KU LEUVEN

FACULTY OF SCIENCE

Geometry of Singular Foliations through their Universal Lie ∞-algebroid

Yenthe DESCHRYVER

Supervisor: Prof. M. Zambon

Thesis presented in fulfillment of the requirements for the degree of Master of Science in Mathematics

Academic year 2020-2021

Acknowledgements

Non si può descrivere la passione,
la si può solo vivere.

Enzo Ferrari

I would like devote this part of the thesis to thank some people without whom I would not stand where I am today.

First of all I would like to thank professor Marco Zambon. His enthusiastic teaching style first drew me toward differential geometry. Also, his help and guidance where paramount in the writing of this thesis, I am very grateful for the weekly (online) meetings to discuss the material and his patience with me during these meetings.

I would also like to thank my readers: professor Raf Cluckers and Dr. Feng Hao. The selected topics course by professor Cluckers during last academic year really broadened my skillset as a mathematics student and my general interest in mathematics.

Of course, I also thank everyone at the geometry section and especially the professors that I had as teachers for the several geometry courses during the bachelor and the master. Their enthusiasm for the subject definitely also infected me (not unlike the coronavirus).

I also owe an infinite amount of thanks to my parents, my brother and grandparents. Without their support during my studies, I would certainly not be where I am today. During the past five years they did everything to make life at home as comfortable as it was. Mom, dad and Yoren: a very big thank you!

Also, a very big thank you to my friends for making the past years so much fun and the support along the way. Especially Milan Donvil for all the mathematics discussions from the first bachelor year up until now and certainly for the help with LaTeXwhile writing this thesis!

Lastly, I would like to thank some very important teachers I had during my studies in high school. Without Joeri Weckx and Wim Nivelle my life would certainly be very different, without them 'the switch' would not have been possible. After this switch I had the pleasure of meeting Dirk Wouters as a teacher. His passion for mathematics, relentless support, encouragement and belief in me where the sole reason I decided to pursue mathematics at university. I owe him a huge amount of thanks!

Dankwoord

Non si può descrivere la passione,
la si può solo vivere.

Enzo Ferrari

Ik zou dit deel van de thesis willen gebruiken om een aantal mensen te bedanken zonder wie ik niet de persoon zou zijn die ik vandaag de dag ben.

Eerst en vooral zou ik professor Marco Zambon willen bedanken. Zijn enthousiasme tijdens de lessen is wat mijn interesse in differentiaalmeetkunde echt aanwakkerde. Ik ben hem ook zeer dankbaar voor de wekelijkse (online) meetings om over de inhoud van deze thesis te spreken en zijn geduld tijdens deze meetings.

Ik zou ook graag mijn twee lezers willen bedanken: professor Raf Cluckers en Dr. Feng Hao. De selected topics course van professor Cluckers tijdens het vorige academiejaar hebben van mij zeker een betere wiskundestudent gemaakt en heeft bijgedragen aan mijn algemene interesse in de wiskunde.

Graag zou ik ook alle leden van de afdeling meetkunde bedanken. In het bijzonder de professoren die ik als leraren gehad heb tijdens de bachelor en master. Hun enthousiasme voor het vak heeft zeker aanstekelijk gewerkt.

Natuurlijk mag ik de belangrijkste personen niet vergeten: mijn ouders, broer en (over)grootouders. Zonder hun ondersteuning tijdens mijn studies zou ik zeker niet staan waar ik nu sta. Gedurende de afgelopen vijf jaar hebben ze geen enkele moeite gespaard om mij (en mijn broer) het oh zo comfortabele leven thuis te geven. Het is niet altijd even makkelijk geweest maar toch waren ze er telkens en daarom: mama, papa, Yoren, bompa, bomma, opa, oma, moemoe en vava (ook al zijn sommigen onder jullie er ondertussen spijtig genoeg niet meer): enorm bedankt voor alles!

Last but not least wil ik ook nog een aantal leerkrachten uit het middelbaar bedanken. Zonder Joeri Weckx en Wim Nivelle zou het niet mogelijk geweest zijn om van studierichting te veranderen, hiervoor ben ik ze dan ook beiden enorm dankbaar. De misschien wel belangrijkste leerkracht die ik wil bedanken is Dirk Wouters. Zonder zijn enthousiasme voor wiskunde, de ondersteuning, de aanmoediging, zijn geloof in mij en niet te vergeten de mapoefeningen, zou mijn enthousiasme/liefde voor wiskunde niet geworden zijn tot wat het vandaag de dag is. Bedankt om mij op deze weg te sturen!

Summary

During the 1950s and 1960s the problem of integrating singular distributions on smooth manifolds arose in control theory. From this research the concept of singular foliations arose. Singular foliations are partitioning's of manifolds into smoothly immersed connected submanifolds that we call the leaves of the foliation. They appear naturally in the study of differential geometry: Lie group actions, symplectic foliations on Poisson manifolds,... An important difference with regular foliations is that the dimension of the leaves may vary. A classic result from differential geometry is Frobenius's theorem. It states that when given a foliation, one can associate to it a constant rank distribution. From this distribution one gets a submodule of the vectorfields on the manifold by taking (global) compactly supported sections of the distribution. Furthermore Frobenius's theorem ensures that the geometric point of view i.e., the partitioning into leaves, and the algebraic point of view i.e., through the submodule of the vector fields, are equivalent. For singular foliations the situation is more complicated. In the singular case the algebraic point of view carries inherently more information. This is due to the fact that when given a partitioning into leaves, there may be infinitely many choices of submodules that induce this partitioning. In this context the problem of defining invariants to singular foliations arose. In [AS09] Androulidakis and Skandalis gave two such invariants: the isotropy Lie algebra (a more local invariant) and the holonomy groupoid (which can give more global information). The aim of this thesis is to introduce the necessary concepts and results to understand a recently discovered invariant by S. Lavau, C. Laurent-Gengoux and T. Strobl in [LGLS20]. Throughout the whole thesis we will provide more detailed proofs from the results in [LGLS20] than the ones found in the original publication. In their work they constructed the universal Lie ∞-algebroid of a singular foliation. This object involves constructing a 'higher structure' on a so called geometric resolution of a singular foliation. By this we mean constructing 'higher brackets' between sections of a particular complex of vector bundles associated to the foliation, in such a way that these brackets satisfy 'higher Jacobi identities'. Our main focus will be on defining all the involved objects and necessary lemmas, propositions and theorems following, for the most part, our main source [LGLS20]. Once we have defined all the necessary concepts we will shift our focus to answering the following question: 'can all rank r singular foliations be locally induced by a rank r Lie algebroid?'. For this we again follow our main source [LGLS20].

List of Symbols

Introduction

Foliations

Regular foliations are classical objects of study in differential geometry. They are treated in most introductory texts on differential geometry, see for instance [Lee12]. A foliation $\mathcal{F}$ describes the partition of a smooth manifold M into smoothly immersed disjoint submanifolds, called leaves, of the same dimension that satisfy the foliation property (that is they fit together nicely like illustrated in figure 1). An important result in the study of regular foliations is Frobenius's theorem: it allows to couple a distribution $D \subset TM$ to a foliation $\mathcal{F}$ and from this distribution we also get a submodule $\Gamma_c(D) \subset \mathfrak{X}(M)$ that completely describes the foliation $\mathcal{F}$. All of these results will be described in the first section of chapter 1 of this thesis.

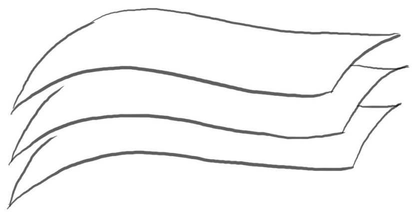


Figure 1: Example of a regular foliation of $\mathbb{R}^3$ by hypersurfaces.

In the 1950s and 1960s a lot of work was done in the field of control theory. In particular the study of the solvability of first-order differential equations under the influence of certain external parameters. It turned out that this can be modelled by considering the flow of a family of vector fields on a smooth manifold. In his work from 1963 [Her63] R. Hermann described a relation between control theory and differential geometry: he worked on the problem of integrating a family of vector fields on a manifold M into a *singular foliation*. A singular foliation is a partition of a manifold into leaves but unlike the regular case we do not require the leaves to have a fixed dimension.

Over the years this sparked more research into integrability problems of families of vector fields and their associated generalized distributions (these are distributions $D \subset TM$ for which $\dim D_m$ need not to be constant for all $m \in M$). Some prominent figures in the development of this research are Hermann, Nagano, Stefan and Sussmann. For a description of the historical development of this research we refer the reader to [Lav18]. The main results of this research were some Frobenius like theorems which give conditions under which a family of vector fields or a generalized distribution is integrable.

Nowadays singular foliations are objects studied purely in the setting of differential geometry as they arise frequently: the action of a Lie group G on a smooth manifold M, the symplectic foliation on a Poisson manifold,... Because of the dimension jump/drop that may occur for singular foliations their description becomes quite complicated. For regular foliations we can take a strictly geometric point of view or an equivalent algebraic point of view by the module $\mathcal{F} = \Gamma_c(D)$, this is not possible for singular foliations. For this class of foliations an algebraic description will carry more information than just the geometric picture (see for instance [AS09]). In the second section of the thesis, we will describe the two main definitions used in the literature today. Firstly, we will look at the point of view [AS09], [AZ13] take (not an exhaustive list of works that use this definition). There a singular foliation is considered as being a locally finitely generated involutive submodule $\mathcal{F} \subset \mathfrak{X}_c(M)$. After this we will explore the point of view [LGLS20] takes; here a singular foliation is considered as being a locally finitely generated involutive subsheaf of $\mathfrak{X}$ (the sheaf of vector fields on a manifold). We will also show that these two notions are equivalent by following the arguments given in [Gar19].

Finally, we will also explain geometric resolutions of singular foliations. These are important objects for the further development of the material. In this section we will give a detailed proof of the first point in proposition 2.3 in [LGLS20]. The proof of this particular point is given without details in [LGLS20]. For this purpose, we will give a proof of Hilbert's syzygy theorem following and adapting the argument in [Har97]. Furthermore, we will give a proof of lemma 3.19 in [LGLS20]. The proof of this lemma is left out in the original publication but is used at several point throughout the paper [LGLS20].

Higher Structures and the Universal Lie ∞-algebroid of a Singular Foliation

Throughout the years it became clear that singular foliations are not as well-behaved as their regular counterparts. Hence the need to define certain invariants associated to them arose. In their work [AS09] Androulidakis and Skandalis mentioned two first invariants: the *isotropy Lie algebra* $\mathfrak{g}_m$ and the *holonomy groupoid*. Both of these can give some geometrical information of the foliation.

In this thesis we will explore a recently discovered invariant, the *universal Lie ∞-algebroid* of a singular foliation, it was first proposed in Sylvain Lavau's PhD thesis [Lav16] and recently published by T. Strobl, S. Lavau and C. Laurent-Gengoux in [LGLS20]. Lie ∞-algebroids can be seen as a combination of two objects: a Lie algebroid and an L_∞-algebra. Lie algebroids are quite familiar objects in differential geometry. A Lie algebroid is a vector bundle $A \to M$ for which $\Gamma(A)$ is a Lie algebra together with an anchor map $\rho : A \to TM$ that satisfies a Leibniz identity and hence also is a Lie algebra homomorphism. On the other hand L_∞-algebras where first studied in theoretical physics while studying string theory, supergravity, quantum field theory,... see for instance: [Sta92], [Zwi93], [KS06] and [LS93]. These objects consist of a graded vector space $E = \oplus_{i\in\mathbb{Z}} E_{-i}$ and a family of skew-symmetric brackets $(\{\cdots\}_k)_{k\geq 1}$ called the k-ary brackets that satisfy so-called 'higher Jacobi identities'. Lie ∞-algebroids are then a combination of these two notions; both Lie ∞-algebroids and L_∞-algebras will be introduced in chapter 2. Here we will also explain the duality between Lie ∞-algebroids and objects from graded geometry called NQ-manifolds. Using this duality we may view Lie ∞-algebroids as NQ-manifolds (E, Q).

In chapter 3 we will then give one of the main results of [LGLS20] that is the following theorem.

Theorem (Theorem 2.7 in [LGLS20])**.** *Let $\mathcal{F}$ be a singular foliation on a manifold M which admits a geometric resolution (E, d, ρ). Then there exists a universal Lie ∞-algebroid of $\mathcal{F}$, the linear part of which is the geometric resolution.*

Throughout this chapter we will provide some small details and calculations to prove the statements. These where not always given completely in [LGLS20]. There also is a 'uniqueness' result proven in [LGLS20]. The most interesting consequence of this uniqueness result is that any two universal Lie ∞-algebroids of a singular foliation $\mathcal{F}$ are homotopy equivalent and any two such homotopy equivalences are homotopic. This allows one to essentially 'guess' a Lie ∞-algebroid structure on any geometric resolution of $\mathcal{F}$ and immediately conclude this is the universal one.

From the theorem above it also follows that we can only look at a special class of singular foliations, namely the ones that admit geometric resolutions. In chapter 3 we will also explain the main steps in the proof of the existence result which will be considered as a deformation problem. We will leave out the very technical details and solely focus on how one solves the associated deformation problem.

The geometry of singular foliations

In the final chapter we will then use the theory of universal Lie ∞-algebroids to answer the following question, following section 4 in [LGLS20]:

does there always exist a Lie algebroid of minimal rank which locally induces the foliation $\mathcal{F}$?

For this we exploit cohomologies that arise out of the universal Lie ∞-algebroid. In particular we will focus on the isotropy L_∞-algebra and show that the isotropy Lie algebra $\mathfrak{g}_m$ from [AS09] can be recovered from this object. In this chapter we will provide more detailed proofs of lemma 4.13, proposition 4.27 and proposition 4.29 in [LGLS20] by providing the necessary calculations which are left out in the original publication. We will end this chapter by giving an original example of a foliation which answers the question above negatively.

Contents

Chapter 1

Singular Foliations

In this chapter we will introduce the notion of a *singular foliation*. In the first section we will briefly recall regular foliations. Our focus will be on ending with Frobenius's theorem which is an important result in the study of regular foliations.

In the second section we will introduce singular foliations. This kind of foliation allows for the dimension of the leaves to jump. While for regular foliations a geometric point of view and an algebraic point of view are equivalent it turns out an algebraic approach is preferred for singular foliations. We will introduce singular foliations in the two main ways they are used in the literature: as modules and as sheaves.

After laying the foundations through definitions we will then explain several associated constructions that are necessary for the next chapters.

1.1 Regular Foliations

In this section we will first review some material about regular foliations. It is based on Chapter 19 in Lee's book [Lee12] to which we also refer for all results left without proof.

1.1.1 Distributions and Involutivity

Let M be a smooth manifold.

Definition 1.1.1 ([Lee12])**.** A **distribution on** M **of rank** k is a rank k subbundle of the tangent bundle TM. It is called a **smooth distribution** if it is smooth subbundle.

Perhaps the most intuitive way to think about distributions is by specifying for each point $p \in M$ a k-dimensional linear subspace $D_p \subset T_pM$ and then letting $D = \cup_{p \in M} D_p$. From the local frame criterion for subbundles it then follows that D is a smooth distribution if and only if each point $p \in M$ has a neighborhood U on which there are smooth vector fields $X_1, \ldots, X_k : U \to TM$ such that $X_1(q), \ldots, X_k(q)$ for a basis for D_q for each $q \in U$. We then say that the distribution D is locally spanned by the vector fields $X_1, \ldots, X_k$.

Definition 1.1.2 ([Lee12])**.** Assume that $D \subset TM$ is a smooth distribution. A nonempty immersed submanifold $N \subset M$ is called an **integral manifold of** D if $T_pN = D_p$ at each point $p \in N$.

Example 1.1.3. Consider $M = \mathbb{R}^3$ with coordinates x, y, z. Now consider the distribution $D = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle$. The integral manifolds to this distribution are the planes $\{z = \text{constant}\} \subset \mathbb{R}^3$. ♦

The following example shows that not all distributions have integral manifolds.

Example 1.1.4. Consider again $\mathbb{R}^3$ with coordinates x, y, z and the distribution spanned by the following two vector fields

$$X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y}.$$

This distribution has no integral manifolds: suppose N is an integral manifold through the origin, both X and Y are tangent to N and any integral curve of X and Y that starts in N has to stay in N for short time. Because the x-axis is an integral curve of X, the integral manifold N has to contain a small part of it. Also, for sufficiently small x, it contains an open subset of the line parallel to the y-axis and passing through the point $(x, 0, 0)$ because this corresponds to an integral curve of Y. Therefore, N contains an open subset of the xy-plane. However, for any point p not on the x-axis the tangent plane to the xy-plane at that point is not equal to D_p. Therefore, no such integral manifold can exist. ♦

We now continue with two definitions

Definition 1.1.5 ([Lee12])**.** Suppose D is a smooth distribution on M. We say that D is **involutive** if the Lie bracket of two smooth local sections is again a smooth local section of D.

If D is a smooth distribution on M then one can show that D is involutive if and only if $\Gamma(D)$ is a Lie subalgebra of $\mathfrak{X}(M)$.

Definition 1.1.6 ([Lee12])**.** A smooth distribution D on M is said to be **integrable** if each point of M is contained in an integral manifold of D.

1.1.2 Frobenius's Theorem

Definition 1.1.7 ([Lee12])**.** Given a rank-k distribution $D \subset TM$ we say that a coordinate chart (U, φ) on M is a **flat chart for** D if $\varphi(U)$ is a cube in $\mathbb{R}^n$ and at points in U, D is spanned by the first k coordinate vector fields $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}$. In any such chart each slice of the form $x_{k+1} = c_{k+1}, \ldots, c_n = c_n$ for constants $c_{k+1}, \ldots, c_n$ is an integral manifold of D.

Note that the definition above captures the 'nicest possible way' for integral manifolds to fit together: locally they all fit together like parallel subspaces of $\mathbb{R}^n$, this is illustrated in figure 1.1.

Definition 1.1.8 ([Lee12])**.** Suppose $D \subset TM$ is a distribution then we call it **completely integrable** if there exists a flat chart for D in a neighborhood of each point of M.

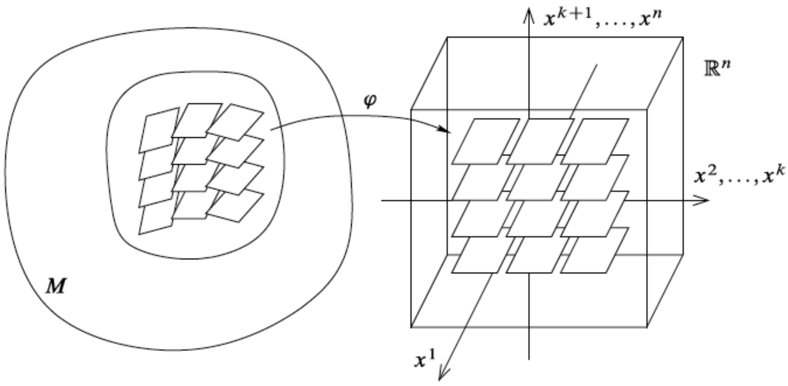


Figure 1.1: Flat chart for a distribution (mind the slightly different notation compared to definition 1.1.7), source [Lee12]

Note that we have the following sequence of implications

$$\text{completely integrable} \Rightarrow \text{integrable} \Rightarrow \text{involutive}.$$

The Frobenius's theorem shows that these are in fact equivalences.

Theorem 1.1.9. *(Frobenius's Theorem) Every involutive distribution is completely integrable.*

For a proof see the proof of theorem 19.12 in [Lee12]

1.1.3 Foliations

We now come to the notion of a foliation; it captures the behavior of 'dividing up' a smooth manifold into k-dimensional submanifolds that fit together in a nice way.

Definition 1.1.10 ([Lee12])**.** Let M be a smooth n-manifold and let $\mathcal{F}$ be any collection of k-dimensional submanifolds of M. A smooth chart (U, φ) for M is said to be **flat for** $\mathcal{F}$ if $\varphi(U)$ is a cube in $\mathbb{R}^n$ and each submanifold in $\mathcal{F}$ intersects U in either the empty set or a countable union of k-dimensional slices of the form $x_{k+1} = c_{k+1}, \ldots, x_n = c_n$.

This concept is illustrated nicely in figure 1.2, this figure uses different notation (X corresponds to our φ and U is not necessarily mapped to a cube) but the idea is still clear.

Definition 1.1.11 ([Lee12])**.** We define a **regular foliation of dimension** k **on** M to be a collection $\mathcal{F}$ of disjoint, connected, nonempty, immersed k-dimensional submanifolds of M that we call the **regular leaves of the foliation**, whose union is M and such that in a neighborhood of each point $p \in M$ there exists a flat chart for $\mathcal{F}$.

In the above definition we emphasize the word *regular* so there does not arise any confusion later when we will start to talk about *foliations* by which we will mean singular foliations (to be defined in due course).

We will now provide some examples of regular foliations.

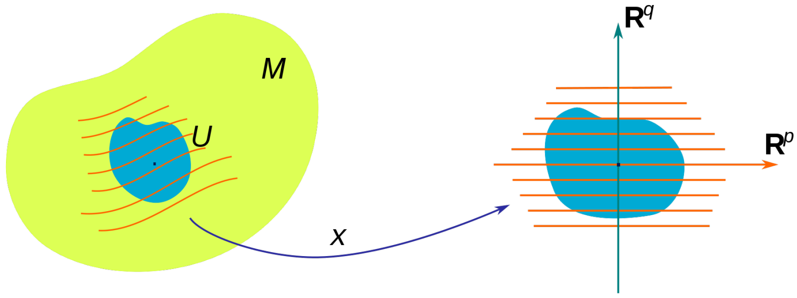


Figure 1.2: Geometrical illustration of flat chart as defined in definition 1.1.10 (again mind the different notation), source Wikipedia

Example 1.1.12 (Example 19.18 in [Lee12])**.** Let M and N be connected smooth manifolds then the collection $\mathcal{F} = \{M \times \{q\} \mid q \in N\}$ forms a foliation of the product manifold $M \times N$. A particular example of this is when we consider the torus $T^2 = S^1 \times S^1$. We can now form two collections of submanifolds:

$$\mathcal{F}_1 = \{S^1 \times \{q\} \mid q \in S^1\}$$
$$\mathcal{F}_2 = \{\{p\} \times S^1 \mid p \in S^1\}.$$

They form two different foliations of the torus as shown in figure 1.3, here $\mathcal{F}_1$ corresponds to picture (a) while $\mathcal{F}_2$ corresponds to (b).

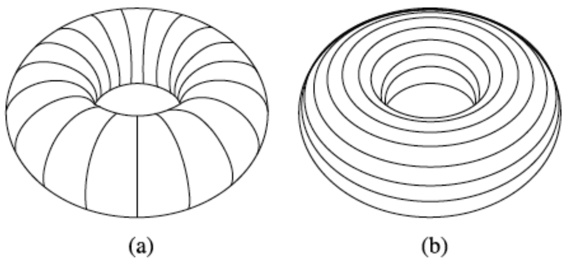


Figure 1.3: Foliations of the torus T^2 from example 1.1.12 , source [Lee12]

♦

Example 1.1.13 (Example 19.18(e) in [Lee12])**.** There is another interesting example of a foliation on the torus with 1-dimensional leaves called the Kronecker foliation. It consists of the images of all curves of the form

$$\gamma_\theta(t) = \left(e^{it}, e^{i(\alpha t + \theta)}\right),$$

as θ ranges over $\mathbb{R}$. If $\alpha \in \mathbb{Q}$ each leaf is an embedded circle, if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ each leaf is dense. An illustration of this is given in figure 1.4. ♦

The global Frobenius's theorem now establishes a one-to-one correspondence between involutive distributions on the one hand and foliations on the other.

Theorem 1.1.14 (Global Frobenius's Theorem)**.** *Let D be an involutive distribution on a smooth manifold M. The collection of all maximal connected integral manifolds of D forms a foliation of M.*

For a proof we refer to the proof of theorem 19.21 in [Lee12].

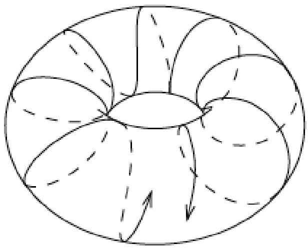


Figure 1.4: Foliation of the torus T^2 from example 1.1.13, source [Lee12]

1.2 Singular Foliations

This section is based on the book by Dufour and Zung (see section 1.2 [DN05]), Alfonso Garmendia's PhD thesis [Gar19], the paper by Laurent-Gengoux, Lavau and Strobl (see section 3.1 in [LGLS20]), the paper by Androulidakis and Zambon (see [AZ13]) and the paper by Androulidakis and Skandalis (see [AS09]). It will contain the basic definitions and examples of singular foliations which will be the central object of the thesis. There are two main definitions in use for singular foliations and we will explain both of them: the first one considers singular foliations as being finitely generated involutive submodules of the compactly supported vector fields $\mathfrak{X}_c(M)$, the other one considers singular foliations as sheaves. We will proceed by first explaining the first definition.

Throughout this section it will be assumed that M is a smooth (real) manifold unless stated otherwise.

1.2.1 Singular Foliations Through Distributions and Submodules

By a Stefan-Sussman singular foliation we mean a partition $\mathcal{F} = \{\mathcal{F}_\alpha\}_{\alpha\in A}$ of a manifold M into a disjoint union of smoothly immersed connected submanifolds, which we call the leaves of the foliation, which satisfy the local foliation property at each point $p \in M$. This means that when we denote by $\mathcal{F}_p$ the leaf that contains p, by d the dimension of $\mathcal{F}_p$ and by m the dimension of M then there is a smooth local chart of M with coordinates $y_1, \dots, y_m$ on a neighborhood U of p with $U = \{-\varepsilon < y_1 < \varepsilon, -\varepsilon < y_2 < \varepsilon, \dots, -\varepsilon < y_m < \varepsilon\}$. In such a way that the d-dimensional disk $\{y_{d+1} = \cdots = y_m = 0\}$ coincides with the path connected component of the intersection $\mathcal{F}_p \cap U$. Furthermore each d-dimensional disk $\{y_{d+1} = c_{d+1}, \dots, y_m = c_m\}$ (with the $c_i \in \mathbb{R}$) is wholly contained in some leaf $\mathcal{F}_\alpha$ of $\mathcal{F}$. Like for regular foliations we begin by considering some type of distribution.

Definition 1.2.1 ([DN05])**.** A **singular distribution** on a manifold M is the assignment, to each point $x \in M$, of a vector subspace D_x of the tangent space T_xM. The dimension of D_x may depend on x.

Example 1.2.2. Let $\mathcal{F}$ be a Stefan-Sussman singular foliation like explained above then it has a natural associated tangent distribution $D^{\mathcal{F}}$. This distribution is defined at each x by taking the tangent space $D^{\mathcal{F}}_x$ to the leaf of $\mathcal{F}$ which contains x, at x. ♦

Definition 1.2.3 ([DN05])**.** We call a singular distribution D **smooth** if for all $p \in M$ and any $X_0 \in D_p$, there exists a smooth vector field X defined in a neighborhood U of p which is tangent to the distribution: $X(y) \in D_y$ for all $y \in U$ and it extends X_0 in the sense that $X_0 = X(p)$. If the dimension of D_p does not depend on p we say that D is a **smooth regular distribution**

From now on when we talk about distributions on a manifold, we see them in the sense of definition 1.2.3. As explained in section 1.2.1 of [Lav16] an equivalent way of saying that a distribution is smooth is by saying that there exists (a possibly infinite) family of vector fields $\{X_k\}_{k\in I}$ such that for all $y \in U$ we have that $D_y = \mathrm{span}\{X_k(y)\}$. In [DLPR10] it is shown however that the generating family of vector fields can always be chosen to be finite. An important remark to be made here is that this *does not imply* that the $C^\infty(M)$-module of sections $\Gamma(D)$ is finitely generated. This is shown in section 5 of [DLPR10] where they propose the following counterexample (there are technical details involved for which we refer to the original publication):

Example 1.2.4. Define the vector field $X = \chi(x)\frac{\partial}{\partial x}$ on $M = \mathbb{R}$ with the function χ a rapidly vanishing function in a neighborhood of the origin, for example

$$\chi(x) = \begin{cases} e^{-\frac{1}{x}} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases},$$

then the associated distribution $\mathcal{D}$ looks like

$$\mathcal{D}_x = \begin{cases} T_x\mathbb{R} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}.$$

One can show (see [DLPR10] proposition 5.3) that $\Gamma(\mathcal{D})$ is not finitely generated in a neighborhood of the origin.

♦

Definition 1.2.5 ([DN05])**.** The distribution D is called **locally finitely generated** if for every point p there exists a neighborhood U such that the $C^\infty(U)$-module $\Gamma_U(D)$ is finitely generated.

Definition 1.2.6 ([DN05])**.** Given a distribution D on a manifold M an **integral submanifold** is a connected immersed submanifold N of M such that for all $y \in N$ the tangent space T_yN is a subspace of D_y. We call it a maximal integral submanifold when it is not contained in any other integral submanifold. The maximum dimension of the tangent space to $y \in N$ is exactly the dimension of D_y.

Notice the resemblance with the notion of integral manifold above. This time the situation is more complicated because of the possibility that the dimension varies. When we consider a smooth regular distribution as defined above we just recover the definition from the previous section.

Definition 1.2.7 ([DN05])**.** A distribution D on M is called **integrable** when each $p \in M$ is contained in a maximal integral manifold of maximum dimension of D.

The following example shows that when considering distributions that are possibly singular, Frobenius's theorem fails.

Example 1.2.8. Define the following distribution on $\mathbb{R}^2$ with coordinates x, y

$$D_{(p,q)} = \begin{cases} T_{(p,q)}\mathbb{R}^2 \text{ if } p > 0, \\ \langle \frac{\partial}{\partial x} \rangle \text{ if } p \leq 0. \end{cases}$$

Sections of this distribution consist of vector fields of the form $X = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}$ where $g(x, y) = 0$ when $x \leq 0$. Now also consider $Y = \tilde{f}\frac{\partial}{\partial x} + \tilde{g}\frac{\partial}{\partial y}$ to be a section of D then the Lie bracket between X and Y is given by

$$[X, Y] = \left(f\frac{\partial \tilde{f}}{\partial x} - \tilde{f}\frac{\partial f}{\partial x} - \tilde{g}\frac{\partial f}{\partial y} + g\frac{\partial \tilde{f}}{\partial y} \right) \frac{\partial}{\partial x} + \left(f\frac{\partial \tilde{g}}{\partial x} - \tilde{f}\frac{\partial g}{\partial x} + g\frac{\partial \tilde{g}}{\partial y} - \tilde{g}\frac{\partial g}{\partial y} \right) \frac{\partial}{\partial y}.$$

Now clearly when $g(x, y) = 0$ and $\tilde{g}(x, y) = 0$ when $x \leq 0$. We also have that

$$\frac{\partial \tilde{g}}{\partial x} - \tilde{f}\frac{\partial g}{\partial x} + g\frac{\partial \tilde{g}}{\partial y} - \tilde{g}\frac{\partial g}{\partial y} = 0,$$

when $x \leq 0$ so we get another section of D. This shows that D is an involutive distribution. We now argue that it is not integrable. On the right halve plane $x > 0$ we have that the integral submanifold is the open half-plane. For $x < 0$ we have that the leaves are horizontal because their tangent space is spanned by $\frac{\partial}{\partial x}$. This still holds for $x = 0$ and so the leaves are the horizontal rays. When we consider these rays as subspaces of $\mathbb{R}$ they are not open (the right end is closed) and hence they are not submanifolds. ⧫

The example above thus illustrates that we need some other extra conditions for a singular distribution to be integrable. Important progress on this question was made by Nagano (for the analytical case), Hermann, Lobry, Stefan, Sussmann and others. The road to these results is quite a bumpy one and many (wrong) results were published. For a chronological exposition and resume of important results in this domain we refer to [Lav18]. The first proper result in the smooth case is due to Hermann and bears his name today.

