

In particular, if we apply Lemma 7.1.2 to the lower central series, which consists of characteristic subgroups, we obtain the following corollary.

Corollary 7.1.3. *Let G be a group. If $G/\gamma_i(G)$ has the R_∞-property for some positive integer $i > 0$, then G itself and the groups $G/\gamma_j(G)$ for any integer $j \geq i$ also have the R_∞-property.*

With this corollary in mind, the following definition was introduced in [DG16].

Definition 7.1.4. Let G be a group. The *R_∞-nilpotency index* is the least integer c such that $G/\gamma_{c+1}(G)$ has the R_∞-property. If no such integer exists, then we say the R_∞-nilpotency index is infinite.

The R_∞-nilpotency index has been studied on free groups [DG14], surface groups [DG16] and Baumslag-Solitar groups [DLG20]. In Section 7.2 we prove that any non-abelian right-angled Artin group has the R_∞-property and give bounds for the R_∞-nilpotency index.

7.2 The R_∞-property for RAAGs

This section is based on [Wit23].

Recall from Section 4.2 of Chapter 4 that we defined for any graph $\mathcal{G} = (V, E)$ a finitely presented group

$$A(\mathcal{G}) = \left\langle V \;\middle|\; [v, w];\ v, w \in V,\ \{v, w\} \notin E \right\rangle,$$

called the (free) partially commutative group or right-angled Artin group (RAAG), associated to $\mathcal{G}$. The goal of this section is to study the R_∞-property and the R_∞-nilpotency index for the groups $A(\mathcal{G})$ for an arbitrary graph $\mathcal{G}$. Recall that we also defined for any non-empty graph $\mathcal{G}$ and any integer $c > 1$ the c-step nilpotent quotient of $A(\mathcal{G})$ as

$$A(\mathcal{G}, c) = \frac{A(\mathcal{G})}{\gamma_{c+1}(A(\mathcal{G}))}.$$

In [DS21] it was proven that $A(\mathcal{G})$ has the R_∞-property for certain subclasses of graphs, among which the class of non-empty transvection-free graphs. Using the notation from Section 4.1, a graph $\mathcal{G} = (V, E)$ is called *transvection-free* if for any distinct vertices $v, w \in V$ it does not hold that $v \prec w$. Moreover, from their proof it follows that the R_∞-nilpotency index for such graphs is equal to either 2 or 3. In Section 7.2.4 we improve this result by showing that the R_∞-nilpotency index is in fact equal to 2 and that this is true on the following larger class of graphs.

Definition 7.2.1. A graph $\mathcal{G} = (V, E)$ is called *transposition-free* if for any two distinct vertices $v, w \in V$, the transposition of v and w (leaving all other vertices fixed) is not an automorphism of $\mathcal{G}$.

Using the notation from Section 4.1, it follows that a graph is transposition-free if and only if $\forall v, w \in V$ with $v \neq w$ it does not hold that $v \sim w$. Equivalently, all coherent components are singletons. From this it is easy to see that being transposition-free is a weaker assumption than being transvection-free since $v \sim w \Leftrightarrow (v \prec w \ \wedge \ v \succ w)$. As mentioned before, in Section 7.2.4, we prove the following result. Recall that a graph is said to be non-empty if its set of edges is non-empty.

Theorem 7.2.2. *If $\mathcal{G}$ is a non-empty transposition-free graph, then $A(\mathcal{G})$ has R_∞-nilpotency index 2.*

Using the fact that a graph with trivial automorphism group is also transposition-free together with Theorem 4.1.11, we find immediately the following corollary.

Corollary 7.2.3. *The group $A(\mathcal{G})$ has R_∞-nilpotency index 2 for almost all unlabelled graphs $\mathcal{G}$.*

Let $\mathcal{G} = (V, E)$ be a graph. Recall from Section 4.1 the definition of its associated quotient graph $\overline{\mathcal{G}} = (\Lambda_\mathcal{G}, \overline{E}, \Phi)$ where $\Lambda_\mathcal{G}$ are the coherent components of $\mathcal{G}$. The R_∞-nilpotency index strongly depends on the sizes of adjacent coherent components in the quotient graph $\overline{\mathcal{G}}$. To this extend we define for any non-empty graph $\mathcal{G}$, the positive integers

$$\xi(\mathcal{G}) = \min\left\{|\lambda| + |\mu| \;\middle|\; \lambda, \mu \in \Lambda_\mathcal{G},\, \{\lambda, \mu\} \in \overline{E}\right\} \tag{7.1}$$

and

$$\Xi(\mathcal{G}) = \min\left\{c(\lambda, \mu) \;\middle|\; \lambda, \mu \in \Lambda_\mathcal{G},\, \{\lambda, \mu\} \in \overline{E}\right\} \tag{7.2}$$

where for any $\lambda, \mu \in \Lambda_\mathcal{G}$:

$$c(\lambda, \mu) = \begin{cases} \max\{2|\lambda| + |\mu|, |\lambda| + 2|\mu|\} & \text{if } \lambda \neq \mu \\ 2|\lambda| & \text{if } \lambda = \mu. \end{cases} \tag{7.3}$$

Note that given an arbitrary graph $\mathcal{G}$, these numbers are relatively easy to compute as the time complexity is polynomial with respect to the number of vertices. The main theorem of this section is then formulated as follows:

Theorem 7.2.4. *If $\mathcal{G}$ is a non-empty graph, then $A(\mathcal{G})$ has the R_∞-property with R_∞-nilpotency index c satisfying*

$$\xi(\mathcal{G}) \leq c \leq \Xi(\mathcal{G}).$$

Note that a graph $\mathcal{G}$ is transposition-free if and only if each coherent component has size one. In this case, the above theorem thus tells us that a transposition-free graph has R_∞-nilpotency index c equal to 2 or 3. Therefore Theorem 7.2.2 is really a stronger statement than Theorem 7.2.4 in the transposition-free case.

As a second remark, note that the upper bound $\Xi(\mathcal{G})$ is always less or equal than $2r$ with r the number of vertices of the graph. We thus get that the R_∞-nilpotency index of $A(\mathcal{G})$ is always less or equal than twice the number of vertices of $\mathcal{G}$.

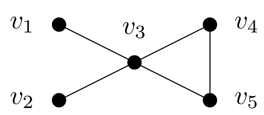
It also agrees with the known result for free groups, which states that the R_∞-nilpotency index of a free group of finite rank at least 2 is equal to two times its rank [DG14]. Indeed, the free group of rank r is isomorphic to $A(\mathcal{G})$ with $\mathcal{G}$ the complete graph on r vertices. This graph has only one coherent component of size r and thus $\xi(\mathcal{G}) = \Xi(\mathcal{G}) = 2r$.

The proof of Theorem 7.2.4 is done using a Lie-theoretic approach. As discussed in Section 4.3.2, any automorphism φ on $A(\mathcal{G}, c) = A(\mathcal{G})/\gamma_{c+1}(A(\mathcal{G}))$ induces a graded automorphism $\overline{\varphi}$ on the partially commutative nilpotent Lie algebra $\mathfrak{n}^{\mathbb{Q}}(\mathcal{G}, c)$. An existing result establishes an equivalence between $R(\varphi) = \infty$ and the occurence of an eigenvalue 1 for the Lie algebra automorphism $\overline{\varphi}$.

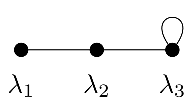
First, the theory of linear algebraic groups is used to get the automorphism $\overline{\varphi}$ into a 'nice' form without altering its eigenvalues. Specifically, some positive power of this 'nice' automorphism becomes diagonal on the vertices. Next, Lemma 4.6.13 along with the theory of partially commutative Lyndon words, is used to construct an eigenvector with eigenvalue 1 as a folded Lie bracket of eigenvectors in $\mathfrak{n}_1^{\mathbb{C}}(\mathcal{G}, c)$. This construction can be carried out for any edge $\{\lambda, \mu\} \in \overline{E}$ in the quotient graph $\overline{\mathcal{G}}$. The resulting eigenvector with eigenvalue 1 lies in the i-th subspace $\mathfrak{n}_i^{\mathbb{C}}(\mathcal{G}, c)$, where i depends on the cardinalities of the coherent components λ and μ. This dependence is encapsulated in the definition of the number $\Xi(\mathcal{G})$, as it is necessary that $\mathfrak{n}_i^{\mathbb{C}}(\mathcal{G}, c) \neq \{0\}$ (or equivalently $c \geq i$) for this eigenvector to be non-zero.

At last, let us give an example to make the reader more familiar with the definitions of the numbers $\xi(\mathcal{G})$ and $\Xi(\mathcal{G})$.

Example 7.2.5. Let $\mathcal{G} = (V, E)$ be the graph determined by $V = \{v_1, v_2, v_3, v_4, v_5\}$ and $E = \{\{v_1, v_3\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_3, v_5\}, \{v_4, v_5\}\}$, as illustrated below.



The coherent components are given by $\lambda_1 = \{v_1, v_2\}$, $\lambda_2 = \{v_3\}$ and $\lambda_3 = \{v_4, v_5\}$. The set of edges in the quotient graph is given by $\overline{E} = \{\{\lambda_1, \lambda_2\}, \{\lambda_2, \lambda_3\}, \{\lambda_3\}\}$ and the quotient graph can be drawn as follows.



We thus find that

$$\xi(\mathcal{G}) = \min\{|\lambda_1| + |\lambda_2|, |\lambda_2| + |\lambda_3|, |\lambda_3| + |\lambda_3|\}$$
$$= \min\{2 + 1, 1 + 2, 2 + 2\} = 3$$

and

$$\begin{aligned}\Xi(\mathcal{G}) &= \min\{2|\lambda_1| + |\lambda_2|, |\lambda_2| + 2|\lambda_3|, 2|\lambda_3|\} \\ &= \min\{2\cdot 2 + 1, 1 + 2\cdot 2, 2\cdot 2\} = 4.\end{aligned}$$

By Theorem 7.2.4 we conclude that the R_∞-nilpotency index of $A(\mathcal{G})$ is either 3 or 4.

7.2.1 Notation and methodology

To prove the main results, we use a Lie theoretic approach. Recall from Section 4.3.2 that for any field K one has a functor

$$L^K : \mathbf{Grp} \to \mathbf{LieAlg}_K$$

which assigns to a group a graded Lie algebra by use of its lower central series. The following lemma relates this functor with the R_∞-property on finitely generated nilpotent groups. It was stated in [DG14] as a generalization of corollary 4.2 from [Rom11a].

Lemma 7.2.6. *Let G be a finitely generated nilpotent group and $\varphi \in \mathrm{Aut}(G)$. Then $R(\varphi) = \infty$ if and only if 1 is an eigenvalue of $L^{\mathbb{C}}(\varphi)$.*

Let $\mathcal{G} = (V, E)$ be a graph and $c > 1$. By equation (4.5), we have, for any field K, a natural isomorphism

$$\mathfrak{n}^K(\mathcal{G}, c) \cong L^K\left(A(\mathcal{G}, c)\right),$$

where $\mathfrak{n}^K(\mathcal{G}, c)$ is the c-step nilpotent partially commutative Lie algebra (see Section 4.2) and the isomorphism is induced by the identity on the vertices of $\mathcal{G}$. Moreover, the isomorphism preserves the grading on both Lie algebras. Thus, we see that any automorphism φ of $A(\mathcal{G}, c)$ induces an automorphism $L^K(\varphi)$ on $L^K(A(\mathcal{G}, c))$ which, under the above isomorphism, induces a graded automorphism on $\mathfrak{n}^K(\mathcal{G}, c)$, which we will write as $\overline{\varphi}$ for the remainder of Section 7.2. We write $\overline{\varphi_i} \in \mathrm{GL}(\mathfrak{n}_i^K(\mathcal{G}, c))$ for the linear map obtained by restricting $\overline{\varphi}$ to $\mathfrak{n}_i^K(\mathcal{G}, c)$.

Hence, to prove Theorem 7.2.4, we need to show for any $\varphi \in \mathrm{Aut}(A(\mathcal{G}, c))$ that $\overline{\varphi} \in \mathrm{Aut}(\mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c))$ has an eigenvalue 1 if $c \geq \Xi(\mathcal{G})$ and does not have an eigenvalue 1 if $c < \xi(\mathcal{G})$. The fact that the automorphism $\overline{\varphi}$ is induced by an automorphism of the group $A(\mathcal{G}, c)$ is reflected in the coefficients of the characteristic polynomial of each $\overline{\varphi}_i$. This is argued in the following remark.

Remark 7.2.7. Let H be a finitely generated abelian group and write $\tau(H)$ for its torsion subgroup (which is characteristic in H). Let the field K be of characteristic 0. Then there is a natural isomorphism between $\frac{H}{\tau(H)} \otimes_{\mathbb{Z}} K$ and $H \otimes_{\mathbb{Z}} K$. If f is an automorphism of H, it also induces an automorphism on the free $\mathbb{Z}$-module $H/\tau(H)$, which can be represented by a matrix in $\mathrm{GL}(n,\mathbb{Z})$ with respect to some basis for $H/\tau(H)$. The induced linear map on $\frac{H}{\tau(H)} \otimes_{\mathbb{Z}} K \cong H \otimes_{\mathbb{Z}} K$ can therefore be represented by the same matrix and will thus have a characteristic polynomial with integer coefficients and constant term equal to ± 1. Applying this to the abelian groups $L_i^K(G)$ from above for a field K of characteristic 0, we see that for all $i > 0$, $\overline{\varphi_i}$ is integer-like (see Definition 3.4.18).

Recall from Remark 4.6.5 that we have an isomorphism of linear algebraic groups

$$\pi_{\mathrm{ab}} : \mathrm{Aut}_g\left(\mathfrak{n}^K(\mathcal{G},c)\right) \to G^K(\mathcal{G}).$$

Thus, we can consider the inverse of this map which we will write as

$$\varpi_c : G^K(\mathcal{G}) \to \mathrm{Aut}_g\left(\mathfrak{n}^K(\mathcal{G},c)\right) \tag{7.4}$$

where we use the subscript c to emphasize the codomain of the map. Recall from Theorem 4.6.6 and the discussion below it, that the linear algebraic group $G^K(\mathcal{G})$ is given by the product

$$G^K(\mathcal{G}) := \overline{P}(\mathrm{Aut}(\overline{\mathcal{G}})) \cdot \left(\prod_{\lambda\in\Lambda_{\mathcal{G}}} \mathrm{GL}(\mathrm{span}_K(\lambda))\right) \cdot U^K(\mathcal{G})$$

where $U^K(\mathcal{G})$ is the unipotent radical of $G^K(\mathcal{G})$.

Remark 7.2.8. Let K be any subfield of $\mathbb{C}$ and $g \in G^K(\mathcal{G})$ a semi-simple element, i.e. an element whose matrix w.r.t. the basis V is diagonalizable over $\mathbb{C}$. Then g lies in some linearly reductive subgroup (see Definition 4.5.14) of the linear algebraic group $G^K(\mathcal{G})$. Note as well that

$$R^K(\mathcal{G}) := \overline{P}(\mathrm{Aut}(\overline{\mathcal{G}})) \cdot \prod_{\lambda\in\Lambda_{\mathcal{G}}} \mathrm{GL}(\mathrm{span}_K(\lambda))$$

is a linearly reductive subgroup of $G^K(\mathcal{G})$. Indeed, it suffices to show that its Zariski-connected component at the identity is linearly reductive. This Zariski connected component is equal to $\prod_{\lambda\in\Lambda_{\mathcal{G}}} \mathrm{GL}(\mathrm{span}_K(\lambda))$. This group is linearly reductive as it is a direct product of general linear groups, which are known to be linearly reductive. The group $R^K(\mathcal{G})$ also has the property that

$$G^K(\mathcal{G}) \cong R^K(\mathcal{G}) \ltimes U^K(\mathcal{G}),$$

with $U^K(\mathcal{G})$ the unipotent radical of $G^K(\mathcal{G})$. Thus, Theorem 4.5.16 applies and there exists an $h \in G^K(\mathcal{G})$ such that $hgh^{-1} \in R^K(\mathcal{G})$. In particular, this is valid for $K = \mathbb{Q}$ which will be of use in the next section.

At last, let us mention that we will use standard cycle notation in the following sections. Thus, for a finite set X and any distinct elements $x_1, x_2, \dots, x_k \in X$, we let $(x_1 \dots x_n)$ denote the permutation in $\mathrm{Perm}(X)$ which is determined by

$$x_1 \mapsto x_2,\ \dots,\ x_{n-1} \mapsto x_n,\ x_n \mapsto x_1$$

and $y \mapsto y$ for all $y \in X \setminus \{x_1, \dots, x_n\}$.

7.2.2 Automorphisms with the same characteristic polynomial

The goal of this section is to find for any automorphism φ of the group $A(\mathcal{G}, c)$ an automorphism $f \in \mathrm{Aut}_g(\mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c))$ for which it is clear what the eigenvectors and eigenvalues on $\mathfrak{n}_1^{\mathbb{C}}(\mathcal{G}, c)$ are and such that f and the induced automorphism $\overline{\varphi} \in \mathrm{Aut}_g(\mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c))$ have the same characteristic polynomial. Moreover, the fact that $\overline{\varphi}$ is integer-like (see Remark 7.2.7), will give us relations among the eigenvalues of f on $\mathfrak{n}_1^{\mathbb{C}}(\mathcal{G}, c)$. At the end of the section, we also prove a corollary of these results which says that any integer-like automorphism of the rational Lie algebra $\mathfrak{n}^{\mathbb{Q}}(\mathcal{G}, c)$ has a characteristic polynomial which can be realized as the characteristic polynomial of an automorphism of the group $A(\mathcal{G}, c)$.

Let $\mathcal{G} = (V, E)$ be a graph and fix an order on the vertices inside each coherent component as $\lambda = \{v_{\lambda 1}, \dots, v_{\lambda |\lambda|}\}$. For any integer $n > 0$, let I_n denote the $n \times n$ identity matrix. For any k-cycle $\sigma = (\lambda_1 \dots \lambda_k) \in \mathrm{Perm}(\Lambda_{\mathcal{G}})$ on the coherent components with $|\lambda_1| = \ldots = |\lambda_k| = n$ and any $B \in \mathrm{GL}(n, K)$, we define the linear map $T(\sigma, B) \in \mathrm{GL}(\mathrm{span}_K(V))$ as represented by the matrix B if $k = 1$ and by the matrix

$$\begin{pmatrix} 0 & & & B \\ I_n & 0 & & \\ & \ddots & \ddots & \\ & & I_n & 0 \end{pmatrix}$$

if $k > 1$ with respect to the ordered vertices

$$v_{\lambda_1 1}, \dots, v_{\lambda_1 n}, v_{\lambda_2 1}, \dots, v_{\lambda_2 n}, \dots, v_{\lambda_k 1}, \dots, v_{\lambda_k n}$$

and by the identity on all other vertices. Let $\psi \in \mathrm{Aut}(\overline{\mathcal{G}})$ be an automorphism of the quotient graph with disjoint cycle decomposition $\psi = \sigma_1 \circ \ldots \circ \sigma_d$ (where 1-cycles are not omitted) and write n_i for the size of the coherent components in the cycle σ_i. Note that for any matrices $B_i \in \mathrm{GL}(n_i, K)$, the linear map $\prod_{i=1}^{d} T(\sigma_i, B_i)$ is an element of $G^K(\mathcal{G})$.

Lemma 7.2.9. *Let $\mathcal{G} = (V, E)$ be a graph and $g \in G^{\mathbb{Q}}(\mathcal{G})$ a semi-simple integer-like element. Then there exists a $\psi \in \mathrm{Aut}(\overline{\mathcal{G}})$ with disjoint cycle decomposition $\psi = \sigma_1 \circ \ldots \circ \sigma_d$ (where 1-cycles are not omitted) and for each cycle σ_i consisting of coherent components of size n_i, an integral semi-simple matrix $B_i \in \mathrm{GL}(n_i, \mathbb{Z})$ such that g is conjugated to $\prod_{i=1}^{d} T(\sigma_i, B_i)$ in $G^{\mathbb{Q}}(\mathcal{G})$.*

Proof. By Remark 7.2.8, there exists a $\psi \in \mathrm{Aut}(\overline{\mathcal{G}})$ and a $C \in \prod_{\lambda \in \Lambda_{\mathcal{G}}} \mathrm{GL}(\mathrm{span}_{\mathbb{Q}}(\lambda))$ such that g is conjugated to $\overline{P}(\psi)C$ in $G^{\mathbb{Q}}(\mathcal{G})$. As a consequence, $\overline{P}(\psi)C$ is also semi-simple and integer-like. Let $\psi = \sigma_1 \circ \ldots \circ \sigma_d$ be a disjoint cycle decomposition of ψ with $\sigma_i = (\lambda_{i1}, \ldots, \lambda_{ik_i})$ and $|\lambda_{i1}| = \ldots = |\lambda_{ik_i}| = n_i$. For any $i \in \{1, \ldots, d\}$, define the subspaces $W_i = \mathrm{span}_{\mathbb{Q}}(\lambda_{i1} \cup \ldots \cup \lambda_{ik_i})$. It follows that each W_i is invariant under $\overline{P}(\psi)C$. Define $h_i := (\overline{P}(\psi)C)|_{W_i}$. It follows that for each $i \in \{1, \ldots, d\}$, h_i is semi-simple and using Lemma 6.1.4 that h_i is integer-like.

Note that for any $\lambda \in \{\lambda_{i1}, \ldots, \lambda_{ik_i}\}$ we have $h_i(\mathrm{span}_{\mathbb{Q}}(\lambda)) = \mathrm{span}_{\mathbb{Q}}(\sigma_i(\lambda))$. As a consequence we get that the set $\lambda_{i1} \cup h_i(\lambda_{i1}) \cup \ldots \cup h_i^{k_i-1}(\lambda_{i1})$ is a basis for W_i. Clearly, with respect to this basis, h_i is represented by a matrix $\tilde{B}_i \in \mathrm{GL}(n_i, \mathbb{Q})$ if $k_i = 1$ and by the matrix

$$\begin{pmatrix} 0 & & & \tilde{B}_i \\ I_{n_i} & 0 & & \\ & \ddots & \ddots & \\ & & I_{n_i} & 0 \end{pmatrix}$$

if $k_i > 1$ for some $\tilde{B}_i \in \mathrm{GL}(n_i, \mathbb{Q})$. This is equivalent to saying that $\overline{P}(\psi)C$ is conjugated to $\prod_{i=1}^{d} T(\sigma_i, \tilde{B}_i)$ by an element of $\prod_{\lambda \in \Lambda_{\mathcal{G}}} \mathrm{GL}(\mathrm{span}_{\mathbb{Q}}(\lambda)) \subset G^{\mathbb{Q}}(\mathcal{G})$. Again, we have that each $T(\sigma_i, \tilde{B}_i)|_{W_i}$ is semi-simple and integer-like and the same must hold for its k_i'th power $\left(T(\sigma_i, \tilde{B}_i)|_{W_i}\right)^{k_i}$ which is represented by the matrix $\mathrm{diag}(\tilde{B}_i, \ldots, \tilde{B}_i)$ with respect to the basis $\lambda_{i1} \cup \ldots \cup \lambda_{ik_i}$. We thus get that $\tilde{B}_i$ is semi-simple and using Lemma 6.1.4 that $\tilde{B}_i$ is integer-like. As a consequence, there exists a matrix $Q_i \in \mathrm{GL}(n_i, \mathbb{Q})$ such that

$$B_i := Q_i \tilde{B}_i Q_i^{-1} \in \mathrm{GL}(n, \mathbb{Z}).$$

Clearly, B_i is also semi-simple. Define

$$Q \in \prod_{\lambda \in \Lambda_{\mathcal{G}}} \mathrm{GL}(\mathrm{span}_{\mathbb{Q}}(\lambda)) \subset G^{\mathbb{Q}}(\mathcal{G})$$

as represented by the matrix $\operatorname{diag}(Q_i, \dots, Q_i)$ with respect to the basis $\lambda_{i1} \cup \dots \cup \lambda_{ik_i}$ on the subspace W_i for each $i \in \{1, \dots, d\}$. It is then straightforward to check that

$$Q \left(\prod_{i=1}^{d} T(\sigma_i, \tilde{B}_i) \right) Q^{-1} = \prod_{i=1}^{d} T(\sigma_i, B_i).$$

We have thus proven that g is conjugated to $\prod_{i=1}^{d} T(\sigma_i, B_i)$ in $G^{\mathbb{Q}}(\mathcal{G})$ with $B_i \in \mathrm{GL}(n_i, \mathbb{Z})$ semi-simple. □

For any cycle $\sigma = (\lambda_1 \dots \lambda_k) \in \operatorname{Perm}(\Lambda_{\mathcal{G}})$ consisting of coherent components of size n and any n-tuple $\alpha = (\alpha_1, \dots, \alpha_n) \in (K^*)^n$ we will use the shortened notation

$$T(\sigma, \alpha) = T(\sigma, \operatorname{diag}(\alpha_1, \dots, \alpha_n)).$$

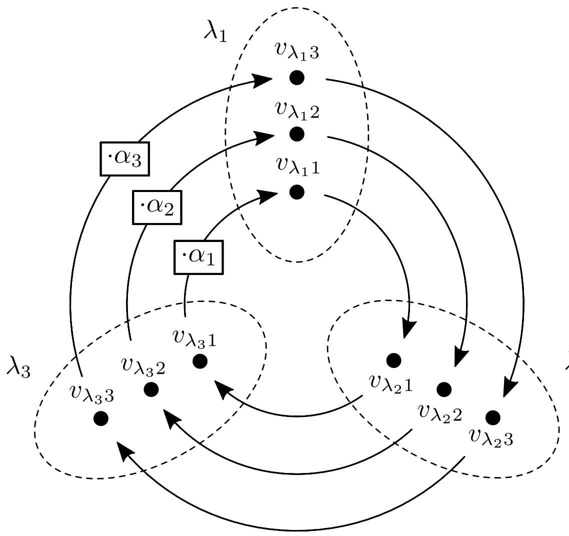


Figure 7.1: Visual representation of the map $T(\sigma, \alpha)$ for a 3-cycle σ consisting of coherent components of size 3.