Theorem 1.2.9. *(Hermann, 1962) Any finitely generated submodule of $\mathfrak{X}(M)$ defines an integrable distribution if it is involutive.*

Remark 1.2.10. It deserves to be noted that the converse of Hermann's theorem is false. Indeed, consider the following counterexample due to Balan (it was contained in unpublished notes, we refer to [Lav18] for this example). Consider $M = \mathbb{R}^2$ and define the vector fields

$$X = \varphi(x, y)\frac{\partial}{\partial x}$$
$$Y = (x^2 + y^2)\frac{\partial}{\partial y},$$

where the function $\varphi(x, y)$ is defined as

$$\varphi(x, y) = \begin{cases} e^{-\frac{1}{x^2+y^2}} & \text{for } (x, y) \neq (0, 0), \\ 0 & \text{for } (x, y) = (0, 0). \end{cases}$$

Let $\mathcal{F} = \langle X, Y \rangle_{C^\infty(M)}$ then the distribution that $\mathcal{D}$ that $\mathcal{F}$ induces is given by

$$\mathcal{D}_{(x,y)} = \begin{cases} T_{(x,y)}\mathbb{R}^2 & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0). \end{cases}$$

Obviously, this distribution is integrable: one maximal integral submanifold of dimension 2 i.e., the plane without the origin and the origin itself as 0-dimensional integral submanifold. Now choose $(x,y) \neq (0,0)$ then a small computation shows that

$$[X,Y](x,y) = 2x\frac{\varphi(x,y)}{x^2+y^2}X - \frac{2y}{x^2+y^2}Y.$$

Now one can show that the function $(x,y) \mapsto 2x\frac{\varphi(x,y)}{x^2+y^2}$ is smooth at the origin (this is due to the rapid vanishing of φ as (x,y) approaches the origin). However, when considering the function $(x,y) \mapsto f(x,y) := \frac{2y}{x^2+y^2}$ we encounter some problems as

$$\lim_{x\to 0} f(x,0) \neq \lim_{y \to 0} f(0,y),$$

i.e. the limit as $(x,y) \to (0,0)$ of $f(x,y)$ does not exist and so it is not a smooth function. This also means that $[X,Y]$ is not contained in $\mathcal{F}$ because it has non-smooth coefficients. Of course that means that $\mathcal{F}$ induces and integrable distribution but is itself not involutive.

Theorem 1.2.9 leads us to the following definition of a singular foliation in terms of submodules of the compactly supported vector fields $\mathfrak{X}_c(M)$.

Definition 1.2.11 ([Gar19])**.** A $C_c^\infty(M)$-submodule $\mathcal{F} \subset \mathfrak{X}_c(M)$ is **finitely generated** if there exists a finite set of vector fields $Y_1, \dots, Y_n \in \mathfrak{X}(M)$ such that

$$\mathcal{F} = \langle Y_1, \dots, Y_r \rangle_{C_c^\infty(M)}.$$

Definition 1.2.12 ([Gar19])**.** A submodule $\mathcal{F} \subset \mathfrak{X}_c(M)$ is **locally finitely generated** if every point $m \in M$ has a neighborhood $U \subset M$ such that

$$\iota_U^{-1}\mathcal{F} := \{X|_U \mid X \in \mathcal{F} \text{ and } \operatorname{supp}(X) \subset U\},$$

is finitely generated as a $C_c^\infty(U)$-module.

Definition 1.2.13 ([AZ13], [AS09])**.** A **singular foliation** on a manifold M is a locally finitely generated submodule $\mathcal{F} \subset \mathfrak{X}_c(M)$ such that $[\mathcal{F}, \mathcal{F}] \subset \mathcal{F}$.

One may wonder why we require the condition of the vector fields in the submodule to have compact support. The reason for this is that want the map

$$\{\text{regular foliations}\} \to \{\text{singular foliations}\},$$

that maps a regular distribution D to some submodule[1] of $\mathfrak{X}(M)$, to be injective. This means that when we consider a regular foliation $\mathcal{F}$ we want it to correspond to a singular foliation in a unique way: there is only one submodule $\mathcal{F} \subset \mathfrak{X}_c(M)$ such that it generates the regular foliation. The following example shows that this does not need to be the case when we consider the whole of $\mathfrak{X}(M)$.

[1] Actually we map D to $\Gamma_c(D)$ but this is what we want to explain.

Example 1.2.14. Consider for example the foliation on $\mathbb{R}^2$ described by the distribution $D = \langle \frac{\partial}{\partial x} \rangle$ (the leaves thus look like horizontal lines in the plane). Now we can make two choices of submodules of $\mathfrak{X}(M)$ that both generate the given foliation, for example

$$\mathcal{F} = \Gamma(D) = \Big\{ f(x,y)\frac{\partial}{\partial x} \mid f \in C^\infty(\mathbb{R}^2) \Big\},$$
$$\mathcal{F}' = C_c^\infty(\mathbb{R}^2)\frac{\partial}{\partial x}.$$

♦

The following proposition, that can be found in [AZ16], shows that we can make the map above injective if we let $\mathcal{F} = \Gamma_c(D)$.

Proposition 1.2.15. *Let $\mathcal{F}$ be a singular foliation whose evaluation at points of M delivers a constant rank distribution D, then necessarily $\mathcal{F} = \Gamma_c(D)$.*

Proof. The proof comes from [AZ16] lemma 1.7. Let k denote the rank of the distribution D. Note that for all $p \in M$ there exists a subset $\mathbb{Y} = \{Y_1, \dots, Y_k\} \subset \mathcal{F}$ for which $\mathbb{Y}(p)$ is a basis for D_p (here $\mathbb{Y}(p)$ denotes the evaluation of each element in $\mathbb{Y}$ at p). Let V be an open neighborhood of $p \in M$ on which the set $\mathbb{Y}$ is linearly independent. Now we can construct an open cover $\{V_\alpha\}_{\alpha\in A}$ where each V_α is as V above. Fix an element $X \in \Gamma_c(D)$ then we have to show that $X \in F$. Since X has compact support $\mathrm{supp}(X)$ there are finitely many V_α's covering it. Hence we may assume that our open cover $\{V_\alpha\}_{\alpha\in A}$ is of such nature that only finitely many V_α's intersect $\mathrm{supp}(X)$ (this is a form of being locally finite and we will denote it as that). Now let $\{\varphi_\alpha\}_{\alpha\in A}$ be a partition of unity subordinate to the open cover $\{V_\alpha\}_{\alpha\in A}$, i.e. $\sum_{\alpha\in A} \varphi_\alpha(p) = 1$ for all $p \in M$ and $\mathrm{supp}(\varphi_\alpha) \subset V_\alpha$. Since $\varphi_\alpha X$ is supported on V_α for all $\alpha \in A$ there exist smooth functions $h_\alpha^i \in C^\infty(M)$ for which $\varphi_\alpha = \sum_{i=1}^k h_\alpha^i Y_\alpha^i \in \mathcal{F}$. Hence by the locally finiteness property the sum $X = \sum_{\alpha\in A} \varphi_\alpha X$ is essentially a finite sum and so lies in $\mathcal{F}$. □

1.2.2 Singular Foliations as Sheaves

In this section we will introduce singular foliations in the language of sheaves as this is the point of view [LGLS20] takes. Before proceeding we will briefly recall the definition of sheaves and related concepts, this information is mainly based on [Har97] and [Vak17].

Definition 1.2.16 ([Har97])**.** Let X be a topological space. A **presheaf** $\mathcal{G}$ with values in a category $\mathbf{C}$[2] is an assignment $U \mapsto \mathcal{G}(U)$ which associates to any open U in X an object $\mathcal{G}(U)$ in $\mathbf{C}$ such that for every inclusion $V \subset U$ of open sets we get a restriction morphism

$$\rho_V^U : \mathcal{G}(U) \to \mathcal{G}(V),$$

in the category $\mathbf{C}$. Furthermore, for every open U in X the morphism ρ_U^U must be the identity and for a sequence of inclusions of open sets $W \subset V \subset U$ we have $\rho_W^U = \rho_W^V \circ \rho_V^U$.

[2]We assume $\mathbf{C}$ to be a set-like category, which roughly speaking means that $\mathbf{C}$ has properties similar to **Set**.

Remark 1.2.17. We can make this definition more compact when using more category theory. If X is a topological space we can attach a category $\mathbf{Open}_X$ to it: objects are just the open sets of X and if V and U are objects in $\mathbf{Open}_X$ then there exists a unique morphism $V \to U$ in $\mathbf{Open}_X$ if $V \subset U$ and no morphism otherwise. Now let $\mathbf{Open}_X^{\mathrm{opp}}$ denote the opposite category then a presheaf on X is a functor

$$\mathcal{G} : \mathbf{Open}_X^{\mathrm{opp}} \to \mathbf{C}.$$

Definition 1.2.18 ([Har97])**.** Let $\mathcal{G}$ be a presheaf on a topological space X then we say that $\mathcal{G}$ is a **sheaf** if the following conditions are satisfied.

1. Let U be an open subset of X and let $\{U_i | i \in I\}$ be an open cover of U. Let $f, g \in \mathcal{G}(U)$ then $f = g$ if and only if the restrictions of f and g to the U_i are equal for all $i \in I$.

2. Let U be an open subset of X and let $\{U_i | i \in I\}$ be an open cover of U. Let $f_i \in \mathcal{G}(U_i)$ for every $i \in I$ and assume that the restrictions of f_i and f_j are equal on $U_i \cap U_j$ for all $i, j \in I$. Then there exists an element $f \in \mathcal{G}(U)$ whose restriction is equal to f_i for every $i \in I$. By the first property this f must be unique.

3. The object $\mathcal{G}(\emptyset)$ is a final object in $\mathbf{C}$.

Remark 1.2.19. Note that for most categories $\mathbf{C}$ the last condition follows from the first two.

Example 1.2.20. A smooth manifold together has two natural sheaves: the sheaf of rings C^∞ i.e. the smooth functions and the sheaf of vector fields $\mathfrak{X}$ that is also a C^∞-module.

Definition 1.2.21 ([Har97])**.** A **subsheaf** $\mathcal{G}'$ of a sheaf $\mathcal{G}$ is a sheaf such that $\mathcal{G}'(U) \subset \mathcal{G}(U)$ is a sub object in $\mathbf{C}$ (e.g. subgroup, submodule,...).

As a final note we say that a sheaf of modules $\mathcal{G}$ on a manifold (M, C^∞) (i.e. a presheaf that takes values in the category of C^∞-modules that is also a sheaf) is locally finitely generated[3] if for all $p \in M$ there exists a neighborhood U such that there is some $n > 0$ and a surjective morphism of sheaves $\varphi : (C^\infty)^n |_U \to \mathcal{G}|_U$. Using this terminology, we can define a singular foliation in the following way.

Definition 1.2.22 ([LGLS20])**.** A **singular foliation** is a subsheaf $\mathcal{F} : U \mapsto \mathcal{F}(U)$ of the sheaf of vector fields $\mathfrak{X}$ that is locally finitely generated as a C^∞-module and is closed with respect to the Lie bracket of vector fields.

The notions of singular foliations as a submodule and the one as sheaf seem very different at first sight. We will now explain that these two definitions are indeed the same thing. This is entirely based on section 1.5 of [Gar19]. More precisely we will have the following theorem.

Theorem 1.2.23 (Theorem 1.5.1 in [Gar19])**.** *For any smooth manifold M, we have the following:*

- *there is a bijection between submodules of $\mathfrak{X}_c(M)$ and subsheaves of $\mathfrak{X}$,*

[3]One can define this concept for any ringed space but we will only need it for smooth manifolds.

- *the condition of being locally finitely generated is invariant under this bijection,*
- *the involutivity condition is invariant under this bijection.*

It is useful to define the following object to proof theorem 1.2.23.

Definition 1.2.24 (Definition 4.1 in [ZA16] and Definition 1.5.2 in [Gar19])**.** Given a submodule $\mathcal{F} \subset \mathfrak{X}_c(M)$, the **global hull** of $\mathcal{F}$ is given by

$$\widehat{\mathcal{F}} := \{X \in \mathfrak{X}(M) \mid fX \in \mathcal{F}, \forall f \in C_c^\infty(M)\}.$$

Given a submodule $\mathcal{S} \subset \mathfrak{X}(M)$ one can define its **compact elements**

$$(\mathcal{S})_c := \{X \in \mathcal{S} \mid \operatorname{supp}(X) \text{ is compact}\} = \langle \mathcal{S} \rangle_{C_c^\infty(M)}.$$

We can show the following important property of these objects.

Lemma 1.2.25 (Lemma 1.5.3 in [Gar19])**.** *For a submodule $\mathcal{F} \subset \mathfrak{X}_c(M)$ and a subsheaf $\mathcal{S}$ of $\mathfrak{X}$ and $U \subset M$ open, we get the following equalities:*

$$\left(\widehat{\mathcal{F}}\right)_c = \mathcal{F},$$
$$(\widehat{(\mathcal{S}(U))_c}) = \mathcal{S}(U).$$

Proof. It is clear that the first equality holds by definition and that $\mathcal{S}(U) \subset (\widehat{(\mathcal{S}(U))_c})$. So, we proceed by showing the other inclusion. Take $X \in (\widehat{(\mathcal{S}(U))_c})$ and $\{\varphi_i\}_{i \in I}$ a partition of unity for U with functions that have compact support. There exists a cover $\{U_j\}_{j \in J}$ of U such that the sum $\sum_i \varphi_i X$ is finite in each U_j. Moreover we have that $\varphi_i X \in \mathcal{U}$. Therefore $X|_{U_j} = \sum_i \varphi_i X|_{U_j} \in \mathcal{S}(U_j)$. Now by the gluing property of sheaves (property 2 described in definition 1.2.18) there exists an element $Y \in \mathcal{S}(U)$ such that $Y|_{U_i} = X|_{U_j}$ and by the locality of sheaves (point 1 in definition 1.2.18) we conclude that $X = Y \in \mathcal{S}(U)$. □

So when given a subsheaf $\mathcal{S}$ of $\mathfrak{X}$ it is straightforward to define a submodule of $\mathfrak{X}_c(M)$: define $\mathcal{F} := (\mathcal{S}(M))_c$. We now show that the reverse can also be done, i.e. recovering a subsheaf of $\mathfrak{X}$ from a given submodule of $\mathfrak{X}_c(M)$. For this we first establish the notation that when $\mathcal{F} \subset \mathfrak{X}_c(M)$ and $U \subset M$ open, we denote

$$\iota_U^{-1}\mathcal{F} := \{X|_U \mid X \in \mathcal{F}, \quad \operatorname{supp}(X) \subset U\}.$$

Lemma 1.2.26 (Lemma 1.5.4 in [Gar19])**.** *Let $\mathcal{S}$ be a subsheaf of $\mathfrak{X}$. Denote $\mathcal{F} = (\mathcal{S}(M))_c$ then for any open set $U \subset M$ we have $(\mathcal{S}(U))_c = \iota_U^{-1}\mathcal{F}$ and therefore also*

$$\mathcal{S}(U) = \widehat{\iota_U^{-1}\mathcal{F}}.$$

Proof. We will show that $(\mathcal{S}(U))_c = \iota_U^{-1}\mathcal{F}$ for all $U \subset M$ open. From this the result immediately follows by lemma 1.2.25. Take $X \in (\mathcal{S}(U))_c$, then $X \in \mathcal{S}(U)$ and U together with $M \setminus \operatorname{supp}(X)$ are a cover of M. Since $\mathcal{S}$ is a sheaf we can use the gluing property to conclude that there exists an element $Y \in \mathcal{S}(M) = \widehat{\mathcal{F}}$ for which $Y|_U = X$ and $Y|_{M \setminus \operatorname{supp}(X)} = 0$. Note that Y has compact support, then $Y \in \mathcal{F}$ and $\operatorname{supp}(Y) \subset U$. Therefore we have by definition $X = Y|_U \in \iota_U^{-1}\mathcal{F}$. Conversely, let $X \in \iota_U^{-1}\mathcal{F}$ then by definition there exists a $Y \in \mathcal{F} = (\mathcal{S}(M))_c \subset \mathcal{S}(M)$ such that $Y|_U = X$ and $\operatorname{supp}(Y) \subset U$. So, we conclude that $X = Y|_U \in (\mathcal{S}(U))_c$. □

Lemma 1.2.26 provides us with a way to associate, to a given singular foliation $\mathcal{F} \subset \mathfrak{X}_c(M)$, a *presheaf* in the following manner:

$$\mathcal{S}^{\mathcal{F}}(U) := \widehat{\iota_U^{-1}\mathcal{F}}. \tag{1.1}$$

To conclude that the assignment

$$\{\text{subsheaf } \mathcal{S} \text{ of } \mathfrak{X}\} \to \{\text{submodule } \mathcal{F} \subset \mathfrak{X}_c(M)\},$$

is invertible, i.e. to a singular foliation we can associate a *subsheaf* of $\mathfrak{X}$, it suffices to show that the presheaf determined by equation (1.1) indeed gives a sheaf. This is done in the following lemma.

Lemma 1.2.27 (Lemma 1.5.5. in [Gar19])**.** *Given a submodule $\mathcal{F} \subset \mathfrak{X}_c(M)$, the presheaf $\mathcal{S}^{\mathcal{F}}$ as determined by equation (1.1) is a sheaf.*

Proof. To prove this we need to check the conditions in definition 1.2.18. Because $\mathcal{S}^{\mathcal{F}}$ is a sub-presheaf of the sheaf $\mathfrak{X}$ the locality (point 1 in definition 1.2.18) is immediately satisfied. We proceed by showing the gluing axiom. Let $U \subset M$ be an open subset and $\{U_i\}_{i\in I}$ an open cover of U for which $U_i \subset U$ for all $i \in I$. Let $X_i \in \mathcal{S}^{\mathcal{F}}(U_i)$ such that $X_i|_{U_i\cap U_j} = X_j|_{U_i\cap U_j}$. Since $\mathfrak{X}$ is a sheaf, there exists a vector field $X \in \mathfrak{X}(U)$ for which $X|_{U_i} = X_i$. It now suffices to show that $X \in \mathcal{S}^{\mathcal{F}}(U)$. Take $f \in C_c^\infty(U)$ then by compactness of $\operatorname{supp}(f)$ there exist finitely many U_i in the family $\{U_i\}_{i\in I}$ that cover $\operatorname{supp}(f)$. After a possible renumbering we may assume that these are $U_1, \dots, U_k$. Now let $U_0 := U \setminus \operatorname{supp}(f)$. There exists a partition of unity $\varphi_0, \varphi_1, \dots, \varphi_k \in C_c^\infty(U)$ subordinate to the cover $U_0, U_1, \dots, U_k$. For all $j > 0$ the functions φ_j have compact support on U_j, then $\varphi_j f X = \varphi_j f X_j \in \iota_U^{-1}\mathcal{F}$. Therefore, we immediately have that

$$fX = \sum_{j>0} \varphi_j f X \in \iota_U^{-1}\mathcal{F}.$$

Since f was chosen arbitrary we have that $X \in \widehat{\iota_U^{-1}\mathcal{F}} = \mathcal{S}^{\mathcal{F}}(U)$. □

From lemma 1.2.27 we conclude that the correspondence

$$\{\text{subsheaf } \mathcal{S} \text{ of } \mathfrak{X}\} \longleftrightarrow \{\text{submodule } \mathcal{F} \subset \mathfrak{X}_c(M)\},$$

is a bijection. This also shows the first point in theorem 1.2.23. For the second and third point of theorem 1.2.23 we refer to lemma 1.5.8 and proposition 1.5.9 in [Gar19]

1.2.3 Examples of Singular Foliations

In this section we will provide some examples of singular foliations. Throughout we will use definition 1.2.13 and definition 1.2.22 interchangeably.

Example 1.2.28. A first example of a singular foliation is a regular foliation. Let D be the distribution corresponding to the regular foliation then let $\mathcal{F} = \Gamma_c(D)$. ♦

Example 1.2.29. Let G be a Lie group acting on a smooth manifold M, i.e. we are given a group homomorphism $G \to \operatorname{Diff}(M)$. Then from this group action we have an associated infinitesimal action $\rho : \mathfrak{g} \to \mathfrak{X}(M)$ where $\mathfrak{g} = \operatorname{Lie}(G)$, the associated Lie algebra to G. Now when $v_1, \dots, v_n$ is a basis for $\mathfrak{g}$ we can take $\mathcal{F}$ to be the $C_c^\infty(M)$-module generated by $\{\rho(v_1), \dots, \rho(v_n)\}$. ♦

The above example is actually a particular instance of a more general object.

Definition 1.2.30 ([dSW99])**.** A **Lie algebroid** is a triple $(A, [\cdot,\cdot]_A, \rho)$ where A is a vector bundle over M, $[\cdot,\cdot]_A$ is a Lie bracket on $\Gamma(A)$ and $\rho : A \to TM$ is a vector bundle morphism over the identity on M, called the anchor, such that for all $f \in C^\infty(M)$ and sections $x, y \in \Gamma(A)$ we have that

$$[x, fy]_A = f\,[x, y]_A + (\rho(x)(f))\,y.$$

Remark 1.2.31. From the identity $[x, fy]_A = f\,[x, y]_A + (\rho(x)(f))\,y$ it also follows that ρ is a Lie algebra homomorphism between $\Gamma(A)$ and $\mathfrak{X}(M)$. The full proof is quite a long calculation but the main insight is using the following version of the Jacobi identity:

$$[x, [y, fz]_A]_A + [fz, [x, y]_A]_A + [y, [fz, x]_A]_A = 0,$$

for $x, y, z \in \Gamma(A)$ and $f \in C^\infty(M)$. Now using the Leibniz identity to expand all of these brackets and cancelling some terms gives that ρ is indeed a Lie algebra homomorphism.

Example 1.2.32. From a Lie algebroid $(A, [\cdot,\cdot]_A, \rho)$ we get a singular foliation. By using the sheaf point of view we let $\mathcal{F}(U) := \rho\,(\Gamma(A|_U))$. We show that this indeed yields a singular foliation. Let $x_1 = \rho(a_1)$ and $x_2 = \rho(a_2)$ for some $a_1, a_2 \in A$ and U_1 and U_2 two open neighborhoods of x_1 and x_2 respectively s.t. $x_1|_{U_1\cap U_2} = x_2|_{U_1\cap U_2}$. Let f_1, f_2 be a partition of unity subordinate to U_1 and U_2 then $a := f_1 a_1 + f_2 a_2 \in \Gamma\,(A|_{U_1\cup U_2})$. Now consider $\rho(a) = f_1\rho(a_1) + f_2\rho(a_2)$. Now clearly for $p \in U_1 \cap U_2$ we have that $\rho(a)(p) = f_1(p)x_1(p) + f_2(p)x_2(p) = x_1(p)(f_1(p) + f_2(p)) = x_1(p) = x_2(p)$. Similarly for $p \in U_1 \setminus (U_1 \cap U_2)$ we have that $f_2(p) = 0$ and so $\rho(a)(p) = x_1(p)$. Completely similar one can show that for $p \in U_2 \setminus (U_1 \cap U_2)$, $\rho(a)(p) = x_2(p)$. The only thing left to show is that this $\mathcal{F}$ is indeed locally finitely generated. Given a point $p \in M$ we can find an open neighborhood V of p on which A is trivial. Let $a_1, \ldots, a_n \in \Gamma(A|_V)$ be a local frame. Then every $a \in \Gamma(A|_V)$ is of the form $a = \sum f_i a_i$ for $f \in C^\infty(M)$ hence we also have that any $x \in \rho\,(\Gamma(A|_V)$ can we written as $x = \sum f_i \rho(a_i)$. ♦

Example 1.2.33. As noted above the example of a Lie group/algebra action on M can be seen as the foliation arising from a Lie algebroid. From a Lie group action we get an infinitesimal action $\varphi : \mathfrak{g} \to \mathfrak{X}(M)$. Now consider the vector bundle $\mathfrak{g}\times M$ then we will give this the structure of a Lie algebroid. Define the anchor $\rho : \mathfrak{g} \times M \to TM : (v, p) \mapsto \varphi(v)_p$. For $v, w \in \mathfrak{g}$ let $\underline{v}, \underline{w}$ denote the corresponding constant sections $M \to \mathfrak{g}$. Now define

$$[\underline{v}, \underline{w}] := \underline{[v, w]_{\mathfrak{g}}}.$$

Note that this bracket inherits the properties from the Lie bracket $[\cdot,\cdot]_{\mathfrak{g}}$ and so is a Lie bracket itself. Finally, one can extend this bracket to non-constant sections by using the Leibniz identity. For example when we let $G = S^1$ and $M = \mathbb{C}$ we can let G act on M by $t \cdot z := e^{it}z$. When we write $z = x + iy$ we have that the infinitesimal generator φ is given by

$$\varphi : \mathbb{R} \to \mathfrak{X}(\mathbb{C}) : v \mapsto v\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right).$$

The resulting foliation is illustrated in figure 1.5, the regular leaves are concentric circles while the singular leaf consists of the origin.

♦

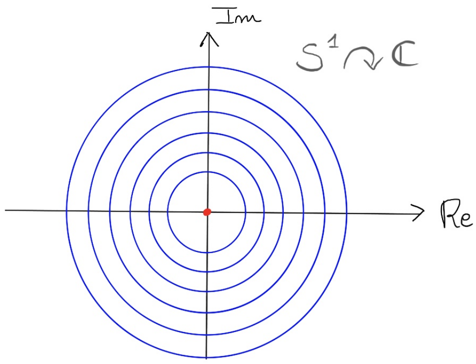


Figure 1.5: Foliation given by the action $(\mathbb{R},+) \curvearrowright \mathbb{C}$ from example 1.2.33.

Example 1.2.34 (Taken from [LGLS20])**.** Consider $M = \mathbb{R}$ and the partition into leaves $(-\infty, 0)$, $\{0\}$ and $(0, +\infty)$. In this case there are infinitely many modules that induce this foliation, they are indexed by a $k \in \mathbb{N}_0$ and have the form

$$\mathcal{F}^k = \left\langle x^k \frac{\partial}{\partial x} \right\rangle_{C_c^\infty(M)}.$$

This example illustrates that the definition using submodules or sheaves inherently carries more information than the 'geometric picture' that comes from the partition into leaves only, which is illustrated in figure 1.6. ♦

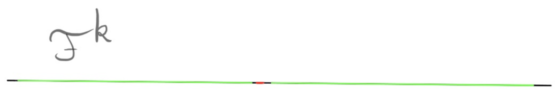


Figure 1.6: Foliation $\mathcal{F}^k$ from example 1.2.34

Example 1.2.35 (Example 3.12 in [LGLS20])**.** To a bivector $\pi \in \mathfrak{X}^2(M)$ we can associate a map $\pi^\sharp : \Omega^1(M) \to TM : df \mapsto \pi(df \wedge \cdot)$. We say that π is foliated when $\pi^\sharp(\Omega^1(M))$ is closed under the Lie bracket. As an application of Dirac geometry we know that when π is a Poisson bivector, T^*M gets a Lie algebroid structure with anchor $\pi^\sharp$. We call the resulting foliation of the Poisson manifold the *symplectic foliation*. This because when π is Poisson, the leaves inherit a non-degenerate Poisson structure which is inverse to a symplectic structure. For more information on this we refer to section 1.3.4 in [LGPV13]. ♦

Example 1.2.36 (Based on Example 3.36 in [LGLS20])**.** Consider the space $\mathbb{C}^n$ and a k-tuple of polynomials $\varphi = (\varphi_1, \ldots, \varphi_k)$ where $\varphi_k \in \mathbb{C}[x_1, \ldots, x_n]$. Let $\mathfrak{X}_{\mathrm{pol}}(\mathbb{C}^n)$ denote the module of polynomial vector fields (the coefficients are polynomials in $\mathbb{C}[x_1, \ldots, x_n]$). Now consider all $X \in \mathfrak{X}_{\mathrm{pol}}(\mathbb{C}^n)$ such that $X\varphi = 0$. We argue that these vector fields form a singular foliation. Let X, Y be such that $X[\varphi] = 0$ and $Y[\varphi] = 0$ then $[X, Y]\varphi = 0$, indeed $[X, Y]\varphi = X(Y[\varphi]) - Y(X[\varphi]) = 0$ and so this set of vector fields is closed under the Lie bracket. If $\mathcal{F}$ denotes all $X \in \mathfrak{X}_{\mathrm{pol}}(\mathbb{C}^n)$ for which $X\varphi = 0$ then it can easily be seen that $\mathcal{F}$ is a $\mathbb{C}[x_1, \ldots, x_n]$-submodule of $\mathfrak{X}_{\mathrm{pol}}(\mathbb{C}^n)$. The only thing left to show is that $\mathcal{F}$ is finitely generated but this is a standard result in commutative algebra: $\mathfrak{X}_{\text{pol}}(\mathbb{C}^n)$ is a finitely generated $\mathbb{C}[x_1, \ldots, x_n]$-module and $\mathbb{C}[x_1, \ldots, x_n]$ is a Noetherian ring (another standard result of commutative algebra), now $\mathfrak{X}_{\text{pol}}(\mathbb{C}^n)$ is also a Noetherian $\mathbb{C}[x_1, \ldots, x_n]$-module and hence all submodules are finitely generated so in particular $\mathcal{F}$ is finitely generated. We will use this example as an example at several point throughout this thesis, for convenience we will denote it by $\mathcal{F}_\varphi$.

Note that one could replace $\mathbb{C}$ by $\mathbb{R}$ and all the results still remain true. Indeed $\mathfrak{X}_{\text{pol}}(\mathbb{R}^n)$ is still a finitely generated $\mathbb{R}[x_1, \ldots, x_n]$-module and $\mathbb{R}[x_1, \ldots, x_n]$ is still Noetherian[4]. Hence $\mathfrak{X}_{\text{pol}}(\mathbb{R}^n)$ is also a Noetherian $\mathbb{R}[x_1, \ldots, x_n]$-module and so all submodules are finitely generated. In particular $\mathcal{F}$ is finitely generated.

Figure 1.7 gives an illustration of $\mathcal{F}_\varphi$ when we let $\varphi(x_1, x_2) = x_1 x_2$. We see the blue and green leaves as the 1-dimensional ones while the origin is the only 0-dimensional leaf.