Let W denote the vector subspace of $\operatorname{span}_K(V)$ spanned by the vertices in $\lambda_1 \cup \dots \cup \lambda_k$. For any non-zero complex number $z = re^{i\theta}$ with $r > 0$ and

$0 \leq \theta < 2\pi$, any positive integer $s > 0$ and any $t \in \{1, \ldots, s\}$, we write the t-th s-root of z as

$$R_{st}(z) := \sqrt[s]{r} e^{i\frac{\theta+(t-1)2\pi}{s}}.$$

The eigenvalues and corresponding eigenvectors of $T(\sigma, \alpha)|_W$ are then given by

$$R_{kj}(\alpha_i) \quad \text{and} \quad w(\sigma, \alpha)_{ij} := \sum_{m=1}^{k} \frac{v_{\lambda_m i}}{R_{kj}(\alpha_i)^m}, \tag{7.5}$$

respectively, with $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, k\}$.

Lemma 7.2.10. *Let φ be an automorphism of $A(\mathcal{G}, c)$ and $\overline{\varphi}$ the induced morphism on $\mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c)$. There exists an automorphism $\psi \in \operatorname{Aut}(\overline{\mathcal{G}})$ with a disjoint cycle decomposition $\psi = \sigma_1 \circ \ldots \circ \sigma_d$ (where 1-cycles are not omitted) and for each cycle σ_i consisting of coherent components of size n_i, an element $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{in_i}) \in (\mathbb{C}^*)^{n_i}$ with $\prod_{j=1}^{n_i} \alpha_{ij} = \pm 1$ such that the Lie algebra automorphisms $\varpi_c \left(\prod_{i=1}^{d} T(\sigma_i, \alpha_i)\right)$ and $\overline{\varphi}$ have the same characteristic polynomial.*

Proof. First we will work over the field $\mathbb{Q}$. With abuse of notation we let $\overline{\varphi}$ also denote the induced automorphism on $\mathfrak{n}^{\mathbb{Q}}(\mathcal{G}, c)$. Let $\overline{\varphi} = \overline{\varphi}_s \overline{\varphi}_u$ be the Jordan-Chevalley decomposition of $\overline{\varphi}$ into its semi-simple and unipotent part. Note that $\overline{\varphi}_s$ has the same characteristic polynomial as $\overline{\varphi}$. Since $\operatorname{Aut}_g(\mathfrak{n}^{\mathbb{Q}}(\mathcal{G}, c))$ is a linear algebraic group over a perfect field, we know that $\overline{\varphi_s}$ still lies in $\operatorname{Aut}_g(\mathfrak{n}^{\mathbb{Q}}(\mathcal{G}, c))$. Since π_{ab} is a linear algebraic group isomorphism to $G^{\mathbb{Q}}(\mathcal{G})$, we have that $\pi_{\text{ab}}(\overline{\varphi}_s) \in G^{\mathbb{Q}}(\mathcal{G})$ is the semi-simple part of $\pi_{\text{ab}}(\overline{\varphi})$. By Remark 7.2.7, we know that $\pi_{\text{ab}}(\overline{\varphi})$ is integer-like which implies that also $\pi_{\text{ab}}(\overline{\varphi}_s)$ is integer-like. Applying Lemma 7.2.9 to $\pi_{\text{ab}}(\overline{\varphi}_s)$, we get an element $h \in G^{\mathbb{Q}}(\mathcal{G})$, an automorphism $\psi \in \operatorname{Aut}(\overline{\mathcal{G}})$ with disjoint cycle decomposition $\psi = \sigma_1 \circ \ldots \circ \sigma_d$ and semi-simple matrices $B_i \in \operatorname{GL}(n_i, \mathbb{Z})$ such that

$$h\pi_{\text{ab}}(\overline{\varphi}_s)h^{-1} = \prod_{i=1}^{d} T(\sigma_i, B_i).$$

Since each B_i is semi-simple, there exists a matrix $Q_i \in \operatorname{GL}(n_i, \mathbb{C})$ and a tuple $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{in_i}) \in (\mathbb{C}^*)^{n_i}$, such that $Q_i B_i Q_i^{-1} = \operatorname{diag}(\alpha_{i1}, \ldots, \alpha_{in_i})$. Since $\det(B_i) = \pm 1$, it follows that $\prod_{j=1}^{n_i} \alpha_{ij} = \pm 1$. $\square$

As a consequence of Lemma 7.2.9, we can show that the characteristic polynomial of any graded integer-like automorphism of $\mathfrak{n}^{\mathbb{Q}}(\mathcal{G}, c)$ can be realised as the characteristic polynomial of an automorphism of the group $A(\mathcal{G}, c)$. First we show some lemmas to construct this automorphism on $A(\mathcal{G}, c)$.

Let $\mathcal{G} = (V, E)$ be a graph and $\Lambda_\mathcal{G}$ the set of coherent components. For any $\lambda \in \Lambda_\mathcal{G}$, we write $\mathrm{GL}(\mathbb{Z}, \lambda)$ for the subgroup of $\mathrm{GL}(\mathrm{span}_\mathbb{C}(\lambda))$ of linear maps that are represented by a matrix in $\mathrm{GL}(|\lambda|, \mathbb{Z})$ with respect to the basis λ. As before, we identify $\mathrm{GL}(\mathbb{Z}, \lambda)$ with its corresponding subgroup in $\mathrm{GL}(\mathrm{span}_K(V))$.

Lemma 7.2.11. *Let $\mathcal{G} = (V, E)$ be a graph with set of coherent components $\Lambda_\mathcal{G}$ and K a subfield of $\mathbb{C}$. For any $B \in \left(\prod_{\lambda\in\Lambda_\mathcal{G}} \mathrm{GL}(\mathbb{Z}, \lambda)\right) \leq \mathrm{GL}(\mathrm{span}_K(V))$, there exists an automorphism $\varphi \in \mathrm{Aut}(A(\mathcal{G}, c))$ such that the induced automorphism $\overline{\varphi} \in \mathrm{Aut}_g(L^K(\mathcal{G}, c))$ satisfies $\pi_{\mathrm{ab}}(\overline{\varphi}) = B$.*

Proof. Recall the ordering on the vertices inside each coherent component $\lambda = \{v_{\lambda 1}, \dots, v_{\lambda|\lambda|}\}$. We write $B = \prod_{\lambda\in\Lambda_\mathcal{G}} B_\lambda$ with $B_\lambda \in \mathrm{GL}(\mathbb{Z}, \lambda)$. For any $\lambda \in \Lambda_\mathcal{G}$, let $(b^\lambda_{ij})_{1\leq i,j\leq|\lambda|}$ denote the matrices which represent the maps B_λ with respect to the basis $v_{\lambda 1}, \dots, v_{\lambda|\lambda|}$. Let $F(V)$ be the free group on the vertices V. By the universal property of free groups, there exists a unique endomorphism $\varphi : F(V) \to F(V)$ which satisfies

$$\varphi(v_{\lambda i}) = \prod_{j=1}^{|\lambda|} v_{\lambda j}^{b^\lambda_{ji}}$$

for any $\lambda \in \Lambda_\mathcal{G}$ and $i \in \{1, \dots, |\lambda|\}$. Since $A(\mathcal{G})$ is a quotient of $F(V)$, we can compose φ with the quotient map to get a morphism $\tilde{\varphi} : F(V) \to A(\mathcal{G})$. Now take any two distinct vertices $v_{\lambda i}, v_{\mu i'} \in V$ such that $\{v_{\lambda i}, v_{\mu i'}\} \notin E$. By Remark 4.1.6 it follows that for all $1 \leq j \leq |\lambda|$, $1 \leq j' \leq |\mu|$ also $\{v_{\lambda j}, v_{\mu j'}\} \notin E$. Thus $v_{\lambda j}$ commutes with $v_{\mu j'}$ in $A(\mathcal{G})$ for any $1 \leq j \leq |\lambda|$, $1 \leq j' \leq |\mu|$. Using this we find that

$$\tilde{\varphi}([v_{\lambda i}, v_{\mu i'}]) = [\tilde{\varphi}(v_{\lambda i}), \tilde{\varphi}(v_{\mu i'})] = \left[\prod_{j=1}^{|\lambda|} v_{\lambda j}^{b^\lambda_{ji}}, \prod_{j'=1}^{|\mu|} v_{\mu j'}^{b^\mu_{j'i'}}\right] = 1.$$

Since this holds for arbitrary distinct vertices $v_{\lambda i}, v_{\mu i'} \in V$, we find that $\tilde{\varphi}$ induces an endomorphism on $A(\mathcal{G})$ and thus also an endomorphism on $A(\mathcal{G}, c)$ which we call φ. It is straightforward to verify that $\pi_{\mathrm{ab}}(\overline{\varphi}) = B$. Thus, the endomorphism induced by φ on the abelianization of $A(\mathcal{G}, c)$ is invertible. Since $A(\mathcal{G}, c)$ is nilpotent, this implies that φ is itself an automorphism of $A(\mathcal{G}, c)$. □

We can generalize Lemma 7.2.11 to the following:

Lemma 7.2.12. *Let $\mathcal{G} = (V, E)$ be a graph and K a subfield of $\mathbb{C}$. Let $\psi \in \mathrm{Aut}(\overline{\mathcal{G}})$ be a quotient graph automorphism with disjoint cycle decomposition $\psi = \sigma_1 \circ \ldots \circ \sigma_d$ (where 1-cycles are not omitted) and for each cycle σ_i consisting of coherent components of size n_i, take any matrix B_i in $\mathrm{GL}(n_i, \mathbb{Z})$. There* *exists an automorphism $\varphi \in \mathrm{Aut}(A(\mathcal{G},c))$ such that the induced automorphism $\overline{\varphi} \in \mathrm{Aut}(L^K(\mathcal{G},c))$ satisfies $\pi_{\mathrm{ab}}(\overline{\varphi}) = \prod_{i=1}^d T(\sigma_i, B_i)$.*

Proof. Note that any graph automorphism $\theta \in \mathrm{Aut}(\mathcal{G})$ naturally induces a group automorphism on $A(\mathcal{G})$ and thus also on $A(\mathcal{G},c)$. They are part of the so called elementary Nielsen automorphisms in the partially commutative setting, see [Ser89]. Now take $\theta = r(\psi)$, i.e. $\theta(v_{\lambda i}) = v_{\psi(\lambda)i}$ for any $\lambda \in \Lambda_{\mathcal{G}}$ and $i \in \{1,\ldots,|\lambda|\}$ and let φ_1 be the automorphism on $A(\mathcal{G},c)$ that it induces. It is not hard to verify that

$$B := \pi_{\mathrm{ab}}(\overline{\varphi_1})^{-1} \cdot \prod_{i=1}^{d} T(\sigma_i, B_i)$$

is an element of $\prod_{\lambda\in\Lambda_{\mathcal{G}}} \mathrm{GL}(\lambda,\mathbb{Z})$. Thus by Lemma 7.2.11, there exists an automorphism φ_2 of $A(\mathcal{G},c)$ such that $\pi_{\mathrm{ab}}(\overline{\varphi_2}) = B$. From this it follows that the automorphism $\varphi = \varphi_1 \circ \varphi_2$ satisfies

$$\pi_{\mathrm{ab}}(\overline{\varphi}) = \pi_{\mathrm{ab}}(\overline{\varphi_1}) \circ \pi_{\mathrm{ab}}(\overline{\varphi_2}) = \pi_{\mathrm{ab}}(\overline{\varphi_1}) \circ B = \prod_{i=1}^{d} T(\sigma_i, B_i).$$

This completes the proof. □

We are now ready to prove the corollary which can be seen as a sort of converse to what is stated in Remark 7.2.7.

Corollary 7.2.13. *Let ϕ be any graded integer-like automorphism of $\mathfrak{n}^{\mathbb{Q}}(\mathcal{G},c)$. Then there exists an automorphism φ of $A(\mathcal{G},c)$ such that its induced automorphism $\overline{\varphi}$ on $\mathfrak{n}^{\mathbb{Q}}(\mathcal{G},c)$ has the same characteristic polynomial as ϕ.*

Proof. Let g be the semi-simple part of $\pi_{\mathrm{ab}}(\phi)$. Applying Lemma 7.2.9 to g, we find that g is conjugated in $G^{\mathbb{Q}}(\mathcal{G})$ to $\prod_{i=1}^d T(\sigma_i, B_i)$ where $\psi \in \mathrm{Aut}(\overline{\mathcal{G}})$ with disjoint cycle decomposition $\psi = \sigma_1 \circ \ldots \circ \sigma_d$ and $B_i \in \mathrm{GL}(n_i,\mathbb{Z})$. By Lemma 7.2.12 there exists an automorphism φ of $A(\mathcal{G},c)$ such that $\pi_{\mathrm{ab}}(\overline{\varphi}) = \prod_{i=1}^d T(\sigma_i, B_i)$. We thus see that the semi-simple part of $\pi_{\mathrm{ab}}(\phi)$ is conjugated to $\pi_{\mathrm{ab}}(\overline{\varphi})$, but since $\pi_{\mathrm{ab}} : \mathrm{Aut}_g(\mathfrak{n}^{\mathbb{Q}}(\mathcal{G},c)) \to \mathrm{GL}(\mathrm{span}_{\mathbb{Q}}(V))$ is a linear algebraic group isomorphism the same must hold for ϕ and φ, i.e. the semi-simple part of ϕ is conjugated to φ. This shows that ϕ and φ have the same characteristic polynomial and completes the proof. □

7.2.3 Non-zero Lie brackets in $\mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c)$

In this section we prove a result that, for a given subset $W \subset \operatorname{span}_{\mathbb{C}}(V)$ satisfying certain assumptions, allows one to construct folded Lie brackets of elements in W which are non-zero in $\mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c)$. We use the same notation as introduced in Section 4.4. Recall that for any subset $W \subset \mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c)$ we have an evaluation map

$$\phi_W^c : \mathrm{BW}(W) \to \mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c)$$

where we use the superscript c to emphasize the codomain of this map.

Define the set of weights on V with sum at most c as

$$\mathcal{E}(V, c) = \left\{ e : V \to \mathbb{N} \;\middle|\; \sum_{v \in V} e(v) \leq c \right\}. \tag{7.6}$$

The next Lemma follows immediately from combining Lemma 4.6.10 and Theorem 4.4.8.

Lemma 7.2.14. *Let $\mathcal{G} = (V, E)$ be a graph, $c > 1$ an integer and $e \in \mathcal{E}(V, c)$ a weight. If $|\operatorname{supp}(e)| \geq 2$ and $\operatorname{supp}(e)$ is connected, then there exists a bracket word $b \in \mathrm{BW}(V)$ of weight e such that $\phi_V^c(b)$ is non-zero.*

As a vector space, $\mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c)$ can be written as a direct sum according to the weights of bracket words in V as follows:

$$\mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c) = \bigoplus_{e \in \mathcal{E}(V,c)} \operatorname{span}_{\mathbb{C}} \left(\{ \phi_V^c(b) \mid b \in \mathrm{BW}(V) \text{ of weight } e \} \right). \tag{7.7}$$

Using this decomposition, we can prove the following generalization of Lemma 7.2.14.

Lemma 7.2.15. *Let $\mathcal{G} = (V, E)$ be a graph and $A \subset V$ a connected subset with $|A| \geq 2$. Let $W \subset \operatorname{span}_{\mathbb{C}}(V) \subset \mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c)$ be a finite set of vectors such that there exists a surjective map $\kappa : W \to A$ which satisfies*

$$\forall w \in W : \; w \in \operatorname{span}_{\mathbb{C}}(\kappa(w) \cup (V \setminus A)) \quad \textit{and} \quad w \notin \operatorname{span}_{\mathbb{C}}(V \setminus A).$$

For any weight $e \in \mathcal{E}(W, c)$ with $\operatorname{supp}(e) = W$, there exists a bracket word $b \in \mathrm{BW}(W)$ of weight e such that $\phi_W^c(b)$ is non-zero.

Proof. Note that the map $\kappa : W \to A$ induces a map on bracket words $\overline{\kappa} : \mathrm{BW}(W) \to \mathrm{BW}(V)$ defined inductively by $\overline{\kappa}(w) = \kappa(w)$ if w is a bracket word of length 1 (and thus an element of W), and $\overline{\kappa}([b_1, b_2]) = [\overline{\kappa}(b_1), \overline{\kappa}(b_2)]$ for any $b_1, b_2 \in \mathrm{BW}(W)$. From the weight $e : W \to \mathbb{N}$ we define a new weight $\tilde{e} \in \mathcal{E}(V, c)$ by

$$\tilde{e} : V \to \mathbb{N} : a \mapsto \begin{cases} \sum_{w \in \kappa^{-1}(a)} e(w) & \text{if } a \in A \\ 0 & \text{else.} \end{cases}$$

From the fact that $\mathrm{supp}(e) = W$ and that κ is surjective, it follows that $\mathrm{supp}(\tilde{e}) = A$. Using Lemma 7.2.14, there exists a bracket word $\tilde{b} \in \mathrm{BW}(V)$ with weight $\tilde{e}$, such that $\phi_V^c(\tilde{b}) \in \mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c)$ is non-zero. By the way we constructed $\tilde{e}$ from e, it is clear that one can choose a bracket word $b \in \mathrm{BW}(W)$ such that $\overline{\kappa}(b) = \tilde{b}$ and such that b has weight e.

For each $w \in W$, let $f_w : V \to \mathbb{C}$ denote the unique function such that $w = \sum_{v \in V} f_w(v)v$, thus f_w expresses the coordinates of w with respect to V. From the assumption on κ, it is clear that for any $w \in W$ we have

$$w = f_w(\kappa(w))\kappa(w) + \sum_{v \in V \setminus A} f_w(v)v$$

with $f_w(\kappa(w)) \neq 0$. Using the linearity of the Lie bracket in $\mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c)$, we can thus rewrite the element $\phi_W(b)$ as a sum

$$\phi_W^c(b) = \left(\prod_{w \in W} f_w(\kappa(w))^{e(w)} \right) \cdot \phi_V^c(\tilde{b}) + \sum_{d \in D} a_d \cdot \phi_V^c(d)$$

for some coefficients $a_d \in \mathbb{C}$ and a subset $D \subset \mathrm{BW}(V)$ of bracket words in V each having a weight with support not contained in A. In particular, each $d \in D$ has weight not equal to $\tilde{e}$, the weight of $\tilde{b}$. Using the vector space direct sum decomposition of $\mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c)$ as given in (7.7) and the fact that $f_w(\kappa(w)) \neq 0$ for all $w \in W$ and $\phi_V^c(\tilde{b}) \neq 0$, it follows that $\phi_W^c(b)$ is non-zero. □

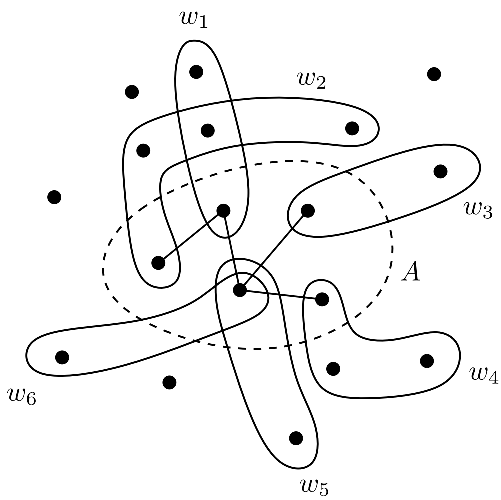


Figure 7.2: A visualization of the condition on the support of the vectors $w_i \in W$ from Lemma 7.2.15. The map κ maps a vector w_i to the unique vertex in the intersection of A with the support of w_i.

In Section 7.2.5 we will apply the above lemma to a subset W of eigenvectors of the automorphism $\varpi_c(T(\sigma_1, \alpha_1) \cdot \ldots \cdot T(\sigma_d, \alpha_d))$, namely a subset of the eigenvectors $w(\sigma, \alpha)_{ij}$ from equation (7.5). This will give us a tool for constructing an eigenvector with eigenvalue 1.

7.2.4 Transposition-free graphs

In this section we prove Theorem 7.2.2. This is an improvement to a result in [DS21], where it is proven (although not stated as a theorem) that the R_∞-nilpotency index of the RAAGs associated to the smaller class of non-empty transvection-free graphs is less than or equal to 3. Note that the author uses the opposite convention to define RAAGs, namely that adjacent vertices commute.

Let $\mathcal{G} = (V, E)$ be a non-empty transposition-free graph. This is equivalent with saying that all its coherent components are singletons and that $\mathcal{G}$ has at least 2 vertices. In this case, the graph $\mathcal{G}$ and its quotient graph $\overline{\mathcal{G}}$ are essentially the same and we get a natural identification $\mathrm{Aut}(\mathcal{G}) \cong \mathrm{Aut}(\overline{\mathcal{G}})$, which we use in what follows.

Applying Lemma 7.2.10 to $\mathcal{G}$ for any $c > 1$, we find that for any automorphism φ of $A(\mathcal{G}, c)$, there exists a graph automorphism $\psi \in \mathrm{Aut}(\mathcal{G}) \cong \mathrm{Aut}(\overline{\mathcal{G}})$ with disjoint cycle decomposition

$$\psi = \sigma_1 \circ \ldots \circ \sigma_d, \qquad \sigma_i = (v_{i1} v_{i2} \ldots v_{ik_i})$$

where $v_{ij} \in V$ and there exist $\alpha_1, \ldots, \alpha_d \in \{-1, 1\}$ such that the induced automorphism $\overline{\varphi} \in \mathrm{Aut}_g(\mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c))$ has the same characteristic polynomial as the automorphism $\varpi_c \left(\prod_{i=1}^{d} T(\sigma_i, \alpha_i) \right)$ where the maps $T(\sigma_i, \alpha_i)$ have matrix representation (± 1) if $k_i = 1$ and

$$\begin{pmatrix} 0 & & & \pm 1 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}$$

if $k_i > 1$ with respect to the basis $v_{i1}, v_{i2}, \ldots, v_{ik_i}$. Combining this with Lemma 7.2.6, we find that in order to prove Theorem 7.2.2, it suffices to show that $\theta := \pi_2^{-1} \left(\prod_{i=1}^{d} T(\sigma_i, \alpha_i) \right)$ has an eigenvector with eigenvalue 1. In the following cases, we can always write down such an eigenvector.