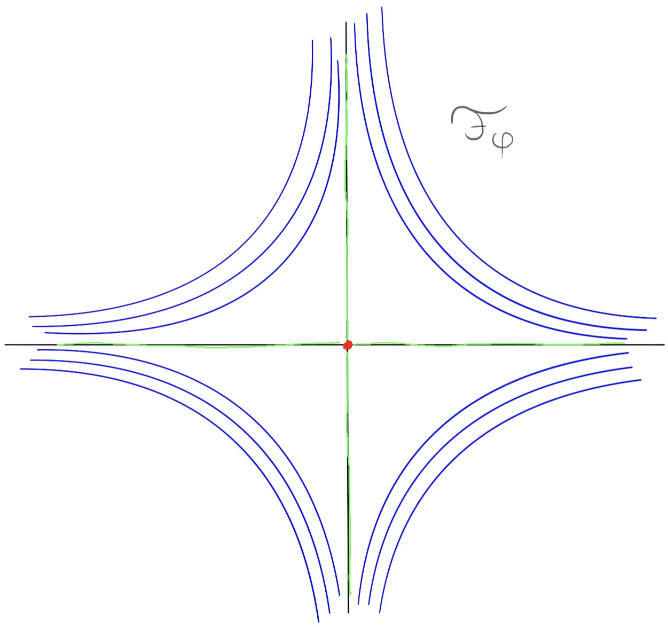


Figure 1.7: Foliation $\mathcal{F}_\varphi$ for $\varphi(x_1, x_2) = x_1 x_2$

♦

1.2.4 Singular Foliations and (almost-)Lie algebroids

Examples 1.2.32, 1.2.33 and 1.2.35 form a large class of examples, it is natural to ask whether *all* singular foliations arise, locally or globally, as the image of a Lie algebroid. Note that this is indeed the case for regular foliations $\mathcal{F}$ for which we can take the vector bundle to be $D = T\mathcal{F}$ and use the Lie bracket $[\cdot, \cdot]$ on $\mathfrak{X}(M)$ to define a Lie bracket on $\Gamma(D)$ and the anchor is just the inclusion. It turns out that the answer to the question when considered in a global setting is negative: for this see the next example which also

[4] In fact when R is a Noetherian ring we have that $R[x_1, \ldots, x_n]$ is Noetherian as an R-module.

contains the definition of a first important invariant associated to a singular foliation at a point. Later in the thesis we will discuss the local question using the tools that where developed.

Example 1.2.37 (Based on Lemma 1.3 in [AZ13]). A first invariant of a singular foliation as introduced by Androulidakis and Skandalis in [AS09] is called the **isotropy Lie algebra** $\mathfrak{g}_x$ at some point $x \in M$. It is defined as the quotient

$$\mathfrak{g}_x := \frac{\mathcal{F}(x)}{I_x\mathcal{F}} = \frac{\{X \in \mathcal{F} \mid X(x) = 0\}}{I_x\mathcal{F}}$$

where $I_x = \{f \in C^\infty(M) \mid f(x) = 0\}$. The vector space $\mathfrak{g}_x$ gets a Lie algebra structure because $I_x\mathcal{F} \subset \mathcal{F}(x)$ is a Lie ideal. We now use this object to show that not all singular foliations are induced (in the sense of example 1.2.32 meaning that $\mathcal{F} = \rho(\Gamma(A))$), globally, by a Lie algebroid.

Let $\mathcal{F}$ be such a foliation coming from a Lie algebroid A. Consider the space $\ker \rho_x$ (called the isotropy of the Lie algebroid at x). Then there is a well-defined linear mapping $\ker \rho_x \to \mathfrak{g}_x$ that maps $\overline{a}$ to $\langle \rho(a) \rangle$ where $a \in \Gamma(A)$ is any extension of $\overline{a}$. Remark that this map is surjective: every element in $\mathfrak{g}_x$ is represented by an $X \in \mathcal{F}$ that vanishes at x; hence $X = \rho(a)$ for some $a \in C^\infty_c(A, M)$ with $\rho_x(a_x) = 0$. Hence we have that $\dim \mathfrak{g}_x \leq \operatorname{rk} A$ for all $x \in M$.

Now let $k \geq 1$ and consider the foliation $\mathcal{F}^k$ of $\mathbb{R}^2$ generated by

$$(x-k)^i y^j \frac{\partial}{\partial x}, \quad (x-k)^i y^j \frac{\partial}{\partial y} \qquad \forall i, j \geq 0 \text{ for which } i + j = k.$$

Now take the foliation $\mathcal{F}$ generated by $\cup_{k\geq 1}\varphi_k\mathcal{F}^k$ where φ_k is some fixed bump function on $\mathbb{R}^2$ with small support concentrated around $(k, 0)$. Then one can show that $\mathfrak{g}_{(k,0)} = \mathbb{R}^{2k+2}$. So the dimension of the spaces $\mathfrak{g}_{(k,0)}$ grows linearly with k and so is certainly not bounded above; so $\mathcal{F}$ cannot come from a Lie algebroid.

♦

We can also look at almost-Lie algebroids. They are very similar to Lie algebroids and the only difference is that the bracket $[\cdot,\cdot]_A$ need not to satisfy the Jacobi identity.

Definition 1.2.38 (As defined in [Hue05]). An **almost-Lie algebroid** over a manifold M is a vector bundle $A \to M$, equipped with a vector bundle morphism $\rho : A \to TM$ called the anchor and skew-symmetric bracket $[\cdot,\cdot]_A$ on $\Gamma(A)$. This bracket must satisfy the Leibniz identity

$$\forall x, y \in \Gamma(A), f \in C^\infty(M) \qquad [x, fy]_A = f\,[x, y]_A + \rho(x)\,[f]\,y,$$

and the anchor must be an algebra morphism w.r.t. the bracket operation[5]

$$\forall x, y \in \Gamma(A) \qquad \rho\left([x, y]_A\right) = [\rho(x), \rho(y)]\,. \tag{1.2}$$

[5]Note this is not a Lie algebra morphism as the Jacobi identity does not need to hold for the bracket $[\cdot,\cdot]_A$.

The following proposition shows that from a singular foliation one can get an almost-Lie algebroid. This also shows that the main failure point in getting a Lie algebroid from a singular foliation is getting the Jacobi identity for the bracket on the sections.

Proposition 1.2.39 ([Lav16] and [LGLS20])**.** *Let M be a smooth manifold and (A, ρ) and anchored vector bundle*[6]*.*

1. *For every almost-Lie algebroid structure on $A \to M$, the image of the anchor map $\rho : \Gamma(A) \to \mathfrak{X}(M)$ is a singular foliation.*
2. *Every finitely generated foliation on M is the image under the anchor map of an almost-Lie algebroid, defined on a trivial bundle.*
3. *Every anchored vector bundle (A, ρ) over M that covers a singular foliation $\mathcal{F}$ can be equipped with an almost-Lie algebroid structure with anchor ρ.*
4. *A singular foliation is the image under the anchor of an almost-Lie algebroid if and only if it is finitely generated.*

Proof. 1. This follows immediately from the definition.

2. Let $X_1, \ldots, X_r$ be generators of a singular foliation $\mathcal{F}$. By definition $\mathcal{F}$ is closed under the Lie bracket of vector fields and so there exist functions $c_{ij}^k \in C^\infty(M)$ such that
$$[X_i, X_j] = \sum_{k=1}^{r} c_{ij}^k X_k \quad \forall i, j \in \{1, \ldots, r\}.$$
Now when $c_{ij}^k \neq -c_{ji}^k$ we may replace c_{ij}^k by $\frac{1}{2}\left(c_{ij}^k - c_{ji}^k\right)$. Now when $g_{ij}^k = \frac{1}{2}\left(c_{ij}^k - c_{ji}^k\right)$ it is easy to see that $g_{ij}^k = -g_{ji}^k$. Remark that this replacing does not change anything:
$$\begin{aligned}[X_i, X_j] = \sum_{i=1}^{r} c_{ij}^k X_k \rightsquigarrow \sum_{i=1}^{r} \frac{1}{2}\left(c_{ij}^k - c_{ji}^k\right) X_k \\ &= \frac{1}{2}\sum_{k=1}^{r} c_{ij}^k X_k - \frac{1}{2}\sum_{k=1}^{r} c_{ji}^k X_k \\ &= \frac{1}{2}[X_i, X_j] - \frac{1}{2}[X_j, X_i] \\ &= [X_i, X_j].\end{aligned}$$
Where we used the antisymmetry of the Lie bracket. We now define $A = \mathbb{R}^r \times M \to M$ and construct an almost-Lie algebroid structure on it. Denote its canonical global sections as $e_1, \ldots, e_r$ then define:
   - the anchor map $\rho(e_i) = X_i$ for $i = 1, \ldots, r$,
   - the bracket using the structure functions c_{ij}^k, $[e_i, e_j]_A = \sum_{k=1}^{r} c_{ij}^k e_k$; this bracket can then be extended to nonconstant sections using the Leibniz rule.

   With this we have by definition $\rho(\Gamma(A)) = \mathcal{F}$.

[6] An **anchored vector bundle** is a vector bundle $A \to M$ together with an anchor map $\rho : A \to TM$ over the identity on M.

3. Let $(U_i)_{i\in I}$ denote a collection of open sets such that $A|_{U_i}$ is trivial for all $i \in I$. We may then choose a partition of unity $(\varphi_i)_{i\in I}$ subordinate to the open cover $(U_i)_{i\in I}$. Just as we did in the previous we can find an anchor ρ and a collection of brackets $[\cdot,\cdot]_{U_i}$ on $\Gamma(A|_{U_i})$. Now we can glue these brackets together using the partition of unity
$$[\cdot,\cdot]_A := \sum_{i\in I} \varphi_i \, [\cdot,\cdot]_{U_i} \,.$$

4. This follows immediately from the previous item.

□

1.2.5 Geometric Resolutions of Singular Foliations

Geometric resolutions of a singular foliations will become important when we define the universal Lie ∞-algebroid of a singular foliation.

Definition 1.2.40 (Definition 2.1 in [LGLS20])**.** Let $\mathcal{F} \subset \mathfrak{X}(M)$ be a singular foliation on a manifold M. A **geometric resolution** of $\mathcal{F}$ is a triple (E, d, ρ) such that

1. $E = \oplus_{i\geq 1} E_{-i}$ is a collection of vector bundles over M,
2. d is a family of vector bundle morphisms $d^{(i)} : E_{-i} \to E_{-i+1}$ over the identity on M,
3. ρ is a vector bundle morphism $\rho : E_{-1} \to TM$ over the identity on M called the anchor of the geometric resolution.

All such that the following sequence of $C^\infty(M)$-modules is exact[7]
$$\cdots \xrightarrow{d^{(4)}} \Gamma(E_{-3}) \xrightarrow{d^{(3)}} \Gamma(E_{-2}) \xrightarrow{d^{(2)}} \Gamma(E_{-1}) \xrightarrow{\rho} \mathcal{F} \to 0.$$

When all E_{-i} are trivial bundles we speak of a resolution by trivial bundles. A geometric resolution is called **minimal at** $m \in M$ if for all $i \geq 2$ the linear maps $d^{(i)}_m : E_{-i}|_m \to E_{-i+1}|_m$ vanish.

By the Serre-Swan theorem, see for instance theorem 12.32 in [Nes20], we know that the $C^\infty(M)$-module of sections of a vector bundle is a projective $C^\infty(M)$-module. This means that geometric resolutions can be seen as projective resolutions of the module $\mathcal{F}$. It is a standard result in commutative algebra (see for instance part XX §1 in [Lan05]) that every module admits a free resolution and hence also a projective resolution. This however does not mean that all singular foliations admit geometric resolutions[8]. Indeed there do exists counterexamples on $\mathbb{R}$ (in the smooth case). However the following theorem, which is part of proposition 2.3 in [LGLS20], gives a class of foliations for which we do have an existence result.

Theorem 1.2.41. *Every algebraic singular foliation on a Zariski open subset of $\mathbb{C}^n$ admits a geometric resolution of length less than or equal to $n + 1$.*

[7] Remark that we used the same notation $d^{(i)}$ for the maps between vector bundles and the induced map on the module of sections.

[8] Not every projective module arises as the module of sections of a vector bundle over M. Indeed, they need not to be finitely generated which is a necessary condition for the converse of the Serre-Swan theorem.

To show theorem 1.2.41 we will make use of Hilbert's syzygy theorem (see for instance [Eis95], corollary 19.7). The proof we give here is an adaptation of the proof originally given by Cartan and Eilenberg as discussed in section 19.1 of [Eis95]. We give an adaptation of their proof because they originally gave it in the setting of local rings. We will need it for the case of graded rings and so some adaptations of necessary lemmas and the statements of some propositions are needed. Before giving the proof we need some preliminary results.

Lemma 1.2.42 (Graded Nakayama Lemma)**.** *Let $R = \oplus_{r\geq 0} R_r$ be a graded ring with the degree 0 component a field k. Let M be a finitely generated graded R-module and $I \lhd R$ a graded homogeneous ideal such that $I \subset R_+ = \oplus_{r>0} R_r$ and $IM = M$. Then we have $M = 0$.*

Proof. Since M is a graded module, we can write $M = \oplus_{n\in\mathbb{Z}} M_n$. Since M is assumed to be finitely generated we can write $M = \langle x_1, \ldots, x_l\rangle_R$ for some homogeneous elements $x_1, \ldots, x_l$. Let $d := \min_{i=1,\ldots,l} \deg x_i$ then $M_d \neq 0$ (there is some element of degree d that generates M) but $M_{d-m} = 0$ for all $m \geq 1$ (there are no elements of degree $< d$ that generate M). Note that R does not contain any elements of negative degree and $I \subset R_+$ so in IM the minimal degree of elements in IM must be $d+1$ which of course also means that M_d is not contained in IM which clearly contradicts $IM = M$. In this way we conclude that there does not exist an integer d for which $M_d \neq 0$ hence $M = 0$. □

Remark 1.2.43. A classic application of Nakayama's lemma is to make sense of a minimal generating set for M (i.e. no smaller subset generates M) . For arbitrary finitely generated modules over arbitrary rings this does not need to be a well-defined notion. In our case, by using Nakayama's lemma, it will be well-defined. Note that since $R_0 = k$ is a field we have that $R/R_+ = k$. Hence M/R_+M is a k-vector space. This means it has a basis $\langle \widehat{x_1}, \ldots, \widehat{x_n}\rangle$ where $x_i \in M$ and $\widehat{x_i}$ is it representative in M/R_+M. For a basis of a k-vector space V we do have a well-defined notion of minimal generating sets. One needs exactly $\dim_k V$ linearly independent elements to generate V. Now consider the submodule $N := \langle x_1, \ldots, x_n\rangle$. By construction of the submodule N we have that $M/N = R_+(M/N)$. Now applying the graded Nakayama lemma 1.2.42 with $I = R_+$ we get $M/N = 0$. Hence we must have $M = \langle x_1, \ldots, x_n\rangle$. I.e. we have lifted a basis of a vector space to a generating set of M and the minimal number of generators is well-defined.

Definition 1.2.44 ([Eis95])**.** A **graded free resolution** of an R-module M is a complex

$$\mathbb{F} : \cdots \to F_n \xrightarrow{\varphi_n} \cdots \to F_1 \xrightarrow{\varphi_1} F_0 \to M \to 0, \tag{1.3}$$

where $R = \oplus_{d\geq 0} R_d$ is graded ring and all the F_i are graded free modules, that is $F_i = \oplus_{d\in I} R_d$ for I some index set. Furthermore, the maps are homogeneous of degree zero. We call the resolution finite of length n if $F_{n+1} = 0$ and $F_i \neq 0$ for $0 \leq 1 \leq n$.

Definition 1.2.45 ([Eis95])**.** A complex

$$\mathbb{F} : \cdots \to F_n \xrightarrow{\varphi_n} F_{n-1} \to \cdots ,$$

of graded modules over a graded ring $R = \oplus_{d\geq 0} R_d$ is called **minimal** if the maps in the complex $\mathbb{F} \otimes R/R_+$ are all 0. That is $\operatorname{im} \varphi_n \subset R_+ F_{n-1}$.

Lemma 1.2.46 ([Eis95]). *A graded free resolution*

$$\mathbb{F} : \cdots \to F_n \xrightarrow{\varphi_n} F_{n-1} \to \cdots \xrightarrow{\varphi_1} F_0$$

over a (positively) graded ring R is a minimal complex if and only if for each n, a basis of F_{n-1} maps onto a minimal generating set of $\operatorname{coker} \varphi_n$.

Proof. Let R_+ be as above and let φ_0 denote the natural map $F_0 \to \operatorname{coker} \varphi_1$. For any $n \geq 0$ consider the surjective map of vector spaces given by

$$\frac{F_{n-1}}{R_+ F_{n-1}} \to \frac{\operatorname{coker} \varphi_n}{R_+ \operatorname{coker} \varphi_n}.$$

By the graded Nakayama lemma 1.2.42 a basis for $\operatorname{coker} \varphi_n / R_+ \operatorname{coker} \varphi_n$ can be lifted to a minimal set of generators for $\operatorname{coker} \varphi_n$, see remark 1.2.43. Thus we have that a basis of F_{n-1} is mapped onto a minimal generating set for $\operatorname{coker} \varphi_n$ if and only if the surjective map from above is in fact an isomorphism which happens exactly when $\operatorname{im} \varphi_n \subset R_+ F_{n-1}$. □

The following result will be the key result in proving Hilbert's syzygy theorem. Before proceeding note that by $\operatorname{pd} M$ we denote the *projective dimension* of M which is the minimal length of projective resolutions of M, $\operatorname{gldim} R$ is the supremum of the projective dimensions of all R-modules. Also recall that the functor $\operatorname{Tor}_i^R(-, N)$ can be computed as the left derived functors of the functors $- \otimes_R N$.

Proposition 1.2.47 ([Eis95]). *Let R be a positively graded ring with $R_0 = k$ a field and M a finitely generated nonzero graded R-module. In this case* $\operatorname{pd} M$ *is the length of any minimal free resolution. Furthermore,* $\operatorname{pd} M$ *is the smallest integer i for which* $\operatorname{Tor}_{i+1}^R(k, M) = 0$ *and thus* $\operatorname{pd} k = \operatorname{gldim} R$.

Proof. As remarked earlier $\operatorname{Tor}_{i+1}^R(k, M)$ can be computed as the left derived functor of $- \otimes_R M$ which means it is the $i+1$-th homology of a projective resolution of M tensored with k. Thus if $n = \operatorname{pd} M$ by definition the projective modules P_{i+1} for $i \geq n$ in the projective resolution are zero. From this it immediately follows that also $\operatorname{Tor}_{i+1}^R(k, M) = 0$. Now assume that

$$\mathbb{F} : \cdots \to 0 \to F_n \xrightarrow{\varphi_n} F_{n-1} \to \cdots \xrightarrow{\varphi_1} F_0,$$

is a graded free resolution of M of length n. Let $i \geq 0$ be the smallest integer for which $\operatorname{Tor}_{i+1}^R(k, M) = 0$ then we immediately have that $n \geq \operatorname{pd} M \geq i$. When $\mathbb{F}$ is a minimal free graded resolution the differentials in the complex $R/R_+ \otimes \mathbb{F}$ are zero and since $R/R_+ = k$ we have that the differentials in the complex $k \otimes \mathbb{F}$ are zero which immediately implies

$$\operatorname{Tor}_{i+1}^R(k, M) = k \otimes F_{i+1}.$$

Hence $\operatorname{Tor}_{i+1}^R(k, M) = 0$ if and only if $F_{i+1} = 0$ and so $i = n$. From theorem 10.94 in [Rot08] it follows that we can also compute $\operatorname{Tor}_{i+1}^R(k, M)$ starting from a projective resolution of k. Clearly $\operatorname{Tor}_{i+1}^R(k, M) = 0$ for all $i \geq \operatorname{pd} k$ and so combined with the above we have that $\operatorname{pd} M \leq \operatorname{pd} k$. Indeed, if $j = \operatorname{pd} k$ and $n = \operatorname{pd} M$ and assume $j \leq n$ then we would have $\operatorname{Tor}_{i+1}^R(k, M) = 0$ for $i \geq j$ but we have just shown that n is the smallest integer for which $\operatorname{Tor}_{i+1}^R(k, M) = 0$ for $i \geq n$ which clearly contradicts. So, we conclude $\operatorname{pd} M \leq \operatorname{pd} k$ and combined with Auslander's theorem (a classical result, see theorem 19.1 in [Eis95]) this gives $\operatorname{gldim} R = \operatorname{pd} k$. □

We give the following definition in the setting where R is a commutative ring and F is a free R-module with basis $\{e_1, \dots, e_s\}$.

Definition 1.2.48 ([Eis95])**.** A **regular sequence** is a sequence $r_1, \dots, r_d \in R$ such that r_i is not a zero-divisor in $R/(r_1, \dots, r_{i-1})R$ for $i = 1, \dots, d$ and $R/(r_1, \dots, r_d)R \neq 0$.

The following definition is an adaptation of definition of a Koszul complex in [Rot08] on page 1004. The notion of a Koszul complex will also come up later in examples 1.2.59 and 1.2.60.

Definition 1.2.49 ([Eis95])**.** Let $x = (x_1, \dots, x_d)$ be a sequence in R then the **Koszul complex** of x is defined as

$$K(x)^\bullet : \cdots \to \wedge^p F \xrightarrow{d_p} \wedge^{p-1} F \to \cdots \to \wedge^2 F \xrightarrow{d_2} F \xrightarrow{d_1} R,$$

where the differentials d_p are defined as

$$d_p \left(e_{i_1} \wedge \cdots \wedge e_{i_p} \right) = \sum_{r=0}^{s} (-1)^{r-1} x_r e_{i_1} \wedge \cdots \wedge \widehat{e_{i_r}} \wedge \cdots \wedge e_{i_p},$$

and especially $d_1(\sum_{i=1}^s c_i e_i) = \sum_{i=1}^s c_i x_i$.

We now use (without proof) that the ideal R_+ of positively graded elements in R is generated by a regular sequence $x = (x_1, \dots, x_n)$ and by corollary 19.3 in [Eis95] the Koszul complex $K(x)^\bullet$ forms a minimal graded free resolution of length n of $k = R/R_+$.

Theorem 1.2.50 (Hilbert's syzygy theorem)**.** *If k is a field, then every finitely generated graded module over $k[x_1, \dots, x_n]$ has a graded free resolution of length $\leq n$.*

Proof. Using that $K(x)^\bullet$ forms a minimal free resolution of length n for k we can combine this with proposition 1.2.47 to conclude that $n = \operatorname{pd} k$ is equal to the global dimension of R which means that the length of graded free resolutions are bounded above by n. □

Now that we have theorem 1.2.50 we are almost done. Our singular foliation $\mathcal{F}$ is a finitely generated module over the ring of functions on Zariski open subsets of $\mathbb{C}^n$ which of course is the ring $R = \mathbb{C}[x_1, \dots, x_n]$. We think that in general there is no way to make certain that $\mathcal{F}$ is generated by homogeneous elements and so we need to proof the following, quite surprising, corollary of Hilbert's syzygy theorem which gets rid of the graded condition and works for arbitrary finitely generated modules over $k[x_1, \dots, x_n]$. Once we have this result, we will give the proof of theorem 1.2.41.

Corollary 1.2.51 ([Eis95])**.** *Every finitely generated module over $k[x_1, \dots, x_n]$ has a finite free resolution.*

The proof is taken from corollary 19.8 in [Eis95].

Proof. Let $S = k[x_1, \dots, x_n]$ and M a finitely generated S-module. It is a standard result from commutative algebra that any module admits a free presentation so we choose a free presentation $F \xrightarrow{\varphi} G \to M \to 0$. We can choose a basis such that $\varphi : F \to G$ is represented by a matrix with polynomial coefficients. By introducing a new variable x_0 we can homogenize all entries in this matrix: let d denote the maximal degree of a polynomial in this matrix, next multiply all entries by the appropriate power of x_0 to take the degree of the entry up to d. In this way we start working over the polynomial ring $T = k[x_0, x_1, \dots, x_n]$ and get a new matrix $\widetilde{\varphi}$ with entries consisting of homogeneous polynomials of degree d. Note that $S \cong T/(1-x_0)$ where $(1-x_0)$ denotes the ideal generated by $1-x_0$ in T. This makes S into a T-module and with this structure it is clear that $\varphi = \widetilde{\varphi} \otimes_T S$ (indeed this just says that replacing x_0 by 1 in $\widetilde{\varphi}$ just gives φ). Now define $\widetilde{M} := \operatorname{coker} \widetilde{\varphi}$ for which we thus have $\widetilde{M} \otimes_T S = M$. By Hilbert's syzygy theorem 1.2.50 there exists a free resolution $\widetilde{\mathbb{F}}$ of $\widetilde{M}$. We now finish the proof by showing that $\widetilde{\mathbb{F}} \otimes_T S$ gives a free resolution for M. This holds in particular when $\widetilde{\mathbb{F}} \otimes_T S$ has no homology (except the 0-th homology) which in turn means that $\operatorname{Tor}_i^T\left(\widetilde{M}, S\right) = 0$ for all $i \geq 1$ (by construction of the Tor-functor). We can compute the modules $\operatorname{Tor}_i^T\left(\widetilde{M}, S\right)$ starting from the following free resolution for S

$$0 \to T \xrightarrow{1-x_0} T \to S \to 0.$$

We tensor this free resolution with $\widetilde{M}$ over T to get the complex from which we can compute the Tor modules. This gives us the following sequence

$$0 \to \widetilde{M} \xrightarrow{1-x_0} \widetilde{M} \to M \to 0,$$

and this sequence has no homology (except again at the 0-th step) when the part $0 \to \widetilde{M} \xrightarrow{1-x_0} \widetilde{M}$ is an exact sequence. This thus means that $\ker(1-x_0) = \{0\}$ i.e. $1-x_0$ is not a zero divisor on $\widetilde{M}$. This however is clear since any element $\widetilde{m} \in \widetilde{M}$ can be written as $\widetilde{m} = \widetilde{m_e} + (\text{degree greater than } e)$ where $\deg \widetilde{m_e} = e$ and so $(1-x_0)\widetilde{m} = \widetilde{m_e} + (\text{degree greater than } e)$ which proves what we wanted to show.

□

We are now ready to give a proof of theorem 1.2.41:

Proof of theorem 1.2.41. We have that $\mathcal{F} \subset \mathfrak{X}_{\text{pol}}(\mathbb{C}^n)$ is an involutive submodule. Since $\mathfrak{X}_{\text{pol}}(\mathbb{C}^n)$ is a finitely generated module over the Noetherian ring $S = \mathbb{C}[x_1, \dots, x_n]$ we have that $\mathfrak{X}_{\text{pol}}(\mathbb{C}^n)$ is a Noetherian S-module. Hence since $\mathcal{F}$ is a submodule of the Noetherian module $\mathfrak{X}_{\text{pol}}(\mathbb{C}^n)$ we have that $\mathcal{F}$ is finitely generated. By corollary 1.2.51 we have that a *finite* free resolution of $\mathcal{F}$ exists and hence also a projective resolution. Note that by definition this implies that $\operatorname{pd} \mathcal{F} < \infty$. In fact, by examining the proof of corollary 1.2.51, we even have that $\operatorname{pd} \mathcal{F} \leq n+1$. We will now show that we are able to construct a projective resolution in which all the projective modules are finitely generated S-modules.

Since $\mathcal{F}$ is finitely generated we have that $\mathcal{F} = \langle X_1, \dots, X_k \rangle_S$. Now consider the free module $P_0 = \oplus_{i=1}^k S$ with basis denoted $e_1, \dots, e_k$. Then there is a unique surjection[9] mapping the e_i to the X_i

$$P_0 \xrightarrow{\varphi_0} \mathcal{F} \to 0.$$

Note that this P_0 is clearly finitely generated as an S-module. Hence P_0 is a Noetherian S-module. Now define $M_0 := \ker \varphi_0$, then $M_0 \subset P_0$ as an S-submodule and so M_0 is finitely generated. One can now do the same steps with M_0 to obtain the next projective

[9] Any module can be written uniquely as the quotient of a free module. In particular finitely generated modules are exactly the ones that are isomorphic to a quotient of a finite rank free module.

module P_1 in the projective resolution which will again be finitely generated by the same arguments. This process can be continued to obtain the complete projective resolution. Indeed, this process stops because $\operatorname{pd}\mathcal{F} \leq n+1$ (as observed above) and so there can at most be $n+1$ projective modules P_i.

We have now shown that when

$$P_l \xrightarrow{\varphi_l} P_{l-1} \xrightarrow{\varphi_{l-1}} \cdots \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi_0} \mathcal{F} \to 0, \quad l \leq n+1,$$

is a projective resolution of $\mathcal{F}$, all modules P_i are finitely generated. Since all Zariski open sets $U \subset \mathbb{C}^n$ are connected, the Serre-Swan theorem asserts that all the P_i arise as the module of sections of some vector bundle E_{-i} hence obtaining a geometric resolution of length at most $n+1$.

□

Remark 1.2.52. In this particular case we are able to construct a projective resolutions with finitely generated projective modules. The key property to do this is that the polynomial ring is Noetherian. This fails in the smooth case: $C^\infty(M)$ is not Noetherian when $\dim M > 0$. Hence in the smooth case a geometric resolution may not always exist (as we already remarked earlier).

1.2.6 Relations Between Geometric Resolutions

In this part we will examine the relation between two geometric resolutions. An object that provides a relaxation of the conditions in the definition of a geometric resolution and thus also gives some slightly more general results is defined in the following definition. Note that all vector bundle morphisms involved are considered to be over the identity on M.

Definition 1.2.53 (Definition 3.16 in [LGLS20])**.** A **complex of vector bundles** (E, d, ρ) over a singular foliation $\mathcal{F}$ is a collection E of vector bundles $(E_{-i})_{i\geq 1}$ over M, a collection d of vector bundle morphisms $d^{(i)} : E_{-i} \to E_{-i+1}$ and a vector bundle morphism $\rho : E_{-1} \to TM$ such that $d^{(i-1)} \circ d^{(i)} = 0$ for all $i \geq 3$, $\rho \circ d^{(2)} = 0$ and $\rho\left(\Gamma(E_{-1})\right) \subset \mathcal{F}$.

Remark that in particular every geometric resolution is a complex of vector bundles over $\mathcal{F}$ and that every complex of vector bundles over $\mathcal{F}$ is a geometric resolution if and only if it is exact on the level of sections and $\rho\left(\Gamma(E_{-1})\right) = \mathcal{F}$. The following definition captures the notion of morphisms and homotopy of morphisms for complexes of vector bundles over $\mathcal{F}$.