C1. There exists an index $i \in \{1, \ldots, d\}$ such that $\alpha_i = 1$. Then it is clear that $v_{i1} + \ldots + v_{ik_i} \in \mathfrak{n}_1^{\mathbb{C}}(\mathcal{G}, 2)$ is an eigenvector with eigenvalue 1.

C2. There exists indices $i \in \{1, \ldots, d\}$, $s, t \in \{1, \ldots, k_i\}$ such that $\alpha_i = -1$ and $\{v_{is}, v_{it}\} \in E$. Depending on whether $k_i = 2|s - t|$ or not, the vector

$$\sum_{m=1}^{k_i/2} [\theta^m(v_{is}), \theta^m(v_{it})] \quad \text{or} \quad \sum_{m=1}^{k_i} [\theta^m(v_{is}), \theta^m(v_{it})],$$

respectively, will be an eigenvector with eigenvalue 1 in $\mathfrak{n}_2^{\mathbb{C}}(\mathcal{G}, 2)$.

C3. There exist indices $i, j \in \{1, \ldots, d\}, s \in \{1, \ldots, k_i\}, t \in \{1, \ldots, k_j\}$ such that $\alpha_i = \alpha_j = -1$, $\{v_{is}, v_{jt}\} \in E$ and $\mathrm{lcm}(k_i, k_j) \left(\frac{1}{k_i} + \frac{1}{k_j} \right)$ is even. Then the vector

$$\sum_{m=1}^{\mathrm{lcm}(k_i, k_j)} [\theta^m(v_{is}), \theta^m(v_{jt})]$$

will be an eigenvector with eigenvalue 1 on $\mathfrak{n}_2^{\mathbb{C}}(\mathcal{G}, 2)$.

In [DS21], the authors consider some other cases where an eigenvector with eigenvalue one is constructed on $\mathfrak{n}_3^{\mathbb{C}}(\mathcal{G}, c)$. In what follows, we will exploit the fact that the graph is assumed to be non-empty transposition-free to show that at least one of the cases C1., C2. or C3. is always true. This is formulated in Proposition 7.2.17 and thus proves Theorem 7.2.2 from the introduction.

First, we prove the following lemma which tells us something about the neighbours of vertices lying in the same cycle of an automorphism $\psi \in \text{Aut}(\mathcal{G})$. Recall the definition of the open neighbourhood $N(v)$ of a vertex $v \in V$ from Section 4.1.

Lemma 7.2.16. *Let $\mathcal{G}$ be a graph and $\psi \in \text{Aut}(\mathcal{G})$ a graph automorphism with disjoint cycle decomposition $\psi = \sigma_1 \circ \sigma_2 \circ \ldots \circ \sigma_d$, where $\sigma_i = (v_{i1}\, v_{i2} \ldots v_{ik_i})$. For any $i, j \in \{1, \ldots, d\}$, $s, t \in \{1, \ldots, k_i\}$ it holds that if $\gcd(k_i, k_j) \mid (s-t)$, then $N(v_{is}) \cap V_j = N(v_{it}) \cap V_j$ where V_j denotes the set of vertices in the cycle σ_j.*

Proof. Assume $\gcd(k_i, k_j) \mid (s-t)$. It follows by the theorem of Bachet-Bézout, that there exist integers $\alpha, \beta \in \mathbb{Z}$ such that $\alpha k_i + t - s = \beta k_j$. Take an arbitrary vertex $w \in N(v_{is}) \cap V_j$. By definition we have that $\{v_{is}, w\} \in E$. Since ψ is a graph automorphism we have as well that $\{\psi^{\alpha k_i + t - s}(v_{is}), \psi^{\beta k_j}(w)\} \in E$. Using the cycle decomposition of ψ, we get the equality $\{\psi^{\alpha k_i + t - s}(v_{is}), \psi^{\beta k_j}(w)\} = \{v_{it}, w\}$. This shows that $w \in N(v_{it})$ and thus that $N(v_{is}) \cap V_j \subseteq N(v_{it}) \cap V_j$. This finishes the proof since the other inclusion is analogous . □

Using the above lemma we can prove our main structural result on a graph and the cycle decomposition of one of its automorphisms.

Proposition 7.2.17. *Let $\mathcal{G}$ be a non-empty transposition-free graph and $\psi \in \text{Aut}(\mathcal{G})$ with disjoint cycle decomposition $\psi = \sigma_1 \circ \sigma_2 \circ \ldots \circ \sigma_d$, where $\sigma_i = (v_{i1}\, v_{i2} \ldots v_{ik_i})$. Then at least one of the following is true:*

(i) $\exists\, i \in \{1, \ldots, d\}$, $s, t \in \{1, \ldots, k_i\} : \{v_{is}, v_{it}\} \in E$.

(ii) $\exists\, i, j \in \{1, \ldots, d\}$, $s \in \{1, \ldots, k_i\}$, $t \in \{1, \ldots, k_j\} : i \neq j$, $\{v_{is}, v_{jt}\} \in E$ *and* $\text{lcm}(k_i, k_j) \cdot \left(\frac{1}{k_i} + \frac{1}{k_j}\right)$ *is even.*

Proof. We prove this statement by contradiction. Assume both (i) and (ii) do not hold. Let us define for any $i \in \{1, \ldots, d\}$ the set of indices

$$\Psi(i) = \{j \in \{1, \ldots, d\} \setminus \{i\} \mid \exists s \in \{1, \ldots, k_i\},\, t \in \{1, \ldots, k_j\},\, \{v_{is}, v_{jt}\} \in E\}.$$

From the assumption that (ii) does not hold, it follows immediately that for any $j \in \Psi(i)$ the integer $\text{lcm}(k_i, k_j) \cdot \left(\frac{1}{k_i} + \frac{1}{k_j}\right)$ is odd. We can write each cycle length k_i uniquely as $k_i = 2^{e_i} m_i$ with $e_i \in \mathbb{N}$, $m_i \in \mathbb{N} \setminus \{0\}$ and $2 \nmid m_i$. Now suppose that $j \in \Psi(i)$ and that $e_i = e_j$. We then have that

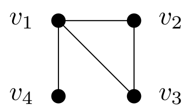
$$\begin{aligned}\operatorname{lcm}(k_i, k_j) \cdot \left(\frac{1}{k_i} + \frac{1}{k_j} \right) &= \frac{\operatorname{lcm}(k_i, k_j)}{k_i} + \frac{\operatorname{lcm}(k_i, k_j)}{k_j} \\ &= \frac{2^{e_i} \operatorname{lcm}(m_i, m_j)}{2^{e_i} m_i} + \frac{2^{e_i} \operatorname{lcm}(m_i, m_j)}{2^{e_i} m_j} \\ &= \frac{\operatorname{lcm}(m_i, m_j)}{m_i} + \frac{\operatorname{lcm}(m_i, m_j)}{m_j}\end{aligned}$$

is even, since both summands $\operatorname{lcm}(m_i, m_j)/m_i$ and $\operatorname{lcm}(m_i, m_j)/m_j$ are odd. Clearly this contradicts the assumption and thus we must conclude that for any $j \in \Psi(i)$, $e_i \neq e_j$.

Take any $i \in \{1, \ldots, d\}$ for which $\Psi(i)$ is non-empty and suppose that $e_j < e_i$ for any $j \in \Psi(i)$. Then we get for any $j \in \Psi(i)$ that $\gcd(k_i, k_j) = 2^{e_j} \gcd(m_i, m_j)$ which divides $2^{e_i - 1} m_i = k_i/2$. By Lemma 7.2.16 we thus have for any $j \in \Psi(i)$ that $N(v_{i1}) \cap V_j = N(v_{ir_i}) \cap V_j$ where we write $r_i := 1 + k_i/2$. Note that by the definition of $\Psi(i)$, we also have that $N(v_{i1}) \cap V_j = \emptyset = N(v_{ir_i}) \cap V_j$ for any $j \in \{1, \ldots, d\} \setminus (\Psi(i) \cup \{i\})$. At last, we also find that $N(v_{i1}) \cap V_i = \emptyset = N(v_{ir_i}) \cap V_i$ as a consequence of the assumption that condition (i) does not hold. Since $\{V_i \mid 1 \leq i \leq d\}$ is a partition of V, we can combine these equalities to get $N(v_{i1}) = N(v_{ir_i})$ which contradicts the fact that $\mathcal{G}$ is transposition-free. Therefore we must have that if $\Psi(i)$ is non-empty, then there exists a $j \in \Psi(i)$ such that $e_j > e_i$.

Since $\mathcal{G}$ is not the empty graph and since we assume that (i) does not hold, we must have that there exists an $i_0 \in \{1, \ldots, d\}$ such that $\Psi(i_0)$ is non-empty. By the above, we know that there is an index $i_1 \in \Psi(i_0)$ such that $e_{i_1} > e_{i_0}$. Note that $\Psi(i_1)$ is also non-empty since $i_0 \in \Psi(i_1)$. We can repeat the argument to find an index $i_2 \in \Psi(i_1)$ such that $e_{i_2} > e_{i_1}$. If we keep repeating this process, we clearly get a contradiction since the set $\{e_i \mid i \in \{1, \ldots, d\}\}$ is finite. □

The above proves that if a graph $\mathcal{G}$ is transposition-free and has more than one vertex, then $A(\mathcal{G})$ has R_∞-nilpotency index 2. The converse does not hold. Indeed, consider the graph on four vertices as drawn below.



This graph is clearly not transposition-free, since the transposition of v_2 with v_3 is a graph automorphism, but $A(\mathcal{G}, 2)$ does have the R_∞-property. This is not so hard to prove using Lemma 7.2.10. For an actual proof, we also refer to [DL23] (where we note that the authors use the opposite convention to define a RAAG). This raises the following question.

Question 7.2.18. For which graphs $\mathcal{G}$ does $A(\mathcal{G})$ have R_∞-nilpotency index equal to 2?

7.2.5 Bounds on the R_∞-nilpotency index of a RAAG

In this section we prove Theorem 7.2.4. We divide the proof into two parts. The first part deals with the lower bound $\xi(\mathcal{G})$ and the second with the upper bound $\Xi(\mathcal{G})$ as defined by equations (7.1) and (7.2), respectively.

Lower bound

First, we show the existence of finitely many polynomials of arbitrary degree for which certain products of their roots are not equal to one. These polynomials will then be used to construct automorphisms on $A(\mathcal{G}, c)$ with finite Reidemeister number. A useful tool to construct these polynomials are the so called Pisot units. A *Pisot number* is a real algebraic integer greater than one such that all its conjugates have absolute value stricly less than one. A *Pisot unit* is a Pisot number which is also an algebraic unit, i.e. an algebraic integer for which the constant term of its minimal polynomial over $\mathbb{Q}$ is equal to ± 1. A proof of the following fact can be found in Lemma 2.7. of [DG14] and in Proposition 3.6.(3) of [Pay09].

Lemma 7.2.19. *Let α_1 be a Pisot unit with conjugates $\alpha_2, \ldots, \alpha_d$. For any $e = (e_1, \ldots, e_d) \in \mathbb{Z}^d$ it holds that if $\prod_{i=1}^{d} \alpha_i^{e_i} = 1$ then $e_1 = \ldots = e_d$.*

We can now prove the following existence result.

Lemma 7.2.20. *Let $n > 0$, $c > 1$ and $d_1, \ldots, d_n > 0$ be any positive integers. There exist monic irreducible polynomials $p_1(X), \ldots, p_n(X) \in \mathbb{Z}[X]$ of degree $d_1, \ldots, d_n$, respectively, such that if $\alpha_{i1}, \ldots, \alpha_{id_i} \in \mathbb{C}$ denote the the zeros of $p_i(X)$, the following are true:*

(i) $\forall\, i \in \{1, \ldots, n\} : \alpha_{i1} \cdot \ldots \cdot \alpha_{id_i} = -1,$

(ii) $\forall e = (e_{11}, \ldots, e_{1d_1}, e_{21}, \ldots, e_{2d_2}, \ldots, e_{n1}, \ldots, e_{nd_n}) \in (\mathbb{N})^{d_1+\ldots+d_n}$ *with* $\sum_{i=1}^{n}\sum_{j=1}^{d_i} e_{ij} \leq c$*:*

$$\prod_{i=1}^{n}\prod_{j=1}^{d_i} \alpha_{ij}^{e_{ij}} = 1 \quad \Rightarrow \quad (\forall i \in \{1, \ldots, n\} : \, e_{i1} = \ldots = e_{id_i}\,).$$

Proof. From Lemma 2.7. in [DG14], it follows that for all $i \in \{1, \ldots, n\}$, there exists a monic $\mathbb{Q}$-irreducible polynomial $q_i(X) \in \mathbb{Z}[X]$ of degree d_i such that if $\beta_{i1}, \ldots, \beta_{id_i}$ denote its roots, we have $\beta_{i1} \cdot \ldots \cdot \beta_{id_i} = -1$ and β_{i1} is a Pisot number. Next, for any $e = (e_{11}, \ldots, e_{1d_1}, e_{21}, \ldots, e_{2d_2}, \ldots, e_{n1}, \ldots, e_{nd_n}) \in (\mathbb{N})^{d_1+\ldots+d_n}$ define the map

$$\varphi_e : \mathbb{Z}^n \to \mathbb{C}^* : (k_1, \ldots, k_n) \mapsto \prod_{i=1}^{n}\prod_{j=1}^{d_i} \left(\beta_{ij}^{k_i}\right)^{e_{ij}},$$

which is a group morphism between the additive group $\mathbb{Z}^n$ and the multiplicative group $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. The kernel $\ker(\varphi_e)$ is a free abelian subgroup of $\mathbb{Z}^n$, with rank less or equal to n. Assume that $\operatorname{rank}(\ker(\varphi_e)) = n$. Take any $i \in \{1, \ldots, n\}$. Then there exists a $k_i \in \mathbb{Z}$ such that the element $(0, \ldots, 0, k_i, 0, \ldots, 0)$, where the non-zero element is on the i-th entry, lies in $\ker(\varphi_e)$. We thus have that $\prod_{j=1}^{d_i} \left(\beta_{ij}^{k_i}\right)^{e_{ij}} = 1$. Note that $\beta_{i1}^{k_i}$ is still a Pisot number with conjugates $\beta_{i2}^{k_i}, \ldots, \beta_{id_i}^{k_i}$. By Lemma 7.2.19 we thus have that $e_{i1} = \ldots = e_{id_i}$. Since i was chosen arbitrarily, we get $\forall i \in \{1, \ldots, n\} : \, e_{i1} = \ldots = e_{id_i}$. This proves that for any $e \in \mathbb{Z}^{d_1+\ldots+d_n}$ with $\neg(\forall i \in \{1, \ldots, n\} : \, e_{i1} = \ldots = e_{id_i})$ the kernel $\ker(\varphi_e)$ has rank strictly less than n. As a consequence

$$\mathcal{H} = \left\{ \ker(\varphi_e) \;\middle|\; \begin{array}{c} e \in (\mathbb{N})^{d_1+\ldots+d_n} : \, \neg(\forall i \in \{1, \ldots, n\} : \, e_{i1} = \ldots = e_{id_i}) \\ \text{and } \sum_{i=1}^{n}\sum_{j=1}^{d_i} e_{ij} \leq c \end{array} \right\}$$

is a finite collection of subgroups of $\mathbb{Z}^n$ of rank strictly less than n. It follows that we can find an element $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ such that $m \notin \bigcup_{H \in \mathcal{H}} H$ and such that each integer m_i is odd. For each $i \in \{1, \ldots, n\}$, we then define the polynomial

$$p_i(X) = (X - \beta_{i1}^{m_i}) \cdot \ldots \cdot (X - \beta_{id_i}^{m_i}).$$

As one can check, these polynomials satisfy all requirements of the lemma. $\square$

We also need the following result which was proven in Proposition 3.7. from [DW23a] and tells us what the eigenvalues are of an automorphism $\varphi \in \operatorname{Aut}(\mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c))$ which is diagonal on the vertices V. Recall the notation

$\mathcal{E}(V,c)$ as introduced in Section 7.2.3 and Proposition 4.6.14 from Section 4.6.3 which we reformulate below.

Proposition 7.2.21. *Let $\mathcal{G}$ be a graph and $c > 1$. Let $\varphi \in \mathrm{Aut}(\mathfrak{n}^{\mathbb{C}}(\mathcal{G},c))$ be a vertex-diagonal automorphism, i.e. an automorphism such that there exists a function $\Psi : V \to \mathbb{C}^*$ with $\forall v \in V : \varphi(v) = \Psi(v)v$. The set of eigenvalues of φ is equal to*

$$\Psi(V) \cup \left\{ \prod_{v \in V} \Psi(v)^{e(v)} \;\middle|\; \begin{array}{l} e \in \mathcal{E}(V,c),\ |\operatorname{supp}(e)| \geq 2, \\ \operatorname{supp}(e) \textit{ is connected in } \mathcal{G} \end{array} \right\}.$$

We are now ready to prove the lower bound on the R_∞-nilpotency index of $A(\mathcal{G})$ of which we recall the definition:

$$\xi(\mathcal{G}) = \min \left\{ |\lambda| + |\mu| \;\middle|\; \lambda, \mu \in \Lambda_{\mathcal{G}},\ \{\lambda, \mu\} \in \overline{E} \right\}.$$

Theorem 7.2.22. *Let $\mathcal{G}$ be a non-empty graph. The group $A(\mathcal{G}, \xi(\mathcal{G}) - 1)$ does not have the R_∞-property.*

Proof. Let $\overline{\mathcal{G}} = (\Lambda_{\mathcal{G}}, \overline{E}, \Phi)$ be the quotient graph of $\mathcal{G}$ and write $c = \xi(\mathcal{G}) - 1$. Using Lemma 7.2.20, we can find for any $\lambda \in \Lambda_{\mathcal{G}}$ a monic irreducible polynomial $p_\lambda(X) \in \mathbb{Z}[X]$ of degree $|\lambda|$ with eigenvalues $\alpha_{\lambda 1}, \ldots, \alpha_{\lambda|\lambda|}$ such that $\alpha_{\lambda 1} \cdot \ldots \cdot \alpha_{\lambda|\lambda|} = -1$ and $\forall e_{\lambda i} \in \mathbb{N}$ with $\sum_{\lambda \in \lambda} \sum_{j=1}^{|\lambda|} e_{\lambda j} \leq c$:

$$\prod_{\lambda \in \Lambda_{\mathcal{G}}} \prod_{j=1}^{|\lambda|} \alpha_{\lambda j}^{e_{\lambda j}} = 1 \quad \Rightarrow \quad \left(\forall \lambda \in \Lambda_{\mathcal{G}} : e_{\lambda 1} = \ldots = e_{\lambda|\lambda|} \right).$$

Recall that we fixed an ordering of the vertices in each coherent component $\lambda = \{v_{\lambda 1}, \ldots, v_{\lambda|\lambda|}\}$. For each $\lambda \in \Lambda_{\mathcal{G}}$, let $B_\lambda \in \mathrm{GL}(\mathbb{Z}, \lambda)$ be the linear map such that its matrix representation w.r.t. the basis $v_{\lambda 1}, \ldots, v_{\lambda|\lambda|}$ is given by the companion matrix of $p_\lambda(X)$. This gives an element $B = \prod_{\lambda \in \Lambda_{\mathcal{G}}} B_\lambda$ in $\prod_{\lambda \in \Lambda_{\mathcal{G}}} \mathrm{GL}(\mathbb{Z}, \lambda)$. By Lemma 7.2.11, there exists an automorphism $\varphi \in \mathrm{Aut}(A(\mathcal{G},c))$ such that, if $\overline{\varphi} \in \mathrm{Aut}_g(\mathfrak{n}^{\mathbb{C}}(\mathcal{G},c))$ denotes the induced automorphism on the Lie algebra, $\pi_{\mathrm{ab}}(\overline{\varphi}) = B$. Equivalently we have $\varpi_c(B) = \overline{\varphi}$. Note that, by Lemma 7.2.6, if $\varpi_c(B) = \overline{\varphi}$ does not have an eigenvalue equal to 1, the Reidemeister number of φ is not equal to ∞ and thus $A(\mathcal{G},c)$ does not have the R_∞-property. Let us prove by contradiction that this is indeed the case.

Assume $\varpi_c(B)$ has an eigenvalue equal to one. Note that companion matrices are diagonalizable and thus that for each $\lambda \in \Lambda_{\mathcal{G}}$ there exists a $Q_\lambda \in \mathrm{GL}(\mathrm{span}_{\mathbb{C}}(\lambda))$ such that the matrix representation of $Q_\lambda B_\lambda Q_\lambda^{-1}$ w.r.t. the basis $v_{\lambda 1}, \ldots, v_{\lambda|\lambda|}$

is given by $\operatorname{diag}(\alpha_{\lambda 1}, \ldots, \alpha_{\lambda|\lambda|})$. It follows that

$$D := \varpi_c \left(\left(\prod_{\lambda \in \Lambda_{\mathcal{G}}} Q_\lambda \right) B \left(\prod_{\lambda \in \Lambda_{\mathcal{G}}} Q_\lambda \right)^{-1} \right)$$

is an automorphism of $\mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c)$ which is diagonal on the vertices and which has the same eigenvalues as $\varpi_c(B)$. We thus have that D has an eigenvalue equal to one. Note that this eigenvalue can not occur on $\operatorname{span}_{\mathbb{C}}(V)$ since $\alpha_{\lambda j} \neq 1$ for all $\lambda \in \Lambda_{\mathcal{G}}$ and $j \in \{1, \ldots, |\lambda|\}$. By Lemma 7.2.21 above, there must therefore exist $e_{\lambda j} \in \mathbb{N}$ such that

$$s := \sum_{\lambda \in \Lambda_{\mathcal{G}}} \sum_{j=1}^{|\lambda|} e_{\lambda j} \leq c, \qquad \prod_{\lambda \in \Lambda_{\mathcal{G}}} \prod_{j=1}^{|\lambda|} \alpha_{\lambda j}^{e_{\lambda j}} = 1$$

and the set $A := \{v_{\lambda j} \mid \lambda \in \Lambda_{\mathcal{G}}, 1 \leq j \leq |\lambda|, e_{\lambda j} \neq 0\}$ has size greater or equal to two and is connected in $\mathcal{G}$. From the properties of the polynomials $p_\lambda(X)$, we have for each $\lambda \in \Lambda_{\mathcal{G}}$ that $e_{\lambda 1} = \ldots = e_{\lambda|\lambda|}$. As a consequence A is equal to a union of coherent components. This gives two cases:

- If there is only one coherent component in this union, say $A = \lambda \in \Lambda_{\mathcal{G}}$, then it follows from A being connected, that $\{\lambda\} \in \overline{E}$. From the definition of $\xi(\mathcal{G})$ we thus have that $\xi(\mathcal{G}) \leq 2|\lambda|$. We must also have that $e_{\lambda 1} = \ldots = e_{\lambda|\lambda|}$ are even integers since $\alpha_{\lambda 1} \cdot \ldots \cdot \alpha_{\lambda|\lambda|} = -1$. We thus have that $2|\lambda| \leq s$ which implies $\xi(\mathcal{G}) \leq 2|\lambda| \leq s$. This gives a contradiction with $s \leq c = \xi(\mathcal{G}) - 1$.
- If, on the other hand, A contains more than one coherent component, then there must exist $\lambda, \mu \subset A$ with $\lambda \neq \mu$ and $\{\lambda, \mu\} \in \overline{E}$ since A is connected. We thus have that $\xi(\mathcal{G}) \leq |\lambda| + |\mu| \leq s$, which again contradicts the fact that $s \leq c = \xi(\mathcal{G}) - 1$.

Thus, we must conclude that $\varpi_c(B) = \overline{\varphi}$ does not have an eigenvalue equal to one. □

Upper bound

For the upper bound (see (7.2) for the definition), one has to show that if $c = \Xi(\mathcal{G})$ then for each automorphism $\varphi \in \operatorname{Aut}(A(\mathcal{G}, c))$, the induced automorphism $\overline{\varphi} \in \operatorname{Aut}_g(\mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c))$ has an eigenvalue one. We will construct such an eigenvalue 1 as a product of eigenvalues of $\overline{\varphi}$ on $\mathfrak{n}_1^{\mathbb{C}}(\mathcal{G}, c)$. If one wants the corresponding eigenvector in $\mathfrak{n}_i^{\mathbb{C}}(\mathcal{G}, c)$ with i as small as possible, one needs to solve the following problem: given two positive integers k, l and elements $\alpha, \beta \in \{-1, 1\}$, how can we obtain 1 as a product of k-roots of α and l-roots of β with the least factors as possible, where we do need to take at least one k-root of α and one l-root of β.

The answer, of course, depends on the integers k, l, α, β, but as it turns out, one always needs either 2 or 3 factors. In case $\alpha = \beta = 1$, the problem is trivial. Since 1 is a k-root of α and 1 is an l-root of β, one needs only two factors: $1 \cdot 1 = 1$. The cases $\alpha = \beta = -1$ and $\alpha = -1, \beta = 1$ are less trivial and are treated in the following two lemmas.

Recall that we defined the roots of unity

$$R_{st}(z) := \sqrt[s]{r} e^{i \frac{\theta + (t-1)2\pi}{s}}$$

for any non-zero complex number $z = re^{i\theta}$ with $r > 0$ and $0 \leq \theta < 2\pi$ and any positive integers $s > 0$ and $t \in \{1, \ldots, s\}$.

Lemma 7.2.23. *Take any positive integers $k, l > 0$ and write $M = \operatorname{lcm}(k, l)$. Then the following are true:*

(i) There exist integers $s \in \{1, \ldots, k\}$ and $t \in \{1, \ldots, l\}$ such that

$$R_{ks}(1) \cdot R_{lt}(1) = 1.$$

(ii) If $M \cdot \left(\frac{1}{k} + \frac{1}{l}\right)$ is even, then there exist integers $s \in \{1, \ldots, k\}$ and $t \in \{1, \ldots, l\}$ such that

$$R_{ks}(-1) \cdot R_{lt}(-1) = 1.$$

(iii) If M/k is odd and M/l is even, then there exist integers $s, r \in \{1, \ldots, k\}$ and $t \in \{1, \ldots, l\}$ such that

$$R_{ks}(-1) \cdot R_{kr}(-1) \cdot R_{lt}(-1) = 1.$$

Proof. Statement (i) is trivial, namely take $s = t = 1$. For (ii), note that M/k and M/l are coprime and thus by the Theorem of Bachet-Bézout, there exist integers $s, t \in \mathbb{Z}$ such that

$$\frac{M}{k} + \frac{M}{l} + 2\left((s-1)\frac{M}{k} + (t-1)\frac{M}{l}\right) \equiv 0 \mod 2M,$$

where we used the assumption that $M/k + M/l$ is even. Moreover, since the equation is modulo $2M$, it is clear that we can take s, t such that $s \in$

$\{1,\ldots,k\}$ and $t \in \{1,\ldots,l\}$. Dividing the left-hand side by M, we thus find that $\frac{1}{k}+\frac{1}{l}+\frac{2(s-1)}{k}+\frac{2(t-1)}{l} = \frac{2s-1}{k}+\frac{2t-1}{l}$ is an even integer. From this it follows that

$$R_{ks}(-1)\cdot R_{lt}(-1) = e^{i\left(\frac{\pi}{k}+(s-1)\frac{2\pi}{k}\right)}e^{i\left(\frac{\pi}{l}+(t-1)\frac{2\pi}{l}\right)} = e^{i\pi\left(\frac{2s-1}{k}+\frac{2t-1}{l}\right)} = 1.$$

For (iii), note that M/k and M/l are coprime and thus by the Theorem of Bachet-Bézout, there exist integers $s,r,t \in \mathbb{Z}$ such that

$$2\frac{M}{k}+\frac{M}{l}+2\left((s+r-2)\frac{M}{k}+(t-1)\frac{M}{l}\right) \equiv 0 \mod 2M,$$

where we used the assumption that M/l is even. Moreover, since the equation is modulo $2M$, it is clear that we can take s,r,t such that $s,r \in \{1,\ldots,k\}$ and $t \in \{1,\ldots,l\}$. Dividing the left-hand side by M, we thus find that $\frac{2}{k}+\frac{1}{l}+\frac{2(s+r-2)}{k}+\frac{2(t-1)}{l} = \frac{2s+2r-2}{k}+\frac{2t-1}{l}$ is an even integer. From this it follows that

$$\begin{aligned} R_{ks}(-1)\cdot R_{kr}(-1)\cdot R_{lt}(-1) &= e^{i\left(\frac{\pi}{k}+s\frac{2\pi}{k}\right)}e^{i\left(\frac{\pi}{k}+r\frac{2\pi}{k}\right)}e^{i\left(\frac{\pi}{l}+t\frac{2\pi}{l}\right)} \\ &= e^{i\pi\left(\frac{2s+2r-2}{k}+\frac{2t-1}{l}\right)} = 1. \end{aligned}$$

□

Lemma 7.2.24. *Take any positive integers $k,l > 0$ and write $M = \operatorname{lcm}(k,l)$. Then the following are true:*

(i) If M/k is even, then then there exist integers $s \in \{1,\ldots,k\}$ and $t \in \{1,\ldots,l\}$ such that
$$R_{ks}(-1)\cdot R_{lt}(1) = 1.$$

(ii) If M/k is odd, then then there exist integers $s,r \in \{1,\ldots,k\}$ and $t \in \{1,\ldots,l\}$ such that

$$R_{ks}(-1)\cdot R_{kr}(-1)\cdot R_{lt}(1) = 1.$$

Proof. For (i), note that M/k and M/l are coprime and thus by the Theorem of Bachet-Bézout, there exist integers $s,r,t \in \mathbb{Z}$ such that

$$\frac{M}{k}+2\left((s-1)\frac{M}{k}+(t-1)\frac{M}{l}\right) \equiv 0 \mod 2M,$$

where we used the assumption that M/k is even. Moreover, since the equation is modulo $2M$, it is clear that we can take s,t such that $s \in \{1,\ldots,k\}$ and

$t \in \{1, \ldots, l\}$. Dividing the left-hand side by M, we thus find that $\frac{1}{k} + \frac{2(s-1)}{k} + \frac{2(t-1)}{l} = \frac{2s-1}{k} + \frac{2t-2}{l}$ is an even integer. From this it follows that

$$R_{ks}(-1) \cdot R_{lt}(1) = e^{i\left(\frac{\pi}{k}+(s-1)\frac{2\pi}{k}\right)} e^{i(t-1)\frac{2\pi}{l}} = e^{i\pi\left(\frac{2s-1}{k}+\frac{2t-2}{l}\right)} = 1.$$

For (ii), note that M/k and M/l are coprime and thus by the Theorem of Bachet-Bézout, there exist integers $s, r, t \in \mathbb{Z}$ such that

$$2\frac{M}{k} + 2\left((s+r-2)\frac{M}{k} + (t-1)\frac{M}{l}\right) \equiv 0 \mod 2M.$$

Moreover, since the equation is modulo $2M$, it is clear that we can take s, r, t such that $s, r \in \{1, \ldots, k\}$ and $t \in \{1, \ldots, l\}$. Dividing the left-hand side by M, we thus find that $\frac{2}{k} + \frac{2(s+r-2)}{k} + \frac{2(t-1)}{l} = \frac{2s-2r-2}{k} + \frac{2t-2}{l}$ is an even integer. From this it follows that

$$\begin{aligned} R_{ks}(-1) \cdot R_{kr}(-1) \cdot R_{lt}(1) &= e^{i\left(\frac{\pi}{k}+(s-1)\frac{2\pi}{k}\right)} e^{i\left(\frac{\pi}{k}+(r-1)\frac{2\pi}{k}\right)} e^{i(t-1)\frac{2\pi}{l}} \\ &= e^{i\pi\left(\frac{2s-2r-2}{k}+\frac{2t-2}{l}\right)} = 1. \end{aligned}$$

□

In combination with the two preceding lemmas, the next lemma gives us an answer to a more general problem: given integers $k, l, n, m > 0$ and complex numbers $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m \in \mathbb{C}^*$ with $\prod_{i=1}^{n} \alpha_i = \pm 1$ and $\prod_{i=1}^{m} \beta_i = \pm 1$, how can we obtain 1 as a product of k-roots of the numbers α_i and l-roots of the numbers β_i, with as little factors as possible, but with at least one k-root of α_i for any $i \in \{1, \ldots, n\}$ and at least one l-root of β_i for any $i \in \{1, \ldots, m\}$.

Lemma 7.2.25. *Let $n > 0$, $k > 0$ be positive integers and $\alpha_1, \ldots, \alpha_n \in \mathbb{C} \setminus \{0\}$. For any integer $s \in \{1, \ldots, k\}$, there exist integers $s_1, \ldots, s_n \in \{1, \ldots, k\}$ such that*

$$R_{ks}\left(\prod_{i=1}^{n} \alpha_i\right) = \prod_{i=1}^{n} R_{ks_i}(\alpha_i).$$

Proof. Note that all elements in the set

$$S = \left\{ \prod_{i=1}^{n} R_{ks_i}(\alpha_i) \;\middle|\; s_1, \ldots, s_n \in \{1, \ldots, k\} \right\}$$

are indeed k-roots of $\prod_{i=1}^{n} \alpha_i$ since

$$\left(\prod_{i=1}^{n} R_{ks_i}(\alpha_i)\right)^k = \prod_{i=1}^{n} R_{ks_i}(\alpha_i)^k = \prod_{i=1}^{n} \alpha_i.$$

By only varying s_1 in $\{1, \dots, k\}$, it is easily verified that S must count at least k elements. We can thus conclude that S is exactly equal to the set of k-roots of $\prod_{i=1}^{n} \alpha_i$. From this observation, the lemma follows immediately. □

We are now ready to prove the upper bound on the R_∞-nilpotency index of $A(\mathcal{G})$ of which we recall the definition:

$$\Xi(\mathcal{G}) = \min \left\{ c(\lambda, \mu) \;\middle|\; \lambda, \mu \in \Lambda_{\mathcal{G}},\ \{\lambda, \mu\} \in \overline{E} \right\}$$

where for any $\lambda, \mu \in \Lambda_{\mathcal{G}}$:

$$c(\lambda, \mu) = \begin{cases} \max\{2|\lambda| + |\mu|, |\lambda| + 2|\mu|\} & \text{if } \lambda \neq \mu \\ 2|\lambda| & \text{if } \lambda = \mu. \end{cases}$$

Theorem 7.2.26. *Let $\mathcal{G} = (V, E)$ be a non-empty graph. The group $A(\mathcal{G}, \Xi(\mathcal{G}))$ has the R_∞-property.*

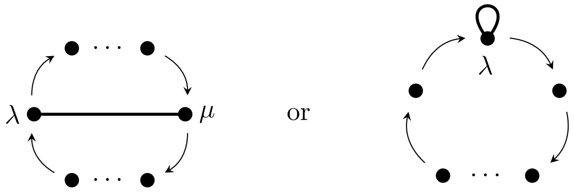
Proof. Let $\overline{\mathcal{G}} = (\Lambda_{\mathcal{G}}, \overline{E}, \Phi)$ be the quotient graph of $\mathcal{G}$. Take $\lambda, \mu \in \Lambda_{\mathcal{G}}$ arbitrarily such that $\{\lambda, \mu\} \in \overline{E}$. It suffices to prove that $A(\mathcal{G}, c)$ has the R_∞-property with $c = c(\lambda, \mu)$ (since we can then take the minimum of $c(\lambda, \mu)$ for all $\{\lambda, \mu\} \in \overline{E}$ in the end).

Take an arbitrary automorphism $\varphi \in \mathrm{Aut}(A(\mathcal{G}, c))$. By Lemma 7.2.6 it satisfies to prove that the induced automorphism $\overline{\varphi} \in \mathrm{Aut}_g(\mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c))$ has an eigenvalue 1. Using Lemma 7.2.10, it is sufficient to prove that for any automorphism $\psi \in \mathrm{Aut}(\overline{\mathcal{G}})$ with disjoint cycle decomposition $\psi = \sigma_1 \circ \ldots \circ \sigma_d$ and any tuples $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{in_i}) \in \mathbb{C}^{n_i}$ where n_i is the size of the coherent components in the cycle σ_i and $\alpha_{i1} \cdot \ldots \cdot \alpha_{in_i} = \pm 1$, it holds that the automorphism $\varpi_c(T(\sigma_1, \alpha_1) \cdot \ldots \cdot T(\sigma_d, \alpha_d))$ has an eigenvalue 1.

Note that every coherent component lies in a unique cycle from the decomposition of ψ. Let $p, q \in \{1, \ldots, d\}$ be the indices such that c_p and c_q denote these cycles containing λ and μ, respectively. For sake of notation, we write $\sigma := \sigma_p$, $\tau := \sigma_q$, $n = |\lambda|$, $m = |\mu|$, $\alpha = (\alpha_1, \ldots, \alpha_n) := \alpha_p$ and $\beta = (\beta_1, \ldots, \beta_m) := \alpha_q$ and let k, l denote the orders of the cycles σ and τ, respectively.

We divide the proof into several cases:

- $\underline{\sigma = \tau}$



Note that in this case also $n = m$, $\alpha = \beta$ and $k = l$.

- First consider the case where $n = m = 1$. Then we must have that $\lambda \neq \mu$, $c(\lambda, \mu) = 3$ and $T(\sigma, \alpha) = T(\sigma, (\pm 1))$. If $\alpha = 1$, then there exists an eigenvector with eigenvalue 1 in $\mathfrak{n}_1^{\mathbb{C}}(\mathcal{G}, c)$ by the same argument as in C1. from Section 7.2.4. If, on the other hand $\alpha = -1$, then there exists an eigenvector with eigenvalue 1 in $\mathfrak{n}_2^{\mathbb{C}}(\mathcal{G}, c)$ by the same argument as in C2. from Section 7.2.4.
- Next, consider the case where $n = m > 1$. Using that $\prod_{i=1}^{n} \alpha_i = \pm 1$, we can combine Lemma 7.2.23 (i) (ii) (with $k = l$) and 7.2.25 to get the existence of indices $s_1, \ldots, s_n, t_1, \ldots, t_n \in \{1, \ldots, k\}$ such that

$$\left(\prod_{i=1}^{n} R_{ks_i}(\alpha_i) \right) \left(\prod_{i=1}^{n} R_{kt_i}(\alpha_i) \right) = 1.$$

Next, consider the set $A := (\lambda \setminus \{v_{\lambda 1}\}) \cup \{v_{\mu 1}\}$ for which it is clear that $|A| \geq 2$. From the fact that $\{\lambda, \mu\} \in \overline{E}$ we have that $\{v, w\} \in E$ for any $v \in \lambda, w \in \mu$, hence A is connected. Define

$$W := \{ w(\sigma, \alpha)_{is_i},\, w(\sigma, \alpha)_{it_i} \mid 1 \leq i \leq n \}$$

where we remind the reader of the definition of $w(\sigma, \alpha)_{ij} \in \operatorname{span}_{\mathbb{C}}(V)$ in (7.5). Define a map

$$\kappa : W \to A : w(\sigma, \alpha)_{ij} \mapsto \begin{cases} v_{\mu i} & \text{if i = 1} \\ v_{\lambda i} & \text{else.} \end{cases}$$

It is clear that A, W and κ satisfy the assumptions of Lemma 7.2.15. Define the weight

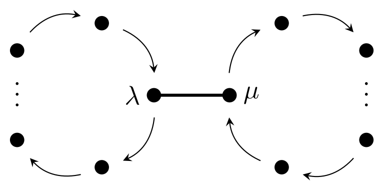
$$e : W \to \mathbb{N} : w(\sigma, \alpha)_{ij} \mapsto \begin{cases} 1 & \text{if } s_i \neq t_i \\ 2 & \text{else,} \end{cases}.$$

for which it holds that

$$\sum_{w \in W} e(w) = 2n \leq c(\lambda, \mu) = c.$$

As a consequence we can apply Lemma 7.2.15 to A, W and κ for the weight e and get a bracket word $b \in \mathrm{BW}(W)$ of weight e such that $\phi_W^c(b) \in \mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c)$ is non-zero. Since each element of W is an eigenvector of the automorphism $\varpi_c(T(\sigma_1, \alpha_1) \cdot \ldots \cdot T(\sigma_d, \alpha_d))$, it follows that $\phi_W^c(b)$ is still an eigenvector. By construction, it is clear that the eigenvalue of $\phi_W^c(b)$ is equal to 1.

- $\underline{\sigma \neq \tau.}$



Note that since $\{\lambda, \mu\} \in \overline{E}$, the set $A := \lambda \cup \mu$ is connected and that since $\lambda \neq \mu$ we have $|A| \geq 2$. By combining Lemmas 7.2.23, 7.2.24 and 7.2.25, we know one of following three cases is true, depending on the integers k and l and the signs of $\prod_{i=1}^n \alpha_i$ and $\prod_{i=1}^m \beta_i$. This dependence is summarized in Table 7.1 below.

P1. There exist $s_1, \ldots, s_n \in \{1, \ldots, k\}, t_1, \ldots, t_m \in \{1, \ldots, l\}$ such that

$$\left(\prod_{i=1}^{n} R_{ks_i}(\alpha_i) \right) \left(\prod_{i=1}^{m} R_{lt_i}(\beta_i) \right) = 1.$$

In this case, define

$$W := \{w(\sigma, \alpha)_{is_i} \mid 1 \leq i \leq n\} \cup \{w(\tau, \beta)_{it_i} \mid 1 \leq i \leq m\}$$

and

$$\kappa : W \to A : w(\sigma, \alpha)_{ij} \mapsto v_{\lambda i},\ w(\tau, \beta)_{ij} \mapsto v_{\mu i}.$$

It is clear that A, W and κ satisfy the assumptions of Lemma 7.2.15. Define the constant weight $e : W \to \mathbb{N} : w \mapsto 1$ for which we have

$$\sum_{w \in W} e(w) = n + m \leq c(\lambda, \mu) = c.$$

As a consequence we can apply Lemma 7.2.15 to A, W and κ for the weight e and get a bracket word $b \in \mathrm{BW}(W)$ of weight e such that $\phi^c_W(b) \in \mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c)$ is non-zero. Since each element of W is an eigenvector of the automorphism $\varpi_c(T(\sigma_1, \alpha_1) \cdot \ldots \cdot T(\sigma_d, \alpha_d))$, it follows that $\phi^c_W(b)$ is still an eigenvector. By construction, it is clear that the eigenvalue of $\phi^c_W(b)$ is equal to 1.

P2. There exist $s_1, \ldots, s_n, r_1, \ldots, r_n \in \{1, \ldots, k\}, t_1, \ldots, t_m \in \{1, \ldots, l\}$ such that

$$\left(\prod_{i=1}^{n} R_{ks_i}(\alpha_i)\right)\left(\prod_{i=1}^{n} R_{kr_i}(\alpha_i)\right)\left(\prod_{i=1}^{m} R_{lt_i}(\beta_i)\right) = 1.$$

In this case, define

$$W := \{w(\sigma, \alpha)_{is_i},\, w(\sigma, \alpha)_{ir_i} \mid 1 \leq i \leq n\} \cup \{w(\tau, \beta)_{it_i} \mid 1 \leq i \leq m\}$$

and

$$\kappa : W \to A : w(\sigma, \alpha)_{ij} \mapsto v_{\lambda i},\, w(\tau, \beta)_{ij} \mapsto v_{\mu i}.$$

It is clear that A, W and κ satisfy the assumptions of Lemma 7.2.15. Define the weight

$$e : W \to \mathbb{N} : w(\tau, \beta)_{ij} \mapsto 1,\, w(\sigma, \alpha)_{ij} \mapsto \begin{cases} 1 & \text{if } s_i \neq r_i \\ 2 & \text{else,} \end{cases}$$

for which it holds that

$$\sum_{w \in W} e(w) = 2n + m \leq c(\lambda, \mu) = c.$$

As a consequence we can apply Lemma 7.2.15 to A, W and κ for the weight e and get a bracket word $b \in \mathrm{BW}(W)$ of weight e such that $\phi^c_W(b) \in \mathfrak{n}^{\mathbb{C}}(\mathcal{G}, c)$ is non-zero. Since each element of W is an eigenvector of the automorphism $\varpi_c(T(\sigma_1, \alpha_1) \cdot \ldots \cdot T(\sigma_d, \alpha_d))$, it follows that $\phi^c_W(b)$ is still an eigenvector. By construction, it is clear that the eigenvalue of $\phi^c_W(b)$ is equal to 1.

P3. There exist $s_1, \ldots, s_n \in \{1, \ldots, k\}, t_1, \ldots, t_m, r_1, \ldots, r_n \in \{1, \ldots, l\}$ such that

$$\left(\prod_{i=1}^{n} R_{ks_i}(\alpha_i)\right)\left(\prod_{i=1}^{m} R_{lt_i}(\beta_i)\right)\left(\prod_{i=1}^{m} R_{lr_i}(\beta_i)\right) = 1.$$

In this case, define

$$W := \{w(\sigma,\alpha)_{is_i} \mid 1 \leq i \leq n\} \\ \cup \{w(\tau,\beta)_{it_i},\, w(\tau,\beta)_{ir_i} \mid 1 \leq i \leq m\}$$

and

$$\kappa : W \to A : w(\sigma,\alpha)_{ij} \mapsto v_{\lambda i},\, w(\tau,\beta)_{ij} \mapsto v_{\mu i}.$$

It is clear that A, W and κ satisfy the assumptions of Lemma 7.2.15. Define the weight

$$e : W \to \mathbb{N} : w(\sigma,\alpha)_{ij} \mapsto 1,\, w(\tau,\beta)_{ij} \mapsto \begin{cases} 1 & \text{if } t_i \neq r_i \\ 2 & \text{else} \end{cases}$$

for which it holds that

$$\sum_{w \in W} e(w) = n + 2m \leq c(\lambda,\mu) = c.$$

As a consequence we can apply Lemma 7.2.15 to A, W and κ for the weight e and get a bracket word $b \in \mathrm{BW}(W)$ of weight e such that $\phi^c_W(b) \in \mathfrak{n}^{\mathbb{C}}(\mathcal{G},c)$ is non-zero. Since each element of W is an eigenvector of the automorphism $\varpi_c(T(\sigma_1,\alpha_1)\cdot\ldots\cdot T(\sigma_d,\alpha_d))$, it follows that $\phi^c_W(b)$ is still an eigenvector. By construction, it is clear that the eigenvalue of $\phi^c_W(b)$ is equal to 1.

This concludes the proof. □

	M/k even M/l even	M/k odd M/l odd	M/k even M/l odd	M/k odd M/l even
$\prod_i^n \alpha_i = 1$ $\prod_i^n \beta_i = 1$	Lem. 7.2.23 (i) $\to P1.$	Lem. 7.2.23 (i) $\to P1.$	Lem. 7.2.23 (i) $\to P1.$	Lem. 7.2.23 (i) $\to P1.$
$\prod_i^n \alpha_i = -1$ $\prod_i^n \beta_i = -1$	Lem. 7.2.23 (ii) $\to P1.$	Lem. 7.2.23 (ii) $\to P1.$	Lem. 7.2.23 (iii) $\to P3.$	Lem. 7.2.23 (iii) $\to P2.$
$\prod_i^n \alpha_i = -1$ $\prod_i^n \beta_i = 1$	Lem. 7.2.24 (i) $\to P1.$	Lem. 7.2.24 (ii) $\to P2.$	Lem. 7.2.24 (i) $\to P1.$	Lem. 7.2.24 (ii) $\to P2.$
$\prod_i^n \alpha_i = 1$ $\prod_i^n \beta_i = -1$	Lem. 7.2.24 (i) $\to P1.$	Lem. 7.2.24 (ii) $\to P3.$	Lem. 7.2.24 (ii) $\to P3.$	Lem. 7.2.24 (i) $\to P1.$

Table 7.1: Summary of which lemma is used in combination with Lemma 7.2.25 to obtain one of the cases P1., P2. or P3.

7.3 The R_∞-property and commensurability

This section is based on [LW24], which is joint work with Maarten Lathouwers.

Let G and H be groups. Recall that G is *abstractly commensurable with* H if there exist finite index subgroups $G' \leq G$ and $H' \leq H$ such that G' is isomorphic to H'. The starting point of this section is the following question:

Question 7.3.1. Is the R_∞-property for groups an abstract commensurability invariant?