Definition 1.2.54 (Definition 3.17 in [LGLS20])**.**

- A morphism φ between two complexes of vector bundles (E, d, ρ) and (E', d', ρ') over $\mathcal{F}$ is a collection of vector bundle morphisms $\varphi_i : E_{-i} \to E'_{-i}$ (over the identity map on M) making the following diagram commutative

$$\begin{array}{ccccccccc} \cdots \longrightarrow & E_{-3} & \xrightarrow{d^{(3)}} & E_{-2} & \xrightarrow{d^{(2)}} & E_{-1} & \xrightarrow{\rho} & TM \\ & \downarrow{\varphi_3} & & \downarrow{\varphi_2} & & \downarrow{\varphi_1} & & \downarrow{\mathbb{1}} \\ \cdots \longrightarrow & E'_{-3} & \xrightarrow{d'^{(3)}} & E'_{-2} & \xrightarrow{d'^{(2)}} & E'_{-1} & \xrightarrow{\rho} & TM \end{array}$$

- two morphisms $\varphi, \psi : (E, d, \rho) \to (E', d', \rho')$ are said to be homotopic if there exists a collection of vector bundle morphisms $h_i : E_{-i} \to E'_{-i-1}$ such that $\varphi_i = \psi_i + d'^{(i+1)} \circ h_i + h_{i-1} \circ d^{(i)}$ for all $i \geq 2$ and $\varphi_1 = \psi_1 + d'^{(2)} \circ h_1$. That is the following diagram commutes

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & E_{-3} & \xrightarrow{d^{(3)}} & E_{-2} & \xrightarrow{d^{(2)}} & E_{-1} & \xrightarrow{\rho} & TM \\
 & \swarrow^{h_3} & \psi_3 \downarrow\ \downarrow \varphi_3 & \swarrow^{h_2} & \psi_2 \downarrow\ \downarrow \varphi_2 & \swarrow^{h_1} & \psi_1 \downarrow\ \downarrow \varphi_1 & & \downarrow \mathbb{1} \\
\cdots & \longrightarrow & E'_{-3} & \xrightarrow[d'^{(3)}]{} & E'_{-2} & \xrightarrow[d'^{(2)}]{} & E'_{-1} & \xrightarrow[\rho']{} & TM
\end{array}$$

- two complexes of vector bundles (E, d, ρ) and (E', d', ρ') over $\mathcal{F}$ are said to be homotopy equivalent if there exist chain maps $\varphi : (E, d, \rho) \to (E', d', \rho')$ and $\psi : (E', d', \rho') \to (E, d, \rho)$ such that $\varphi \circ \psi$ and $\psi \circ \varphi$ are homotopic to the respective identity morphism of complexes of vector bundles.

An important lemma that is used at several points in the paper [LGLS20] but is left without proof is the following one.

Lemma 1.2.55 (Lemma 3.19 in [LGLS20])**.** *Let (E, d, ρ) be a geometric resolution of a singular foliation $\mathcal{F}$. For every complex of vector bundles (E', d', ρ') over $\mathcal{F}$, there exists a morphism of complexes of vector bundles over $\mathcal{F}$ from (E', d', ρ') to (E, d, ρ) and any two such morphisms are homotopy equivalent.*

Proof. Define the $C^\infty(M)$-modules $P_i := \Gamma(E_{-i})$ and $Q_i := \Gamma(E'_{-i})$. Then both of these are projective modules and furthermore the complex $P_\bullet$ (consisting of the modules P_i and the differentials $d^i : P_{i+1} \to P_i$) is a projective resolution of the module $\mathcal{F}$. Furthermore when we define $\mathcal{F}' := \rho\left(\Gamma(E'_{-1})\right)$ then we know, since (E', d', ρ') is a complex of vector bundles over $\mathcal{F}$, $\mathcal{F}' \subset \mathcal{F}$ as submodules. So we have a natural map (the inclusion) $f : \mathcal{F}' \to \mathcal{F}$ and the following diagram where the bottom row is exact:

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d'^2} & Q_2 & \xrightarrow{d'^1} & Q_1 & \xrightarrow{\rho'} & \mathcal{F}' & \longrightarrow & 0 \\
 & & & & & & \downarrow f & & \\
\cdots & \xrightarrow{d^2} & P_2 & \xrightarrow{d^1} & P_1 & \xrightarrow{\rho} & \mathcal{F} & \longrightarrow & 0
\end{array}.$$

We now show that there exists a sequence of maps $f_n : Q_n \to P_n$ such that all the formed squares commute. We do this by induction on $n \geq 1$. First let $n = 1$ then we have the following diagram

$$\begin{array}{ccccc}
 & & Q_1 & & \\
 & \swarrow^{f_1} & \downarrow f \circ \rho' & & \\
P_1 & \xrightarrow{\rho} & \mathcal{F} & \longrightarrow & 0
\end{array}$$

and the projectivity of the module Q_1 together with the fact that ρ is surjective (from exactness) implies the existence of a morphism of modules $f_1 : Q_1 \to P_1$ such that

$\rho \circ f_1 = f \circ \rho'$. Now for the inductive step consider

$$\begin{array}{ccccc} Q_{n+1} & \xrightarrow{d'^n} & Q_n & \xrightarrow{d'^{n-1}} & Q_{n-1} \\ & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} \\ P_{n+1} & \xrightarrow{d^n} & P_n & \xrightarrow{d^{n-1}} & P_{n-1} \end{array}.$$

If we can now show that $f_n \circ d'^n \subset \operatorname{im} d^n$ then we have the diagram

$$\begin{array}{ccccc} & & Q_{n+1} & & \\ & \swarrow & \downarrow{\scriptstyle f_n \circ d'^n} & & \\ P_{n+1} & \xrightarrow{d^n} & \operatorname{im} d^n & \longrightarrow & 0 \end{array},$$

and so the projectivity of Q_{n+1} would give us a map $f_{n+1} : Q_{n+1} \to P_{n+1}$. The inclusion can be shown as follows: from exactness of the bottom row we get $\operatorname{im} d^n = \ker d^{n-1}$ and so it suffices to show that $d^{n-1} \circ f_n \circ d'^n = 0$. But this follows immediately from the commutativity of the square formed by f_n, f_{n-1}, d'^{n-1} and d^{n-1} together with the fact that the top row is a complex and so $d'^{n-1} \circ d'^n = 0$. So this shows the existence of all the module homomorphisms $f_n : Q_n \to P_n$, we denote this chain map by $f_\bullet : Q_\bullet \to P_\bullet$. Now suppose that $g_\bullet : Q_\bullet \to P_\bullet$ is another chain map that satisfies the conditions (i.e. makes diagrams commute and $\rho \circ g_1 = f \circ \rho'$). Then we now construct the terms of a homotopy $s_n : Q_n \to P_{n+1}$ inductively. Define $\delta_\bullet := f_\bullet - g_\bullet$ then $\delta_1 = f_1 - g_1$. Note that $\rho \circ \delta_1 = f \circ \rho' - f \circ \rho' = 0$ and so $\operatorname{im}(\delta_1) \subset \ker(\rho)$. The map $P_2 \to \ker(\rho)$ induced by $d^1 : P_2 \to P_1$ is surjective by exactness and so we get a commutative diagram

$$\begin{array}{ccccc} & & Q_1 & & \\ & \swarrow & \downarrow{\scriptstyle \delta_1} & & \\ P_2 & \xrightarrow{d^1} & \ker(\rho) & \longrightarrow & 0 \end{array}$$

and projectivity of Q_1 gives us a map $s_1 : Q_1 \to P_2$ such that $d^1 \circ s_1 = \delta_1$. Now to construct $s_2 : Q_2 \to P_3$, note that

$$\begin{aligned} d^1 \circ \left(\delta_2 - s_1 \circ d'^1\right) &= d^1 \circ f_2 - d^1 \circ g_2 - d^1 \circ s_1 \circ d'^1 \\ &= f_1 \circ d'^1 - g_1 \circ d'^1 - s_1 \circ d'^1 \\ &= (f_1 - g_1 - d^1 \circ s_1) \circ d'^1 \\ &= (d^1 \circ s_1 - d^1 \circ s_1) \circ d'^1 \\ &= 0, \end{aligned}$$

and so $\operatorname{im}(\delta_2 - s_1 \circ d'^1) \subset \ker(d^1)$ and again the differential $d^2 : P_3 \to P_2$ induces a surjective map $P_3 \to \ker(d^1)$ and so we get the following diagram

$$\begin{array}{ccccc} & & Q_2 & & \\ & \swarrow & \downarrow{\scriptstyle \delta_2 - s_1 \circ d'^1} & & \\ P_3 & \xrightarrow{d^2} & \ker(d^1) & \longrightarrow & 0 \end{array}$$

and again projectivity of Q_2 gives us a map $s_2 : Q_2 \to P_3$ for which $d^2 \circ s_2 = \delta_2 - s_1 \circ d'^1$ which is exactly the homotopy between f_2 and g_2. One can continue this proces inductively to get a chain homotopy between $f_\bullet$ and $g_\bullet$. As all the maps in the construction are $C^\infty(M)$-module homomorphisms they are in particular $C^\infty(M)$-linear and so come from vector bundle morphisms for which the homotopy property is preserved. This shows that there is a morphism of complexes of vector bundles over $\mathcal{F}$ from (E', d', ρ') to (E, d, ρ) and that two such morphisms are homotopic.

□

An immediate consequence of this lemma is the following one.

Corollary 1.2.56 (Lemma 3.20 in [LGLS20]). *Any two geometric resolutions of a singular foliation $\mathcal{F}$ are homotopy equivalent.*

Examples of Geometric Resolutions

We now give some examples of geometric resolutions.

Example 1.2.57 (Example 3.29 in [LGLS20]). Let $\mathcal{F}$ be a regular foliation then $E_{-1} := T\mathcal{F}$, $E_{-i} = 0$ for all $i \geq 2$ together with $\rho : T\mathcal{F} \hookrightarrow TM$ is a geometric resolution.

♦

Example 1.2.58 (Example 3.31 in [LGLS20]). Consider the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ with its three generators denoted h, e, f that satisfy the following relations:

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

We let $\mathfrak{sl}_2(\mathbb{R})$ act on $\mathbb{R}^2$ in the following way:

$$\underline{h} = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, \quad \underline{e} = x\frac{\partial}{\partial y}, \quad \underline{f} = y\frac{\partial}{\partial x}.$$

We now let $\mathcal{F} = \langle \underline{h}, \underline{e}, \underline{f} \rangle_{C_c^\infty(M)}$. The resulting partitioning of $\mathbb{R}^2$ is displayed in figure 1.8, it consists of a 2-dimensional leaf $\mathbb{R}^2 \setminus \{0\}$ (the blue leaf) and a 0-dimensional leaf $\{0\}$ (the red leaf).

The vector fields $\underline{h}, \underline{e}, \underline{f}$ are not linearly independent over $C^\infty(\mathbb{R}^2)$ but it can be shown that every relation between them is a multiple of

$$xy\underline{h} + y^2\underline{e} - x^2\underline{f} = 0.$$

We will now describe a geometric resolution for this foliation. Define E_{-1} to be the trivial bundle of rank 3 generated by the sections $\widetilde{e}, \widetilde{f}, \widetilde{h}$. Define an anchor $\rho : E_{-1} \to TM$ by fixing the images of the generating sections

$$\rho(\widetilde{e}) = \underline{e}, \quad \rho(\widetilde{f}) = \underline{f}, \quad \rho(\widetilde{h}) = \underline{h}.$$

Note that $E_{-1} = \mathbb{R}^2 \times \mathbb{R}^3\,[1] \cong \mathbb{R}^2 \times \mathfrak{sl}_2(\mathbb{R})\,[1]$. Define E_{-2} to be the trivial bundle of rank 1 generated by a section denoted s and define a vector bundle morphism

$$d^{(2)}(s) = xy\widetilde{h} + y^2\widetilde{e} - x^2\widetilde{f}.$$

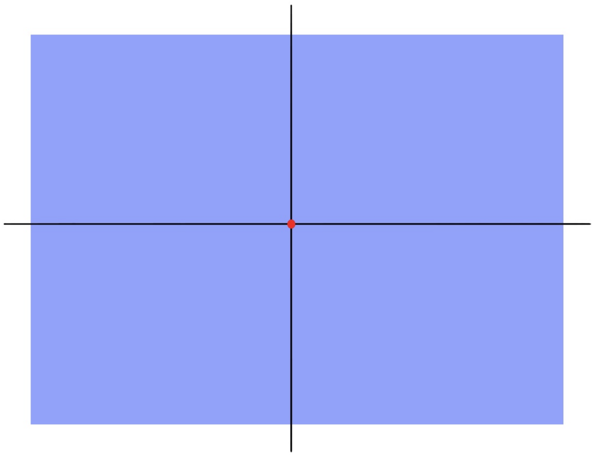


Figure 1.8: Foliation given by the action $\mathfrak{sl}_2(\mathbb{R})$ on $\mathbb{R}^2$ as given in example 1.2.58

Note that $E_{-2} = \mathbb{R}^2 \times \mathbb{R}\,[2]$. Finally for $i \geq 3$ define $E_{-i} = 0$ and $d^{(i)} = 0$. Then the triple (E, d, ρ) provides a geometric resolution of $\mathcal{F}$. Indeed we need to check exactness of the sequence

$$\mathbb{R}\,[2] \xrightarrow{d^{(2)}} \mathfrak{sl}_2(\mathbb{R})\,[1] \xrightarrow{\rho} \mathcal{F} \to 0.$$

Clearly we have that $\rho\left(\mathfrak{sl}_2(\mathbb{R})\,[1]\right) = \mathcal{F}$, furthermore we have that $\rho \circ d^{(2)} = 0$ by construction. Because also every relation between the vector fields $\underline{h}, \underline{e}, \underline{f}$ is multiple of $xh\underline{h} + y^2\underline{e} - x^2\underline{f} = 0$ we also immediately have $\operatorname{im} d^{(2)} \subset \ker \rho$ proving exactness.

♦

Example 1.2.59 (Example 3.33 in [LGLS20])**.** Let φ be a polynomial function on $V = \mathbb{C}^n$ then if $\iota_{d\varphi}$ denotes the contraction by $d\varphi$ we get a complex of trivial vector bundles over V

$$\cdots \xrightarrow{\iota_{d\varphi}} \wedge^3 TV \xrightarrow{\iota_{d\varphi}} \wedge^2 TV \xrightarrow{\iota_{d\varphi}} TV \xrightarrow{\iota_{d\varphi}} \underline{\mathbb{C}}.$$

Here $\underline{W}$ is the notation for the trivial bundle $V \times W$. Let $\mathfrak{X}^i = \Gamma(\wedge^i TV)$ be the sheaf of i-multivector fields on V then taking sections of the complex above gives

$$\cdots \xrightarrow{\iota_{d\varphi}} \mathfrak{X}^3 \xrightarrow{\iota_{d\varphi}} \mathfrak{X}^2 \xrightarrow{\iota_{d\varphi}} \mathfrak{X}^1 \xrightarrow{\iota_{d\varphi}} C^\infty(V). \tag{1.4}$$

This is also a complex since $\iota_{d\varphi} \circ \iota_{d\varphi} = 0$. This is because when $X_1 \wedge \cdots \wedge X_k \in \mathfrak{X}^k$ and $\alpha_1, \ldots, \alpha_k \in \Omega^1$ then

$$X_1 \wedge \cdots \wedge X_k(\alpha_1, \ldots, \alpha_k) = \det\left[\alpha_i(X_j)\right]_{i,j=1}^k.$$

From this it can easily be seen that contracting with $d\varphi$ twice yields two row-equivalent rows and so by the properties of the determinant this becomes zero. We call this the Koszul complex associated to φ. Note than when $x_1, \ldots, x_n$ are coordinates on V we have that $\mathfrak{X}^1$ is generated by the sections $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$. Hence by contracting all of these section by $d\varphi$ we get that the image of $\mathfrak{X}^1 \to C^\infty(V)$ is generated by the functions $\frac{\partial\varphi}{\partial x_1}, \ldots, \frac{\partial\varphi_n}{\partial x_n}$. It can be shown that when φ is a weight-homogeneous polynomial that admits an isolated singularity at the origin, the Koszul complex of φ actually is an exact sequence. Now consider the following complex of vector bundles

$$\cdots \xrightarrow{\iota_{d\varphi}\otimes id} \wedge^3 TV \otimes \underline{V} \xrightarrow{\iota_{d\varphi}\otimes id} \wedge^2 TV \otimes \underline{V} \xrightarrow{\iota_{d\varphi}\otimes id} TV \otimes \underline{V} \xrightarrow{\iota_{d\varphi}\otimes id} \underline{\mathbb{C}} \otimes \underline{V} \cong \underline{V}. \tag{1.5}$$

Then on the level of sections we have $\Gamma\left(\wedge^k TV \otimes \underline{V}\right)$ for which $\Gamma(\wedge^k TV \otimes \underline{V}) \cong \Gamma(\wedge^k TV) \otimes_{C^\infty} \Gamma(\underline{V})$. Now $\Gamma(\underline{V})$ is a free C^∞-module because $\Gamma(\underline{V}) \cong \oplus_{i=1}^n C^\infty$ hence $\mathfrak{X}^k \otimes_{C^\infty} \Gamma(\underline{V}) \cong \mathfrak{X}^k \otimes_{C^\infty} (\oplus_{i=1}^n C^\infty) \cong \oplus_{i=1}^n \mathfrak{X}^k$. From this it also follows that on the level of sections the complex (1.5) is exact. Now consider the sequence

$$\cdots \xrightarrow{d^3} \Gamma\left(\wedge^2 TV \otimes \underline{V}\right) \xrightarrow{d^2} \Gamma\left(TV \otimes \underline{V}\right) \xrightarrow{d^1} \mathcal{F} \to 0,$$

then this sequence is exact at $\mathcal{F}$ if and only if $\operatorname{im} d^1 = \mathcal{F}$. So using that $\Gamma(TV \otimes \underline{V}) \cong \Gamma(TV) \otimes \Gamma(TV)$ and we assume $\Gamma(TV)$ to be generated by the section $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ then we find for $\alpha_i, \beta_j \in C^\infty(V)$,

$$\begin{aligned} d^1\left(\left(\sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}\right) \otimes \left(\sum_{j=1}^n \beta_j \frac{\partial}{\partial x_j}\right)\right) &= (\iota_{d\varphi} \otimes \mathbb{1})\left(\left(\sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}\right) \otimes \left(\sum_{j=1}^n \beta_j \frac{\partial}{\partial x_j}\right)\right) \\ &= \sum_{i,j} \alpha_i \beta_j \iota_{d\varphi}\left(\frac{\partial}{\partial x_i}\right) \otimes \mathbb{1}\left(\frac{\partial}{\partial x_j}\right) \\ &= \sum_{i,j} \alpha_i \beta_j \frac{\partial \varphi}{\partial x_i} \otimes \frac{\partial}{\partial x_j} \\ &= \sum_{i,j} \alpha_i \beta_j \frac{\partial \varphi}{\partial x_i} \frac{\partial}{\partial x_j}. \end{aligned}$$

Where we used that $\frac{\partial \varphi}{\partial x_i} \in C^\infty(V)$ for all $i = 1, \dots, n$. Hence we see that when we let $\mathcal{F}$ be defined as

$$\mathcal{F} = \left\{\frac{\partial \varphi}{\partial x_i} \frac{\partial}{\partial x_j} \mid i, j = 1, \dots, n\right\},$$

then we obtain a geometric resolution for this foliation.

♦

Example 1.2.60 (Example 3.36 in [LGLS20])**.** The following example will be an important example throughout the rest of the thesis. Let φ be a function on $V = \mathbb{C}^n$ such that $\left(\frac{\partial \varphi}{\partial x_1}, \dots, \frac{\partial \varphi}{\partial x_n}\right)$ is a regular sequence. By a theorem of Koszul (see theorem 16.5(i) [Mat87]) this implies that the sequence (1.4) is an exact sequence. Consider the singular foliation consisting of all vector fields X for which $X[\varphi] = 0$. Since (1.4) is exact it has no cohomology in degree -1 which exactly means that

$$\operatorname{im}\left(\iota_{d\varphi} : \mathfrak{X}^2 \to \mathfrak{X}\right) = \ker\left(\iota_{d\varphi} : \mathfrak{X} \to C^\infty(V)\right).$$

Since $X \in \ker\left(\iota_{d\varphi} : \mathfrak{X} \to C^\infty(V)\right)$ exactly means that $X[\varphi] = 0$ this means that there exists a bivector field $\pi \in \mathfrak{X}^2$ of the form

$$\pi = \frac{1}{2} \sum_{i,j} \pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j},$$

such that $\iota_{d\varphi}(\pi) = X$ and so

$$\iota_{d\varphi}(\pi) = \frac{1}{2} \sum_{i,j} \pi_{ij} \left(\frac{\partial \varphi}{\partial x_i} \frac{\partial}{\partial x_j} - \frac{\partial \varphi}{\partial x_j} \frac{\partial}{\partial x_i}\right).$$

From this we conclude that the foliation $\mathcal{F}_\varphi$ is generated as

$$\mathcal{F}_\varphi = \left\{ \frac{\partial \varphi}{\partial x_i}\frac{\partial}{\partial x_j} - \frac{\partial \varphi}{\partial x_j}\frac{\partial}{\partial x_i} \mid 1 \leq i < j \leq n \right\}.$$

From the Koszul complex we also immediately get a geometric resolution by defining $E_{-i} := \wedge^{i+1} TV$ and $d := \iota_{d\varphi}$.

♦

As we already discussed above not *all* smooth foliations admit geometric resolutions. Therefore it might be interesting to give such an example. Before doing so we cite the following result from [LGLS20] (for which the authors credit Marco Zambon).

Proposition 1.2.61 (Proposition 2.5 in [LGLS20])**.** *If a singular foliation $\mathcal{F}$ on a connected manifold M admits a geometric resolution of finite length in a neighborhood of all points in M, then all its regular leaves have the same dimension r. Moreover, for every geometric resolution of finite length (E, d, ρ) of $\mathcal{F}$ over an open subset of M:*

$$r = \sum_{i \geq 1} (-1)^{i-1} \operatorname{rk}(E_{-i}).$$

The following example from [LGLS20] is accredited to Jean-Louis Tu.

Example 1.2.62 (Example 3.38 in [LGLS20])**.** Let χ be a smooth real-valued function on $M := \mathbb{R}$ that vanishes identically on $\mathbb{R}^-$ and is strictly positive on $\mathbb{R}^+_0$. Consider the singular foliation $\mathcal{F}$ generated by the vector field v defined as

$$v := \chi(t)\frac{d}{dt}. \tag{1.6}$$

Now all points of $\mathbb{R}^-_0$ and $\mathbb{R}^+_0$ are regular points. Therefore, there is an uncountable family of 0-dimensional leaves and a 1-dimensional leaf. If a finite geometric resolution where to exist, this clearly contradicts proposition 1.2.61. So, we conclude there does not exists a finite geometric resolution.

One can even show more: in the neighborhood of $t = 0$ there does not even exist an infinite geometric resolution for $\mathcal{F}$. For the proof of this we refer to the aforementioned example in [LGLS20]. In conclusion we have that this particular $\mathcal{F}$ does not admit any smooth geometric resolutions.

♦

Chapter 2

Lie ∞-algebras & Lie ∞-algebroids

In this chapter we will give the definitions of Lie ∞-algebras (also called homotopy Lie algebras, sh-Lie algebras or L_∞-algebras)[1] and Lie ∞-algebroids. The concept of L_∞-algebras as we present it in this text was first given by Stasheff and Lada (altough they credit other authors, for more information see [nLa19]) in [Sta92] and [LS93]. Their work was inspired by work of Zwiebach in [Zwi93] which concerned closed string theory in theoretical physics. The L_∞-algebra structure also comes up in other parts of theoretical physics: supergravity, string field theory, perturbative quantum field theory,... which also means a lot of examples can be found in these parts of physics. The second important structure introduced here are Lie ∞-algebroids will be the most important object in this thesis. We will associate to singular foliations which admit a (finite) geometric resolution a so-called *universal Lie ∞-algebroid* from which we will be able to deduce some geometric properties of the foliation. Altough we will use it in a 'pure mathematics' setting these objects also comes up in several domains of theoretical physics.

2.1 Lie ∞-algebras

We will start by giving the definition of Lie ∞-algebras as we will use it later. First note that when $E = \oplus_{i\geq 1} E_{-i}$ is a graded vector space we call the elements of E_{-i} homogeneous of degree $-i$. For a real vector space V we denote by $S^n(V)$ the n-th symmetric product of V that is defined as

$$S^n(V) = \frac{\otimes^n V}{\langle x_1 \otimes \cdots \otimes x_n - x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)} \mid \sigma \in \mathcal{S}_n \rangle}$$

From this we can also construct the symmetric algebra of V as $S(V) = \oplus_{n\geq 0} S^n(V)$ where we set $S^0(V) = \mathbb{R}$ or any field over which V is defined[2]. By $\mathcal{S}(i, n-i)$ we denote the $(i, n-i)$-unshuffles, these are the $\sigma \in \mathcal{S}_n$ for which $\sigma(1) < \cdots < \sigma(i)$ and $\sigma(i+1) < \cdots < \sigma(n)$. Using this terminology, we can make the following definition.

Definition 2.1.1 (Definition 3.39 in [LGLS20])**.** A **Lie ∞-algebra** is a graded vector space $E = \oplus_{i\geq 1} E_{-i}$ together with a family of graded-symmetric n-multilinear maps $(\nu_n := \{\dots\}_n)_{n\geq 1}$ of degree $+1$ that we call the **n-ary brackets**, which satisfy a set of

[1] We will mainly use the last notational convention.

[2] The same can be done for modules M over a ring R and then we set the 0-th symmetric power to be the ring over which M is defined as a (left/right) module.

compatibility conditions that are called the general Jacobi identities. This means that for all $n \geq 2$ and for every n-tuple of homogeneous elements $x_1, \ldots, x_n \in E$ the following equation is satisfied

$$\sum_{i=1}^{n} \sum_{\sigma \in \mathcal{S}(i,n-i)} \epsilon(\sigma) \{\{x_{\sigma(1)}, \ldots, x_{\sigma(i)}\}_i, x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}\}_{n-i+1} = 0. \tag{2.1}$$

Here $\epsilon(\sigma)$ is the Koszul sign defined by

$$x_{\sigma(1)} \odot \cdots \odot x_{\sigma(n)} = \epsilon(\sigma) x_1 \odot \cdots \odot x_n.$$

We call a Lie ∞-algebra structure a Lie n-algebra when $E_{-i} = 0$ for all $i \geq n+1$.

As is noted in [Lav16] an important remark needs to be made here: in, for example, [LS93] and [LM95], a different convention is used to define an L_∞-algebra. In these aforementioned works an L_∞ structure on a graded vector space E is defined to be a collection of skew symmetric linear maps $(l_n : \otimes^n E \to E)_{n\geq 1}$ of degree $2-n$, i.e. a collection of maps $(\mu_n : \wedge^n E \to E)_{n\geq 1}$[3] of degree $2-n$ such that they satisfy some higher Jacobi identities. In this framework, what we call an L_∞-algebra is actually an $L_\infty[1]$-algebra (a shift in degrees of elements). This convention is somewhat easier to construct examples out of (the examples below indeed use this convention) but will not be the preferable convention for the further theory. Hence, we will only use it to display two examples and from there on we use the definition 2.1.1. One can show through the so-called décalage isomorphism that these two different notions of L_∞ structures are indeed the same, symbolically this isomorphism states that

$$S^n(E[1]) \cong \left(\bigwedge^n E\right)[n], \tag{2.2}$$

between spaces of linear maps this translates into the following isomorphism (where the superscript denotes the degree of the considered maps)

$$\mathrm{Hom}^i\left(\bigwedge^n E, E\right) \cong \mathrm{Hom}^{i+n-1}\left(S^n(E[1]), E[1]\right).$$

From this isomorphism it can be seen immediately that degree $2-n$ maps $\mu_n : \wedge^n E[-1] \to E[-1]$ correspond uniquely to degree $+1$ maps $\nu_n : S^n(E) \to E$ (which directly translates to the brackets we are considering). In particular given a collection of n homogeneous elements $x_1, \ldots, x_n$ and their representatives in $y_1, \ldots, y_n \in E[1]$ (remember that $|y_i| = |x_i| - 1$) then the isomorphism between a graded skew-symmetric bracket $[\cdots]_n$ on E and a graded symmetric bracket on $E[1]$ that we denote by $\{\cdots\}_n$ is given by the following equation

$$[x_1, \ldots, x_n]_n = (-1)^{n(2-n)+\sum_{i=1}^{n}(n-i)(|y_i|+1)}\{y_1, \ldots, y_n\}_n. \tag{2.3}$$

Example 2.1.2. When using the convention from, for example [LS93] and [LM95], an L_∞-algebra with only l_1 and l_2 nontrivial is a differential graded Lie algebra or DGLA for short. The degree 1 map l_1 corresponds to the differential and the degree 0 map l_2 corresponds to the bracket.

♦

[3] By skew symmetry of the l_n they factor through such maps μ_n.

Example 2.1.3. In particular when $E = E_{-1}$ we recover an ordinary Lie algebra (again working in the same setting as example 2.1.2).

♦

Of course, one can also talk about morphisms between L_∞-algebras. They are defined in the following way.

Definition 2.1.4 (Definition 6 in [KS06])**.** Let (E, μ_k) and (E', μ'_k) be two L_∞-algebras in the sense of definition 2.1.1. An L_∞-algebra morphism is a collection of maps

$$f_k : S^k(E) \to E'$$

for which

$$\begin{aligned} &\sum_{\sigma \in \mathcal{S}(k,l=n-k)} \epsilon(\sigma) f_{1+l}\left(\mu_k \otimes \mathbb{1}^{\otimes l}\right)\left(x_{\sigma(I)}\right) \\ &\qquad = \sum_{\substack{\sigma \in \mathcal{S}(k_1,\ldots,k_j) \\ k_1+\cdots+k_j=n \\ j=1,\ldots,n}} \frac{\epsilon(\sigma)}{j!} \mu'_j \left(f_{k_1} \otimes \cdots \otimes f_{k_j}\right)(x_{\sigma(I)}). \end{aligned} \tag{2.4}$$

Here we denote by $\epsilon(\sigma)$ the Koszul sign as above and for $I = (i_1, \ldots, i_n)$, $x_{\sigma(I)}$ denotes $x_{\sigma(i_1)} \odot \cdots \odot x_{\sigma(i_n)}$ for homogenous elements $x_{i_1}, \ldots, x_{i_n}$.

Remark 2.1.5. In the notation in definition 2.1.4 we denote the degree +1 graded symmetric k-ary bracket from definition 2.1.1 by μ_k, not to be confused with notation we used to illustrate the difference between the graded skew-symmetric brackets and the graded symmetric brackets from the paragraph after definition 2.1.1.

2.2 Lie ∞-algebroids

Definition 2.2.1 (Definition 3.40 in [LGLS20])**.** Let M be a smooth manifold and $E = (E_{-i})_{1 \le i < \infty}$ a sequence of vector bundles over M. A **Lie ∞-algebroid** structure on E consists of a Lie ∞-algebra structure on $\Gamma(E)$ and a vector bundle morphism[4] $\rho : E_{-1} \to TM$, called the anchor, such that the brackets $\{\ldots\}_n$ are $C^\infty(M)$-linear in each of their n arguments except if $n = 2$ and at least one of the two entries has degree -1. Then the 2-ary bracket satisfies the following Leibniz identity

$$\{x, fy\}_2 = f\{x, y\}_2 + \rho(x)[f]\, y,$$

for all $x \in \Gamma(E_{-1})$, $y \in \Gamma(E)$ and $f \in C^\infty(M)$.