It is known however that in general the answer to this question is negative. A counterexample is given by the fundamental group of the Klein bottle. This group has the R_∞-property, while it has a subgroup of index 2 isomorphic to $\mathbb{Z}^2$ which is known not to have the R_∞-property, see section 2 of [GW09]. Note that the fundamental group of the Klein bottle is torsion-free virtually abelian, but not nilpotent. A new interesting question arises when we ask our groups to be finitely generated torsion-free nilpotent.

Question 7.3.2. Is the R_∞-property for groups an abstract commensurability invariant within the class of finitely generated torsion-free nilpotent groups?

The reason for considering finitely generated torsion-free nilpotent groups is that there is a well-known abstract commensurability invariant within this class of groups, namely the rational Mal'cev completion (see Section 2.3.1 and Section 7.3.2) and that this rational Mal'cev completion is often used to prove results regarding the R_∞-property in this class of groups.

In an attempt to answer Question 7.3.2, we arrived at the following observation. We recall that a linear self-map on a finite-dimensional vector space is called *integer-like* if its characteristic polynomial has integral coefficients and constant term equal to ± 1.

Proposition 7.3.3. *Let G be a finitely generated torsion-free nilpotent group. All groups that are abstractly commensurable to G have the R_∞-property if and only if every integer-like automorphism on the associated Mal'cev rational Lie algebra has an eigenvalue 1.*

The actual answer to Question 2 turns out to be negative and we will present a counterexample within the class of 2-step nilpotent quotients of right-angled Artin groups. Using weight functions on the edges of the defining graphs, one can define groups which are commensurable to these 2-step nilpotent quotients of right-angled Artin groups, but which have considerably less induced automorphisms on the abelianization. These groups are defined in Section 7.3.3

and the restrictions on their automorphisms are proven in Section 7.3.5. The (to our knowledge) smallest concrete counterexample to Question 7.3.2 is then presented in Example 7.3.29, which proves the following.

Theorem 7.3.4. *There exist finitely generated torsion-free 2-step nilpotent groups G and G_n (with $n \in \mathbb{N} \setminus \{0,1\}$) such that:*

(i) $G \leq G_n$ and $[G_n : G] = n$ for all $n \in \mathbb{N} \setminus \{0,1\}$

(ii) G does not have the R_∞-property

(iii) all G_n have the R_∞-property

On the level of nilmanifolds we obtain the following corollary.

Corollary 7.3.5. *For any $n \in \mathbb{N} \setminus \{0,1\}$, there exist 2-step nilmanifolds M and N, a homeomorphism $f : N \to N$ and an n-to-one covering projection $p : M \to N$ such that*

(i) any map homotopic to f has a fixed point and

(ii) any self-homotopy equivalence of M is homotopic to a fixed point free map.

7.3.1 Nilpotent quotients of RAAGs

In this section, we explore how the R_∞-property and abstract commensurability behave within the class of nilpotent quotients of RAAGs (or free nilpotent partially commutative groups). As it turns out, these groups $A(\mathcal{G}, c)$ (with $c > 1$) contain a lot of information in the following sense: if $A(\mathcal{G}, c)$ has the R_∞–property, then all finitely generated torsion-free nilpotent groups commensurable with it must have the R_∞-property too. In fact, even a wider family of groups are implied to have R_∞, which is stated in the following proposition. At the heart of the proof lies Corollary 7.2.13.

Proposition 7.3.6. *Let $\mathcal{G}$ be a graph and $c > 1$ an integer. The following are equivalent:*

(i) $A(\mathcal{G}, c)$ has the R_∞-property.

(ii) Every integer-like automorphism of $\mathfrak{n}^{\mathbb{Q}}(\mathcal{G}, c)$ has an eigenvalue 1.

(iii) Every group H with $L^{\mathbb{Q}}(H) \cong \mathfrak{n}^{\mathbb{Q}}(\mathcal{G}, c)$ has the R_∞-property.

Proof. $(i) \Rightarrow (ii)$. Take any integer-like automorphism φ of $\mathfrak{n}^{\mathbb{Q}}(\mathcal{G}, c)$. By Corollary 7.2.13, there exists an automorphism ϕ on $A(\mathcal{G}, c)$ such that the induced automorphism $\overline{\phi}$ on $\mathfrak{n}^{\mathbb{Q}}(\mathcal{G}, c)$ has the same characteristic polynomial as φ. Using Lemma 7.2.6 and the assumption, the automorphism $\overline{\phi}$ has an eigenvalue 1 and hence φ has an eigenvalue 1.

$(ii) \Rightarrow (iii)$. Let us write $\tilde{H}$ for the group $H/\gamma_{c+1}(H)$. Using Lemma 4.3.9, we have the isomorphisms

$$L^{\mathbb{Q}}(\tilde{H}) \cong \frac{L^{\mathbb{Q}}(H)}{\gamma_{c+1}(L^{\mathbb{Q}}(H))} \cong \frac{\mathfrak{n}^{\mathbb{Q}}(\mathcal{G}, c)}{\gamma_{c+1}(\mathfrak{n}^{\mathbb{Q}}(\mathcal{G}, c))} \cong \mathfrak{n}^{\mathbb{Q}}(\mathcal{G}, c).$$

Let φ be any automorphism of $\tilde{H}$ and write $\overline{\varphi}$ for the induced graded automorphism on $\mathfrak{n}^{\mathbb{Q}}(\mathcal{G}, c)$ where we use the identification above. Note that $\overline{\varphi}$ is integer-like. Thus, by assumption, it has an eigenvalue 1. Lemma 7.2.6 then implies that $R(\varphi) = \infty$. Since φ was an arbitrarily chosen automorphism of $\tilde{H}$, this proves that $\tilde{H}$ has the R_∞-property and by Lemma 1.1. from [GW09], so does H.

$(iii) \Rightarrow (i)$. This follows if we substitute $A(\mathcal{G}, c)$ for H in the assumption (see isomorphism 4.5 from Chapter 4). □

Remark 7.3.7. Note that any finitely generated torsion-free nilpotent group H, which is abstractly commensurable to $A(\mathcal{G}, c)$, satisfies (iii) of the above proposition. Indeed, we get that $M(H) \cong M(A(\mathcal{G}, c))$ and thus also $L^{\mathbb{Q}}(H) \cong \mathrm{gr}(M(H)) \cong \mathrm{gr}(M(A(\mathcal{G}, c))) \cong L^{\mathbb{Q}}(A(\mathcal{G}, c)) \cong \mathfrak{n}^{\mathbb{Q}}(\mathcal{G}, c)$.

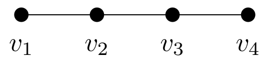
Remark 7.3.8. Note that in (iii) of Proposition 7.3.6 one can, in general, not replace the field $\mathbb{Q}$ with a non-trivial field extension of it (which would be a weaker assumption). Let us give a concrete example. Consider the 2-step nilpotent group H with presentation

$$H = \left\langle \begin{array}{c} x_1, x_2, x_3, x_4, \\ y_1, y_2, y_3 \end{array} \;\middle|\; \begin{array}{ll} [x_1, x_3] = y_1, & [x_1, x_4] = y_2, \\ [x_2, x_4] = y_1^2, & [x_2, x_3] = y_2, \\ [x_3, x_4] = y_3, & [y_i, y_j] = 1 = [x_1, x_2] \end{array} \right\rangle. \tag{7.8}$$

This group does not have the R_∞-property since the automorphism of H that is defined by

$$\begin{array}{ll} x_1 \mapsto x_1^{-1} x_2 & y_1 \mapsto y_1^3 y_2^{-2} \\ x_2 \mapsto x_1^2 x_2^{-1} & y_2 \mapsto y_1^{-4} y_2^3 \\ x_3 \mapsto x_3^{-1} x_4 & y_3 \mapsto y_3^{-1} \\ x_4 \mapsto x_3^2 x_4^{-1} & \end{array}$$

induces an automorphism on $L^{\mathbb{C}}(H)$ which does not have 1 as an eigenvalue (see Lemma 7.2.6). On the other hand, consider the graph $\mathcal{G}$ as drawn below.



By Theorem 7.2.2, $A(\mathcal{G}, 2)$ has the R_∞-property. However, over the field $K = \mathbb{Q}(\sqrt{2})$, we have a graded Lie algebra isomorphism $L^K(H) \cong \mathfrak{n}^K(\mathcal{G}, 2)$ which is determined by

$$\begin{array}{ll} x_1 \mapsto v_1 + v_4 & x_3 \mapsto v_2 + v_3 \\ x_2 \mapsto \sqrt{2}(v_1 - v_4) & x_4 \mapsto \sqrt{2}(v_2 - v_3). \end{array}$$

7.3.2 Rational Mal'cev Lie algebras

Let G be a finitely generated torsion-free nilpotent group. Recall from Section 2.3.1 that such a group has a rational Mal'cev completion $G^{\mathbb{Q}}$ and a real Mal'cev completion $G^{\mathbb{R}}$. The real Mal'cev completion $G^{\mathbb{R}}$ is a Lie group and G lies in $G^{\mathbb{R}}$ as a cocompact lattice. From Section 2.3.3 in Chapter 2, we know that there is an associated rational finite dimensional nilpotent Lie algebra $\mathfrak{n}_G^{\mathbb{Q}}$. Moreover, if H is another finitely generated torsion-free nilpotent group, then $\mathfrak{n}_G^{\mathbb{Q}} \cong \mathfrak{n}_H^{\mathbb{Q}}$ as rational Lie algebras if and only if G and H are abstractly commensurable.

The Lie algebra structure on $\mathfrak{n}_G^{\mathbb{Q}}$ and the group structure on $G^{\mathbb{Q}}$ are related to one another via the BCH-formula (see Section 2.3.2). Any automorphism of G extends uniquely to an automorphism of $G^{\mathbb{Q}}$ which in turn defines an automorphism of the Lie algebra $\mathfrak{n}_G^{\mathbb{Q}}$.

Recall from Section 4.3.2 that to any group G we can also associate a graded rational Lie algebra $L^{\mathbb{Q}}(G)$. In what follows we will relate the Lie algebras $L^{\mathbb{Q}}(G)$ and $\mathfrak{n}_G^{\mathbb{Q}}$ for a finitely generated torsion-free nilpotent group G. Let us also recall from Section 4.3.1 that to any Lie algebra $\mathfrak{g}$ we associate a graded Lie algebra $\mathrm{gr}(\mathfrak{g})$ defined as a sum of the factors of the lower central series.

First, note that the following result was proven in [Qui68, Lemma 3.3].

Lemma 7.3.9 ([Qui68])**.** *Let G be a finitely generated torsion-free nilpotent group. The inclusion of G in $G^{\mathbb{Q}}$ induces a graded Lie algebra isomorphism*

$$L^{\mathbb{Q}}(G) = L(G) \otimes_{\mathbb{Z}} \mathbb{Q} \cong L(G^{\mathbb{Q}}).$$

Next, recall that we write $\log : G^{\mathbb{Q}} \to \mathfrak{n}_G^{\mathbb{Q}}$ for the inverse of the exponential map. Using the BCH-formula and Theorem 2.3.20, we find that log induces for all $i > 0$ a rational vector space isomorphism

$$f_i : \gamma_i(G^{\mathbb{Q}})/\gamma_{i+1}(G^{\mathbb{Q}}) \longrightarrow \gamma_i(\mathfrak{n}_G^{\mathbb{Q}})/\gamma_{i+1}(\mathfrak{n}_G^{\mathbb{Q}})$$

by
$$x\gamma_{i+1}(G^{\mathbb{Q}}) \mapsto \log(x) + \gamma_{i+1}(\mathfrak{n}_G^{\mathbb{Q}}).$$
Hence, $f := \bigoplus_{i>0} f_i$ gives an isomorphism
$$L^{\mathbb{Q}}(G) \longrightarrow \mathrm{gr}(\mathfrak{n}_G^{\mathbb{Q}}).$$
as rational vector spaces. To see that this map is also a Lie algebra isomorphism, one can use the BCH-formula to show that for any $x \in \gamma_i(G^{\mathbb{Q}})$, $y \in \gamma_j(G^{\mathbb{Q}})$, it holds that
$$\log([x, y]) = \log(xyx^{-1}y^{-1}) = [\log(x), \log(y)] + \text{terms in } \gamma_{i+j+1}(\mathfrak{n}_G^{\mathbb{Q}}),$$
which is for example also derived in [Seg83, p.103]. Together with Lemma 7.3.9 we thus find that
$$L^{\mathbb{Q}}(G) \cong \mathrm{gr}(\mathfrak{n}_G^{\mathbb{Q}})$$
as rational graded Lie algebras. Moreover, given an automorphism on G, the induced automorphisms on $L^{\mathbb{Q}}(G)$ and $\mathrm{gr}(\mathfrak{n}_G^{\mathbb{Q}})$ coincide through this isomorphism. In particular, the induced automorphism on $L^{\mathbb{Q}}(G)$ has the same eigenvalues as the induced automorphism on $\mathfrak{n}_G^{\mathbb{Q}}$. Thus, using Lemma 7.2.6 we retrieve the following well-known lemma.

Lemma 7.3.10. *Let G be a finitely generated torsion-free nilpotent group and $\varphi \in \mathrm{Aut}(G)$. Let φ' be the induced Lie algebra automorphism on $\mathfrak{n}_G^{\mathbb{Q}}$. Then*
$$R(\varphi) = \infty \quad \iff \quad \varphi' \textit{ has eigenvalue } 1.$$

Let us now prove the proposition from the beginning of Section 7.3.

Proof of Proposition 7.3.3. Let G be a finitely generated torsion-free nilpotent group, $G^{\mathbb{Q}}$ its Mal'cev completion and $\mathfrak{n}_G^{\mathbb{Q}}$ the rational Mal'cev Lie algebra.

First, assume that every integer-like automorphism of $\mathfrak{n}_G^{\mathbb{Q}}$ has an eigenvalue 1. Let H be a group commensurable with G and φ any automorphism of H. Then φ induces an automorphism on the Mal'cev Lie algebra of H, which is isomorphic to $\mathfrak{n}_G^{\mathbb{Q}}$. Moreover, this automorphism must be integer-like, thus by the assumption it has an eigenvalue 1. Using Lemma 7.3.10, we find that $R(\varphi) = \infty$ and thus that H has the R_∞-property.

Conversely, assume that every group commensurable to G has the R_∞-property. Let φ be an integer-like automorphism of $\mathfrak{n}_G^{\mathbb{Q}}$ and write $\tilde{\varphi}$ for the induced automorphism on $G^{\mathbb{Q}}$. A property of an integer-like automorphism is that there exists a basis $\{x_1, \ldots, x_n\}$ of $\mathfrak{n}_G^{\mathbb{Q}}$ with respect to which φ has a matrix representation in $\mathrm{GL}_n(\mathbb{Z})$. By Proposition 3.2 in [DD16], there exists some finitely generated subgroup H in $G^{\mathbb{Q}}$ with Mal'cev completion $G^{\mathbb{Q}}$ and such that $\tilde{\varphi}(H) = H$. In particular, H is commensurable to G. Thus, by assumption, H has the R_∞-property and $R(\tilde{\varphi}|_H) = \infty$. Lemma 7.3.10 then implies that the induced morphism on $\mathfrak{n}_H^{\mathbb{Q}} = \mathfrak{n}_G^{\mathbb{Q}}$, which must be equal to φ, has an eigenvalue 1. This completes the proof. □

It is not clear to us whether in the statement of Proposition 7.3.3 one can replace 'the associated Mal'cev Lie algebra' with 'the associated graded Lie algebra'. Equivalently, we do not have an answer to the following question.

Question 7.3.11. Does there exist a finite-dimensional nilpotent rational Lie algebra $\mathfrak{n}$ for which every integer-like automorphism has an eigenvalue 1, but for which $\operatorname{gr}(\mathfrak{n})$ does not have this property?

It is not hard to check that the other direction is true, namely if $\operatorname{gr}(\mathfrak{n})$ has the property then so does $\mathfrak{n}$.

7.3.3 Groups from edge-weighted graphs

Let $\mathcal{G} = (V, E)$ be a simple undirected graph and fix a total order $\leq$ on the vertices V. Let $k : E \to \mathbb{N} \setminus \{0\}$ be a weight function on the edges. We define the group $A(\mathcal{G}, 2, k)$ by the presentation

$$A(\mathcal{G},2,k) := \left\langle V \cup E \;\middle|\; [V,E] = [E,E] = 1 \text{ and } \forall v, w \in V \text{ with } v < w : \; [v,w] = \begin{cases} e^{k(e)} & \text{if } e := \{v,w\} \in E \\ 1 & \text{else} \end{cases} \right\rangle.$$

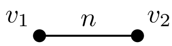
Note that $A(\mathcal{G}, 2, k) \cong A(\mathcal{G}, 2)$ if k is constant with value 1.

Remark 7.3.12. Up to isomorphism, the definition of $A(\mathcal{G}, 2, k)$ is independent of the choice of total order on the vertices. Indeed, let $\tilde{\leq}$ be another total order on the vertices V. Then it is not hard to check that the map θ, which is the identity on V and is defined for all $e = \{v, w\} \in E$ by

$$\theta(e) = \begin{cases} e & \text{if } v < w \Leftrightarrow v \tilde{<} w \\ e^{-1} & \text{otherwise,} \end{cases}$$

induces an isomorphism from the group defined using $\leq$ to the group defined using $\tilde{\leq}$.

Example 7.3.13. Consider the graph $\mathcal{G} = (V, E)$ where $V = \{v_1, v_2\}$ and $E = \{\{v_1, v_2\}\}$ and for any $n \in \mathbb{N}_0$, let $k_n : E \to \mathbb{N}_0$ be the weight function on the singleton set E with value n.



Then for each $n \in \mathbb{N}_0$, the group $A(\mathcal{G}, 2, k_n)$ is isomorphic to the matrix group

$$H_n(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & nx & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| \, x, y, z \in \mathbb{Z} \right\}$$

where the isomorphism is given by

$$v_1 \mapsto \begin{pmatrix} 1 & n & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad v_2 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \{v_1, v_2\} \mapsto \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In particular, when $n = 1$, $A(\mathcal{G}, 2, k_1) \cong A(\mathcal{G}, 2)$ is isomorphic to the Heisenberg group. It is well-known that the groups $H_n(\mathbb{Z})$ for $n \in \mathbb{N}_0$ are, up to isomorphism, all the possible groups abstractly commensurable to $H_1(\mathbb{Z})$. Moreover, $\mathrm{Spec}(H_n(\mathbb{Z})) = (2\mathbb{N}_0) \cup \{\infty\}$ for all $n \in \mathbb{N}_0$ (see for example [Rom11b]). Hence, $H_1(\mathbb{Z})$ is an example of a finitely generated torsion-free 2-step nilpotent group such that all groups commensurable with $H_1(\mathbb{Z})$ have the same (infinite) Reidemeister spectrum.

It will be helpful to address the elements $e^{k(e)} \in A(\mathcal{G}, 2, k)$, so we define

$$E^k := \{e^{k(e)} \mid e \in E\}.$$

We say a vertex $v \in V$ is an *isolated vertex* if $N(v) = \emptyset$, i.e. v has no neighbours. We write $V_{\mathrm{iso}} \subset V$ for the subset of all isolated vertices of $\mathcal{G}$. The next lemma provides insight into the structure of the group $A(\mathcal{G}, 2, k)$. We leave the proof as an exercise to the reader.

Lemma 7.3.14. *Let $\mathcal{G} = (V, E)$ be a graph, $k : E \to \mathbb{N}_0$ a weight on the edges and $G = A(\mathcal{G}, 2, k)$ the associated group. Write $V = \{v_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_m\}$. Then any element of G can be uniquely expressed as a product*

$$\prod_{i=1}^{n} v_i^{z_i} \prod_{j=1}^{m} e^{t_j}$$

with $z_i, t_j \in \mathbb{Z}$*. Moreover, the centre and the factors of the (adapted) lower central series of* G *are given by*

$$\begin{aligned}
Z(G) &= \langle V_{iso} \cup E\rangle \cong \mathbb{Z}^{|V_{iso}\cup E|},\\
\gamma_2(G) &= \left\langle E^k \right\rangle \cong \mathbb{Z}^{|E|},\\
\sqrt{\gamma_2(G)} &= \langle E\rangle \cong \mathbb{Z}^{|E|},\\
\frac{G}{\gamma_2(G)} &\cong \mathbb{Z}^{|V|} \times \bigtimes_{e\in E} \frac{\mathbb{Z}}{k(e)\mathbb{Z}},\\
\frac{G}{\sqrt{\gamma_2(G)}} &= \bigtimes_{v\in V} \left\langle v\sqrt{\gamma_2(G)}\right\rangle \cong \mathbb{Z}^{|V|}\\
\gamma_3(G) &= \sqrt{\gamma_3(G)} = 1.
\end{aligned}$$

In particular, G *is finitely generated torsion-free 2-step nilpotent. Moreover, the identity on* V *induces an injective morphism*

$$\iota : A(\mathcal{G},2) \hookrightarrow A(\mathcal{G},2,k)$$

with $A(\mathcal{G},2) \cong \mathrm{Im}(\iota) = \langle V \cup E^k\rangle$*. In particular,* $A(\mathcal{G},2)$ *is isomorphic to the finite index subgroup* $\langle V \cup E^k\rangle \subset A(\mathcal{G},2,k)$ *with index equal to* $\prod_{e\in E} k(e)$*.*

At last, we give isomorphisms between the associated Lie algebra structures. Recall that $\mathfrak{n}^{\mathbb{Q}}(\mathcal{G},c)$ is the c-step nilpotent partially commutative Lie algebra associated to the graph $\mathcal{G}$, that $L^{\mathbb{Q}}(\mathcal{G})$ is the graded rational Lie algebra associated to the group G and that $\mathfrak{n}^{\mathbb{Q}}_G$ is the rational Mal'cev Lie algebra associated to a finitely generated torsion-free nilpotent group G.

Lemma 7.3.15. *For any finite undirected simple graph* $\mathcal{G} = (V,E)$ *and a weight function* $k : E \to \mathbb{N}_0$ *on its edges, we have*

$$\mathfrak{n}^{\mathbb{Q}}(\mathcal{G},2) \cong L^{\mathbb{Q}}(A(\mathcal{G},2)) \cong \mathfrak{n}^{\mathbb{Q}}_{A(\mathcal{G},2)} \cong \mathfrak{n}^{\mathbb{Q}}_{A(\mathcal{G},2,k)} \cong L^{\mathbb{Q}}(A(\mathcal{G},2,k))$$

and the isomorphisms are induced by the identity on the vertices.

Proof. The first isomorphism is Theorem 4.3.10 and equation (4.5) below it. The third isomorphism follows from the fact that $A(\mathcal{G},2)$ is isomorphic to a finite index subgroup of $A(\mathcal{G},2,k)$ and thus, in particular, that $A(\mathcal{G},2)$ is abstractly commensurable with $A(\mathcal{G},2,k)$. Hence their Mal'cev Lie algebras $\mathfrak{n}^{\mathbb{Q}}_{A(\mathcal{G},2)}$ and $\mathfrak{n}^{\mathbb{Q}}_{A(\mathcal{G},2,k)}$ are isomorphic.

For the second and fifth isomorphism, note that any 2-step nilpotent rational Lie algebra $\mathfrak{n}$ is isomorphic to $\mathrm{gr}(\mathfrak{n})$ and that the isomorphism is canonical up to a choice of vector space complement to $[\mathfrak{n},\mathfrak{n}]$. Since for $\mathfrak{n}^{\mathbb{Q}}_{A(\mathcal{G},2)}$ and $\mathfrak{n}^{\mathbb{Q}}_{A(\mathcal{G},2,k)}$, such a complement is naturally given by the rational span of the vertices, one gets (after combining with Equation (7.3.2)) the isomorphisms $\mathfrak{n}^{\mathbb{Q}}_{A(\mathcal{G},2)} \cong \mathrm{gr}(\mathfrak{n}^{\mathbb{Q}}_{A(\mathcal{G},2)}) \cong L^{\mathbb{Q}}(A(\mathcal{G},2))$ and $\mathfrak{n}^{\mathbb{Q}}_{A(\mathcal{G},2,k)} \cong \mathrm{gr}(\mathfrak{n}^{\mathbb{Q}}_{A(\mathcal{G},2,k)}) \cong L^{\mathbb{Q}}(A(\mathcal{G},2,k))$. As one can check, all these isomorphisms restrict to the identity on the vertices. □

Remark 7.3.16. Note that any automorphism φ of the group $A(\mathcal{G},2,k)$ induces an automorphism $L^{\mathbb{Q}}(\overline{\varphi})$ on $L^{\mathbb{Q}}(A(\mathcal{G},2,k))$ and by the lemma above also an automorphism on $\mathfrak{n}^{\mathbb{Q}}(\mathcal{G},2)$ which we write as $\overline{\varphi}$. In particular, since the isomorphisms of the above lemma restrict to the identity on the vertices, we can say the following: with respect to the basis of vertices, the map that φ induces on the free $\mathbb{Z}$-module $A(\mathcal{G},2,k)/\sqrt{A(\mathcal{G},2,k)}$ has the same matrix representation as the map that $\overline{\varphi}$ induces on the abelianization of $\mathfrak{n}^{\mathbb{Q}}(\mathcal{G},2)$.