Remark 2.2.2. Like for Lie algebroids this Leibniz identity implies that ρ is a Lie algebra homomorphism. Furthermore, it follows that $\rho \circ \{\cdot\}_1|_{E_{-2}} = 0$. To make the notation a bit more clear we write $d := \{\cdot\}_1$. Now indeed if we let $x_1, x_2 \in \Gamma(E)$ be degree homogeneous elements then from the higher Jacobi identity (2.1) for $n = 2$ it follows that

$$d(\{x_1, x_2\}_2) = \{dx_1, x_2\}_2 + (-1)^{|x_1|}\{x_1, dx_2\}_2. \tag{2.5}$$

[4]Over the identity on M.

Now let $e \in \Gamma(E_{-2})$, $x \in \Gamma(E)$ and $f \in C^\infty(M)$. Then we have, by the Leibniz identity for the 2-ary bracket, that

$$\{de, fx\}_2 = f\{de, x\}_2 + \rho(de)\,[f]\,x. \tag{2.6}$$

On the other hand, by $C^\infty(M)$-linearity, we have $fd\left(\{e,x\}_2\right) = d\left(\{e,fx\}\right)$. Furthermore, by equation (2.5) and the Leibniz identity (2.6) one has

$$\begin{aligned} fd\left(\{e,x\}_2\right) &= f\{de,x\}_2 + f\{e,dx\} & (2.7)\\ &= f\{de,x\}_2 + \rho(de)\,[f]\,f - \rho(de)\,[f]\,x + f\{e,dx\}_2 & (2.8)\\ &= \{de,fx\}_2 - \rho(de)\,[f]\,x + f\{e,dx\}_2. & (2.9) \end{aligned}$$

Now again by $C^\infty(M)$-linearity and equation (2.5) we have that

$$d\left(\{e,fx\}_2\right) = \{de,fx\}_2 + f\{e,dx\}_2.$$

Using this result combined with equation (2.9) we recover that

$$\cancel{fd\left(\{e,x\}_2\right)} = \cancel{d\left(\{e,fx\}_2\right)} - \cancel{f\{e,dx\}_2} - \rho(de)\,[f]\,x + \cancel{f\{e,dx\}_2}.$$

So we conclude that for arbitrary $e \in \Gamma(E_{-2})$, $x \in \Gamma(E)$ and $f \in C^\infty(M)$ one has that

$$\rho(de)\,[f]\,x = 0.$$

Hence, we have shown that $\rho \circ \{\cdot\}_1|_{E_{-2}} = 0$.

Note that from the generalized Jacobi identity (2.1) it follows that for $n = 1$, for all $x \in \Gamma(E)$: $\{\{x\}_1\}_1 = 0$ i.e. the unary bracket squares to zero. Since $\{\cdot\}_1 : \Gamma(E) \to \Gamma(E)$ is a multilinear map of degree $+1$ it consists of a family of $C^\infty(M)$-linear maps $d^{(i)} : \Gamma(E_{-i}) \to \Gamma(E_{-i+1})$ which, by $C^\infty(M)$-linearity, come from vector bundle morphisms $d^{(i)} : E_{-i} \to E_{-i+1}$ (see for instance lemma 10.29 in [Lee12]). As $\{\cdot\}_1^2 = 0$ it also follows that $d^{(i)} \circ d^{(i-1)} = 0$ so we get a complex of vector bundles

$$\cdots \xrightarrow{d^{(4)}} E_{-3} \xrightarrow{d^{(3)}} E_{-2} \xrightarrow{d^{(2)}} E_{-1} \xrightarrow{\rho} TM. \tag{2.10}$$

We call (2.10) the **linear part** of the Lie ∞-algebra.

Example 2.2.3. When $M = \{*\}$ we recover the definition of an L_∞-algebra.

♦

Example 2.2.4. Later in this chapter we will see that a Lie algebroid is a special instance of a Lie ∞-algebroid. We include this example here without further details as to continue the direction taken in examples 2.1.2 and 2.1.3 where we reviewed special 'limiting' cases of L_∞-algebras.

♦

2.2.1 NQ-manifolds

In this section we will introduce N-manifolds and Q-manifolds, both of these are studied in graded geometry which is the geometrical framework developed in the 1970s to study supersymmetry in physics. The theory of Lie ∞-algebroids as we explained above is quite complicated to work with and in this part of the text we will explain the duality between Lie ∞-algebroids and NQ-manifolds on which certain notions (e.g. morphisms, homotopy, ...) are more easily accessible. For more background on graded geometry, we refer to [Fai17].

Definition 2.2.5 ([LGLS20])**.** A sequence of finite rank vector bundles $E = (E_{-i})_{i\geq 1}$ over M is called an N**-manifold** and we denote it $E \to M$.

Remark 2.2.6. Strictly speaking this is not the definition of an N-manifold but by Batchelor's theorem we may, after a noncanonical choice of a so-called splitting, assume that a general N-manifold (not defined here) is of the form presented in definition 2.2.5. For a proof of this we refer to theorem 1 in [BP13].

We will now define the functions on the N-manifold: the sheaf of graded commutative C^∞-algebras of smooth sections of the graded symmetric algebra $S(E^*)$ will be denoted as $\mathcal{E}$ and these are the functions on the N-manifold $E \to M$. Some remarks on the degrees of elements: an element $x \in \Gamma(E_{-i})$ is said to be of degree $-i$ while an element of $\Gamma(E^*_{-i})$ are said to be of degree $+i$ and where $E^* = \oplus_{i\geq 1} E^*_{-i}$. Using this definition elements $f \in \mathcal{E}$ get 'two gradings'. Namely the one from inside the graded vector bundle E^* and the one from the graded symmetric algebra $S(E^*)$. When $\odot$ denotes the graded-symmetric tensor product (i.e. the product on the graded symmetric algebra) this boils down to sections of

$$\bigoplus_{i_1+\cdots+i_k=n} E^*_{-i_1} \odot \cdots \odot E^*_{-i_k}$$

being of **degree** n and of **arity** k. We denote the collection of these elements as $\mathcal{E}^{(k)}_n$. For example, the degree 0 functions are just the smooth functions on the base manifold M and the degree 1 functions are sections of E^*_{-1}, degree 2 functions are sections of $E^*_{-2} \oplus S^2(E^*_{-1})$.

Definition 2.2.7 ([LGLS20])**.** Graded derivations of $\mathcal{E}$ are called **vector fields** on the N-manifold $E \to M$. A vector field Q is said to be of **arity** k if for all $f \in \mathcal{E}$ of arity l the arity of $Q[f]$ is $l+k$ (degree of Q is defined completely similarly). A vector field Q of odd degree satisfying $Q^2 := \frac{1}{2}[Q,Q] = 0$ is called **homological** (here $[\cdot,\cdot]$ denotes the commutator).

Remark 2.2.8. Graded manifolds equipped with a homological vector field are called Q-manifolds, sometimes also denoted differential graded manifolds (or dg-manifolds) in the literature.

Example 2.2.9 (Based on Example 4.1 in [LGLS20])**.** Let M be a smooth n-dimensional manifold with tangent bundle TM. We can lift the degree of fiber elements to obtain the suspended tangent bundle[5] $T[1]M$ and obtain an N-manifold[6] $E_{-1} := T[1]M \to$

[5]By this we mean the following: elements in TM are considered to be of degree 0, in the suspended bundle $T[1]M$ we view them as having degree -1.

[6]We have $E_{-i} = 0$ for $i \geq 2$.

M with functions defined as sections of the vector bundle $S(T^*[1]M)$ which, by the décalage isomorphism, is isomorphic to $\bigwedge T^*M$. We of course recognize sections of the bundle $\bigwedge T^*M$ as being the sheaf of differential forms Ω on M. This means that the pair $\Theta_{T[1]M} = (T[1]M, \Omega)$ forms an N-manifold. Just as on ordinary smooth manifolds we can choose coordinates on an N-manifold. The big difference between them is that now certain subsets of the coordinates may have nonzero degrees and are noncommutative with respect to the other coordinates. We refer to [Fai17] for more information on this. Let $(x_i)_{i=1,\dots,n}$ denote coordinates on the base M (these are the degree 0 coordinates) then we get a coordinate system $(x_i, dx_i)_{i=1,\dots,n}$ on $\Theta_{T[1]M}$ where the dx_i are the degree 1 coordinates. Now we already know a nice example of a derivation on the differential forms: the de Rham differential is a derivation of $\Omega(V)$ for $V \subset M$ open. Locally it is given as $d = \sum dx_i \frac{\partial}{\partial x_i}$ and so it is an example of a vector field on $\Theta_{T[1]M}$. For the particular case of the de Rham differential it also holds, by construction, that $\deg(d) = 1$ and $d^2 = 0$ i.e. $[d, d] = 0$ and so it also is a homological vector field. We conclude that $\left(\Theta_{T[1]M}, d\right)$ is a Q-manifold.

♦

Remark 2.2.10. If Q is a vector field on an N-manifold $E \to M$ and $Q^{(k)}$ denotes a vector field of arity k then Q can be written as

$$Q = \sum_{k \geq -1} Q^{(k)}.$$

Definition 2.2.11 (Definition 3.43 in [LGLS20])**.** An NQ**-manifold** is a pair (E, Q) where $E \to M$ is an N-manifold over some base M and where Q is homological vector field of degree $+1$.

Example 2.2.12. Of course the pair $\left(\Theta_{T[1]M}, d\right)$ from example 2.2.9 is an NQ-manifold.

♦

The following examples can be found in [Lav16] as examples 6 and 7 in chapter 1.

Example 2.2.13 (Example 6 in [Lav16])**.** This example was of great importance in the historical development of the notion of Lie ∞-algebroids as being 'higher Lie algebroids' by Voronov [Vor10]. Let $A \to M$ be a Lie algebroid with bracket $[\cdot, \cdot]$. This bracket is in particular a skew-symmetric bracket on $\Gamma(A)$. By equation (2.3) we can translate the skew-symmetry of the bracket to a symmetric bracket on the vector space $\Gamma(A[1])$. Note again that elements of $\Gamma(A[1])$ have degree -1 while they have degree 0 when considered in $\Gamma(A)$ thus a direct application of (2.3) with $n = 2$ yields that for all $x, y \in \Gamma(A[1])$ with representatives $\tilde{x}, \tilde{y} \in \Gamma(A)$ one has

$$\{x, y\} = [\tilde{x}, \tilde{y}].$$

As discussed the space of functions on $A[1]$ is isomorphic to $\Gamma(S(A[1]^*))$, so it is sufficient to define a vector field Q on $C^\infty(M)$ and $\Gamma(A[1]^*)$ and then extend by derivation. Note that Q has degree $+1$ and so it maps $C^\infty(M)$ to $\Gamma(A[1]^*)$ and in turn maps this module to $\Gamma(S^2(A[1]^*))$. Now define the following relations for all $f \in C^\infty(M)$, $\alpha \in \Gamma(A[1]^*)$ and for all $x, y \in \Gamma(A[1])$:

$$\langle Q[f], x\rangle = \rho(x)[f]$$
$$\langle Q[\alpha], x \odot y\rangle = \rho(x)\langle \alpha, y\rangle - \rho(y)\langle \alpha, y\rangle - \langle \alpha, \{x, y\}\rangle.$$

One can now extend Q to the whole of $\Gamma(S(A\,[1]^*))$ by the derivation property. Some small computations reveal something very interesting: there is a one-to-one correspondence between Lie algebroids and degree $+1$ Q-manifold structures on $A\,[1]$. Indeed when $f \in C^\infty(M)$ we have that $Q\,[f] \in \Gamma(A\,[1]^*)$ hence

$$\begin{aligned}\langle Q\,[Q\,[f]]\,, x \odot y\rangle &= \rho(x)\langle Q\,[f]\,, y\rangle - \rho(y)\langle Q\,[f]\,, x\rangle - \langle Q\,[f]\,, \{x, y\}\rangle \\ &= \rho(x)\rho(y)\,[f] - \rho(y)\rho(x)\,[f] - \rho(\{x, y\})\,[f] \\ &= ([\rho(x), \rho(y)] - \rho(\{x, y\}))\,[f]\,,\end{aligned}$$

and using that Q is a degree $+1$ derivation one can show that also

$$\langle Q^2\,[\alpha]\,, x \odot y \odot z\rangle = \Big\langle \alpha, \Big\{\{x, y\}, z\Big\} + \Big\{\{y, z\}, x\Big\} + \Big\{\{z, x\}, y\Big\}\Big\rangle.$$

Hence requiring that $Q^2 = 0$ (i.e. requiring $A\,[1]$ to be a Q-manifold) exactly means that ρ needs to be a Lie algebra homomorphism and that the bracket $\{\cdot, \cdot\}$ on $\Gamma(A\,[1])$ must satisfy the Jacobi identity. All of this corresponds precisely to $A\,[1]$ being a Lie algebroid. This is also a first hint towards theorem 2.2.15 that we will state below.

♦

Example 2.2.14. Combining examples 1.2.35 and 2.2.13 we see that for a Poisson manifold (M, π) one gets a Lie algebroid structure on T^*M which implies that we can associate, to every Poisson manifold, a Q-manifold structure on $T^*M\,[1]$.

♦

For a given NQ-manifold (E, Q) with sheaf of functions $\mathcal{E}$ we know that there is an isomorphism of sheaves $\mathcal{E}_0 \cong C^\infty$. While as we remarked already $\mathcal{E}_1 \cong \Gamma(E^*_{-1})$ and since Q is a degree $+1$ derivation of $\mathcal{E}$ we have a map $Q : C^\infty(M) \to \Gamma(E^*_{-1})$ which is a derivation. If $\langle\cdot, \cdot\rangle$ denotes the duality pairing then the map

$$C^\infty(M) \to C^\infty(M) : f \mapsto \langle Qf, x\rangle, \quad \forall x \in \Gamma(E^*_{-1}),$$

is a derivation of $C^\infty(M)$. So to every $x \in \Gamma(E^*_{-1})$ we get an associated vector field in $\Gamma(TM)$ (this is because vector fields on M can be characterized as being derivations of the algebra of smooth functions on M). I.e. we have a map

$$\tau : \Gamma(E_{-1}) \to \Gamma(TM).$$

Now since this τ is $C^\infty(M)$-linear and E_{-1} and TM are both vector bundles over M we have, by lemma 10.29 in [Lee12], that this τ comes from a vector bundle morphism

$$\rho : E_{-1} \to TM.$$

Note that this ρ satisfies that $\langle Qf, x\rangle = \rho(x)f$, for all $x \in \Gamma(E_{-1})$ and for all $f \in C^\infty(M)$. One can show that for a degree $+1$ vector field we have that

$$Q = \sum_{k\geq 0} Q^{(k)}.$$

The following theorem (originally discussed in[Vor10], for a proof in the notation from this thesis see [Lav16] theorem 1.1.11) will be of great importance as it describes the duality between Lie ∞-algebroids and NQ-manifolds. This duality will help when describing morphisms of Lie ∞-algebroids as we can then consider them as being NQ-manifolds and describe the morphisms in the category of NQ-manifolds.

Theorem 2.2.15 (Theorem 3.44 in [LGLS20])**.** *Let $E = (E_{-i})_{i\geq 1}$ be a sequence of vector bundles over a manifold M. There is a one-to-one correspondence between NQ-manifolds and Lie ∞-algebroid structures on E. The anchor ρ of both is defined like we described above and furthermore we have that:*

1. *The differential d of the linear part of the Lie ∞-algebroid structure is obtained by dualizing the arity 0 component $Q^{(0)}$ of Q, i.e. for all $\alpha \in \Gamma(E^*)$ and $x \in \Gamma(E)$*

$$\langle Q^{(0)}\alpha, x\rangle = (-1)^{\deg \alpha}\langle \alpha, d(x)\rangle. \tag{2.11}$$

2. *The 2-ary bracket $\{\cdot,\cdot\}_2$ and the arity one component $Q^{(1)}$ are related by*

$$\langle Q^{(1)}\alpha, x \odot y\rangle = \rho(x)\langle \alpha, y\rangle - \rho(y)\langle \alpha, x\rangle - \langle \alpha, \{x, y\}_2\rangle,$$

   for all homogeneous elements $x, y \in \Gamma(E)$ and $\alpha \in \Gamma(E^*)$.

3. *For every $n \geq 3$ the n-ary brackets $\{\cdots\}_n : \Gamma(S^n(E)) \to \Gamma(E)$ and the component of arity $n-1$, $Q^{(n-1)} : \Gamma(E^*) \to \Gamma(S^n(E^*))$ are dual to each other.*

Theorem 2.2.15 can be combined with example 2.2.13 from which we also immediately see that a Lie algebroid can be seen as an example of a Lie ∞-algebroid, a fact that we will use later when discussing singular foliations using Lie ∞-algebroids. It also deserves to be noted that when we apply theorem 2.2.15 with $M = \{*\}$ we recover a similar duality theorem for L_∞-algebras (using example 2.2.3).

From here on we will use the notation (E, Q) to denote a Lie ∞-algebroid with Q the homological vector field that gives the brackets as described in theorem 2.2.15.

2.2.2 Lie ∞-algebroid Morphisms and Homotopies

Morphisms

Just as for L_∞-algebras one can define morphisms between Lie ∞-algebroids and homotopies between those. For this the point of view we developed in section 2.2.1 and more specifically theorem 2.2.15 will come in very useful as morphisms are more easily explained in the category of NQ-manifolds. Dualizing then yields the appropriate definitions for Lie ∞-algebroids, we develop these definitions following section 3.4.2 in [LGLS20]. To define morphisms we take inspiration from the smooth manifold case: giving a smooth map $f : M \to N$ between smooth manifolds is equivalent to giving an algebra morphism $f^* : C^\infty(N) \to C^\infty(M)$ between the function spaces on N and M respectively. We view a Lie ∞-algebroid as a pair (E, Q) (as established by theorem 2.2.15) and denote its sheaf of functions by $\mathcal{E}$. Then a Lie ∞-algebroids morphism can be defined in the following way.

Definition 2.2.16 (Definition 3.45 in [LGLS20])**.** A **Lie ∞-algebroid morphism** from a Lie ∞-algebroid (E', Q') to a Lie ∞-algebroid (E, Q) with sheaves of functions $\mathcal{E}'$ and $\mathcal{E}$ respectively, is a graded commutative algebra morphism $\Phi : \mathcal{E} \to \mathcal{E}'$ that satisfies

$$\Phi \circ Q = Q' \circ \Phi. \tag{2.12}$$

We say that Φ intertwines the homological vector fields Q and Q'.

Remark 2.2.17. Note that Φ in particular induces some other morphisms:

- *base morphism:* since Φ is a degree 0 morphism of graded commutative algebras we in particular have that Φ maps $\mathcal{E}_0$ to $\mathcal{E}'_0$. Which exactly means giving an algebra morphism $C^\infty(M) \to C^\infty(M')$, i.e. it induces a smooth map $\varphi : M' \to M$,
- *linear part:* by definition of the sheaves of functions on an $N(Q)$-manifold we have that $\mathcal{E} = \Gamma(S(E^*))$ and $\mathcal{E}' = \Gamma(S(E'^*))$ and so $\Phi : \Gamma(S(E^*)) \to \Gamma(S(E'^*))$. Because Φ is an algebra morphism we may restrict to looking at the restricted map $\Phi : \Gamma(E^*) \to \Gamma(S(E'^*))$. Now we use the following fact from the theory of vector bundles: given vector bundles $A \to M$ and $A' \to M'$, giving a vector bundle morphism $\Xi : A \to A'$ is equivalent to giving the following data: (a) an algebra morphism $\xi^* : C^\infty(M') \to C^\infty(M)$ and (b) a linear map $\Psi : \Gamma(A'^*) \to \Gamma(A^*)$ such that $\Psi(fe) = \xi^*(f)\Psi(e)$ for all $e \in \Gamma(A'^*)$ and $f \in C^\infty(M')$. Now applying this to the restricted morphism $\Phi : \Gamma(E^*) \to \Gamma(S(E'^*))$ we get a vector bundle morphism $\varphi : S(E') \to E$ and so, in particular, a vector bundle morphism $\varphi_0 : E' \to E$ which consist of a sequence of vector bundle morphisms $\varphi_{0,\bullet} : E'_\bullet \to E_\bullet$.

Remark 2.2.18. We can restrict equation (2.12) to terms of arity 0 and apply theorem 2.2.15 to see that $Q^{(0)}$ and $Q'^{(0)}$ correspond to d and d' respectively. Then equation (2.12) states that φ_0 (from remark 2.2.17) is a chain map between the linear parts of the Lie ∞-algebroids (E', Q') and (E, Q) respectively,

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d'} & E'_{-3} & \xrightarrow{d'} & E'_{-2} & \xrightarrow{d'} & E'_{-1} & \xrightarrow{\rho'} & TM' \\
 & & \downarrow{\scriptstyle \varphi_{0,-3}} & & \downarrow{\scriptstyle \varphi_{0,-2}} & & \downarrow{\scriptstyle \varphi_{0,-1}} & & \downarrow{\scriptstyle \varphi_*} \\
\cdots & \xrightarrow{d} & E_{-3} & \xrightarrow{d} & E_{-2} & \xrightarrow{d} & E_{-1} & \xrightarrow{\rho} & TM
\end{array}$$

Consider a $C^\infty(M)$-linear map $\Phi : \mathcal{E} \to \mathcal{E}'$ (not necessarily a Lie ∞-algebroid morphism) then Φ is said to be of arity/degree k if it maps functions of arity l in $\mathcal{E}$ to functions of arity $k + l$ in $\mathcal{E}'$. By $\Phi^{(k)}$ we denote the component of Φ that is of arity k, i.e. $\Phi^{(k)} : \Gamma(E^*) \to \Gamma\left(S^{k+1}(E'^*)\right)$. One can decompose any Ψ as above as

$$\Phi = \sum_{k \in \mathbb{Z}} \Phi^{(k)}. \tag{2.13}$$

Note that by $C^\infty(M)$-linearity the arity k component $\Phi^{(k)}$ comes from a bundle morphism $\Phi^{(k)} : E^* \to S^{k+1}(E'^*)$. Now using that

$$\mathrm{Hom}_{C^\infty(M)}\left(E^*, S^{k+1}(E'^*)\right) \cong S^{k+1}(E'^*) \otimes_{C^\infty(M)} E,$$

(see for instance [Lan05] XVI, §6 corollary 5.5) we see that $\Phi^{(k)}$ gives rise to a section of the bundle $S^{k+1}(E'^*) \otimes E$ (we omit writing the ring over which these modules are defined) that we denote by φ_k and we call it the **k-th Taylor coefficient of Φ**. By definition of these Taylor coefficients, we have that for all $\alpha \in \Gamma(E^*)$

$$\Phi^{(k)}(\alpha) = \langle \varphi_k, \alpha \rangle.$$

Hence together with the decomposition (2.13) we have that any Φ is uniquely determined by it's Taylor coefficients. Indeed we have that for all $k, n \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_k \in \Gamma(E^*)$ the following holds

$$\Phi^{(n)}(\alpha_1 \odot \cdots \odot \alpha_k) = \sum_{i_1 + \cdots + i_k = n} \Phi^{(i_1)}(\alpha_1) \odot \cdots \odot \Phi^{(i_k)}(\alpha_k).$$

Remark 2.2.19. A more concrete interpretation of a Lie ∞-algebroid morphism can be given by using definition 2.1.4. Let (E, Q) and (E', Q') be two Lie ∞-algebroids over the same base manifold M and $\Phi : (E', Q') \to (E, Q)$ a Lie ∞-algebroid morphism between them. By dualizing the Taylor coefficients one obtains a collection of maps $\Phi_k : S^{k+1}(E') \to E$. For these maps we can formulate some concrete conditions under which they constitute a Lie ∞-algebroid morphism. The first one is a compatibility relation with the anchor maps: $\rho' = \rho \circ \Phi_0$. Secondly, since $\Gamma(E')$ and $\Gamma(E)$ are L_∞-algebras, we require $(\Phi_k)_{k\geq 0}$ to be L_∞-algebra morphisms in the sense of definition 2.1.4.

Remark 2.2.20. Note that the sequence of vector bundle morphisms $\varphi_{0,\bullet} : E'_\bullet \to E_\bullet$ from remark 2.2.17 come from the vector bundle morphism $S^1(E') \to E$ which is exactly the arity 0 component $\Phi^{(0)}$.

Example 2.2.21. Let $(A, [\cdot,\cdot]_A, \rho_A)$ and $(B, [\cdot,\cdot]_B, \rho_B)$ be two Lie algebroids then in example 2.2.13 we saw that we get two Lie ∞-algebroids $(A[1], Q_A)$ and $(B[1], Q_B)$. In the literature a morphism between Lie algebroids is defined as a vector bundle morphism $\mu : B \to A$ (over the identity on M) such that $\rho_A \circ \mu = \rho_B$ and $\mu([x, y]_B) = [\mu(x), \mu(y)]_A$ for elements $x, y \in \Gamma(B)$. Writing out the definition of a morphism of Lie ∞-algebroids $(B[1], Q_B) \to (A[1], Q_A)$ now yields exactly the same result.

♦

Let (E, Q) and (E', Q') be two Lie ∞-algebroids over M with sheaves of functions $\mathcal{E}$ and $\mathcal{E}'$ respectively. Define the following degree one operator on the space of linear maps $\mathrm{Lin}(\mathcal{E}, \mathcal{E}')$ from $\mathcal{E}$ to $\mathcal{E}'$

$$Q_{\mathcal{E},\mathcal{E}'} : \mathrm{Lin}(\mathcal{E}, \mathcal{E}') \to \mathrm{Lin}(\mathcal{E}, \mathcal{E}') : \Psi \mapsto Q' \circ \Psi - (-1)^{|\Psi|}\Psi \circ Q.$$

Here $|\Psi|$ denotes the degree of the algebra map $\Psi : \mathcal{E} \to \mathcal{E}'$. Note that for all $\Psi \in \mathrm{Lin}(\mathcal{E}, \mathcal{E}')$ one has

$$\begin{aligned} Q^2_{\mathcal{E},\mathcal{E}'}(\Psi) &= Q' \circ Q_{\mathcal{E},\mathcal{E}'}(\Psi) - (-1)^{|\Psi|} Q_{\mathcal{E},\mathcal{E}'}(\Psi) \circ Q \\ &= Q' \circ \left(Q' \circ \Psi - (-1)^{|\Psi|}\Psi \circ Q\right) - (-1)^{\left|Q_{\mathcal{E},\mathcal{E}'}(\Psi)\right|} \left(Q' \circ \Psi - (-1)^{|\Psi|}\Psi \circ Q\right) \circ Q \\ &= Q'^2 \circ \Psi - (-1)^{|\Psi|} Q' \circ \Psi \circ Q - (-1)^{|\Psi|+1} Q' \circ \Psi \circ Q - (-1)^{2|\Psi|+1}\Psi \circ Q^2 \\ &= 0. \end{aligned}$$

Here we used that for the homological vector field Q we have that $Q^2 = Q'^2 = 0$, that $Q_{\mathcal{E},\mathcal{E}'}$ is a degree $+1$ map and finally that by linearity of Ψ we have that $\Psi \circ Q^2 = 0$. This means that $Q_{\mathcal{E},\mathcal{E}'}$ defines a degree $+1$ differential for which we thus have a notion of cocycle which in turn can be used to find a condition for a graded commutative algebra morphism to be a Lie ∞-algebroid morphism. This is contained in the following lemma.

Lemma 2.2.22 (Lemma 3.47 in [LGLS20]). *Let (E, Q) and (E', Q') be Lie ∞-algebroids, a graded algebra morphism $\Phi : \mathcal{E} \to \mathcal{E}'$ is a Lie ∞-algebroid morphism if and only if it is degree zero $Q_{\mathcal{E},\mathcal{E}'}$-cocycle.*

Proof. We already know that a morphism between N-manifolds necessarily has degree 0. Now note that Φ is a $Q_{\mathcal{E},\mathcal{E}'}$-cocycle exactly when $Q_{\mathcal{E},\mathcal{E}'}(\Psi) = 0$ which, by definition of $Q_{\mathcal{E},\mathcal{E}'}$ and taking into account that Ψ has degree zero, precisely states that

$$Q' \circ \Phi - \Phi \circ Q = 0,$$

i.e. Φ is a Lie ∞-algebroid morphism. □

Definition 2.2.23 (Definition 3.48 in [LGLS20])**.** For every graded algebra morphism $\Phi : \mathcal{E} \to \mathcal{E}'$, a homogeneous map $W : \mathcal{E} \to \mathcal{E}'$ of degree k that satisfies

$$W(F \odot G) = W(F) \odot \Phi(G) + (-1)^{k|F|}\Phi(F) \odot W(G),$$

for all homogeneous elements $F, G \in \mathcal{E}$ is called a **Φ-derivation of degree** k. We denote the space of Φ-derivations by $\mathfrak{X}(\mathcal{E} \xrightarrow{\Phi} \mathcal{E}')$ and its restriction to $C^\infty(M)$-linear ones by $\mathfrak{X}_{\mathrm{vert}}(\mathcal{E} \xrightarrow{\Phi} \mathcal{E}')$.

Example 2.2.24. When $\Phi : \mathcal{E} \to \mathcal{E}'$ is a degree k graded algebra morphism then $Q_{\mathcal{E},\mathcal{E}'}(\Phi)$ is a Φ-derivation $k+1$. For details see lemma 3.49 in [LGLS20].

♦

Note that a Φ-derivation does not need to be a morphism of algebras. However, it is still completely determined (in a unique way) by its Taylor coefficients $w_i \in \Gamma(S^{i+1}(E'^*) \otimes E)$ where now $i \geq -1$;

$$W^{(n)}(\alpha_1 \odot \cdots \odot \alpha_k) = \sum_{j=1}^{k} \sum_{i_1+\cdots+i_k=n} \epsilon_j \Phi^{(i_1)}(\alpha_1) \odot \cdots \odot \langle w_{i_j}, \alpha_j \rangle \odot \cdots \odot \Phi^{(i_k)}(\alpha_k), \quad (2.14)$$

where

$$\epsilon_j = (-1)^{|W|(|\alpha_1|+\cdots+|\alpha_{j-1}|)}.$$

And again for all $\alpha \in \Gamma(E^*)$ we define $\langle w_k, \alpha \rangle = W^{(k)}(\alpha)$. Conversely, starting from a graded algebra morphism $\Phi : \mathcal{E} \to \mathcal{E}'$ and a given section $w \in \Gamma(S^\bullet(E'^*) \otimes E)$, there is a unique Φ-derivation that we denote w^Φ whose arity n component satisfies (2.14). For this we let w_k be the restriction of w to $\Gamma(S^{k+1}(E'^*) \otimes E)$.

Note that when $\Phi : (E', Q') \to (E, Q)$ is a Lie ∞-algebroid morphism then example 2.2.24 immediately yields the following lemma .

Lemma 2.2.25 (Lemma 3.50 in [LGLS20])**.** *For every Lie ∞-algebroid morphism Φ : $(E', Q') \to (E, Q)$ the graded space $\mathfrak{X}(\mathcal{E} \xrightarrow{\Phi} \mathcal{E}')$ equipped with $Q_{\mathcal{E},\mathcal{E}'}$ becomes a complex.*

Proof. As explained earlier $Q_{\mathcal{E},\mathcal{E}'}$ is a degree $+1$ operator on the graded space of Φ-derivations and we already showed that $Q_{\mathcal{E},\mathcal{E}'}$ squares to zero. □

Homotopies

In this section we will follow section 3.4.3 in [LGLS20] from which we also take all definitions. The following will be used in the definition of a homotopy between Lie ∞-algebroid morphisms.