7.3.4 A preferred total order on vertices and edges

In the next section, we will give, to some extend, a description of the automorphisms of the groups $A(\mathcal{G},2,k)$. For this description, we need a nice total order on the vertices and edges which we introduce in this section. After this section we will then always use this preferred order to also define the groups $A(\mathcal{G},2,k)$.

Let $\mathcal{G} = (V,E)$ be a simple undirected graph. We use the definitions and notations introduced in Section 4.1. Recall that we have a naturally defined preorder $\prec$ on the vertices. This preorder induces a preorder on the edges as well by

$$\{v,w\} \prec \{v',w'\} \iff (v \prec v' \text{ and } w \prec w') \text{ or } (v \prec w' \text{ and } w \prec v'),$$

for all $\{v,w\},\{v',w'\} \in E$. Similarly as with the vertices, we can use this preorder on E to get an equivalence relation $\sim$ defined by

$$e \sim e' \iff e \prec e' \text{ and } e' \prec e.$$

for any $e, e' \in E$. We write the set of equivalence classes as

$$\Delta_{\mathcal{G}} = E/\sim.$$

Recall the definition of the set $\overline{E}$ from Section 4.1 as

$$\overline{E} := \{\{[v],[w]\} \mid \{v,w\} \in E\}.$$

The next lemma shows us, among other, that the map

$$\Delta_{\mathcal{G}} \to \overline{E} : [\{v, w\}] \mapsto \{[v], [w]\} \tag{7.9}$$

is a bijection. As a consequence, we can identify $\Delta_{\mathcal{G}}$ with $\overline{E}$.

Lemma 7.3.17. *For any $e = \{v, w\} \in E$ it holds that*

$$[e] = \Big\{\{v', w'\} \;\Big|\; v' \in [v], w' \in [w]\Big\}.$$

Proof. The $\supseteq$-inclusion follows directly by using the definition of $\prec$ and $\sim$ on E. For the other inclusion let us fix some $e' = \{v', w'\} \in [e]$. Hence, it holds that

$$(v' \prec v \text{ and } w' \prec w) \text{ or } (v' \prec w \text{ and } w' \prec v) \text{ and}$$

$$(v \prec v' \text{ and } w \prec w') \text{ or } (v \prec w' \text{ and } w \prec v').$$

Note that if for example $(v' \prec v$ and $w' \prec w)$ and $(v \prec w'$ and $w \prec v')$ then $v \prec w' \prec w \prec v' \prec v$ and thus it follows that $v \sim w \sim v' \sim w'$. Hence, in this particular case e' is definitely contained in the right-hand side. Considering all the possible cases, one can conclude the rest of the proof. ☐

The preorders $\prec$ on the vertices and the edges induce partial orders $\overline{\prec}$ on $\Lambda_{\mathcal{G}}$ and $\Delta_{\mathcal{G}}$ by setting for all $v, w \in V$ and $e, e' \in E$:

$$[v] \,\overline{\prec}\, [w] \iff v \prec w$$

$$[e] \,\overline{\prec}\, [e'] \iff e \prec e'.$$

Using these partial orders, we can fix total orders

$$V/\sim \;= \Lambda_{\mathcal{G}} = \{\lambda_1, \ldots, \lambda_r\},$$

$$E/\sim \;= \Delta_{\mathcal{G}} = \{\mu_1, \ldots, \mu_s\},$$

with the property that $i \leq j$ if $\lambda_i \overline{\prec} \lambda_j$ and $i \leq j$ if $\mu_i \overline{\prec} \mu_j$. We can refine these orders to total orders on V and E which we will both denote with $\leq$.

Example 7.3.18. We illustrate the introduced orders using a concrete example. We consider the graph $\mathcal{G}$ given by the figure below.

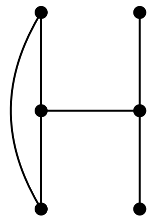
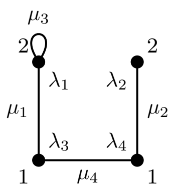
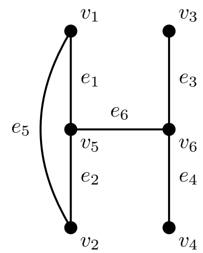


Figure 7.3: The graph $\mathcal{G}$.

The total orders on $\Lambda_{\mathcal{G}}$, $\Delta_{\mathcal{G}}$, V and E can then be chosen as illustrated in the following figure.



(a) The graph quotient graph $\overline{\mathcal{G}}$ with total ordering on $\Lambda_{\mathcal{G}}$ and $\Delta_{\mathcal{G}}$.



(b) The graph $\mathcal{G}$ with total ordering on V and E.

Indeed, the relations on $\Lambda_{\mathcal{G}}$ are

$$\lambda_1 \overline{\prec} \lambda_3,\ \lambda_2 \overline{\prec} \lambda_3 \text{ and } \lambda_2 \overline{\prec} \lambda_4$$

and thus the above total order satisfies that $i \leq j$ if $\lambda_i \overline{\prec} \lambda_j$. (Note that we could have also interchanged λ_1 and λ_2.). The only relations on the edges are

$$\{v_1, v_5\} \sim \{v_2, v_5\} \text{ and } \{v_3, v_6\} \sim \{v_4, v_6\}$$

and thus the induced partial order $\overline{\prec}$ on $\overline{E}$ is the identity relation. Hence, we fixed the total order

$$[\{v_1, v_5\}] < [\{v_3, v_6\}] < [\{v_1, v_2\}] < [\{v_5, v_6\}]$$

on $\overline{E}$ and refined it to the total order on E:

$$\{v_1, v_5\} < \{v_2, v_5\} < \{v_3, v_6\} < \{v_4, v_6\} < \{v_1, v_2\} < \{v_5, v_6\}.$$

7.3.5 Automorphisms of $A(\mathcal{G}, 2, k)$

In this section, we describe to some extend the automorphism group of the group $A(\mathcal{G}, 2, k)$. In particular, we will show how the weight function k imposes conditions on the graph automorphisms that induce an automorphism on the group $A(\mathcal{G}, 2, k)$.

First, let us introduce some notation. Let $(B = \{b_1, \dots, b_n\}, \leq)$ be a finite totally ordered set (with $b_1 < \dots < b_n$) and $\theta : B \to \mathbb{Z}$ a map. Then we define $D(\theta) \in \mathbb{Z}^{n \times n}$ to be the diagonal matrix with the image of θ (ordered by $\leq$ on B) on its diagonal, i.e.

$$D(\theta) := \operatorname{diag}(\theta(b_1), \dots, \theta(b_n)) = \begin{pmatrix} \theta(b_1) & 0 & \dots & 0 \\ 0 & \theta(b_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \theta(b_n) \end{pmatrix}.$$

If B is a basis of a $\mathbb{Z}$-module M, then we will sometimes abuse notation and denote with $D(\theta)$ also the linear map on M induced by $D(\theta)(b_i) := \theta(b_i)b_i$ (for all $i = 1, \dots, n$).

Lemma 7.3.19. *Fix a finite undirected simple graph $\mathcal{G} = (V, E)$ with weights $k : E \to \mathbb{N}_0$ on its edges and write $G = A(\mathcal{G}, 2, k)$. Let $\varphi \in \operatorname{Aut}(G)$ and denote with $A \in \operatorname{GL}_{|E|}(\mathbb{Z})$ the matrix of $\varphi|_{\gamma_2(G)}$ with respect to the $\mathbb{Z}$-basis $(E^k, <)$. The matrix of $\varphi|_{\sqrt{\gamma_2(G)}}$ with respect to $(E, \leq)$ is given by $D(k)AD(k)^{-1}$. In particular:*

$$D(k)AD(k)^{-1} \in \operatorname{GL}_{|E|}(\mathbb{Z}).$$

Proof. Since the induced automorphisms on $\gamma_2(G)$ and $\sqrt{\gamma_2(G)}$ are restrictions of φ, we obtain the following commuting diagram.

$$\begin{array}{ccc} \gamma_2(G) & \xrightarrow{\varphi|_{\gamma_2}} & \gamma_2(G) \\ \Big\downarrow \iota & & \Big\downarrow \iota \\ \sqrt{\gamma_2(G)} & \xrightarrow[\varphi|_{\sqrt{\gamma_2}}]{} & \sqrt{\gamma_2(G)} \end{array}$$

Recall that $\gamma_2(G)$ and $\sqrt{\gamma_2(G)}$ are both free abelian of rank $|E|$. By Lemma 7.3.14, the sets E and $E^k = \{e^{k(e)} \mid e \in E\}$ ordered with the total order $\leq$ give a basis for the $\mathbb{Z}$-modules $\sqrt{\gamma_2(G)}$ and $\gamma_2(G)$, respectively. Hence, by using these bases we get the commuting diagram of matrices

$$\begin{array}{ccc} \mathbb{Z}^{|E|} & \xrightarrow{A} & \mathbb{Z}^{|E|} \\ {\scriptstyle D}\downarrow & & \downarrow{\scriptstyle D} \\ \mathbb{Z}^{|E|} & \xrightarrow[B]{} & \mathbb{Z}^{|E|} \end{array}$$

where $A, B \in \mathrm{GL}_{|E|}(\mathbb{Z})$ are the matrix representations of $\varphi|_{\gamma_2(G)}$ and $\varphi|_{\sqrt{\gamma_2(G)}}$, respectively, and $D \in \mathbb{Z}^{|E|\times|E|}$ is the matrix representation of the inclusion ι with respect to the bases E^k and E. Note that $D = D(k)$ since a basis vector $e^{k(e)}$ of $\gamma_2(G)$ is sent to the basis vector e of $\sqrt{\gamma_2(G)}$ but raised to the power $k(e)$. Thus the commutative diagram implies that $BD(k) = D(k)A$. Since $k(e) \neq 0$ (for all $e \in E$), these matrices are invertible over $\mathbb{Q}$. Over $\mathbb{Q}$, we indeed obtain that $B = D(k)AD(k)^{-1}$. □

Remark 7.3.20. Any automorphism of $A(\mathcal{G}, 2, k)$ is completely determined by the image of the vertices (even though they do not necessarily generate $A(\mathcal{G}, 2, k)$). Indeed, by Lemma 7.3.19 the image of E is fixed by what happens on $\gamma_2(A(\mathcal{G}, 2, k))$, which in turn is determined by the image of the vertices. Since V and E generate $A(\mathcal{G}, 2, k)$ the claim follows.

Using the description of the projections onto the abelianization of the automorphisms of the Lie algebra $\mathfrak{n}^{\mathbb{Q}}(\mathcal{G}, 2)$ from Section 4.6.2, we can describe to some extent the automorphisms of $A(\mathcal{G}, 2, k)$ and in particular their induced maps on $\gamma_2(A(\mathcal{G}, 2, k))$. In order to do so, we need the following notation.

Let $\mathcal{G} = (V, E)$ be a graph with an automorphism $\sigma \in \mathrm{Aut}(\mathcal{G})$. Then σ induces a permutation σ_E on the edge set by

$$\sigma_E : E \to E : \{v, w\} \mapsto \{\sigma(v), \sigma(w)\}.$$

Given a total order $\leq$ on the vertices V, we say that an edge $\{v, w\} \in E$ is an *inversion of* σ if $v < w$, but $\sigma(v) > \sigma(w)$. We then define the map $\varepsilon_\sigma : E \to \{-1, 1\}$ by setting for all $e \in E$

$$\varepsilon_\sigma(e) := \begin{cases} -1 & \text{if } e \text{ is an inversion of } \sigma \\ 1 & \text{otherwise.} \end{cases}$$

With this notation at hand, and by combining Theorem 4.6.6 with Remark 7.3.16, we can say the following about the automorphisms of $A(\mathcal{G}, 2, k)$. Recall from Section 7.3.4 that we fixed a total order $\leq$ on both the vertices and the edges, which were a refinement of the orders on $\Lambda_\mathcal{G} = \{\lambda_1, \dots, \lambda_r\}$ and $\Delta_\mathcal{G} = \{\mu_1, \dots, \mu_s\}$, respectively.

Lemma 7.3.21. *Let $\mathcal{G} = (V, E)$ be a simple undirected graph with weight function $k : E \to \mathbb{N}_0$ on its edges and write $G = A(\mathcal{G}, 2, k)$. For any automorphism $\varphi \in \operatorname{Aut}(G)$ there exists a $\sigma \in \operatorname{Aut}(\mathcal{G})$ such that*

(i) with respect to the ordered $\mathbb{Z}$-basis $(V, \leq)$, the induced automorphism on $G/\sqrt{\gamma_2(G)}$ is represented by a matrix of the form

$$P(\sigma) \cdot \begin{pmatrix} A_1 & A_{12} & \dots & A_{1r} \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & A_{r-1r} \\ 0 & \dots & 0 & A_r \end{pmatrix}$$

where $A_i \in \mathrm{GL}_{|\lambda_i|}(\mathbb{Z})$, $A_{ij} \in \mathbb{Z}^{|\lambda_i| \times |\lambda_j|}$ and $A_{ij} = 0$ if $\lambda_i \not\prec \lambda_j$.

(ii) with respect to the ordered $\mathbb{Z}$-basis $(E^k, \leq)$, the induced automorphism on $\gamma_2(G)$ is represented by a matrix of the form

$$P(\sigma_E) \cdot D(\varepsilon_\sigma) \cdot \begin{pmatrix} B_1 & B_{12} & \dots & B_{1s} \\ 0 & B_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & B_{s-1s} \\ 0 & \dots & 0 & B_s \end{pmatrix}$$

where $B_i \in \mathrm{GL}_{|\mu_i|}(\mathbb{Z})$, $B_{ij} \in \mathbb{Z}^{|\mu_i| \times |\mu_j|}$ and $B_{ij} = 0$ if $\mu_i \not\prec \mu_j$.

The permutation $\sigma \in \operatorname{Aut}(\mathcal{G})$ is not necessarily unique, but the induced permutation $\overline{\sigma}$ on the quotient graph $\overline{\mathcal{G}}$ is unique. Moreover, the assignment $\varphi \mapsto \overline{\sigma}$ defines a group morphism from $\operatorname{Aut}(A(\mathcal{G}, 2, k))$ to $\operatorname{Aut}(\overline{\mathcal{G}})$.

Proof. Fix $\varphi \in \operatorname{Aut}(G)$ and denote with $\overline{\varphi} \in \operatorname{Aut}_g(\mathfrak{n}^{\mathbb{Q}}(\mathcal{G}, 2))$ the induced graded automorphism (see Remark 7.3.16). By Theorem 4.6.6 and Remark 4.6.7, there exist $A_i \in \mathrm{GL}_{|\lambda_i|}(\mathbb{Q})$, $A_{ij} \in \mathbb{Q}^{|\lambda_i| \times |\lambda_j|}$ and $\sigma \in \operatorname{Aut}(\mathcal{G})$ such that the matrix of $\pi_{\mathrm{ab}}(\overline{\varphi})$ with respect to the ordered basis of vertices $(V, \leq)$ equals

$$P(\sigma) \cdot \underbrace{\begin{pmatrix} A_1 & A_{12} & \dots & A_{1r} \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & A_{r-1r} \\ 0 & \dots & 0 & A_r \end{pmatrix}}_{=:\, U}$$

where $A_{ij} = 0$ if $\lambda_i \not\prec \lambda_j$. Since $\overline{\varphi}$ is induced by the automorphism $\varphi \in \operatorname{Aut}(G)$, the matrix $P(\sigma)U$ is by construction equal to the matrix of the induced

automorphism on $G/\sqrt{\gamma_2(G)}$ with respect to the ordered $\mathbb{Z}$-basis of vertices. In particular, $P(\sigma)U \in \mathrm{GL}_{|V|}(\mathbb{Z})$. Since $P(\sigma) \in \mathrm{GL}_{|V|}(\mathbb{Z})$, this implies that $U \in \mathrm{GL}_{|V|}(\mathbb{Z})$. Hence, we obtain that $A_i \in \mathrm{GL}_{|\lambda_i|}(\mathbb{Z})$ and $A_{ij} \in \mathbb{Z}^{|\lambda_i| \times |\lambda_j|}$.
Now we derive the matrix of the induced automorphism on $\gamma_2(G)$ with respect to the ordered $\mathbb{Z}$-basis $(E^k, \leq)$. Fix any edge $e = \{v, w\} \in E$ (with $v < w$) and note that the Lie bracket $[v, w] = e^{k(e)}$ is sent by the Lie algebra morphism $P(\sigma)$ to

$$[\sigma(v), \sigma(w)] = \sigma_E(e)^{\varepsilon_\sigma(e) k(\sigma_E(e))}.$$

Thus, the matrix (with respect to $(E^k, \leq)$) of the morphism on $\gamma_2(\mathfrak{n}^{\mathbb{Q}}(\mathcal{G}, 2))$ that is induced by $P(\sigma)$ equals

$$P(\sigma_E) D(\varepsilon_\sigma).$$

So it suffices to look at the morphism on $\gamma_2(\mathfrak{n}^{\mathbb{Q}}(\mathcal{G}, 2))$ induced by U. Fix any edge $e = \{v, w\}$ with $v < w$. Note that the only non-zero blocks in the column of U corresponding with $[v]$ are those blocks in the rows corresponding with a $[v']$ where $[v'] \overline{\prec} [v]$ (or equivalently $v' \prec v$). Hence, there are $a_{v'}, b_{w'} \in \mathbb{Z}$ (for all $v' \prec v$ and $w' \prec w$) such that

$$Uv = \sum_{\substack{v' \in V \\ v' \prec v}} a_{v'} v' \quad \text{and} \quad Uw = \sum_{\substack{w' \in V \\ w' \prec w}} b_{w'} w'.$$

Thus we obtain that

$$\begin{aligned} U e^{k(e)} = U[v, w] &= \left[\sum_{\substack{v' \in V \\ v' \prec v}} a_{v'} v', \sum_{\substack{w' \in V \\ w' \prec w}} b_{w'} w' \right] \\ &= \sum_{\substack{v' \in V \\ v' \prec v}} \sum_{\substack{w' \in V \\ w' \prec w}} a_{v'} b_{w'} [v', w'] \\ &= \sum_{\substack{\{v', w'\} \in E \\ \{v', w'\} \prec e}} a_{v'} b_{w'} [v', w'] \\ &\in \left\{ \sum_{\substack{e' \in E \\ e' \prec e}} a_{e'} {e'}^{k(e')} \;\middle|\; a_{e'} \in \mathbb{Z} \right\}. \end{aligned}$$

Since $\mu_i \not\overline{\prec} \mu_j$ if $i > j$, it follows that the matrix (with respect to $(E^k, \leq)$) of the morphism on $\gamma_2(\mathfrak{n}^{\mathbb{Q}}(\mathcal{G}, 2))$ that is induced by U is a block upper triangular matrix

$$\begin{pmatrix} B_1 & B_{12} & \dots & B_{1s} \\ 0 & B_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & B_{s-1s} \\ 0 & \dots & 0 & B_s \end{pmatrix}$$

where $B_{ij} \in \mathbb{Z}^{|\mu_i| \times |\mu_j|}$ and $B_{ij} = 0$ if $\mu_i \not\preceq \mu_j$. Hence, we obtain that the matrix of $\overline{\varphi}|_{\gamma_2(\mathfrak{n}^{\mathbb{Q}}(\mathcal{G},2))}$ has the form

$$P(\sigma_E) \cdot D(\varepsilon_\sigma) \cdot \begin{pmatrix} B_1 & B_{12} & \dots & B_{1s} \\ 0 & B_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & B_{s-1s} \\ 0 & \dots & 0 & B_s \end{pmatrix}.$$

Since this matrix coincides, by construction, with the matrix of $\varphi|_{\gamma_2(G)}$ it follows similarly as before that $B_i \in \mathrm{GL}_{|\mu_i|}(\mathbb{Z})$.

The assignment $\varphi \mapsto \overline{\varphi}$ from $\mathrm{Aut}(G)$ to $\mathrm{Aut}(\mathfrak{n}^{\mathbb{Q}}(\mathcal{G},2))$ is a group morphism since $L^{\mathbb{Q}}(G) \cong \mathfrak{n}^{\mathbb{Q}}(\mathcal{G},2)$ and $L^{\mathbb{Q}} : \mathbf{Grp} \to \mathbf{LieAlg}_{\mathbb{Q}}$ is a functor. Using the morphism $p \circ \pi_{\mathrm{ab}}$ from Section 4.6.2, the assignment $\overline{\varphi} \mapsto \overline{\sigma}$ satisfies the law of a group morphism. Together, we thus find that $\varphi \mapsto \overline{\sigma}$ defines a group morphism from $\mathrm{Aut}(A(\mathcal{G},2,k))$ to $\mathrm{Aut}(\overline{\mathcal{G}})$. □

As stated at the end of the lemma above, any automorphism $\varphi \in \mathrm{Aut}(A(\mathcal{G},2,k))$ induces a unique permutation $\overline{\sigma} \in \mathrm{Aut}(\overline{\mathcal{G}})$. In general, one does not obtain every permutation of $\mathrm{Aut}(\overline{\mathcal{G}})$, as the weight function $k : E \to \mathbb{N}_0$ imposes restrictions on what $\overline{\sigma}$ can be. Before describing these conditions, we introduce some necessary invariants of integer matrices and recall their relation with the Smith normal form.

Definition 7.3.22. Let $A \in \mathbb{Z}^{n \times n}$ be a matrix and $l \in \{1, \dots, n\}$. Then the *l-th determinant divisor* $d_l(A)$ of A equals the greatest common divisor of the determinants of the $l \times l$ minors of A.

In particular, if $A = \mathrm{diag}(a_1, \dots, a_n)$ is a diagonal matrix, then $d_l(A)$ equals the greatest common divisor of all l-fold products of diagonal elements, i.e. for any $l \in \{1, \dots, n\}$ it holds that

$$d_l(\mathrm{diag}(a_1, \dots, a_n)) = \gcd \left\{ \prod_{i \in I} a_i \,\middle|\, I \subset \{1, \dots, n\} \text{ with } |I| = l \right\}.$$

The determinant divisors completely determine the Smith normal form of a matrix in $\mathbb{Z}$. This is summarized by the following lemma. Proofs of these facts and an introduction to the Smith normal form of a matrix can be found in most handbooks about basic algebra, for example in [Nor12, Part 1, section 1].

Lemma 7.3.23. *Let $A, B \in \mathbb{Z}^{n\times n}$ be two matrices, then the following are equivalent:*

(i) *A and B have the same Smith normal form.*

(ii) *there exist $P, Q \in \mathrm{GL}_n(\mathbb{Z})$ such that $A = PBQ$.*

(iii) *$d_i(A) = d_i(B)$ for all $i = 1, \ldots, n$.*

Let $\mathcal{G} = (V, E)$ be a graph with weight function $k : E \to \mathbb{N}_0$ on its edges and quotient graph $\overline{\mathcal{G}} = (\Lambda_{\mathcal{G}}, \overline{E}, \Psi)$. Recall from Section 7.3.4 that any edge $e = \{v, w\} \in E$ gives both an equivalence class $\mu = [e] \in \Delta_{\mathcal{G}}$ and an edge in the quotient graph $\{[v], [w]\} \in \overline{E}$ which can be identified under the map from Equation (7.9). The edges in the equivalence class μ have weights $k|_\mu$ and thus the diagonal matrix with these weights on its diagonal equals $D(k|_\mu)$. For any such $\mu \in \Delta_{\mathcal{G}}$ and $l \in \{1, \ldots, |\mu|\}$ we denote with $d_l(\mu)$ the l-th determinant divisor of the matrix $D(k|_\mu)$, i.e.

$$d_l(\mu) := \gcd\left\{ \prod_{e\in I} k(e) \;\middle|\; I \subseteq \mu \text{ with } |I| = l \right\}.$$

Now we are ready to formulate a connection between the possible graph automorphisms $\sigma \in \mathrm{Aut}(\mathcal{G})$ that can occur in the matrix representation from Lemma 7.3.21 and the weight function k on the edges of $\mathcal{G}$.

Lemma 7.3.24. *Let $\mathcal{G} = (V, E)$ be a finite undirected simple graph with weight function $k : E \to \mathbb{N}_0$ on its edges. Suppose that $\varphi \in \mathrm{Aut}(A(\mathcal{G}, 2, k))$ and let $\sigma \in \mathrm{Aut}(\mathcal{G})$ be a graph automorphism which can occur in the matrix representation of φ given by Lemma 7.3.21. Let $e \in E$ be any edge and write $\mu = [e] \in \Delta_{\mathcal{G}}$, then for any $l = 1, \ldots, |\mu|$ it holds that*

$$d_l(\mu) = d_l(\sigma_E(\mu)).$$

In particular, it holds that:

$$\begin{cases} \gcd\limits_{e'\in\mu}(k(e')) &= \gcd\limits_{e'\in\sigma_E(\mu)}(k(e')) \\ \prod\limits_{e'\in\mu} k(e') &= \prod\limits_{e'\in\sigma_E(\mu)} k(e'). \end{cases}$$

Proof. By Lemma 7.3.21, we know that, with respect to the ordered basis $(E^k, \leq)$, the matrix representation of φ restricted to $\gamma_2(A(\mathcal{G}, 2, k))$ is of the form

$$P(\sigma_E) \cdot D(\varepsilon_\sigma) \cdot \underbrace{\begin{pmatrix} B_1 & B_{12} & \dots & B_{1s} \\ 0 & B_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & B_{s-1s} \\ 0 & \dots & 0 & B_s \end{pmatrix}}_{=:\, B}$$

where $B_i \in \mathrm{GL}_{|\mu_i|}(\mathbb{Z})$, $B_{ij} \in \mathbb{Z}^{|\mu_i| \times |\mu_j|}$ and $B_{ij} = 0$ if $\mu_i \not\preceq \mu_j$. Lemma 7.3.19 implies that

$$D(k) \cdot P(\sigma_E) \cdot D(\varepsilon_\sigma) \cdot B \cdot D(k)^{-1} \in \mathrm{GL}_{|E|}(\mathbb{Z}). \tag{7.10}$$

Remark that $D(k) \cdot P(\sigma_E) = P(\sigma_E) \cdot D(k \circ \sigma_E)$. Indeed, viewing these matrices as linear maps on $\mathrm{span}_{\mathbb{Q}}(E)$, it holds for any edge $e \in E$ that

$$\begin{aligned} D(k)P(\sigma_E)\, e &= D(k)\, \sigma_E(e) \\ &= k(\sigma_E(e))\, \sigma_E(e) \\ &= k(\sigma_E(e)) P(\sigma_E)\, e \\ &= P(\sigma_E) k(\sigma_E(e))\, e \\ &= P(\sigma_E) D(k \circ \sigma_E)\, e. \end{aligned}$$

Using this and the fact that $D(k \circ \sigma_E)$ and $D(\varepsilon_\sigma)$ commute, we find that

$$\begin{aligned} &D(k) \cdot P(\sigma_E) \cdot D(\varepsilon_\sigma) \cdot B \cdot D(k)^{-1} \\ &= \Big(P(\sigma_E) \cdot D(\varepsilon_\sigma)\Big) \cdot \Big(D(k \circ \sigma_E) \cdot B \cdot D(k)^{-1}\Big), \end{aligned}$$

which, by Equation (7.10) must lie in $\mathrm{GL}_{|E|}(\mathbb{Z})$. Since $P(\sigma_E) \cdot D(\varepsilon_\sigma)$ is clearly a matrix in $\mathrm{GL}_{|E|}(\mathbb{Z})$, we find that

$$C := D(k \circ \sigma_E) \cdot B \cdot D(k)^{-1} \in \mathrm{GL}_{|E|}(\mathbb{Z}).$$

Recall that B is a block upper triangular matrix which implies that C is also block upper triangular. Note that if a block upper triangular matrix lies in $\mathrm{GL}_n(\mathbb{Z})$, then so does every one of its blocks on the diagonal. Thus, by considering the block on the diagonal corresponding to $[e] = \mu = \mu_i$, we find that

$$D((k \circ \sigma_E)|_\mu) \cdot B_i \cdot D(k|_\mu) \in \mathrm{GL}_{|\mu|}(\mathbb{Z}).$$

Since B_i also lies in $\mathrm{GL}_{|\mu|}(\mathbb{Z})$, Lemma 7.3.23 implies that $D((k \circ \sigma_E)|_\mu)$ and $D(k|_\mu)$ must have the same Smith normal form or, equivalently, that for any $l = 1, \dots, |\mu|$ we have

$$d_l(\sigma(\mu)) = d_l(D((k \circ \sigma_E)|_\mu)) = d_l(D(k|_\mu)) = d_l(\mu).$$

This proves the main claim of the lemma. By noticing for all $\mu \in M$ that

$$\begin{cases} d_1(\mu) & = \gcd\limits_{e' \in \mu}(k(e')) \\ \\ d_{|\mu|}(\mu) & = \prod\limits_{e' \in \mu} k(e'), \end{cases}$$

we obtain the last claim of the statement. □

Recall that $\Delta_\mathcal{G} = E/\sim$ is the set of equivalence classes of edges and that for any $\sigma \in \mathrm{Aut}(\mathcal{G})$ we write $\sigma_E : E \to E$ for the induced permutation on the edges. We define the following subgroup of $\mathrm{Aut}(\mathcal{G})$

$$\mathrm{Aut}(\mathcal{G}, k) := \left\{ \sigma \in \mathrm{Aut}(\mathcal{G}) \;\middle|\; \begin{array}{c} \forall \mu \in \Delta_\mathcal{G}, 1 \leq l \leq |\mu| : \\ d_l(\mu) = d_l(\sigma_E(\mu)) \end{array} \right\}. \tag{7.11}$$

For what follows, we use the notation as introduced in Section 4.6.2. For any field $K \subset \mathbb{C}$, define the following subgroup of $\mathrm{GL}(\mathrm{span}_K(V))$:

$$G^K(\mathcal{G}, k) := P(\mathrm{Aut}(\mathcal{G}, k)) \cdot \left(\prod_{\lambda \in \Lambda_\mathcal{G}} \mathrm{GL}(\mathrm{span}_K(\lambda)) \right) \cdot U^K(\mathcal{G}).$$

Clearly $G^K(\mathcal{G}, k)$ is a subgroup of $G^K(\mathcal{G})$ as defined in Theorem 4.6.6. Moreover, we can prove the following lemma about $G^K(\mathcal{G}, k)$, which will be useful in the following section.

Lemma 7.3.25. *Let K be a subfield of $\mathbb{C}$. The group $G^K(\mathcal{G}, k)$ is a linear algebraic group that decomposes as a semi-direct product $R^K(\mathcal{G}, k) \rtimes U^K(\mathcal{G})$ where*

$$R^K(\mathcal{G}, k) := P(\mathrm{Aut}(\mathcal{G}, k)) \cdot \left(\prod_{\lambda \in \Lambda_\mathcal{G}} \mathrm{GL}(\mathrm{span}_K(\lambda)) \right)$$

is a linearly reductive group and $U^K(\mathcal{G})$ is the unipotent radical of $G^K(\mathcal{G}, k)$.

Proof. To prove that $G^K(\mathcal{G}, k)$ is a linear algebraic group, it suffices to show that it is Zariski closed as a subset of $\mathrm{GL}(\mathrm{span}(V))$ (with coordinates with respect to the basis of vertices V). This can be easily seen to be the case since it can be written as a finite union of Zariski closed subsets:

$$\bigcup_{\sigma \in \mathrm{Aut}(\mathcal{G},k)} P(\sigma) \cdot G_0^K(\mathcal{G}),$$

where we know that $G_0^K(\mathcal{G})$ is Zariski closed as it is the irreducible component at the identity of the linear algebraic group $G^K(\mathcal{G})$ (see Theorem 4.6.6). Second, $R^K(\mathcal{G}, k)$ is seen to be linearly reductive following the same argument as given in Remark 7.2.8. Lastly, $U^K(\mathcal{G})$ is the unipotent radical of $G^K(\mathcal{G}, k)$ since $U^K(\mathcal{G})$ is the unipotent radical of $G^K(\mathcal{G})$ (see Theorem 4.6.6). Indeed, $G^K(\mathcal{G}, k)$ and $G^K(\mathcal{G})$ have the same Zariski-connected component at the identity and thus the same unipotent radical. □

At last, from Lemma 7.3.24, we can now conclude the following corollary. The field in this corollary is $K = \mathbb{Q}$.

Corollary 7.3.26. *Let $\mathcal{G} = (V, E)$ be a finite undirected simple graph with weight function $k : E \to \mathbb{N}_0$ on its edges. For any automorphism $\varphi \in \mathrm{Aut}(A(\mathcal{G}, 2, k))$ it holds that $\pi_{\mathrm{ab}}(\overline{\varphi})$ lies in $G^{\mathbb{Q}}(\mathcal{G}, k)$, where $\overline{\varphi}$ denotes the induced automorphism on $\mathfrak{n}^{\mathbb{Q}}(\mathcal{G}, 2)$.*

7.3.6 The R_∞-property for the groups $A(\mathcal{G}, 2, k)$

The following theorem relates the structure of the graph $\mathcal{G} = (V, E)$ and the weight function $k : E \to \mathbb{N}_0$ to the R_∞-property for the groups $A(\mathcal{G}, 2)$ and $A(\mathcal{G}, 2, k)$.

Theorem 7.3.27. *Let $\mathcal{G} = (V, E)$ be a finite undirected simple graph and denote with $\mathcal{G}_0 = (V_0, E_0)$ the subgraph of $\mathcal{G}$ containing the vertices belonging to a coherent component of size one. Then the following holds.*

(i) If $E_0 = \emptyset$, then $A(\mathcal{G}, 2)$ does not have the R_∞-property.

(ii) If $\mathcal{G}_0$ has edges and $m \in \mathbb{N} \setminus \{0, 1\}$, then there exists a weight function $k : E \to \mathbb{N}_0$ such that $A(\mathcal{G}, 2, k)$ has the R_∞-property and $[A(\mathcal{G}, 2, k) : A(\mathcal{G}, 2)] = m$.

(iii) If $\mathcal{G}_0$ has edges and $\mathcal{G} = \mathcal{G}_0$, then $A(\mathcal{G}, 2)$ has the R_∞-property. In particular, all finitely generated torsion-free 2-step nilpotent groups commensurable with $A(\mathcal{G}, 2)$ have the R_∞-property.

Proof. (i) If $V_0 = \emptyset$, then all coherent components contain at least two vertices. Hence, it follows that $\xi(\mathcal{G}) \geq 4$. By Theorem 7.2.4 we get that the R_∞-nilpotency index of $A(\mathcal{G})$ is at least 4. Thus $A(\mathcal{G}, 2) = A(\mathcal{G})/\gamma_2(A(\mathcal{G}))$ does not have the R_∞-property.

Recall that $\sim$ is the equivalence relation on V defined in Section 4.1. Let us assume that $V_0 \neq \emptyset$. By Lemma 7.2.20, it follows that there exist monic polynomials $p_\lambda(X) \in \mathbb{Z}[X]$ (for $\lambda \in \Lambda_\mathcal{G} \setminus (V_0/\sim)$) of degree $|\lambda| \geq 2$ such that, if $\alpha_{\lambda 1}, \ldots, \alpha_{\lambda |\lambda|} \in \mathbb{C}$ are the zeros of $p_\lambda(X)$, it holds for all $\lambda, \lambda' \in \Lambda_\mathcal{G} \setminus (V_0/\sim)$ (possibly equal), $i = 1, \ldots, |\lambda|$ and $j = 1, \ldots, |\lambda'|$ that

$$\alpha_{\lambda i} \neq \pm 1,\ \alpha_{\lambda i}\alpha_{\lambda' j} \neq 1 \quad \text{and} \quad \alpha_{\lambda 1} \ldots \alpha_{\lambda |\lambda|} = -1. \tag{7.12}$$

Denote for any $\lambda \in \Lambda_\mathcal{G}$ with $A_\lambda \in \mathrm{GL}_{|\lambda|}(\mathbb{Z}) \subset \mathrm{GL}(\mathrm{span}_\mathbb{Q}(\lambda))$ the linear map for which the matrix with respect to the $\mathbb{Z}$-basis $(\lambda, \leq)$ is the companion matrix of $p_\lambda(X)$ if $\lambda \notin V_0/\sim$ or is $\begin{pmatrix}-1\end{pmatrix}$ if $\lambda \in V_0/\sim$. Combining these linear maps, leads to the linear map

$$A := \prod_{\lambda \in \Lambda_\mathcal{G}} A_\lambda \in \left(\prod_{\lambda \in \Lambda_\mathcal{G}} \mathrm{GL}(\mathbb{Z}, \lambda) \right).$$

Recall that we defined a projection map

$$\pi_{\mathrm{ab}} : \mathrm{Aut}(\mathfrak{n}^\mathbb{Q}(\mathcal{G}, 2)) \to G^\mathbb{Q}(\mathcal{G}). \tag{7.13}$$

Lemma 7.2.11 tells us that there exists an automorphism $\varphi \in \mathrm{Aut}(A(\mathcal{G}, 2))$ such that the induced automorphism $\overline{\varphi} \in \mathrm{Aut}_g(\mathfrak{n}^\mathbb{Q}(\mathcal{G}, 2))$ projects onto $A = \pi_{\mathrm{ab}}(\overline{\varphi})$. By construction, we know that

$$\overline{\varphi}|_{\mathrm{span}_\mathbb{Q}(V)} = \overline{\varphi}|_{\mathrm{span}_\mathbb{Q}(V_0)} \times \overline{\varphi}|_{\mathrm{span}_\mathbb{Q}(V \setminus V_0)}$$

is diagonalizable. Hence, we can fix a basis of eigenvectors $\mathcal{V}_1 \subset \mathrm{span}_\mathbb{Q}(V_0)$ for $\overline{\varphi}|_{\mathrm{span}_\mathbb{Q}(V_0)}$ and $\mathcal{V}_2 \subset \mathrm{span}_\mathbb{Q}(V \setminus V_0)$ for $\overline{\varphi}|_{\mathrm{span}_\mathbb{Q}(V \setminus V_0)}$. Thus, $\mathcal{V} := \mathcal{V}_1 \cup \mathcal{V}_2$ is a basis of eigenvectors for $\overline{\varphi}|_{\mathrm{span}_\mathbb{Q}(V)}$. Note that $[\mathcal{V}, \mathcal{V}]$ is a generating set of $\gamma_2(\mathfrak{n}^\mathbb{Q}(\mathcal{G}, 2))$. Since $[\mathrm{span}_\mathbb{Q}(V_0), \mathrm{span}_\mathbb{Q}(V_0)] = \mathrm{span}_\mathbb{Q}(E_0)$ is trivial, it follows that

$$[\mathcal{V}, \mathcal{V}] = [\mathcal{V}_1, \mathcal{V}_2] \cup [\mathcal{V}_2, \mathcal{V}_2].$$

Note that the non-zero brackets in $[\mathcal{V}_1, \mathcal{V}_2]$ and $[\mathcal{V}_2, \mathcal{V}_2]$ are eigenvectors of $\overline{\varphi}$. Moreover, the corresponding eigenvalues of the latter are products of two (possibly equal) $\alpha_{\lambda i}$ in Equation (7.12) and thus are not equal to 1. Since $\overline{\varphi}|_{\mathrm{span}_\mathbb{Q}(V_0)} = -\mathrm{Id}$, the eigenvalues corresponding with the eigenvectors in $[\mathcal{V}_1, \mathcal{V}_2]$ are of the form $-\alpha_{\lambda i}$ and are thus by Equation (7.12) also not equal to 1. The non-zero brackets in $[\mathcal{V}, \mathcal{V}]$ form a basis of eigenvectors of $\overline{\varphi}$ for $\operatorname{span}_{\mathbb{Q}}(E) = \gamma_2(\mathfrak{n}^{\mathbb{Q}}(\mathcal{G}, 2))$. Hence, $\mathcal{V} \cup [\mathcal{V}, \mathcal{V}]$ forms a basis of $\mathfrak{n}^{\mathbb{Q}}(\mathcal{G}, 2)$ of eigenvectors of $\overline{\varphi}$ with all eigenvalues not equal to 1. By Lemma 7.2.6 it now follows that $R(\varphi) < \infty$ which concludes the proof.

(ii) Fix any $m \in \mathbb{N} \setminus \{0, 1\}$ and any edge $e_0 = \{v_0, w_0\} \in E_0$ (with $v_0, w_0 \in V_0$). Note that the equivalence class $[e_0] = \{\{v_0, w_0\}\} \subset E$ is a singleton, by definition. Define $k : E \to \mathbb{N}_0$ by setting

$$k : E \to \mathbb{N}_0 : e \mapsto \begin{cases} m & \text{if } e = \{v_0, w_0\} \\ 1 & \text{if } e \neq \{v_0, w_0\} \end{cases}.$$

By Lemma 7.3.14 it follows that

$$[A(\mathcal{G}, 2, k) : A(\mathcal{G}, 2)] = \prod_{e \in E} k(e) = m.$$

We prove that $A(\mathcal{G}, 2, k)$ has the R_∞-property.

First, recall the definition of $\operatorname{Aut}(\mathcal{G}, k)$ in Equation (7.11) and note that with the above choice of k, for any $\sigma \in \operatorname{Aut}(\mathcal{G}, k)$, it holds that $\sigma_E(e_0) = e_0$. Next, take any automorphism $\varphi \in \operatorname{Aut}(A(\mathcal{G}, 2, k))$.

By Corollary 7.3.26, we know that $\pi_{\mathrm{ab}}(\overline{\varphi})$ lies in the linear algebraic group $G^{\mathbb{Q}}(\mathcal{G}, k) \subset G^{\mathbb{Q}}(\mathcal{G})$. Thus, the same must hold for its semi-simple part $\pi_{\mathrm{ab}}(\overline{\varphi})_s$. By the same argument as given in Remark 7.2.8 and Lemma 7.3.25, there must exist an $h \in G^{\mathbb{Q}}(\mathcal{G}, k)$ such that $g := h\, \pi_{\mathrm{ab}}(\overline{\varphi})_s\, h^{-1}$ lies in the linearly reductive subgroup $R^{\mathbb{Q}}(\mathcal{G}, k) \subset G^{\mathbb{Q}}(\mathcal{G}, k)$. Thus, g is represented by a matrix of the form

$$P(\sigma) \begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & A_r \end{pmatrix},$$

where $\sigma \in \operatorname{Aut}(\mathcal{G}, k)$ (in particular $\sigma_E(e_0) = e_0$) and $A_i \in \mathrm{GL}_{|\lambda_i|}(\mathbb{Q})$ for all $i = 1, \dots, r$. Note that from the fact that $\sigma_E(e_0) = e_0$ and $[v_0], [w_0]$ are singletons, the subspace $\operatorname{span}_{\mathbb{Q}}\{v_0, w_0\}$ is preserved by g. Now, since $\pi_{\mathrm{ab}}(\overline{\varphi})$ is integer-like, so is g and since g is given by a matrix over $\mathbb{Q}$ of the above form, it follows that each A_i is integer-like (see Lemma 6.1.4). In particular, if A_i has dimension 1×1, it must be equal to (± 1). This implies that if we restrict g to $\operatorname{span}_{\mathbb{Q}}\{v_0, w_0\}$, we get with respect to v_0, w_0, the matrix representation

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},$$

with $a, b \in \{-1, 1\}$. Let us prove that in either case, the automorphism $\varpi_2(g) \in \mathrm{Aut}_g(\mathfrak{n}^{\mathbb{Q}}(\mathcal{G}, 2))$ has an eigenvalue 1, where we recall the definition of ϖ_2 from Equation (7.4).

In the first case, if either a or b is equal to 1, clearly g has an eigenvalue 1 and thus also $\varpi_2(g)$ has. If both a and b are equal to -1, then the Lie bracket $[v_0, w_0]$ will be an eigenvector of $\varpi_2(g)$ with eigenvalue 1. Note that $[v_0, w_0]$ is non-zero since $\{v_0, w_0\} \in E$.

In the second case, if $a = b = 1$, then $v_0 + w_0$ will be an eigenvector of g with eigenvalue 1. If $a = b = -1$, then $v_0 - w_0$ will be an eigenvector of g with eigenvalue 1. In both cases we thus also get that $\varpi_2(g)$ has an eigenvalue 1. If a and b do not have the same sign, then the (non-zero) Lie bracket $[v_0, w_0]$ will be an eigenvector of $\varpi_2(g)$ with eigenvalue 1.

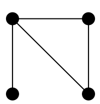
We can thus conclude that $\varpi_2(g)$ always has an eigenvalue 1. Note that since g and $\pi_{\mathrm{ab}}(\overline{\varphi})$ have the same eigenvalues, so do $\varpi_2(g)$ and $\varpi_2(\pi_{\mathrm{ab}}(\overline{\varphi})) = \overline{\varphi}$. We can conclude by Lemma 7.2.6 that $R(\varphi) = \infty$ and thus, since φ was an arbitrary automorphism of $A(\mathcal{G}, 2, k)$, that $A(\mathcal{G}, 2, k)$ has the R_∞-property.

(iii) If $\mathcal{G}_0 = \mathcal{G}$, then all coherent components are singletons (or equivalently $\mathcal{G}$ is transposition-free). From Theorem 7.2.2 it thus follows that $A(\mathcal{G})$ has R_∞-nilpotency index equal to 2. Thus $A(\mathcal{G}, 2)$ has the R_∞–property. By Proposition 7.3.6 the result follows.

□

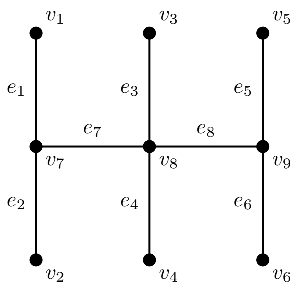
We give some examples where we can say more than the statement in Theorem 7.3.27.

Example 7.3.28 ($\mathcal{G}_0 \subsetneq \mathcal{G}$ has edges and $A(\mathcal{G}, 2)$ has the R_∞-property)**.** Let us consider the following graph $\mathcal{G}$:

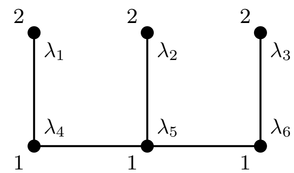


In [DL23, Theorem 4.4 and Example 4.5], it was proven that $A(\mathcal{G}, 2)$ has the R_∞–property (where the authors use the opposite convention to define RAAGs). Proposition 7.3.6 now tells us that all groups commensurable with $A(\mathcal{G}, 2)$ have the R_∞–property. In particular, for any weight function $k : E \to \mathbb{N}_0$ on the edges of $\mathcal{G}$ it holds that $A(\mathcal{G}, 2, k)$ has the R_∞–property.

Example 7.3.29 ($A(\mathcal{G}, 2)$ does not have the R_∞-property, but $A(\mathcal{G}, 2, k)$ does)**.** Consider the following graph, where we fixed a total order of V, E and $\Lambda_\mathcal{G} = V/\sim$ as introduced in Section 7.3.4.

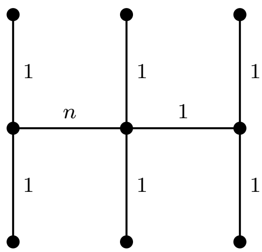


(a) The graph $\mathcal{G}$ with total orders for V and E.