Definition 2.2.26 ([LGLS20])**.** Let $B \to M$ be a vector bundle, a **piecewise smooth path in** $\Gamma(B)$ is a map $\psi : M \times I \to B$ such that, for all fixed $t \in I = [0, 1]$, the map $m \mapsto \psi(m, t)$ is a section of B and there is a subdivision $0 = t_0 < \cdots < t_k = 1$ of I such that the map $\psi : M \times (t_i, t_{i+1}) \to B$ is a smooth map.

Definition 2.2.27 (Definition 3.51 in [LGLS20])**.** Let (E, Q) and (E', Q') be two Lie ∞-algebroids over M. A path $t \mapsto \Phi_t$ valued in Lie ∞-algebroid morphisms from E' to E (i.e. Φ_t is a Lie ∞-algebroid morphism from E' to E for all t) is said to be **continuous**

piecewise-C^∞ when for all $k \in \mathbb{N}$, its Taylor coefficients $t \mapsto \varphi_k(t)$ of arity k is a piecewise-C^∞ path valued in $\Gamma\left(S^{k+1}(E'^*) \otimes E\right)$ (in the sense of definition 2.2.26), which is also continuous-even at the junction points. Given such a piecewise-C^∞ path $t \mapsto \Phi_t$ valued in Lie ∞-algebroid morphisms from (E', Q') to (E, Q), we say that a path $t \mapsto H_t$, with H_t a Φ_t derivation, is **piecewise smooth** if its Taylor coefficients $t \mapsto h_k(t)$ of arity k is are piecewise-smooth paths valued in $\Gamma\left(S^{k+1}(E'^*) \otimes E\right)$ for all k.

Remark 2.2.28. Note that the partition of I for which $\varphi_k(t)$ is a piecewise-C^∞ path valued in $\Gamma(S^{k+1}(E'^* \otimes E)$ may depend on k. The derivative $\frac{d\Phi_t}{dt}$ is well-defined for all $t \in I$ which do not delimit these subdivisions. Note that these special points form a countable set. When considering Lie n-algebroids we can take this subdivision to be the same for all $k \geq 0$. Indeed the Taylor coefficients φ_k can be identified with degree 0 elements in $\Gamma(S(E'^*) \otimes E))$. Now note that $\varphi_k : \Gamma(S^k(E'^*)) \to \Gamma(E)$ is a degree 0 morphism that takes k inputs, hence using that $E = E_{-1} \oplus \cdots \oplus E_{-n}$ we see that the maximal degree of an element in $\Gamma(E)$ must be $-n$, so for degree reasons φ_k must vanish for k big enough. Said differently this means that the arity k components of the degree 0 part of the bundle $S(E'^*) \otimes E$ vanish for k big enough. Hence we can take the subdivision to be the same for k sufficiently big.

One can show that $\frac{d\Phi_t}{dt}$ is a degree zero Φ_t-derivation for all t for which it is well-defined. Furthermore it satisfies that $Q_{\mathcal{E},\mathcal{E}'}\left(\frac{d\Phi_t}{dt}\right) = 0$, i.e. it a cocycle in the complex from lemma 2.2.25. This inspires the following definition behind which the rough idea is to let homotopies be curves of Lie ∞-algebroid morphisms whose derivatives are coboundaries for the complex of Φ-derivations.

Definition 2.2.29 (Definition 3.53 in [LGLS20])**.** Let $\Phi, \Psi : (E', Q') \to (E, Q)$ be two Lie ∞-algebroid morphisms over the identity. A **homotopy between Φ and Ψ** is a pair (Φ_t, H_t) consisting of the following data:

1. a continuous piecewise-C^∞ path $t \mapsto \Phi_t$ valued in Lie ∞-algebroid morphisms between E' and E for which
$$\Phi_0 = \Phi \quad \text{and} \quad \Phi_1 = \Psi,$$
2. a piecewise smooth path $t \mapsto H_t$ valued in Φ_t-derivations of degree -1 such that the following equation is satisfied, for all $t \in I$ for which it is well-defined
$$\frac{d\Phi_t}{dt} = Q_{\mathcal{E},\mathcal{E}'}(H_t). \tag{2.15}$$

Remark 2.2.30. A more precise statement of equation (2.15) is to say that the following equality holds for all $k \in \mathbb{N}$ and $t \in I$ for which it is well-defined

$$\frac{d\Phi_t^{(k)}}{dt} = (Q_{\mathcal{E},\mathcal{E}'}(H_t))^{(k)} = \sum_{i=0}^{k}\left((Q')^{(i)} \circ H_t^{(k-i)} + H_t^{(k-i)} \circ Q^{(i)}\right). \tag{2.16}$$

We now prove the following proposition.

Proposition 2.2.31. *Homotopy of Lie ∞-algebroid morphisms is an equivalence relation, denoted by $\sim$, that is compatible with composition.*

Proof. We first show that $\sim$ indeed defines an equivalence relation.

- **reflexivity:** clearly $\Phi \sim \Phi$ by taking $\Phi_t = \Phi$ and $H_t = 0$ for all $t \in I$.
- **symmetry:** once we have $\Phi \sim \Psi$ by a homotopy (Φ_t, H_t) one can take the homotopy $(\Phi_{1-t}, -H_{1-t})$ from which it also follows that $\Psi \sim \Phi$.
- **transitivity:** suppose we have two homotopies $\Phi \sim \Psi$ and $\Psi \sim \Xi$ by the homotopies (Θ_t, F_t) and (Π_t, G_t) respectively. Now can 'glue' these two homotopies together by rescaling the time parameter: in general this will not give a differentiable path at $t = 1/2$ but it is continuous at this point (which is all we need).

Now for the compatibility with composition; let $\Phi, \Psi : \mathcal{E} \to \mathcal{E}'$ be homotopic Lie ∞-algebroid morphisms from (E', Q') to (E, Q), by the homotopy (Φ_t, H_t), and $\Phi', \Psi' : \mathcal{E}' \to \mathcal{E}''$ be homotopic Lie ∞-algebroid morphisms between (E'', Q'') and (E', Q') by the homotopy (Φ'_t, H'_t). Now we can form the homotopy $(\Phi'_t \circ \Phi_t, H'_t \circ \Phi_t + \Phi'_t \circ H_t)$ between the morphisms $\Phi' \circ \Phi$ and $\Psi' \circ \Psi$. □

Remark 2.2.32. Altough the definition of this kind of homotopy is quite complicated, every homotopy of Lie ∞-algebroids gives a chain homotopy between the linear parts. Indeed, when considering the same setting as in definition 2.2.29 equation (2.16) for $k = 0$ yields that

$$\frac{d\Phi_t^{(0)}}{dt} = Q'^{(0)} \circ H_t^{(0)} + H_t^{(0)} \circ Q^{(0)}.$$

Now integrating this equation gives us that

$$\begin{aligned}\int_0^1 \frac{d\Phi_t^{(0)}}{dt} dt &= \int_0^1 \left(Q'^{(0)} \circ H_t^{(0)} + H_t^{(0)} \circ Q^{(0)}\right) \\ &= Q'^{(0)} \circ \left(\int_0^1 H_t^{(0)} dt\right) + \left(\int_0^1 H_t^{(0)} dt\right) \circ Q^{(0)}.\end{aligned}$$

By the fundamental theorem of calculus we have

$$\int_0^1 \frac{d\Phi_t^{(0)}}{dt} dt = \Phi_1^{(0)} - \Phi_0^{(0)} = \Psi^{(0)} - \Phi^{(0)}.$$

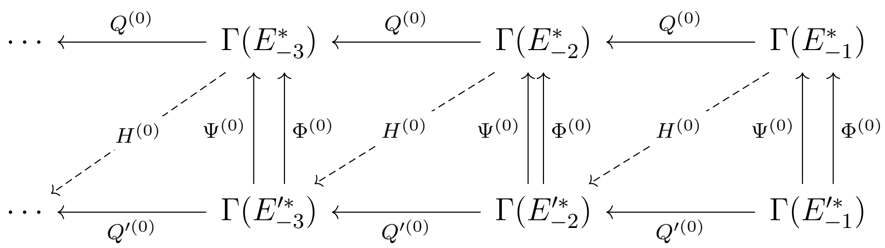
Now define the map

$$H^{(0)} := \int_0^1 H_t^{(0)} dt. \tag{2.17}$$

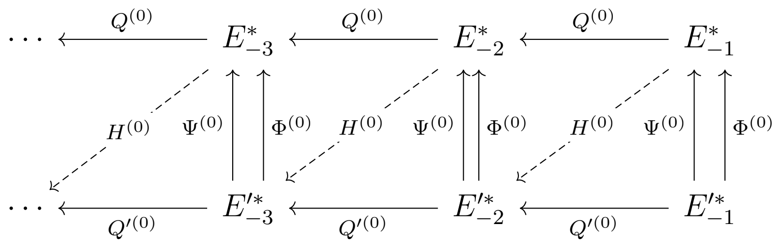
So combining these observations we see that

$$\Psi^{(0)} - \Phi^{(0)} = Q'^{(0)} \circ H^{(0)} + H^{(0)} \circ Q^{(0)}. \tag{2.18}$$

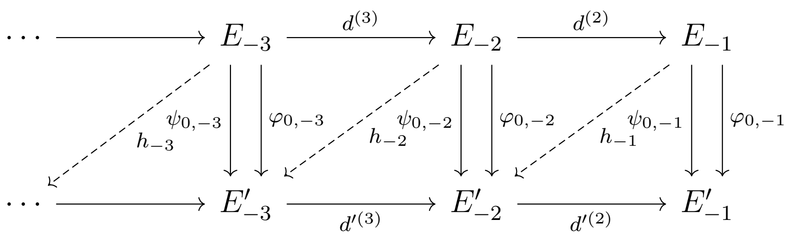
As already noted in remark 2.2.20 the arity zero components $\Psi^{(0)}$ and $\Phi^{(0)}$ induce the chain maps between the linear maps. Combined with equation (2.18) this gives the following commutative diagram.



Since H_t is a Φ_t-derivation for all t we have that $H^{(0)}$ from (2.17) is a $C^\infty(M)$-linear map (integrating does not change this property) and all other maps involved are also $C^\infty(M)$-linear so we can form the following diagram between vector bundles.



In this diagram we can dualize all maps and differentials: by remark 2.2.20 we have that $(\Psi^{(0)})^* = \psi_{0,\bullet}$, $(\Phi^{(0)})^* = \varphi_{0,\bullet}$ and $(H^{(0)})^* = h_\bullet$; furthermore by theorem 2.2.15 we have that the dual of $Q^{(0)}$ is the differential d and similarly for $Q'^{(0)}$. Hence we get the following diagram



From equation (2.18) we get that, by dualizing,

$$\begin{aligned}\psi_{0,\bullet} - \varphi_{0,\bullet} = (\Psi^{(0)})^* - (\Phi^{(0)})^* &= \left(Q'^{(0)} \circ H^{(0)}\right)^* + \left(H^{(0)} \circ Q^{(0)}\right)^* \\ &= h_\bullet \circ d' + d \circ h_\bullet.\end{aligned}$$

This exactly means that $h_\bullet$ is a chain homotopy between the chain maps $\psi_{0,\bullet}$ and $\varphi_{0,\bullet}$.

Chapter 3

Universal Lie ∞-algebroid of a Singular Foliation

In this chapter we will state the main results of [LGLS20]. These results consist of two theorems: an existence result and a uniqueness result. We will give a 'non-technical' description of the proof of the existence result. For this we build on the notions developed in earlier chapters.

3.1 Main Results

Definition 3.1.1 (Definition 2.6 in [LGLS20])**.** Let $\mathcal{F}$ be a singular foliation on a manifold M. We call a Lie ∞-algebroid (E, Q) over M a **universal Lie ∞-algebroid** of $\mathcal{F}$ if the linear part of (E, Q) is a geometric resolution[1] of $\mathcal{F}$.

Using this definition we can state one of the main results of [LGLS20].

Theorem 3.1.2 (Theorem 2.7 in [LGLS20])**.** *Let $\mathcal{F}$ be a singular foliation on a manifold M which admits a geometric resolution (E, d, ρ). Then there exists a universal Lie ∞-algebroid of $\mathcal{F}$ the linear part of which is the geometric resolution.*

The following theorem and its corollary show that there is also some sense of uniqueness to this universal Lie ∞-algebroid.

Theorem 3.1.3 (Theorem 2.8 in [LGLS20])**.** *Let (E, Q) be a universal Lie ∞-algebroid of a singular foliation $\mathcal{F}$ on a smooth manifold. For every Lie ∞-algebroid (E', Q') defining a sub-singular foliation[2] of $\mathcal{F}$. There is a Lie ∞-algebroid morphism from (E', Q') to (E, Q) over the identity on M and any two such Lie ∞-algebroid morphisms are homotopic.*

Corollary 3.1.4 (Corollary 2.9 in [LGLS20])**.** *Two universal Lie ∞-algebroids of the singular foliation $\mathcal{F}$ are homotopy equivalent and two such homotopy equivalences are homotopic.*

Remark 3.1.5. Theorem 3.1.3 and corollary 3.1.4 also imply the following very useful result: once you have found *any* Lie ∞-algebroid structure on the geometric resolution

[1] See definition 1.2.40.

[2] This means that $\rho'\left(\Gamma(E'_{-1})\right) \subset \mathcal{F}$.

of a foliation, you are automatically sure this is the universal one. This essentially allows one to 'guess' (in some sense) a Lie ∞-algebroid structure and then conclude this is the universal one.

We can also rephrase these observations in a categorical language: consider a category where objects are Lie ∞-algebroids whose induced singular foliation is a sub-singular foliation of $\mathcal{F}$ and where morphisms are homotopy classes of morphisms of Lie ∞-algebroid morphisms. Then theorem 3.1.3 implies that any universal Lie ∞-algebroid over $\mathcal{F}$ is a final object in this category. This also provides inspiration for corollary 3.1.4; indeed, any two final objects in a category are *unique up to a unique isomorphism*, see for instance proposition 5.5 in [Alu09].

3.1.1 Proof of Theorem 3.1.2

Instead of explaining the whole proof of theorem 3.1.2 we will explain the main idea, this because the whole proof is of a quite technical nature. For this we follow [LGLS20] section 3.5.

Throughout this section we will assume that $\mathcal{F}$ is a singular foliation on a *smooth* manifold M (but the ideas used in the proof can also be used on a neighborhood when considering analytic or holomorphic manifolds). Of course, we also need to assume that $\mathcal{F}$ admits a geometric resolution (E, d, ρ) for otherwise there would be nothing to show.

By the first part of theorem 2.2.15 we see that a complex of vector bundles

$$\cdots \xrightarrow{d} E_{-3} \xrightarrow{d} E_{-2} \xrightarrow{d} E_{-1}, \tag{3.1}$$

is in a one-to-one correspondence with an NQ-manifold with the homological vector field $Q^{(0)}$ of arity 0, dual to the differential d. We will now use a deformation of such an NQ-manifold to expand $Q^{(0)}$ to a homological vector field Q. With this we mean that, given $Q^{(0)}$, we want to search for arity $k \geq 1$ degree $+1$ vector fields $Q^{(k)}$ such that in the end we can form the vector field

$$Q = \sum_{k\geq 0} Q^{(k)}. \tag{3.2}$$

Of course, we want the degree $+1$ vector field that we obtain in this way to be homological i.e., we want that $[Q, Q] = 0$. Using the expansion (3.2) we can rewrite this condition as a system of equations. We first expand the left-hand side of the following equation

$$\sum_{k\geq 0}\sum_{l\geq 0}\left[Q^{(k)}, Q^{(l)}\right] = 0. \tag{3.3}$$

Which gives the following result

$$\begin{array}{l}
\left[Q^{(0)}, Q^{(0)}\right] + \left[Q^{(0)}, Q^{(1)}\right] + \left[Q^{(0)}, Q^{(2)}\right] + \left[Q^{(0)}, Q^{(3)}\right] + \cdots \\
\left[Q^{(1)}, Q^{(0)}\right] + \left[Q^{(1)}, Q^{(1)}\right] + \left[Q^{(1)}, Q^{(2)}\right] + \left[Q^{(1)}, Q^{(3)}\right] + \cdots \\
\left[Q^{(2)}, Q^{(0)}\right] + \left[Q^{(3)}, Q^{(0)}\right] + \left[Q^{(2)}, Q^{(2)}\right] + \left[Q^{(2)}, Q^{(3)}\right] + \cdots \\
\left[Q^{(3)}, Q^{(0)}\right] + \left[Q^{(3)}, Q^{(1)}\right] + \left[Q^{(3)}, Q^{(2)}\right] + \left[Q^{(3)}, Q^{(3)}\right] + \cdots \\
+ \quad \cdots \quad + \quad \cdots \quad + \quad \cdots \quad + \quad \cdots \quad + \quad \cdots \qquad = 0
\end{array} \tag{3.4}$$

Now grouping the terms in (3.4) by arity (see the colors) and using that $\left[Q^{(k)}, Q^{(l)}\right] = \left[Q^{(l)}, Q^{(k)}\right]$ we get the following system of equations

$$\left[Q^{(0)}, Q^{(0)}\right] = 0, \tag{3.5}$$

$$\left[Q^{(0)}, Q^{(1)}\right] = 0, \tag{3.6}$$

$$\left[Q^{(0)}, Q^{(n)}\right] = -\frac{1}{2} \sum_{\substack{1\le i,j\le n-1 \\ i+j=n}} \left[Q^{(i)}, Q^{(j)}\right]. \tag{3.7}$$

Equation (3.5) is satisfied by assumption because $d^2 = 0$ and $d^* = Q^{(0)} = Q$. We already explained above that from a homological vector field we get an anchor map $\rho : E_{-1} \to TM$. Furthermore from equation (3.6) and theorem 2.2.15 we have that $Q^{(1)}$ gives a family of binary brackets for which the Leibniz identity is satisfied and such that d is a derivation of these brackets.

A particular way of looking at deformation problems is by looking at a differential graded Lie algebra (DGLA); these are defined in the following way.

Definition 3.1.6 ([FM07])**.** A **differential graded Lie algebra (DGLA)** is a $\mathbb{Z}$-graded vector space $L = \oplus_{i\in Z} L_i$ together with a bilinear bracket $[\cdot,\cdot] : L \times L \to L$ and a linear map $d : L \to L$ that satisfies the following conditions:

1. the bracket $[\cdot,\cdot]$ is homogeneous skew-symmetric, i.e. $[L_i, L_j] \subset L_{i+j}$ and for all homogeneous $a, b \in L$
$$[a, b] = -(-1)^{|a||b|} [b, a],$$
2. for all homogeneous $a, b, c \in L$ the graded Jacobi identity holds
$$[a, [b, c]] = [[a, b], c] + (-1)^{|a||b|} [b, [a, c]],$$
3. d is a degree 1 differential and is a derivation of the bracket; i.e. $d(L_i) \subset L_{i+1}$, $d^2 = 0$ and
$$d[a, b] = [da, b] + (-1)^{|a|} [a, db].$$

In the case of the deformation problem of the NQ-manifold we can look at the DGLA[3]

$$\left(\mathfrak{X}(E), [\cdot,\cdot], \left[Q^{(0)}, \cdot\right]\right).$$

Note that we have the following consequence from the graded Jacobi identity

$$\left[Q^{(0)}, \left[Q^{(0)}, Q^{(n)}\right]\right] + \left[Q^{(n)}, \left[Q^{(0)}, Q^{(0)}\right]\right] + \left[Q^{(0)}, \left[Q^{(n)}, Q^{(0)}\right]\right] = 0.$$

Hence using equation (3.5) we see that $\left[Q^{(0)}, \left[Q^{(0)}, Q^{(n)}\right]\right] = 0$. In particular this means that $\left[Q^{(0)}, \cdot\right]$ squares to zero which allows us to talk about cohomology. Furthermore this observation implies that the right-hand side of equation (3.7) is a $\left[Q^{(0)}, \cdot\right]$-closed term. Now suppose for the sake of argument that the right hand side of equation (3.7) is also a $\left[Q^{(0)}, \cdot\right]$-exact term. Then given $Q^{(0)}, Q^{(1)}, \ldots, Q^{(n-1)}$ we are able to define an

[3] A bit lower on this page we show that $\left[Q^{(0)}, \cdot\right]$ is indeed a differential.

arity n component $Q^{(n)}$. This allows us to construct the homological vector field Q in an inductive manner. Indeed, if the right-hand side of (3.7) is exact with respect to the given differential this means that there exists a degree +1 element, denoted Q_n such that

$$\left[Q^{(0)}, Q_n\right] = -\frac{1}{2} \sum_{\substack{1 \leq i,j \leq n-1 \\ i+j=n}} \left[Q^{(i)}, Q^{(j)}\right].$$

Furthermore since the bracket $[\cdot,\cdot]$ is additive with respect to the arity[4] we have that the arity of the right hand side is n and so Q_n has to have arity n also. Hence, we can write $Q_n = Q^{(n)}$.

When considering the DGLA $\left(\mathfrak{X}(E), [\cdot,\cdot], \left[Q^{(0)}, \cdot\right]\right)$ we have an induced complex

$$\mathfrak{X}^1(E) \xrightarrow{[Q^{(0)},\cdot]} \mathfrak{X}^2(E) \xrightarrow{[Q^{(0)},\cdot]} \mathfrak{X}^3(E) \xrightarrow{[Q^{(0)},\cdot]} \cdots.$$

Thus combing the explanation above with this complex we see that the obstruction for finding $Q^{(n)}$ lives in $H^2\left(\mathfrak{X}(E), \left[Q^{(0)}, \cdot\right]\right)$ which is defined as

$$H^2\left(\mathfrak{X}(E), \left[Q^{(0)}, \cdot\right]\right) = \frac{\{X \in \mathfrak{X}^2(E) \mid \left[Q^{(0)}, X\right]\}}{\{[Q^{(0)}, Y] \mid Y \in \mathfrak{X}^1(E)\}}$$

If we are able to show that $H^2\left(\mathfrak{X}(E), \left[Q^{(0)}, \cdot\right]\right) = 0$ we are done and can extend the homological vector field.

It is exactly the vanishing of the second cohomology group that is used in [LGLS20]. However there the authors chose to consider a particular Lie subalgebra of $(\mathfrak{X}(E), [\cdot,\cdot])$ that is more practical to consider in the case of NQ-manifolds. For this we first need some new terminology.

Definition 3.1.7 ([LGLS20]). A **vertical vector field** on an NQ-manifold (E, Q) over a manifold M with sheaf of functions $\mathcal{E}$ is a $C^\infty(M)$-linear derivation of $\mathcal{E}$.

From now on we consider the following differential graded Lie subalgebra

$$\left(\mathfrak{X}_{\text{vert}}(E), [\cdot,\cdot], \left[Q^{(0)}, \cdot\right]\right).$$

By theorem 2.2.15 it can be seen that all of the $Q^{(n)}$ are vertical vector fields except for $Q^{(1)}$, the component defining the binary bracket and anchor. However the Lie bracket between $Q^{(1)}$ and $Q^{(n)}$ is vertical for $n \geq 2$. Hence all of the obstruction classes for $n > 2$ live in $H^2\left(\mathfrak{X}_{\text{vert}}(E), \left[Q^{(0)}, \cdot\right]\right)$. We can specify a bit more: again, using that the Lie bracket adds the arity and that the right hand side of equation (3.7) has arity n, the obstruction lies in

$$H^2\left(\mathfrak{X}^{(n)}_{\text{vert}}(E), \left[Q^{(0)}, \cdot\right]\right), \quad n > 2. \tag{3.8}$$

Here $\mathfrak{X}^{(n)}_{\text{vert}}(E)$ denotes the vertical vector fields of arity n. As is evident from this discussion the case $n = 2$ needs special care. Note that requirement (3.7) for $n = 2$ reads

[4] By this we mean that the arity of $\left[Q^{(i)}, Q^{(j)}\right]$ is $i + j$. This is not hard to see given the definition of arity; it essentially follows from the fact that the degree of the product of two polynomials is also additive.

$\left[Q^{(0)}, Q^{(2)}\right] = \frac{1}{2}\left[Q^{(1)}, Q^{(1)}\right]$. The right hand side of this equation is vertical and so one also needs

$$\left[Q^{(1)}, Q^{(1)}\right] \in \mathfrak{X}_{\mathrm{vert}}(E). \tag{3.9}$$

If this condition is met than its class in $H^2\left(\mathfrak{X}^{(n)}_{\mathrm{vert}}(E), \left[Q^{(0)}, \cdot\right]\right)$ is defined and so needs to vanish.

Now we proceed by outlining the rest of the proof where we do not always give the details and just provide the main steps.

Cohomology of vertical vector fields for geometric resolutions

Here we follow subsection 3.5.2 in [LGLS20]. By $\mathfrak{X}^{(n)}_{\mathrm{vert}}(E)_k$ we denote the space of vertical vector fields on E of arity n and degree k. Since vector fields of arity n can be seen as $C^\infty(M)$-linear maps $E^* \to S^{(n+1)}(E^*)$ we have that $\mathfrak{X}^{(n)}_{\mathrm{vert}}(E)_\bullet \cong \Gamma\left(\mathrm{Hom}^\bullet_{C^\infty(M)}(E^*, S^{n+1}(E^*)\right)$. We have already seen that there is an isomorphism

$$\Gamma\left(\mathrm{Hom}^\bullet_{C^\infty(M)}(E^*, S^{n+1}(E^*)\right) \cong \Gamma\left(S^{n+1}(E^*) \otimes E\right)_\bullet.$$

Hence there also is a natural isomorphism

$$\mathfrak{X}^{(n)}_{\mathrm{vert}}(E)_\bullet \cong \Gamma\left(S^{n+1}(E^*) \otimes E\right)_\bullet.$$

We now define the following map.

Definition 3.1.8 ([LGLS20]). We define the **root map** as

$$\mathrm{rt} : \mathfrak{X}^{(n)}_{\mathrm{vert}}(E)_\bullet \to \Gamma\left(S^{n+1}(E^*)\right)_\bullet \otimes_{C^\infty(M)} \mathcal{F}\left[-1\right],$$

obtained by applying the map $\mathbb{1} \otimes \rho$ to the component of a vertical vector field in $S^{n+1}(E^*)_\bullet \otimes E_{-1}$.

Remark 3.1.9. Note the shift in degree for the elements in $\mathcal{F}$. This is needed to make rt into a degree 0 map.

Remark 3.1.10. We can also characterize the root map as follows: let $f \in C^\infty(M)$, $d_{\mathrm{dR}}f \in \Gamma(T^*M)$ the differential of f and ρ^* the dual of the anchor. Note that when $x \in \Gamma(E_{-1})$ one has $\langle x, \rho^* d_{\mathrm{dR}} f\rangle = \rho(x)\left[f\right]$ and so also for every vertical vector field

$$W\left(\rho^* d_{\mathrm{dR}} f\right) = \mathrm{rt}(W)\left[f\right]. \tag{3.10}$$

Before stating the following proposition we recall a definition from homological algebra.

Definition 3.1.11 ([Eis95]). Let $(A^\bullet, d_A)$ and $(B^\bullet, d_B)$ be two cochain complexes. A cochain map $f : A^\bullet \to B^\bullet$ is called a **quasi-isomorphism** when the induced morphism on cohomology is an isomorphism. That is

$$H^n\left(A^\bullet, d_A\right) \xrightarrow{\cong} H^n\left(B^\bullet, d_B\right).$$

The authors in [LGLS20] now proceed by stating and proving the following proposition. We will only state it since we are only interested in its corollary.

Proposition 3.1.12 (Proposition 3.64 in [LGLS20]). *If (E, d, ρ) is a geometric resolution of $\mathcal{F}$, then*

$$\mathrm{rt} : \left(\mathfrak{X}^{(n)}_{vert}(E)_\bullet, \left[Q^{(0)}, \cdot\right]\right) \to \left(\Gamma(S^{n+1}(E^*))_\bullet \otimes_{C^\infty(M)} \mathcal{F}\left[-1\right], Q^{(0)} \otimes \mathbb{1}\right),$$

is a quasi-isomorphism.

The proof of this proposition is quite technical and for this we refer to section 3.5.2 in [LGLS20]. We now state a lemma from [LGLS20] that is used in the proof of proposition 3.1.12 and the corollary that we are interested in.

Lemma 3.1.13 (Lemma 3.66 in [LGLS20]). *Let (E, d, ρ) be a geometric resolution of $\mathcal{F}$ and R a vertical vector field of degree i and arity n which is $Q^{(0)}$-closed and $\mathrm{rt}(R) = 0$. Then $R = \left[Q^{(0)}, W\right]$ for some vertical vector field W of arity n which has no component in $\Gamma\left(S^{n+1}(E^*)_i \otimes E_{-1}\right)$.*

The proof of this lemma is of a technical nature but it uses one main observation. From the geometric resolution (E, d, ρ) we get an exact sequence on the level of sections

$$\cdots \xrightarrow{d} \Gamma(E_{-2}) \xrightarrow{d} \Gamma(E_{-1}) \xrightarrow{\rho} \mathcal{F} \to 0. \tag{3.11}$$

Since $S^{n+1}(E^*)_k$ is a vector bundle over M we have that $\Gamma\left(S^{n+1}(E^*)_k\right)$ is a projective $C^\infty(M)$-module (see for instance theorem 12.32 in [Nes20]). In particular projective modules are flat modules. This means that we can tensor the exact sequence (3.11) with $\Gamma\left(S^{n+1}(E^*)_k\right)$ such that the resulting sequence is still exact. In this way we obtain the following exact sequence

$$\cdots \xrightarrow{\mathbb{1}\otimes d} \Gamma\left(S^{n+1}(E^*)_k\right) \otimes_{C^\infty(M)} \Gamma(E_{-2}) \xrightarrow{\mathbb{1}\otimes d} \Gamma\left(S^{n+1}(E^*)_k\right) \otimes_{C^\infty(M)} \Gamma(E_{-1}) \xrightarrow{\mathbb{1}\otimes\rho} \Gamma\left(S^{n+1}(E^*)_k\right) \otimes_{C^\infty(M)} \mathcal{F} \to 0.$$

Which can be rewritten as the exact sequence[5]

$$\cdots \xrightarrow{\mathbb{1}\otimes\rho} \Gamma\left(S^{n+1}(E^*)_k \otimes E_{-2}\right) \xrightarrow{\mathbb{1}\otimes\rho} \Gamma\left(S^{n+1}(E^*)_k \otimes E_{-1}\right) \xrightarrow{\mathbb{1}\otimes\rho} \Gamma\left(S^{n+1}(E^*)_k\right) \otimes_{C^\infty(M)} \mathcal{F} \to 0.$$

It is the exactness of the above sequence that is key to proving lemma 3.1.13.

Corollary 3.1.14 (Corollary 3.67 in [LGLS20]). *If (E, d, ρ) is a geometric resolution of a foliation $\mathcal{F}$ we have the following:*

1. $H^k\left(\mathfrak{X}^{(n)}_{vert}(E), \left[Q^{(0)}, \cdot\right]\right) \cong H^{k+1}\left(\Gamma(S^{n+1}(E)) \otimes_{C^\infty(M)} \mathcal{F}, Q^{(0)} \otimes \mathbb{1}\right)$,
2. $H^2\left(\mathfrak{X}^{(n)}_{vert}(E), \left[Q^{(0)}, \cdot\right]\right) = 0$ *when* $n \geq 3$.

Proof. 1. When dropping the degree shift in $\mathcal{F}\left[-1\right]$ the degree of the complex shifts up by one and so proposition 3.1.12 immediately gives the result.

2. Note that when we apply the first result with $k = 2$ we get that

$$H^2\left(\mathfrak{X}^{(n)}_{\mathrm{vert}}(E), \left[Q^{(0)}, \cdot\right]\right) = H^3\left(\Gamma(S^{n+1}(E)) \otimes_{C^\infty(M)} \mathcal{F}, Q^{(0)} \otimes \mathbb{1}\right)$$

Remark that when $n \geq 3$ elements of degree 3 in $\Gamma(S^{n+1}(E^*))$ vanish and so the result immediately follows.

□

[5] This is due to the fact that $\Gamma(\cdot)$ is a strong monoidal functor.

Extension of an almost Lie algebroid to a graded almost Lie algebroid on a geometric resolution

In this part we will follow 3.5.3 in [LGLS20]. The main focus will be on the following problem: given a geometric resolution (E, d, ρ) we automatically get E and $Q^{(0)}$ (as described in the deformation problem above). We now want to search for a $\left[Q^{(0)}, \cdot\right]$-closed $Q^{(1)}$ that induces our given ρ.

Coming back to the geometric resolution (E, d, ρ); we know that by definition of a geometric resolution we have that $\rho(\Gamma(E_{-1})) = \mathcal{F}$. Hence from proposition 1.2.39 we see that E_{-1} can be equipped with an almost-Lie algebroid structure. This means we have an anchor map and a binary bracket on the sections of E_{-1} that satisfies the Leibniz identity. This corresponds precisely to a degree +1 vector field of arity one, denoted $Q^{(1)}_{E_{-1}}$. This does not yet define an almost-Lie algebroid structure; the second condition is still missing (the algebra morphism). It can be shown that equation (1.2) is satisfied if and only if (see appendix A)

$$\left[Q^{(1)}_{E_{-1}}, Q^{(1)}_{E_{-1}}\right] \in \mathfrak{X}_{\mathrm{vert}}(E_{-1}). \tag{3.12}$$

So, to start our deformation problem we need to extend $Q^{(1)}_{E_{-1}}$ on E_{-1} to a vector field $Q^{(1)}$ on the whole bundle E such that $\left[Q^{(0)}, Q^{(1)}\right] = 0$ (we say $Q^{(1)}$ needs to be $Q^{(0)}$-closed). By means of a connection it is possible to lift $Q^{(1)}_{E_{-1}}$ to a vector field $Q^{(1)}_{E}$ on E[6]. It may be tempting to set $Q^{(1)} = Q^{(1)}_{E}$ but it is not guaranteed that our choice of lift indeed satisfies the $Q^{(0)}$-closed condition. To remediate this problem, we define

$$Q^{(1)} = Q^{(1)}_{E} + V, \tag{3.13}$$

where V is a vertical vector field in $\oplus_{i\geq 2}\Gamma\left(S^2(E^*)_{i+1} \otimes E_{-i}\right)$. This thus means that we choose some V to define $Q^{(1)}$. Hence we also get a collection of $Q^{(1)}$'s that depend on different choices of V. We will now continue by showing that there is some particular choice of V for which the corresponding $Q^{(1)}$ has the right properties we will describe below. So an important property to keep in mind is that *the vector field $Q^{(1)}$ depends on the choice of V (this is not immediately clear from the notation).*

Note that by (3.12) $\left[Q^{(1)}_{E}, Q^{(1)}_{E}\right] \in \mathfrak{X}_{\mathrm{vert}}(E)$ and so

$$\begin{aligned}\left[Q^{(1)}, Q^{(1)}\right] &= \left[Q^{(1)}_{E} + V, Q^{(1)}_{E} + V\right] \\ &= \left[Q^{(1)}_{E}, Q^{(1)}_{E}\right] + \cancel{\left[Q^{(1)}_{E}, V\right]} + \cancel{\left[V, Q^{(1)}_{E}\right]} + [V, V].\end{aligned}$$

Hence $Q^{(1)}$ satisfies the condition (3.9). Now we show that one can always choose V in such a way as to make $Q^{(1)}$ a $Q^{(0)}$-closed element. In this way we obtain a structure on E, given by $Q^{(0)} + Q^{(1)}$, that is called a graded almost Lie algebroid.

Definition 3.1.15 (Definition 3.68 in [LGLS20]). A **graded almost Lie algebroid** is a complex of vector bundles in the sense of equation (3.1) equipped with a bracket

$$[\cdot, \cdot] : \Gamma(E_{-i}) \times \Gamma(E_{-j}) \to \Gamma(E_{-i-j+1}),$$

[6] We do not provide details as to why this is possible.

satisfying the following three axioms

$$[x, fy] = f\,[x, y] + \rho(x)\,[f]\,y, \tag{3.14}$$
$$d\,[x, y] = [d(x), y] + (-1)^i\,[x, d(y)]\,, \tag{3.15}$$
$$\rho([x, y]) = [\rho(x), \rho(y)]\,, \tag{3.16}$$

for all $x \in \Gamma(E_{-i})$, $y \in \Gamma(E_{-j})$ and $f \in C^\infty(M)$. As usual it is understood that $\rho(x) = 0$ when $x \in \Gamma(E_{-i})$ and $i \geq 2$.

We will now prove the following proposition.

Proposition 3.1.16 (Proposition 3.69 in [LGLS20]). *Every geometric resolution (E, d, ρ) and every almost Lie algebroid structure on $E_{-1} \subset E$ can be extended to a graded almost Lie algebroid structure on E.*

The proof uses three lemmas.

Lemma 3.1.17 (Lemma 3.70 in [LGLS20]). *A graded almost Lie algebroid structure is in one-to-one correspondence with a graded manifold E equipped with a degree one vector field $Q = Q^{(0)} + Q^{(1)}$ of arity at most one such that equations (3.5), (3.6) and (3.9) are satisfied.*

Proof. The proof of this lemma is not provided in [LGLS20]. We will only describe the very rough idea. It revolves mainly about using theorem 2.2.15. Given a graded manifold E with vector field $Q = Q^{(0)} + Q^{(1)}$ one can dualize $Q^{(0)}$ to obtain the map d. Equation (3.5) then states that $d^2 = 0$, i.e. d is a differential and so (E, d) becomes a complex of vector bundles. From $Q^{(1)}$ we obtain an anchor map $\rho : E_{-1} \to TM$ and a binary bracket. Equation (3.6) states that d is a derivation of the bracket which is equation (3.15). Equation (3.14) comes free in this construction. Finally from (3.9) we get that $\left[Q^{(1)}, Q^{(1)}\right] \in \mathfrak{X}_{\text{vert}}(E)$ and so ρ defines an algebra morphism (again see appendix A) with respect to the bracket which is (3.16). □

Lemma 3.1.18 (Lemma 3.71 in [LGLS20]). *The vector field $\left[Q^{(0)}, Q^{(1)}\right]$ with $Q^{(1)}$ as in (3.13) defines an element in $H^2\left(\mathfrak{X}^{(1)}_{vert}(E), \left[Q^{(0)}, \cdot\right]\right)$.*

Proof. As we already explained in the part right after example 2.2.14 we have that $Q^{(1)} f = \rho^*\left(d_{\text{dR}} f\right)$ for $f \in C^\infty(M)$. Hence we have

$$\begin{aligned}\left[Q^{(0}, Q^{(1)}\right](f) &= \left(d^{(2)}\right)^* \circ \rho^*\left(d_{\text{dR}} f\right)\\ &= \left(\rho \circ d^{(2)}\right)^*\left(d_{\text{dR}} f\right).\end{aligned}$$

Now because of the definition of a geometric resolution one has $\rho \circ d^{(2)} = 0$ and so $\left[Q^{(0)}, Q^{(1)}\right] \in \mathfrak{X}_{\text{vert}}$. □

Lemma 3.1.19 (Lemma 3.72 in [LGLS20]). *For every choice of V in (3.13) one has* $\text{rt}\left(\left[Q^{(0)}, Q^{(1)}\right]\right) = 0$.

Proof. We already calculated that $\left[Q^{(1)}, Q^{(1)}\right] \in \mathfrak{X}_{\text{vert}}(E)$. Hence $\left[Q^{(0)}, \left[Q^{(1)}, Q^{(1)}\right]\right]$ is also a vertical vector field. Now let $f \in C^{\infty}(M)$ then using the Jacobi identity

$$\begin{aligned}
0 &= \frac{1}{2}\left[Q^{(0)}, \left[Q^{(1)}, Q^{(1)}\right]\right](f) \\
&= \left[\left[Q^{(0)}, Q^{(1)}\right], Q^{(1)}\right](f) \\
&= \left[Q^{(0)}, Q^{(1)}\right]\left(\rho^{*}\left(d_{\mathrm{dR}} f\right)\right).
\end{aligned}$$

Using the useful characterization of the root map (3.10) we see that $\operatorname{rt}\left(\left[Q^{(0)}, Q^{(1)}\right]\right)[f] = 0$ for arbitrary $f \in C^{\infty}(M)$ and so we conclude the lemma. □

We are now ready to prove proposition 3.1.16 based on the proof of proposition 3.69 in [LGLS20].

Proof of proposition 3.1.16. For every choice of V in equation (3.13), lemma 3.1.18 implies that $\left[Q^{(0)}, Q^{(1)}\right]$ is a vertical vector field. By lemma 3.1.19 we have $\operatorname{rt}\left(\left[Q^{(0)}, Q^{(1)}\right]\right) = 0$ and so by lemma 3.1.13 there exists some vertical vector field W with no component in $\Gamma(S^2(E_{-1}^{*}) \otimes E_{-1})$ for which

$$\left[Q^{(0)}, Q^{(1)}\right] = \left[Q^{(0)}, W\right]. \tag{3.17}$$

Recall that our $Q^{(1)}$ inherently depends on the choice of V. Hence we can define some new $Q^{(1)}$ (that we may also denote by the same symbol) by replacing V by $V - W$. Note that we clearly have $\left[Q^{(0)}, Q^{(0)}\right] = 0$. Now from equations (3.13) and (3.17) we have that

$$\left[Q^{(0)}, Q_E^{(1)} + V - W\right] = 0.$$

Taking inspiration from this equation we define $Q^{(1)} := Q_E^{(1)} + V - W$. Then clearly $\left[Q^{(1)}, Q^{(1)}\right] \in \mathfrak{X}_{\text{vert}}(E)$ because all involved vector fields are vertical and $\left[Q^{(0)}, Q^{(1)}\right] = 0$ by construction. This means we have shown that all equations mentioned in lemma 3.1.17 are met and so E together with $Q = Q^{(0)} + Q^{(1)}$ defines a graded almost Lie algebroid structure. □

Extension of an almost Lie algebroid to a Lie ∞-algebroid structure on a geometric resolution

In this final part everything will come together to show the following proposition.

Proposition 3.1.20 (Proposition 3.76 in [LGLS20])**.** *Every graded almost Lie algebroid $(E, Q^{(0)} + Q^{(1)})$ over a geometric resolution (E, d, ρ) can be extended to a Lie ∞-algebroid structure on E.*

Before giving the proof we show the following lemma.

Lemma 3.1.21 (Lemma 3.77 in [LGLS20])**.** *For the arity one part $Q^{(1)}$ of the odd vector field characterizing a graded almost Lie algebroid as in lemma 3.1.17, one has*

$$\operatorname{rt}\left(\left[Q^{(1)}, Q^{(1)}\right]\right) = 0. \tag{3.18}$$

Proof. It is easy to see that the Jacobi identity implies $\left[\left[Q^{(1)}, Q^{(1)}\right], Q^{(1)}\right] = 0$. Now using that $\left[Q^{(1)}, Q^{(1)}\right]$ is vertical we can do the following calculation where $f \in C^\infty(M)$:

$$\begin{aligned}
0 &= \left[\left[Q^{(1)}, Q^{(1)}\right], Q^{(1)}\right](f) \\
&= \left[Q^{(1)}, Q^{(1)}\right]\left(Q^{(1)}\right)(f) + Q^{(1)}\left(\left[Q^{(1)}, Q^{(1)}\right](f)\right) \\
&= \left[Q^{(1)}, Q^{(1)}\right]\left(Q^{(1)}\right)(f) + Q^{(1)}(0) \\
&= \left[Q^{(1)}, Q^{(1)}\right](\rho^*(d_{\mathrm{dR}} f)) .
\end{aligned}$$

Thus comparing with equation (3.10) we have $\mathrm{rt}\left(\left[Q^{(1)}, Q^{(1)}\right]\right) = 0$. □

With this lemma we are ready to prove proposition 3.1.20 following the proof outlined in [LGLS20]

Proof of proposition 3.1.20. We are given a graded manifold E and vector field $Q^{(0)} + Q^{(1)}$ that together constitute a graded almost Lie algebroid. This thus means that $Q^{(0)}$ and $Q^{(1)}$ satisfy the equations (3.5), (3.6)[7] and (3.9) by lemma 3.1.17. These are also the equations needed for the deformation problem we described. To solve the extension problem, we need the equations (3.7) to have solutions: a necessary and sufficient condition for this is for the cohomology classes (3.8) to vanish for all $n \geq 2$. Using corollary 3.1.14 we have for $n \geq 3$ that

$$H^2\left(\mathfrak{X}^{(n)}_{\mathrm{vert}}(E), \left[Q^{(0)}, \cdot\right]\right) = 0.$$

Thus, the only case left to study is the $n = 2$ case. By proposition 3.1.16 $\left[Q^{(1)}, Q^{(1)}\right] \in \mathfrak{X}_{\mathrm{vert}}(E)$ and $Q^{(0)}$-closed. In this setting it may happen that $H^2\left(\mathfrak{X}^{(2)}_{\mathrm{vert}}(E), \left[Q^{(0)}, \cdot\right]\right)$ is non-trivial. However, note that it would also be sufficient for the cohomology class of $\left[Q^{(1)}, Q^{(1)}\right]$ to vanish. By lemma 3.1.21 together with the quasi-isomorphism from proposition 3.1.12 we have that the cohomology class of $\left[Q^{(1)}, Q^{(1)}\right]$ indeed vanishes. □

So, to recap: we have shown that the deformation problem initially outlined can be solved. Indeed starting with the graded almost-Lie algebroid $(E, Q^{(0)} + Q^{(1)})$ that one gets from the gemetric resolution we can define, inductively, arity n components of degree $+1$ denoted by $Q^{(n)}$ that satisfy the equations outlined for our deformation problem. This allows us to define the homological vector field Q of degree $+1$ as

$$Q = \sum_{n \geq 0} Q^{(n)}.$$

Hence, we have a Lie ∞-algebroid structure on E by theorem 2.2.15. This means we have shown the following corollary to proposition 3.1.20.

Corollary 3.1.22 (Corollary 3.79 in [LGLS20])**.** *Every geometric resolution (E, d, ρ) and every almost Lie algebroid structure on $E_{-1} \subset E$ can be extended to a Lie ∞-algebroid structure on E.*

[7] Here we already assume that $Q^{(1)}$ is chosen to be $[Q^{(0)}, \cdot]$-closed. We showed this was possible in the proof of proposition 3.1.16

Adding in proposition 1.2.39 which states that for a geometric resolution (E, d, ρ) of a singular foliation $\mathcal{F}$ the bundle $E_{-1} \subset E$ can be equipped with an almost Lie algebroid structure we have the following corollary.

Corollary 3.1.23 (Corollary 3.80 in [LGLS20]). *Every geometric resolution (E, d, ρ) of a singular foliation $\mathcal{F}$ admits a Lie ∞-algebroid structure over it.*

Notice that this is exactly the content of theorem 3.1.2.

3.2 Examples of Universal Lie ∞-algebroids

Example 3.2.1 (Example 3.101 in [LGLS20]). In example 1.2.14 a geometric resolution for the singular foliation $\mathcal{F}_\varphi$ was described. We can define brackets to get a universal Lie ∞-algebroid of $\mathcal{F}_\varphi$. First some notation: we let $I = (i_1, \ldots, i_j)$ denote a multi-index, I^{i_s} denotes the multi-index I where we dropped i_s, $\bullet$ denotes concatenation of multi-indices, $\epsilon(i_1, \ldots, i_k)$ denotes the sign of the permutation needed to bring $i_1, \ldots, i_k$ up to the front in $I_1 \bullet \cdots \bullet I_k$ in that given order and finally ∂_I denotes $\frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_j}}$. Now one can define brackets as follows

$$\{\partial_{I_1}, \ldots, \partial_{I_k}\}_k = \sum_{i_1 \in I_1, \ldots, i_k \in I_k} \epsilon(i_1, \ldots, i_k) \frac{\partial^k \varphi}{\partial x_{i_1} \cdots \partial x_{i_k}} \partial_{I_1^{i_1} \bullet \cdots \bullet I_k^{i_k}}.$$

For example, we can calculate the 2-ary bracket

$$\left\{ \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} \wedge \frac{\partial}{\partial x_5} \wedge \frac{\partial}{\partial x_6} \right\}_2.$$

For short notation we write the first entry as ∂_{123} and the second one as ∂_{456}. Now note that both of these trivector fields are actually degree -2 elements. Indeed $E_{-2} = \wedge^2 T\mathbb{C}^n$ so $\Gamma(E_{-2}) = \mathfrak{X}^3$. Now for $k = 2$ one can show that $\epsilon(i, j) = (-1)^{i+j-1}$ and so the RHS of the formula above becomes

$$\sum_{\substack{i=1,2,3 \\ j=4,5,6}} (-1)^{i+j-1} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \partial_{\{1,2,3\}^i \bullet \{4,5,6\}^j}.$$

So now we can calculate the complete 2-ary bracket but as this would become quite a long formula that would not bring much more insight, we focus on for example the $i = 1, j = 4$ term:

$$\frac{\partial^2 \varphi}{\partial x_1 \partial x_4} \left(\frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_5} \wedge \frac{\partial}{\partial x_6} \right).$$

Now as $\{\cdot, \cdot\}_2$ is a degree $+1$ bracket we have that $\deg\{\partial_I, \partial_J\}_2 = \deg \partial_I + \deg \partial_J + 1$ so in this case we have that $\deg\{\partial_{123}, \partial_{456}\}_2 = -2 - 2 + 1 = -3$. And indeed, the term above is a four vector field and by definition of the geometric resolution this corresponds to a degree -3 element.

♦

Example 3.2.2 (Example 3.97 in [LGLS20]). In example 1.2.58 we displayed a geometric resolution for the singular foliation arising from the action of $\mathfrak{sl}_2(\mathbb{R})$ on $\mathbb{R}^2$. We now give a Lie ∞-algebroid structure on the given geometric resolution. Then we may also immediately conclude that this is the universal one. Since we saw that $E_{-1} \cong \mathfrak{sl}_2(\mathbb{R})$ [1] we can define the bracket between two elements in E_{-1} to just be the Lie bracket of the corresponding elements in $\mathfrak{sl}_2(\mathbb{R})$ and then adjust the degree. This procedure gives a way to compute the bracket between constant sections of E_{-1} and by the Leibniz identity we can extend this to the whole bundle. Now we define a bracket between elements of E_{-1} and E_{-2} for which we first note that

$$\{\widetilde{e}, ds\} = xy\{\widetilde{e}, \widetilde{h}\} + \rho(\widetilde{e})(xy)\widetilde{h} + \rho(\widetilde{e})(y^2)\widetilde{e} - x^2\{\widetilde{e}, \widetilde{f}\} = 0.$$

Indeed by the definition of the differential d we have that $ds = xy\widetilde{h} + y^2\widetilde{e} - x^2\widetilde{f}$ and so we also have that, by the Leibniz identity,

$$\begin{aligned}\{\widetilde{e}, xy\widetilde{h} + y^2\widetilde{e} - x^2\widetilde{f}\} = xy\{\widetilde{e}, \widetilde{h}\} &+ \rho(\widetilde{e})(xy)\widetilde{h} \\ &+ y^2\{\widetilde{e}, \widetilde{e}\} + \rho(\widetilde{e})(y^2)\widetilde{e} \\ &- x^2\{\widetilde{e}, \widetilde{f}\} - \rho(\widetilde{e})(x^2)\widetilde{f}.\end{aligned}$$

Now we use that $\rho(\widetilde{e}) = \underline{e} = x\frac{\partial}{\partial y}$ and so $\rho(\widetilde{e})(x^2) = 0$ and also $\{\widetilde{e}, \widetilde{e}\} = 0$ because, when we consider $\widetilde{e}$ as an element of $\mathfrak{sl}_2$ its bracket with itself vanishes.

Since the map d is injective on a dense open subset and is a derivation of the bracket $\{\cdot, \cdot\}$, this implies that $\{\widetilde{e}, s\} = 0$. A completely similar reasoning can then be used to also recover $\{\widetilde{f}, s\} = 0$ and $\{\widetilde{h}, s\} = 0$. Since this is a bracket defined only on constant sections, we need to extend it to a bracket between sections of E_{-1} and E_{-2} by the Leibniz property. There is no k-ary bracket for $k \geq 3$.

Note that this foliation $\mathcal{F}$ is given by a Lie algebra action and so there also is a transformation Lie algebroid $A = \mathbb{R}^2 \times \mathfrak{sl}_2(\mathbb{R})$. In example 2.2.13 it was shown that a Lie algebroid can be seen as a Lie ∞-algebroid denoted (A, Q), and so by theorem 3.1.3 there exists a Lie ∞-algebroid morphism $\Phi : (A, Q_A) \to (E, Q)$ which in this case can be seen to be the inclusion $\mathbb{1} \oplus 0 : A \to E = E_{-1} \oplus E_{-2}$. ♦

Chapter 4

Geometry of Singular Foliations

In this chapter we will exploit the universal Lie ∞-algebroids associated to singular foliations $\mathcal{F}$ to get information about the geometry of $\mathcal{F}$. We will do this by means of several cohomologies that can be associated to the Lie ∞-algebroids.

4.1 Universal Foliated Cohomology

A first example of a cohomology associated to a singular foliation is the so-called universal foliated cohomology.

Lemma 4.1.1 (Lemma 4.1 in [LGLS20])**.** *Let $\mathcal{F}$ be a singular foliation on M. Let (E, Q) and (E', Q') be two universal Lie ∞-algebroids of $\mathcal{F}$ with sheaves of functions $\mathcal{E}$ and $\mathcal{E}'$ respectively. The cohomologies of $(\mathcal{E}.Q)$ and $(\mathcal{E}', Q')$ are canonically isomorphic as graded commutative algebras.*

Proof. In chapter 3 we noted that there exist two Lie ∞-algebroid morphisms $\varphi : \mathcal{E}' \to \mathcal{E}$ and $\psi : \mathcal{E} \to \mathcal{E}'$ such that $\varphi \circ \psi \sim \mathbb{1}_{\mathcal{E}}$ and $\psi \circ \varphi \sim \mathbb{1}_{\mathcal{E}'}$ (here $\sim$ denotes homotopy equivalence of morphisms of Lie ∞-algebroids). So in particular $\Phi := \varphi \circ \psi$ and $\Psi := \mathbb{1}_{\mathcal{E}}$ are homotopic and by proposition 3.57 in [LGLS20] they are inverses on the level of cohomology. Moreover this does not depend on the choice of φ since another map $\tilde{\varphi} : \mathcal{E}' \to \mathcal{E}$ and φ are homotopic and so would define the same isomorphism on the level of cohomology. □

This lemma ensures that the following is well-defined.

Definition 4.1.2 (Definition 4.2 in [LGLS20])**.** Let $\mathcal{F}$ be a singular foliation on M that admits a geometric resolution. We call the cohomology of $(\mathcal{E}, Q)$, where $\mathcal{E}$ is the sheaf of functions of any universal Lie ∞-algebroid (E, Q) of the given foliation $\mathcal{F}$, the **universal foliated cohomology** of $\mathcal{F}$ and denote it by $H_{\mathfrak{U}}(\mathcal{F})$.

For the 0-th cohomology there is a nice interpretation that we will explain now. For the higher cohomologies this becomes more difficult. Note that by definition

$$H^0_{\mathfrak{U}}(\mathcal{F}) = \frac{\ker\left(Q : \mathcal{E}_0 \to \mathcal{E}_1\right)}{\operatorname{im}\left(Q : \mathcal{E}_{-1} \to \mathcal{E}_0\right)},$$

and by definition $\mathcal{E}_{-1} = \{0\}$ so $\operatorname{im}\left(Q : \mathcal{E}_{-1} \to \mathcal{E}_0\right) = \{0\}$. Hence it remains that

$$H^0_{\mathfrak{U}}(\mathcal{F}) = \ker\left(Q : \mathcal{E}_0 \to \mathcal{E}_1\right).$$

Now as noted above we know that $\mathcal{E}_0 = C^\infty(M)$ so pick an $f \in C^\infty(M)$ then $Qf \in \Gamma(E_{-1}^*)$ (also noted earlier). Now if $f \in H^0_{\mathfrak{U}}(\mathcal{F})$ we need Qf to be the zero section in E_{-1}^*. Clearly this is the case when for all $x \in \Gamma(E_{-1})$ we have that $\langle Qf, x\rangle = 0$. But as we remarked earlier we have that for all $x \in \Gamma(E_{-1})$ and $f \in C^\infty(M)$ the following holds $\langle Qf, x\rangle = \rho(x)f$ so using this we see that $\langle Qf, x\rangle = 0$ if and only if $\rho(x)f = 0$ for all $x \in \Gamma(E_{-1})$. So, since $\rho(\Gamma(E_{-1})) = \mathcal{F}$ it follows that this is equivalent to f being constant along the leaves of $\mathcal{F}$. By this we conclude that $H^0_{\mathfrak{U}}(\mathcal{F})$ consists of those $f \in C^\infty(M)$ that are constant along the leaves of $\mathcal{F}$.

There also is another cohomology that will be of interest. Define the following space

$$\Omega(\mathcal{F}) := \bigoplus_{k\geq 0} \mathrm{Hom}_{\mathcal{O}}\left(\wedge^k_{\mathcal{O}}\mathcal{F}, \mathcal{O}\right).$$

We will call this the space of **longitudinal forms** that we equip with the following differential (here the hat means that that specific entry is left out)

$$\begin{aligned} d_L(\alpha)(X_0, \dots, X_k) = &\sum_{i=0}^{k}(-1)^i X_i\left(\alpha(X_0, \dots, \widehat{X_i}, \dots, X_k)\right) \\ &+ \sum_{0\leq i<j\leq k}(-1)^{i+j}\alpha\left([X_i, X_j], X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_k\right), \end{aligned}$$

where $\alpha \in \Omega^k(\mathcal{F})$ and $X_0, \dots, X_k \in \mathcal{F}$. Equipped with this differential we can define the cohomology of the complex $(\Omega(\mathcal{F}), d_L)$ and call it the **longitudinal cohomology of $\mathcal{F}$** and denote it by $H^\bullet(\mathcal{F})$.

Remark 4.1.3. Note that each $\alpha \in \Omega^k(\mathcal{F})$ induces a k-form on the regular leaves of $\mathcal{F}$. Indeed, if $m \in M$ is a regular point of the foliation then it is contained in a regular leaf L_m and an $\alpha \in \Omega^k(\mathcal{F})$ induces the unique k-form α_{L_m} defined as

$$\alpha(X_1, \dots, X_k)|_m = \alpha_{L_m}(X_1(m), \dots, X_k(m)).$$

This need not to be true for the singular points of the foliation: consider $\mathcal{F} = \langle x\frac{\partial}{\partial x}\rangle$ and define $\alpha \in \Omega^1(\mathcal{F})$ as

$$\alpha : \wedge^1_{\mathcal{O}}\mathcal{F} \to \mathcal{O} : Fx\frac{\partial}{\partial x} \mapsto F.$$

Then the singular leaf $\{0\}$ has dimension 0 and so it does not permit a 1-form satisfying the above.

By definition of the universal Lie ∞-algebroid of $\mathcal{F}$ we have that $\rho(\Gamma(E_{-1})) = \mathcal{F}$ and so it makes sense to define the following map

$$\rho^* : \Omega(\mathcal{F}) \to \mathcal{E} : \rho^*\alpha(x_1, \dots, x_k) \mapsto \alpha(\rho(x_1), \dots, \rho(x_k)). \tag{4.1}$$

Here we have $x_1, \dots, x_k \in \Gamma(E_{-1})$ and $\rho^*\alpha \in \Gamma(E_{-1}^*)$.
Although the universal foliated cohomology and the longitudinal cohomology do not seem to be related to one another at first sight the following lemma states otherwise. For the proof we refer to [LGLS20]

Lemma 4.1.4 (Lemma 4.5 in [LGLS20])**.** *Let $\mathcal{F}$ be a singular foliation on M that admits a universal Lie ∞-algebroid. There is a canonical algebra morphism $\rho^* : H^\bullet(\mathcal{F}) \to H^\bullet_{\mathfrak{U}}(\mathcal{F})$.*

4.2 Isotropy Lie ∞-algebra

Androulidakis and Skandalis defined a rather simple invariant of a singular foliation in the *isotropy Lie algebra* at a point $m \in M$ on a manifold that has a singular foliation $\mathcal{F}$. It is defined as the quotient $\mathcal{F}(m)/I_m\mathcal{F}$. Here $\mathcal{F}(m)$ denotes the sections of $\mathcal{F}$ that vanish at m and $I_m \subset C^\infty(M)$ the functions vanishing at m. Note that $I_m\mathcal{F}$ forms a Lie ideal and so the quotient indeed is a Lie algebra. In this section we will provide a generalization of this notion by defining the *(homotopy) isotropy functor* by which we get, associated to each point $m \in M$, an L_∞-algebra that we will call the isotropy L_∞-algebra of $\mathcal{F}$ at $m \in M$. The isotropy Lie algebra as defined by Androulidakis and Skandalis will be recovered from this L_∞-algebra.

We begin by fixing some notation: let $V \to M$ be some vector bundle then by $\mathfrak{i}_m V$ we will denote the fiber of V above m. If $\varphi : V \to V'$ is a morphism of vector bundles over M then $\mathfrak{i}_m\varphi$ denotes the restriction of φ to the fibers. Now let (E, d, ρ) be a geometric resolution of a singular foliation $\mathcal{F}$ on M. Then the complex

$$\cdots \xrightarrow{\mathfrak{i}_m d^{(4)}} \mathfrak{i}_m E_{-3} \xrightarrow{\mathfrak{i}_m d^{(3)}} \mathfrak{i}_m E_{-2} \xrightarrow{\mathfrak{i}_m d^{(2)}} \ker(\mathfrak{i}_m\rho) \to 0, \tag{4.2}$$

may have cohomology as the following example illustrates.