(b) Quotient graph $\overline{\mathcal{G}} = (\Lambda_\mathcal{G}, \overline{E}, \Psi)$ with total order on $\Lambda_\mathcal{G}$

We define a weight function $k_n : E \to \mathbb{N}_0$ as done on the following figure.



For any $n \in \mathbb{N}\setminus\{0, 1\}$ we consider the group $A(\mathcal{G}, 2, k_n)$ as defined in Section 7.3.3. By the proof of Theorem 7.3.27 (ii) we already know that $A(\mathcal{G}, 2, k)$ has the R_∞-property for any $n \in \mathbb{N} \setminus \{0, 1\}$.

We now prove that $A(\mathcal{G}, 2)$ does not have the R_∞-property by giving an explicit automorphism $\varphi \in \mathrm{Aut}(A(\mathcal{G}, 2))$ with $R(\varphi) < \infty$. Let us write $\sigma := (v_1\, v_5)(v_7\, v_9)(v_2\, v_6) \in \mathrm{Aut}(\mathcal{G})$. Define the matrix $A \in \mathrm{GL}_2(\mathbb{Z})$ by

$$A := \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

and the matrix $B \in \mathrm{GL}_9(\mathbb{Z})$ by

$$B = \begin{pmatrix} 0 & 0 & A & 0 & 0 & 0 \\ 0 & A & 0 & 0 & 0 & 0 \\ \mathbb{1}_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$= P(\sigma) \cdot \begin{pmatrix} \mathbb{1}_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & A & 0 & 0 & 0 & 0 \\ 0 & 0 & A & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

where the second equality for B will illustrate Lemma 7.3.21. Using Corollary 7.2.13, there exists an automorphism $\varphi \in \mathrm{Aut}(A(\mathcal{G},2))$ such that the induced automorphism $\overline{\varphi} \in \mathrm{Aut}(\mathfrak{n}^{\mathbb{Q}}(\mathcal{G},2))$ satisfies that $\pi_{\mathrm{ab}}(\overline{\varphi})$ has matrix representation $B \in \mathrm{GL}_9(\mathbb{Z})$ with respect to the ordered basis of vertices $(V, \leq)$. One can derive that the matrix of $\varphi|_{\gamma_2(A(\mathcal{G},2))}$ with respect to the $\mathbb{Z}$-basis $(E, \leq)$ equals

$$C = \begin{pmatrix} 0 & 0 & -A & 0 & 0 \\ 0 & -A & 0 & 0 & 0 \\ \mathbb{1}_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= P(\sigma_E)D(\varepsilon_\sigma) \begin{pmatrix} \mathbb{1}_2 & 0 & 0 & 0 & 0 \\ 0 & -A & 0 & 0 & 0 \\ 0 & 0 & -A & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\sigma_E := (e_1\, e_5)(e_7\, e_8)(e_2\, e_6)$ and

$$\varepsilon_\sigma(e_i) = \begin{cases} -1 & \text{if } i = 7, 8 \\ 1 & \text{else} \end{cases}.$$

In particular, the eigenvalues of the induced automorphism $\overline{\varphi} \in \mathrm{Aut}(\mathfrak{n}^{\mathbb{Q}}(\mathcal{G},2))$ are precisely the eigenvalues of the matrices B and C. The eigenvalues of A are given by: $\alpha_1 := \frac{3-\sqrt{5}}{2}$ and $\alpha_2 := \frac{3+\sqrt{5}}{2}$. One can calculate that B has eigenvalues

$$\alpha_1, \alpha_2, \pm\sqrt{\alpha_1}, \pm\sqrt{\alpha_2}, -1 \text{ and } \pm i$$

and the eigenvalues of C are

$$-\alpha_1, -\alpha_2, \pm\sqrt{\alpha_1}i, \pm\sqrt{\alpha_2}i \text{ and } \pm i.$$

Since 1 is not an eigenvalue of $\overline{\varphi}$, Lemma 7.2.6 implies that $R(\varphi) < \infty$ and thus $A(\mathcal{G}, 2)$ does not have the R_∞-property.

Bibliography

[AA67] V. I. Arnold and A. Avez. *Problèmes ergodiques de la mécanique classique*, volume 9 of *Monographies Internationales de Mathématiques Modernes.* Gauthier-Villars, Éditeur, Paris, 1967.

[Ano62] D. V. Anosov. Roughness of geodesic flows on compact Riemannian manifolds of negative curvature. *Dokl. Akad. Nauk SSSR*, 145:707–709, 1962.

[Ano63] D. V. Anosov. Ergodic properties of geodesic flows on closed Riemannian manifolds of negative curvature. *Dokl. Akad. Nauk SSSR*, 151:1250–1252, 1963.

[Ano69] D.V. Anosov. *Geodesic flow on closed Riemannian manifolds with negative curvature.* Proc. Steklov Inst., 90, 1969.

[Ano85] D.V. Anosov. The Nielsen numbers of maps of nil-manifolds. *Uspekhi. Mat. Nauk*, 40(4(224)):pp. 133–134, 1985. English transl.: Russian Math. Surveys, 40 (no. 4), 1985, pp. 149–150.

[AS67] D. V. Anosov and Ja. G. Sinaĭ. Certain smooth ergodic systems. *Uspehi Mat. Nauk*, 22(5(137)):107–172, 1967.

[Aus60] Louis Auslander. Bieberbach's Theorem on Space Groups and Discrete Uniform Subgroups of Lie Groups. *Ann. of Math. (2)*, 71((3)):pp. 579–590, 1960.

[Bab95] László Babai. Automorphism groups, isomorphism, reconstruction. In *Handbook of combinatorics, Vol. 1, 2*, pages 1447–1540. Elsevier Sci. B. V., Amsterdam, 1995.

[BBN$^+$78] Harold Brown, Rolf Bülow, Joachim Neubüser, Hans Wondratschek, and Hans Zassenhaus. *Crystallographic groups of four-dimensional space.* Wiley Monographs in Crystallography. Wiley-Interscience [John Wiley & Sons], New York-Chichester-Brisbane, 1978.

[Ber10] Grégory Berhuy. *An introduction to Galois cohomology and its applications*, volume 377 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2010. With a foreword by Jean-Pierre Tignol.

[Ber22] Niels Bertier. Anosov diffeomorphisms on manifolds associated to graphs, 2022. Master's thesis at the Faculty of Science, KU Leuven.

[BG86] R. M. Bryant and J. R. J. Groves. Algebraic groups of automorphisms of nilpotent groups and Lie algebras. *J. London Math. Soc. (2)*, 33(3):453–466, 1986.

[Bie11] Ludwig Bieberbach. Über die Bewegungsgruppen der Euklidischen Räume I. *Math. Ann.*, 70(3):pp. 297–336, 1911.

[Bie12] Ludwig Bieberbach. Über die Bewegungsgruppen der Euklidischen Räume II. *Math. Ann.*, 72(3):pp. 400–412, 1912.

[BK81] Peter Buser and Hermann Karcher. *Gromov's almost flat manifolds*, volume 81 of *Astérisque*. Société Mathématique de France, Paris, 1981.

[Bor91] Armand Borel. *Linear algebraic groups*, volume 126 of *Graduate Texts in Mathematics*. Springer-Verlag, second edition, 1991.

[Bou72] Nicolas Bourbaki. *Eléments de Mathématiques, Groupes et Algèbres de Lie, CH. I, II et III*. Hermann, Paris, 1971/72.

[Bum98] Daniel Bump. *Algebraic geometry*. World Scientific Publishing Co., Inc., River Edge, NJ, 1998.

[Cor16] Yves Cornulier. Gradings on Lie algebras, systolic growth, and cohopfian properties of nilpotent groups. *Bull. Soc. Math. France*, 144(4):693–744, 2016.

[CS01] Carlos Cid and Tilman Schulz. Computation of five- and six-dimensional Bieberbach groups. *Experiment. Math.*, 10(1):109–115, 2001.

[dC92] Manfredo Perdigão do Carmo. *Riemannian geometry*. Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, portuguese edition, 1992.

[DD14] Karel Dekimpe and Jonas Deré. Existence of anosov diffeomorphisms on infra-nilmanifolds modeled on free nilpotent lie groups. *Topological Methods in Nonlinear Analysis*, 2014.

[DD16] Karel Dekimpe and Jonas Deré. Expanding maps and non-trivial self-covers on infra-nilmanifolds. *Topol. Methods Nonlinear Anal.*, 47(1):347–368, 2016.

[Dek96] Karel Dekimpe. *Almost-Bieberbach Groups: Affine and Polynomial Structures*, volume 1639 of *Lect. Notes in Math.* Springer–Verlag, 1996.

[Dek01] Karel Dekimpe. Hyperbolic automorphisms and Anosov diffeomorphisms on nilmanifolds. *Trans. Amer. Math. Soc.*, 353(7):pp. 2859–2877, 2001.

[Dek12] Karel Dekimpe. What an infra-nilmanifold endomorphism really should be .... *Topol. Methods Nonlinear Anal.*, 40(1):111–136, 2012.

[Dek18] Karel Dekimpe. A users' guide to infra-nilmanifolds and almost-Bieberbach groups. In *Handbook of group actions. Vol. III*, volume 40 of *Adv. Lect. Math. (ALM)*, pages 215–262. Int. Press, Somerville, MA, 2018.

[Der16a] Jonas Deré. A new method for constructing Anosov Lie algebras. *Trans. Amer. Math. Soc.*, 368(2):1497–1516, 2016.

[Der16b] Jonas Deré. Periodic and eventually periodic points of affine infra-nilmanifold endomorphisms. *Discrete Contin. Dyn. Syst.*, 36(10):5347–5368, 2016.

[Der17] Jonas Deré. Gradings on Lie algebras with applications to infra-nilmanifolds. *Groups Geom. Dyn.*, 11(1):105–120, 2017.

[DG14] Karel Dekimpe and Daciberg L. Gonçalves. The R_∞ property for free groups, free nilpotent groups and free solvable groups. *Bull. Lond. Math. Soc.*, 46(4):737–746, 2014.

[DG16] Karel Dekimpe and Daciberg L. Gonçalves. The R_∞ property for nilpotent quotients of surface groups. *Trans. London Math. Soc.*, 3(1):28–46, 2016.

[Dir13] Peter Gustav Lejeune Dirichlet. *Beweis des Satzes, dass jede unbegrenzte arithmetische Progression, deren erstes Glied und Differenz ganze Zahlen ohne gemeinschaftlichen Factor sind, unendlich viele Primzahlen enthält*, page 342–359. Cambridge Library Collection - Mathematics. Cambridge University Press, 2013.

[DK92] G. Duchamp and D. Krob. The lower central series of the free partially commutative group. *Semigroup Forum*, 45(3):385–394, 1992.

[DK00] J. J. Duistermaat and J. A. C. Kolk. *Lie groups*. Universitext. Springer-Verlag, Berlin, 2000.

[DL23] Karel Dekimpe and Maarten Lathouwers. The reidemeister spectrum of 2-step nilpotent groups determined by graphs. *Communications in algebra*, 51(6):2384–2407, 2023.

[DLG20] Karel Dekimpe and Daciberg L. Lima Gonçalves. The R_∞-property for nilpotent quotients of Baumslag-Solitar groups. *J. Group Theory*, 23(3):545–562, 2020.

[DM05] S. G. Dani and Meera G. Mainkar. Anosov automorphisms on compact nilmanifolds associated with graphs. *Trans. Amer. Math. Soc.*, 357(6):2235–2251, 2005.

[DM23] Jonas Deré and Meera Mainkar. Anosov diffeomorphisms on infra-nilmanifolds associated to graphs. *Math. Nachr.*, 296(2):610–629, 2023.

[DS21] Karel Dekimpe and Pieter Senden. The R_∞-property for right-angled Artin groups. *Topology Appl.*, 293:Paper No. 107557, 36, 2021.

[DV09] Karel Dekimpe and Kelly Verheyen. Anosov diffeomorphisms on nilmanifolds modeled on a free nilpotent lie group. *Dynamical Systems – an international journal*, 24(1):pp. 117–121, 2009.

[DW23a] Jonas Deré and Thomas Witdouck. Classification of K-forms in nilpotent Lie algebras associated to graphs. *Comm. Algebra*, 51(12):5209–5234, 2023.

[DW23b] Jonas Deré and Thomas Witdouck. On closed manifolds admitting an Anosov diffeomorphism but no expanding map. *J. Pure Appl. Algebra*, 227(4):Paper No. 107247, 26, 2023.

[DW24] Jonas Deré and Thomas Witdouck. A characterization of Anosov rational forms in nilpotent Lie algebras associated to graphs. *To appear in Monatshefte für Mathematik*, 2024. `https://doi.org/10.1007/s00605-024-01978-8`.

[ER63] P. Erdős and A. Rényi. Asymmetric graphs. *Acta Math. Acad. Sci. Hungar.*, 14:295–315, 1963.

[FGH13] David J. Fisher, Robert J. Gray, and Peter E. Hydon. Automorphisms of real Lie algebras of dimension five or less. *J. Phys. A*, 46(22):225204, 18, 2013.

[Fra70] J. Franks. Anosov diffeomorphisms. *Global Analysis: Proceedings of the Symposia in Pure Mathematics*, 14,:pp. 61–93, 1970.

[GL16] Andrey Gogolev and Jean-François Lafont. Aspherical products which do not support Anosov diffeomorphisms. *Ann. Henri Poincaré*, 17(11):3005–3026, 2016.

[GRH14] Andrey Gogolev and Federico Rodriguez Hertz. Manifolds with higher homotopy which do not support Anosov diffeomorphisms. *Bull. Lond. Math. Soc.*, 46(2):349–366, 2014.

[Gro78] M. Gromov. Almost flat manifolds. *J. Differential Geometry*, (13):pp. 231–241, 1978.

[Gro81] M. Gromov. Groups of polynomial growth and expanding maps. *Institut des Hautes Études Scientifiques*, (53):pp. 53–73, 1981.

[GS84] Fritz Grunewald and Dan Segal. Reflections on the classification of torsion-free nilpotent groups. In *Group theory*, pages 121–158. Academic Press, London, 1984.

[GW09] Daciberg L. Gonçalves and Peter Wong. Twisted conjugacy classes in nilpotent groups. *J. Reine Angew. Math.*, 633:11–27, 2009.

[Hir71] Morris W. Hirsch. Anosov maps, polycyclic groups and homology. *Topology*, (10):pp. 177–183, 1971.

[HK03] Boris Hasselblatt and Anatole Katok. *A first course in dynamics.* Cambridge University Press, New York, 2003. With a panorama of recent developments.

[Hoc81] Gerhard P. Hochschild. *Basic theory of algebraic groups and Lie algebras*, volume 75 of *Graduate Texts in Mathematics.* Springer-Verlag, New York-Berlin, 1981.

[Hop26] Heinz Hopf. Zum Clifford-Kleinschen Raumproblem. *Math. Ann.*, 95(1):313–339, 1926.

[Hum81] James E. Humphreys. *Linear Algebraic Groups.* Graduate Texts in Mathematics. Springer-Verlag, New York, 1981.

[Jac55] N. Jacobson. A note on automorphisms and derivations of Lie algebras. *Proc. Amer. Math. Soc.*, 6:281–283, 1955.

[Jia83] Boju Jiang. *Nielsen Fixed Point Theory*, volume 14 of *Contemp. Math.* American Mathematical Society, 1983.

[KH95] Anatole Katok and Boris Hasselblatt. *Introduction to the modern theory of dynamical systems*, volume 54 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1995. With a supplementary chapter by Katok and Leonardo Mendoza.

[Kil91] Wilhelm Killing. Ueber die Clifford-Klein'schen Raumformen. *Math. Ann.*, 39(2):257–278, 1891.

[KL93] D. Krob and P. Lalonde. Partially commutative Lyndon words. In *STACS 93 (Würzburg, 1993)*, volume 665 of *Lecture Notes in Comput. Sci.*, pages 237–246. Springer, Berlin, 1993.

[Lal93] Pierre Lalonde. Bases de Lyndon des algèbres de Lie libres partiellement commutatives. volume 117, pages 217–226. 1993. Conference on Formal Power Series and Algebraic Combinatorics (Bordeaux, 1991).

[Lal95] Pierre Lalonde. Lyndon heaps: an analogue of Lyndon words in free partially commutative monoids. *Discrete Math.*, 145(1-3):171–189, 1995.

[Lau95] Michael R. Laurence. A generating set for the automorphism group of a graph group. *J. London Math. Soc. (2)*, 52(2):318–334, 1995.

[Lau08] Jorge Lauret. Rational forms of nilpotent Lie algebras and Anosov diffeomorphisms. *Monatsh. Math.*, 155(1):15–30, 2008.

[Laz54] Michel Lazard. Sur les groupes nilpotents et les anneaux de Lie. *Ann. Sci. École Norm. Sup. (3)*, 71:101–190, 1954.

[Lot97] M. Lothaire. *Combinatorics on words*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1997. With a foreword by Roger Lyndon and a preface by Dominique Perrin, Corrected reprint of the 1983 original, with a new preface by Perrin.

[LR85] Kyung Bai Lee and Frank Raymond. Rigidity of almost crystallographic groups. *Contemp. Math.*, 44,:pp. 73–78, 1985.

[LW08] Jorge Lauret and Cynthia E. Will. On Anosov automorphisms of nilmanifolds. *J. Pure Appl. Algebra*, 212(7):1747–1755, 2008.

[LW09] Jorge Lauret and Cynthia E. Will. Nilmanifolds of dimension $\leq$ 8 admitting Anosov diffeomorphisms. *Trans. Amer. Math. Soc.*, 361(5):2377–2395, 2009.

[LW24] Maarten Lathouwers and Thomas Witdouck. R_∞-property for groups commensurable to nilpotent quotients of RAAGs, 2024. https://arxiv.org/abs/2402.15320.

[Lyn74] Roger C. Lyndon. Non Euclidean crystallographic groups. In *Proceedings of the Second International Conference on the Theory of Groups (Australian Nat. Univ., Canberra, 1973)*, volume Vol. 372 of *Lecture Notes in Math.*, pages 437–442. Springer, Berlin-New York, 1974.

[Mai06] Meera G. Mainkar. Anosov automorphisms on certain classes of nilmanifolds. *Glasg. Math. J.*, 48(1):161–170, 2006.

[Mai12] Meera G. Mainkar. Anosov Lie algebras and algebraic units in number fields. *Monatsh. Math.*, 165(1):79–90, 2012.

[Mal49a] Anatoliĭ I. Mal'cev. On a class of homogeneous spaces. *Izvestiya Aka. Nauk SSSR., Ser. Mat.*, (13):pp. 9–32, 1949.

[Mal49b] Anatoly I. Maltsev. Nilpotent torsion–free groups. *Izvestiya Aka. Nauk SSSR., Ser. Mat.*, (13):pp. 201–212, 1949.

[Man74] Anthony Manning. There are no new Anosov diffeomorphisms on tori. *Amer. J. Math.*, 96(3):pp. 422–429, 1974.

[Mil22] James S. Milne. Fields and galois theory (v5.10), 2022. Available at www.jmilne.org/math/.

[Nor12] Christopher. Norman. *Finitely Generated Abelian Groups and Similarity of Matrices over a Field.* Springer Undergraduate Mathematics Series. Springer London, London, 1st ed. 2012. edition, 2012.

[Pas77] Donald S. Passman. *The Algebraic Structure of Group Rings.* Pure and Applied Math. John Wiley & Sons, Inc. New York, 1977.

[Pay09] Tracy L. Payne. Anosov automorphisms of nilpotent Lie algebras. *J. Mod. Dyn.*, 3(1):121–158, 2009.

[PS00] Wilhelm Plesken and Tilman Schulz. Counting crystallographic groups in low dimensions. *Experiment. Math.*, 9(3):407–411, 2000.

[Qui68] Daniel G. Quillen. On the associated graded ring of a group ring. *J. Algebra*, 10:411–418, 1968.

[Rom06] Steven Roman. *Field theory*, volume 158 of *Graduate Texts in Mathematics.* Springer, New York, second edition, 2006.

[Rom11a] V. Roman'kov. Twisted conjugacy classes in nilpotent groups. *J. Pure Appl. Algebra*, 215(4):664–671, 2011.

[Rom11b] V. Roman'kov. Twisted conjugacy classes in nilpotent groups. *J. Pure Appl. Algebra*, 215(4):664–671, 2011.

[Rot98] Joseph Rotman. *Galois theory*. Universitext. Springer-Verlag, New York, second edition, 1998.

[Ruh82] Ernst A. Ruh. Almost flat manifolds. *J. Differential Geometry*, (17):pp. 1–14, 1982.

[Sas58] Shigeo Sasaki. On the differential geometry of tangent bundles of Riemannian manifolds. *Tohoku Math. J. (2)*, 10:338–354, 1958.

[Sas62] Shigeo Sasaki. On the differential geometry of tangent bundles of Riemannian manifolds. II. *Tohoku Math. J. (2)*, 14:146–155, 1962.

[Seg83] Daniel Segal. *Polycyclic Groups*. Cambridge University Press, 1983.

[Ser89] Herman Servatius. Automorphisms of graph groups. *J. Algebra*, 126(1):34–60, 1989.

[Ser97] Jean-Pierre Serre. *Galois cohomology*. Springer-Verlag, Berlin, 1997. Translated from the French by Patrick Ion and revised by the author.

[Shi73] Kenichi Shiraiwa. Manifolds which do not admit Anosov diffeomorphisms. *Nagoya Math. J.*, 49:111–115, 1973.

[Shu70] M. Shub. Expanding maps. *Global Analysis: Proceedings of the Symposia in Pure Mathematics XIV*, 14,:pp. 273–277, 1970.

[Sma63] S. Smale. Stable manifolds for differential equations and diffeomorphisms. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)*, 17:97–116, 1963.

[Sma67] S. Smale. Differentiable dynamical systems. *Bull. Amer. Math. Soc.*, 73,:pp. 747–817, 1967.

[Spr81] T.A. Springer. *Linear Algebraic Groups*. Progress in Mathematics. Birkhäuser, Boston, 1981.

[ST87] I.N. Stewart and D.O. Tall. *Algebraic Number Theory, second edition.* Chapman and Hall Mathematics Series. Chapman and Hall, London, 1987.

[Ste89] Ian Stewart. *Galois theory*. Chapman and Hall, Ltd., London, second edition, 1989.

[Sul23] D. Sulca. On the degree of polynomial subgroup growth of nilpotent groups. *Math. Z.*, 303(1):Paper No. 8, 26, 2023.

[Wad15] Richard D. Wade. The lower central series of a right-angled Artin group. *Enseign. Math.*, 61(3-4):343–371, 2015.

[Wit23] Thomas Witdouck. The R_∞-property for right-angled Artin groups and their nilpotent quotients, 2023. `https://arxiv.org/abs/2304.01077`.

[Yan83] Koichi Yano. There are no transitive Anosov diffeomorphisms on negatively curved manifolds. *Proc. Japan Acad. Ser. A Math. Sci.*, 59(9):445, 1983.

[Zas48] H Zassenhaus. über einen algorithmus zur bestimmung der raumgruppen. *Commentarii Mathematici Helvetici*, (21):pp. 117–141, 1948.

List of publications

- Jonas Deré and Thomas Witdouck. On closed manifolds admitting an Anosov diffeomorphism but no expanding map. *J. Pure Appl. Algebra*, 227(4):Paper No. 107247, 26, 2023.
- Jonas Deré and Thomas Witdouck. Classification of K-forms in nilpotent Lie algebras associated to graphs. *Comm. Algebra*, 51(12):5209–5234, 2023.
- Jonas Deré and Thomas Witdouck. A characterization of Anosov rational forms in nilpotent Lie algebras associated to graphs. *To appear in Monatshefte für Mathematik*, 2024. https://doi.org/10.1007/s00605-024-01978-8
- Thomas Witdouck. The R_∞-property for right-angled Artin groups and their nilpotent quotients, 2023. https://arxiv.org/abs/2304.01077 (Preprint)
- Maarten Lathouwers and Thomas Witdouck. R_∞-property for groups commensurable to nilpotent quotients of RAAGs, 2024. https://arxiv.org/abs/2402.15320 (Preprint)

FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS
ALGEBRAIC TOPOLOGY AND GROUP THEORY
Etienne Sabbelaan 53
BE-8500 Kortrijk
thomas.witdouck@kuleuven.be