Example 4.2.1. Recall the geometric resolution from example 1.2.58. Over the origin $0 \in \mathbb{R}^2$ we can form the following sequence

$$\mathfrak{i}_0\mathbb{R}[2] \xrightarrow{\mathfrak{i}_0 d^{(2)}} \ker(\mathfrak{i}_0\rho) \to 0. \tag{4.3}$$

By definition of the anchor map as $\rho(\tilde{e}) = \underline{e}, \rho(\tilde{f}) = \underline{f}$ and $\rho(\tilde{h}) = \underline{h}$ it is not hard to see that $\mathfrak{i}_0\rho$ vanishes and hence that also $\ker(\mathfrak{i}_0\rho) = \mathfrak{sl}_2[1]$. Similarly by definition of $d^{(2)}$ one can see that $\mathfrak{i}_0 d^{(2)} = 0$ and hence the sequence (4.3) has cohomology.

♦

Remark 4.2.2. Note that example 4.2.1 also illustrates that the geometric resolution from example 1.2.58 is minimal at the origin.

This means we can define the following graded vector space

$$H^\bullet(\mathcal{F}, m) = \bigoplus_{i \geq 1} H^{-i}(\mathcal{F}, m). \tag{4.4}$$

Here $H^{-i}(\mathcal{F}, m)$ denotes the degree $-i$ cohomology of the complex (4.2). At first sight it is not immediately clear why this construction is independent of the chosen geometric resolution. To show this we have the following lemma.

Lemma 4.2.3 (Lemma 4.8 in [LGLS20])**.** *Let $\mathcal{F}$ be a singular foliation that admits geometric resolutions in the neighborhood of a point $m \in M$. Then the following holds*

1. *the cohomology of the complex (4.2) is independent of the choice of geometric resolution for $\mathcal{F}$,*

2. *For every geometric resolution (E, d, ρ) of $\mathcal{F}$ which is minimal at m and every $i \geq 2$, the vector space $H^{-i}(\mathcal{F}, m)$ is canonically isomorphic to $\mathfrak{i}_m E_{-i}$. Furthermore $H^{-1}(\mathcal{F}, m)$ is canonically isomorphic to the kernel of $\mathfrak{i}_m\rho : \mathfrak{i}_m E_{-1} \to T_m M$.*

Proof. 1. An immediate consequence of lemma 1.2.55 is that two geometric resolutions (E, d, ρ) and (E', d', ρ') are homotopy equivalent. All the maps that are involved in this homotopy equivalence are $C^\infty(M)$-linear and so they restrict nicely to the fibers where they give a homotopy equivalence between the complexes $(\mathfrak{i}_m E, \mathfrak{i}_m d, \mathfrak{i}_m \rho)$ and $(\mathfrak{i}_m E', \mathfrak{i}_m d', \mathfrak{i}_m \rho')$ which immediately gives the result.

2. Let (E, d, ρ) be a geometric resolution of $\mathcal{F}$ which is minimal at $m \in M$. By definition of minimality we have that for all $i \geq 2$ the maps $\mathfrak{i}_m d^{(i)} : \mathfrak{i}_m E_{-i} \to \mathfrak{i}_m E_{-i+1}$ vanish and so $\ker\left(\mathfrak{i}_m d^{(i)}\right) = \mathfrak{i}_m E_{-i}$ and $\operatorname{im}\left(\mathfrak{i}_m d^{(i+1)}\right) = \{0\}$ by which it immediately follows that $H^{-i}(\mathcal{F}, m)$ and $\mathfrak{i}_m E_{-i}$ are isomorphic. The statement for $i = 1$ is obvious. □

Proposition 4.2.4 (Proposition 4.10 in [LGLS20]). *Let $\mathcal{F}$ be a singular foliation that admits geometric resolutions on the neighborhood of some $m \in M$. Then the following are equivalent:*

1. *There is a neighborhood of $m \in M$ on which $\mathcal{F}$ is a Debord foliation,*
2. *$H^{-i}(\mathcal{F}, x) = 0$ for all $i \geq 2$ and for all x in a neighborhood of m,*
3. *$H^{-2}(\mathcal{F}, m) = 0$.*

Proof. $\boxed{1 \Rightarrow 2}$ In the smooth case $\mathcal{F}$ is a Debord foliation if and only if it is a projective $C^\infty(M)$-module. Hence it admits a geometric resolution of length 1[1]. Hence we certainly have that $H^{-i}(\mathcal{F}, y) = 0$ for all $i \geq 2$ and y in a neighborhood of m since $\mathfrak{i}_y E_{-i} = 0$ for all $i \geq 2$.

$\boxed{2 \Rightarrow 3}$ This is quite obvious.

$\boxed{3 \Rightarrow 1}$ Assume that $H^{-2}(\mathcal{F}, m) = 0$ then by definition we have that $\ker \mathfrak{i}_m d^{(2)} = \operatorname{im} \mathfrak{i}_m d^{(3)}$. Now use the following result from linear algebra: perturbing the elements of a matrix does not decrease the rank of the matrix (see appendix B for a small proof of this fact). Using this we see that there exists some neighborhood U of $m \in M$ such that:

1. the dimension of the image of $\mathfrak{i}_x d^{(3)}$ at every $x \in U$ has to be greater than or equal to its dimension at m,
2. the dimension of $\ker \mathfrak{i}_x d^{(2)}$ at every $x \in U$ has to be lower than or equal to its dimension at m.

But notice that, by definition of a complex, we always have that $\operatorname{im} \mathfrak{i}_x d^{(3)} \subset \ker \mathfrak{i}_x d^{(2)}$ and so we always have $\dim \operatorname{im} \mathfrak{i}_x d^{(3)} \leq \dim \ker \mathfrak{i}_x d^{(2)}$ for all $x \in U$. Now note that from our observations above it also follows that $\dim \operatorname{im} \mathfrak{i}_x d^{(3)} \geq \operatorname{im} \mathfrak{i}_m d^{(3)} = \dim \ker \mathfrak{i}_m d^{(2)} \geq \dim \ker \mathfrak{i}_x d^{(2)}$ which can only hold when $\ker \mathfrak{i}_x d^{(2)} = \operatorname{im} \mathfrak{i}_x d^{(3)}$ for all $x \in U$. From this it immediately follows that $H^{-2}(\mathcal{F}, x) = 0$ for all $x \in U$. This implies that the map $d^{(2)}|_U : E_{-2}|_U \to E_{-1}|_U$ has constant rank and so $d^{(2)}(E_{-2}|_U) \subset E_{-1}|_U$ as subbundles. From this it follows that $E'_{-1} := \left(E_{-1}/d^{(2)}(E_{-1})\right)|_U$ is a vector bundle and because $\rho \circ d^{(2)} = 0$ the anchor restricts to E'_{-1} to define a morphism of $C^\infty(M)$-modules $\rho(E'_{-1}) \to \mathcal{F}$ which, by construction, is an isomorphism. Because $\Gamma(E'_{-1})$ is projective it follows that $\mathcal{F}$ is projective and hence Debord on U. □

[1] As explained above, a geometric resolution of the $C^\infty(M)$-module $\mathcal{F}$ can be thought of as finding a projective resolution for $\mathcal{F}$ in the category of $C^\infty(M)$-modules (but again note it is not exactly the same as not all projective modules come from vector bundle sections).

Intermezzo: Isotropy Functor

Let M be a smooth manifold then for all $k \in \mathbb{N} \cup \{\infty\}$ the Lie k-algebroids together with the Lie ∞-algebroid morphisms form a category that we denote by $\mathbf{Lie - k - algoid_M}$ (one can also consider the category where we mod out the arrows by homotopy equivalences to get the category $\mathbf{hLie - k - algoid_M}$ of homotopy equivalent Lie ∞-algebroids). When M is a point we recover L_∞-algebras which themselves form a category that we denote by $\mathbf{Lie - k - alg}$ (here there also is the homotopy equivalent counterpart $\mathbf{hLie - k - alg}$). Our aim is to define the *isotropy functor* at a point $m \in M$

$$h\mathfrak{J}_m : \mathbf{hLie - k - algoid_M} \to \mathbf{hLie - k - alg}. \tag{4.5}$$

We proceed by defining this functor on objects and then on arrows in the category without worrying about homotopy equivalences for now.

- **On objects:** let (F, Q_F) be a Lie k-algebroid over M with anchor ρ. According to the axioms of Lie k-algebroids the k-ary bracket restricts to the graded vector space

  $$K^\bullet(F, m) = \ker \mathfrak{i}_m \rho \oplus \bigoplus_{i \geq 2} \mathfrak{i}_m F_{-i}.$$

  Here we use $\ker \mathfrak{i}_m \rho$ to ensure that the 2-ary bracket between elements in $K^\bullet(\mathcal{F}, m)$ is well-defined. This has to do with the chosen extension to a local section of an element in $\mathfrak{i}_m E_{-1}$. When restricting to $\ker \mathfrak{i}_m \rho$ the Leibniz identity ensures that the bracket is independent of the chosen extension.

- **On arrows:** let $\Phi : \Gamma(S(F^*))) \to \Gamma(S(E^*)))$ be an arbitrary Lie k-algebroid morphism from (E, Q_E) to (F, Q_F). By $C^\infty(M)$-linearity this restricts to a morphism

  $$\mathfrak{i}_m \Phi : S(\mathfrak{i}_m F^*) \to S(\mathfrak{i}_m E^*).$$

  Considering only the linear part of Φ we see that this linear part must be a chain map (because it must commute with $Q_E^{(0)}$ and $Q_F^{(0)}$ and they are dual to the differentials in the complex) so we get a graded algebra morphism

  $$\mathfrak{J}_m(\Phi) : S(K^\bullet(F, m)^*) \to S(K^\bullet(E, m)^*),$$

  which, by definition, is a Lie k-algebra morphism.

In this way we have completely defined the functor $\mathfrak{J}_m$. We now pass to the categories $\mathbf{hLie - k - algoid_M}$ and $\mathbf{hLie - k - alg}$ to define the isotropy functor. For this we need the following lemma.

Lemma 4.2.5. *Let $\Phi, \Psi : (E, Q) \to (E', Q')$ be two homotopic Lie ∞-algebroid morphisms over M. For every point $m \in M$, $\mathfrak{J}_m(\Phi), \mathfrak{J}_m(\Psi) : \mathfrak{J}_m(E, Q) \to \mathfrak{J}_m(E', Q')$ are homotopic L_∞-algebra morphisms.*

Using lemma 4.2.5 it is clear that $\mathfrak{J}_m$ passes to the category of homotopy equivalent Lie ∞-algebroids

$$h\mathfrak{J}_m : \mathbf{hLie - k - algoid_M} \to \mathbf{hLie - k - alg}.$$

Back to the isotropy Lie ∞-algebra

We will now use the isotropy functor to define the **isotropy L_∞-algebra** of a singular foliation at a point $m \in M$. Let $\mathcal{F}$ be a foliation on a manifold M with universal Lie ∞-algebroid (E, Q) then we may apply $\mathfrak{J}_m$ to (E, Q) and get a L_∞-algebra structure on the complex

$$\cdots \xrightarrow{\mathfrak{i}_m d^{(4)}} \mathfrak{i}_m E_{-3} \xrightarrow{\mathfrak{i}_m d^{(3)}} \mathfrak{i}_m E_{-2} \xrightarrow{\mathfrak{i}_m d^{(2)}} \ker(\mathfrak{i}_m \rho) \to 0. \tag{4.6}$$

Because $\mathfrak{J}_m$ maps homotopy equivalences of Lie ∞-algebroids to homotopy equivalences of L_∞-algebras, we may restrict to 'looking up to homotopy'. All universal Lie ∞-algebroids of $\mathcal{F}$ are unique up to homotopy so, after applying the isotropy functor, any other choice of universal Lie ∞-algebroid results in a homotopy equivalent L_∞-algebra. By this observation, combined with lemma 4.2.3 we have that we may choose the universal Lie ∞-algebroid in a particular way as to induce an L_∞-algebra structure on the cohomology of (4.6) i.e. choosing the Lie ∞-algebroid to be minimal at $m \in M$ (this is (locally) always possible).

Definition 4.2.6 (Definition 4.11 in [LGLS20])**.** Let (E, Q) be a universal Lie ∞-algebroid of $\mathcal{F}$ which is minimal at $m \in M$. Then $h\mathfrak{J}_m(E, Q)$ is an L_∞-algebra structure on $H^\bullet(\mathcal{F}, m)$, which we denote by $(H^\bullet(\mathcal{F}, m), Q_m)$ and call the **isotropy L_∞-algebra of $\mathcal{F}$ at** m.

At first sight this isotropy L_∞-algebra seems to depend on the choice of universal Lie ∞-algebroid of $\mathcal{F}$ but the following proposition ensures this is not the case.

Proposition 4.2.7 (Proposition 4.12 in [LGLS20])**.** *Any two isotropy Lie ∞-algebras at m of $\mathcal{F}$, constructed out of two universal Lie ∞-algebroids of $\mathcal{F}$ minimal at m, are isomorphic through an isomorphism whose linear part is the identity on $H^\bullet(\mathcal{F}, m)$. Furthermore the restricted 2-ary bracket is a graded Lie algebra bracket on $H^\bullet(\mathcal{F}, m)$ which does not depend on any choices made in the construction.*

Before proving this proposition, we show a lemma which will be helpful in the proof but for which the proof in [LGLS20] is very brief and without details. It concerns L_∞-algebras whose 1-ary bracket vanishes which is the case for an isotropy L_∞-algebra as defined above. This follows from minimality of the universal Lie ∞-algebroid. Indeed the 1-ary bracket on the isotropy L_∞-algebra is dual to the map $\mathfrak{i}_m d$. Since the universal Lie ∞-algebroid is minimal at $m \in M$ the maps $\mathfrak{i}_m d$ vanish and so the 1-ary bracket also does.

Lemma 4.2.8 (Lemma 4.13 in [LGLS20])**.** *Let (V, Q) and (V', Q') be two L_∞-algebras whose 1-ary bracket is equal to zero. Then the following holds:*

1. *its 2-ary bracket is a graded Lie algebra bracket,*
2. *the linear part of any L_∞-algebra morphism from (V, Q) to (V', Q') is a graded Lie algebra morphism of the 2-ary bracket,*
3. *the L_∞-algebras (V, Q) and (V', Q') are isomorphic to one another if and only if they are homotopy equivalent.*

Proof. 1. Since we are actually working with an $L_\infty[1]$-algebra we want to show that the space $(V[-1], [\cdot,\cdot])$ is a graded Lie algebra. Here the bracket $[\cdot,\cdot]$ is the one we get from the binary bracket $\{\cdot,\cdot\}_2$ on V (from now on we will denote this binary bracket without the subscript 2); in the following way

$$[v_1, v_2] = (-1)^{|v_1|}\{x_1, x_2\} = (-1)^{|x_1|+1}\{x_1, x_2\}.$$

Here v_1 and v_2 are representatives of the elements $x_1, x_2 \in V$ in $V[-1]$ respectively. Showing that the bracket $[\cdot,\cdot]$ is a graded Lie algebra bracket amounts to showing that the following equation is satisfied for all $v_1, v_2, v_3 \in V[-1]$

$$[v_1, [v_2, v_3]] = [[v_1, v_2], v_3] + (-1)^{|v_1||v_2|}[v_2, [v_1, v_3]]. \tag{4.7}$$

We will now do the following steps: first we show a Jacobi-like identity for the binary bracket $\{\cdot,\cdot\}$ and secondly we will show that equation (4.7) induces the identity found in the first step, showing by computation that the found identity and (4.7) are actually equivalent.

Consider the $n = 3$ higher Jacobi identity

$$\sum_{\sigma\in\mathcal{S}(2,1)} \epsilon(\sigma)\{\{x_{\sigma(1)}, x_{\sigma(2)}\}, x_{\sigma(3)}\} = 0. \tag{4.8}$$

It is easy to compute that $\mathcal{S}(2,1) = \{e, (23), (123)\}$ and so we can write out (4.8) out in full with the corresponding Koszul signs

$$\{\{x_1, x_2\}, x_3\} + (-1)^{|x_1||x_2|}\{\{x_1, x_3\}, x_2\} + (-1)^{|x_1||x_2|+|x_1||x_3|}\{\{x_2, x_3\}, x_1\} = 0. \tag{4.9}$$

Note that $\{\{x_1, x_3\}, x_2\} = (-1)^{|x_1||x_2|}\{\{x_3, x_1\}, x_2\}$ so replacing this term in (4.9) and multiplying everything by $(-1)^{|x_1||x_3|}$ we get that

$$(-1)^{|x_1||x_3|}\{\{x_1, x_2\}, x_3\} + (-1)^{|x_2||x_3|}\{\{x_3, x_1\}, x_2\} + (-1)^{|x_1||x_2|}\{\{x_2, x_3\}, x_1\} = 0. \tag{4.10}$$

On the other hand, we have that

$$\begin{aligned}[v_1, [v_2, v_3]] &= \left[v_1, -(-1)^{|x_2|}\{x_2, x_3\}\right] \\ &= -(-1)^{|x_2|} \cdot -(-1)^{|x_1|}\{x_1, \{x_2, x_3\}\} \\ &= (-1)^{|x_1|+|x_2|+|x_1||x_2|+|x_1||x_3|+|x_1|}\{\{x_2.x_3\}, x_1\}.\end{aligned}$$

$$\begin{aligned}[[v_1, v_2], v_3] &= \left[-(-1)^{|x_1|}\{x_1, x_2\}, v_3\right] \\ &= (-1)^{|x_1|} \cdot (-1)^{|x_1|+|x_2|+1}\{\{x_1, x_2\}, x_3\} \\ &= -(-1)^{|x_2|}\{\{x_1, x_2\}, x_3\}.\end{aligned}$$

$$\begin{aligned}(-1)^{(|x_1|+1)(|x_2|+1)}[v_2, [v_1, v_3]] &= (-1)^{|x_1||x_2|+|x_1|+|x_2|}\left[v_2, -(-1)^{|x_1|}\{x_1, x_3\}\right] \\ &= -(-1)^{|x_1|} \cdot -(-1)^{|x_2|}(-1)^{|x_1||x_2|+|x_1|+|x_2|}\{x_2, \{x_1, x_3\}\} \\ &= (-1)^{|x_1|+|x_2|}(-1)^{|x_1||x_2|+|x_1|+|x_2|}(-1)^{|x_2|(|x_1|+|x_3|+1)}\{\{x_1, x_3\}, x_2\} \\ &= (-1)^{|x_2||x_3|+|x_2|+|x_1||x_3|}\{\{x_3, x_1\}, x_2\}.\end{aligned}$$

Now we apply (4.7) and replace all terms with the $[\cdot,\cdot]$-bracket by its $\{\cdot,\cdot\}$-bracket counterpart. This gives us the following identity

$$\begin{aligned}(-1)^{|x_1|+|x_2|+|x_1||x_2|+|x_1||x_3|+|x_1|}\{\{x_2.x_3\},x_1\} = &-(-1)^{|x_2|}\{\{x_1,x_2\},x_3\} \\ &+(-1)^{|x_2||x_3|+|x_2|+|x_1||x_3|}\{\{x_3,x_1\},x_2\}.\end{aligned} \tag{4.11}$$

We now multiply equation (4.11) by $(-1)^{-|x_2|-|x_1||x_3|}$ and after canceling some terms in the exponents it is not hard to see that we recover equation (4.10). So to conclude: we have shown that equation (4.7) implies equation (4.10); of course one could do the process above in reverse to show that equation (4.10) implies equation (4.7). So we see that these two equations are exactly the same but written down with elements in different vector spaces. Since we derived equation (4.10) from the higher Jacobi identity we finally have that $[\cdot,\cdot]$ indeed satisfies the graded Jacobi identity (4.7) and so $(V,[\cdot,\cdot])$ forms a graded Lie algebra which is what we wanted to show.

2. This follows immediately from the definition of an L_∞-algebra morphism together with the observation that the 1-ary brackets are zero. Indeed, writing out the definition 2.1.4 for $k=2$ and $n=2$ we have the following (where μ_k denotes the k-ary bracket on V and μ'_k the k-ary bracket on V')

$$\begin{aligned}&\sum_{\sigma\in\mathcal{S}(2,0)}\epsilon(\sigma)f_1\left(\mu_2\otimes\mathbb{1}^{\otimes 0}\right)\left(x_{\sigma(I)}\right) \\ &\qquad=\sum_{\substack{\sigma\in\mathcal{S}(k_1,\dots,k_j)\\ k_1+\dots+k_j=2\\ j=1,2}}\frac{\epsilon(\sigma)}{j!}\mu'_j\left(f_{k_1}\otimes\dots\otimes f_{k_j}\right)(x_{\sigma(I)}).\end{aligned} \tag{4.12}$$

Hence using $\mathcal{S}(2,0)=\{\mathbb{1}\}$ and $\mathcal{S}(1,1)=\mathcal{S}_2=\{\mathbb{1},\sigma\}$ this yields

$$f_1(\mu_2)(x_I)=\mu'_1(f_2)(x_I)+\frac{1}{2}\mu'_2(f_1\otimes f_1)(x_I)+\frac{1}{2}(-1)^{|x_1||x_2|}\mu'_2(f_1\otimes f_1)(x_{\sigma(I)})$$

Using that $\mu'_1\equiv 0$ now yields that

$$f_1(\mu_2)(x_I)=\frac{1}{2}\mu'_2(f_1\otimes f_1)(x_I)+\frac{1}{2}(-1)^{|x_1||x_2|}\mu'_2(f_1\otimes f_1)(x_{\sigma(I)}).$$

Now using that $f_1:V\to V'$ is a degree 0 map we see that $|x_1|=|f_1(x_1)|$ and the same for $|x_2|$. This allows us to write

$$f_1(\mu_2)(x_I)=\frac{1}{2}\mu'_2(f_1\otimes f_1)(x_I)+\frac{1}{2}(-1)^{|f_1(x_1)||f_1(x_2)|}\mu'_2(f_1\otimes f_1)(x_{\sigma(I)}).$$

Using graded symmetry of the binary bracket μ'_2 we have

$$\mu'_2(f_1\otimes f_1)(x_I)=(-1)^{|f_1(x_1)||f_1(x_2)|}\mu'_2(f_1\otimes f_1)(x_{\sigma(I)}),$$

and so

$$f_1(\mu_2)(x_I)=\mu'_2(f_1\otimes f_1)(x_I).$$

I.e. we recover that

$$f_1(\mu_2(x_1, x_2)) = \mu_2'(f_1(x_1), f_1(x_2)),$$

This last equation exactly states that the linear part $f_1 : V \to V'$ is a graded Lie algebra homomorphism.

3. Let $\Phi : S(V'^*) \to S(V^*)$ be an L_∞-algebra morphism. This is a morphism between the spaces of functions on V' and V respectively. Just like in the smooth manifold case one can show that a morphism between the spaces of functions is invertible if and only if the underlying/induced smooth map between the manifolds is invertible. Hence one can show that Φ is invertible if and only if the linear part $\varphi : V \to V'$ is invertible. Clearly, when $\Phi : (V, Q) \to (V', Q')$ is a part of a homotopy equivalence, the linear part $\varphi : V \to V'$ is also part of a homotopy equivalence. Because the 1-ary brackets correspond to the differentials in the complexes V and V' these differentials are zero. A homotopy equivalence between complexes with the zero differential clearly has to be invertible. □

Proof of proposition 4.2.7. By the functorial properties of the isotropy functor $h\mathfrak{J}_m$ two isotropy L_∞-algebras at some point $m \in M$ of the foliation $\mathcal{F}$ and which are minimal at m are homotopy equivalent. By (3) of lemma 4.2.8 they are isomorphic and $\{\cdot,\cdot\}_2$ yields a graded Lie algebra structure by (1) of lemma 4.2.8. □

So after all we are left with a graded Lie algebra $(H^\bullet(\mathcal{F}, m), \{\cdot,\cdot\}_2)$. In particular this bracket restricts to $H^{-1}(\mathcal{F}, m)$; indeed $\{\cdot,\cdot\}_2$ has degree $+1$ and so when $x, y \in H^{-1}(\mathcal{F}, m)$ we have

$$\deg(\{x, y\}_2) = \deg(x) + \deg(y) + 1 = -1.$$

This makes $(H^{-1}(\mathcal{F}, m), \{\cdot,\cdot\}_2)$ into an ordinary Lie algebra. As the name of the isotropy L_∞-algebra already hinted, the isotropy Lie algebra from Androulidakis and Skandalis can indeed be recovered from this L_∞-algebra as the following proposition states.

Proposition 4.2.9 (Proposition 4.14 in [LGLS20])**.** *The isotropy Lie algebra of $\mathcal{F}$ at $m \in M$ is isomorphic to the degree -1 component of the isotropy L_∞-algebra $H^\bullet(\mathcal{F}, m)$ of $\mathcal{F}$ at m.*

Proof. We are going to construct a Lie algebra (iso)morphism $\tau : H^{-1}(\mathcal{F}, m) \to \mathfrak{g}_m$ where $\mathfrak{g}_m = \mathcal{F}(m)/I_m\mathcal{F}$ is the isotropy Lie algebra as defined by Androulidakis and Skandalis. To start pick an element $e \in \ker \mathfrak{i}_m\rho \subset \mathfrak{i}_m E_{-1}$ and let $\tilde{e}$ be a (local) extension of e to a section of E_{-1} i.e. $\tilde{e}(m) = e$. By definition of a geometric resolution $\rho(\Gamma(E_{-1})) = \mathcal{F}$ and it easy to see that $\rho(\tilde{e}) \in \mathcal{F}(m)$. The element $\rho(\tilde{e})$ has an equivalence class in $\mathcal{F}(m)/I_m\mathcal{F}$ that we denote by $\tau(e)$. This is also how we define the map τ. We now check that this is indeed a well-defined map. For this let $\hat{e}$ denote another extension of e to $\Gamma(E_{-1})$. Then clearly we must have that

$$\tilde{e} = \hat{e} + (\text{section vanishing at } m),$$

but one can also show that $I_m\Gamma(E_{-1}) = \{X \in \Gamma(E_{-1}) \mid X(m) = 0\}$ and so $\tilde{e}$ and $\hat{e}$ can only differ by an element in $I_m\Gamma(E_{-1})$. Now using that the anchor is $C^\infty(M)$-linear together with $\rho(\Gamma(E_{-1}) = \mathcal{F}$ we see that $\rho(I_m\Gamma(E_{-1})) = I_m\mathcal{F}$ and so $\rho(\tilde{e})$ and $\rho(\tilde{e})$ differ by an element in $I_m\mathcal{F}$ which exactly means that they define the same element $\tau(e) \in \mathfrak{g}_m$. We now proceed by showing that τ is a bijection.

- Surjectivity: again, the crucial observation to be made here is that $\rho\left(\Gamma(E_{-1})\right) = \mathcal{F}$. Let $\overline{X} \in \mathfrak{g}_m$ and X a lift of $\overline{X}$ to $\mathcal{F}(m)$ which you can clearly take (at least locally) to be of the form $\rho(\widetilde{e})$ where $\widetilde{e}$ is again an extension of element $e \in \ker \mathfrak{i}_m \rho$ to $\Gamma(E_{-1})$.

- Injectivity: pcik an element $e \in \mathfrak{i}_m E_{-1}$ and let $\widetilde{e} \in \Gamma(E_{-1})$ again be a local extension of e. Note that $\tau(e) = \overline{\rho(\widetilde{e})} \in \mathfrak{g}_m$ and so $\tau(e) = 0$ if and only if $\rho(\widetilde{e}) \in I_m \mathcal{F}$. This precisely means that

$$\rho(\widetilde{e}) = \sum_{i=1}^{k} f_i X_i, \quad f_i \in I_m \text{ and } X_i \in \mathcal{F}.$$

Again, we use that $\rho\left(\Gamma(E_{-1})\right) = \mathcal{F}$ so we can choose sections $\widetilde{e}_i \in \Gamma(E_{-1})$ such that $\rho(\widetilde{e}_i) = X_i$ for all $i = 1, \ldots, k$. Hence, we have that

$$\rho(\widetilde{e}) = \sum_{i=1}^{k} f_i X_i \Leftrightarrow \rho\left(\widetilde{e} - \sum_{i=1}^{k} f_i \widetilde{e}_i\right) = 0.$$

Because the complex (E, d, ρ) is a geometric resolution we have that $\operatorname{im} d^{(2)} = \ker \rho$. The equation above implies that there exists an element $h \in \Gamma(E_{-1})$ such that

$$\widetilde{e} - \sum_{i=1}^{k} f_i \widetilde{e}_i = d^{(2)} h.$$

Evaluating this expression at $m \in M$ gives that

$$\widetilde{e}(m) = \sum_{i=1}^{k} f_i(m) \widetilde{e}_i(m) + \left(\mathfrak{i}_m d^{(2)}\right)(h(m)),$$

and since $f_i \in I_m$ we have $f_i(m) = 0$ for all $i = 1, \ldots, k$. Because the geometric resolution was chosen to be minimal at $m \in M$ we also have that $\mathfrak{i}_m d^{(2)} \equiv 0$ showing that $e = \widetilde{e}(m) = 0$. This shows τ is injective.

□

Examples of Isotropy L_∞-algebras

We will now display some examples of isotropy L_∞-algebras.

Example 4.2.10 (Example 4.20 in [LGLS20])**.** Let $\mathcal{F}$ be a regular foliation on a manifold M. We know by Frobenius's theorem that $\mathcal{F}$ can be seen as $\mathcal{F} = \Gamma_c(TF)$ for TF the associated tangent distribution. Hence we can form the minimal geometric resolution $E_{-1} := T[1]F \subset T[1]M$, $E_{-i} = 0$ for all $i > 1$ and the anchor map just the inclusion map. In particular we have that around every point $m \in M$ one has $H^\bullet(\mathcal{F}, m) = 0$. Note that this also implies that the isotropy Lie algebra $\mathfrak{g}_m$ is zero for regular foliations, a fact that was already shown in lemma 1.1 in [AZ13]. We can also use exactly the same argument to show that for a regular point $m \in M$ of a *singular foliation* the isotropy L_∞-algebra is identically zero.

♦

Example 4.2.11 (Example 4.22 in [LGLS20]). Consider the foliation given by the action of $\mathfrak{sl}_2(\mathbb{R})$ on $\mathbb{R}^2$ as discussed earlier in examples 1.2.58 and 3.2.2. Using the universal Lie ∞-algebroid structure found in example 3.2.2 we can use the definition of the isotropy L_∞-algebra to compute it. From example 4.2.1 we see that the we get an isotropy Lie 2-algebra

$$H^\bullet(\mathcal{F},0) = \mathbb{R}\,[2] \oplus \mathfrak{sl}_2(\mathbb{R})\,[1]\,.$$

We have a bracket on $H^{-1}(\mathcal{F},0) \cong \mathfrak{sl}_2(\mathbb{R})\,[1]$ and for degree reasons all other brackets vanish.

♦

Example 4.2.12 (Example 4.26 in [LGLS20]). We have encountered the foliation $\mathcal{F}_\varphi$ earlier; in example 3.2.1 we gave a universal Lie ∞-algebroid for $\mathcal{F}_\varphi$ which we will now use to get an isotropy L_∞-algebra at the origin. All (first order) partial derivatives of φ vanish at the origin. So the geometric resolution from example 1.2.60 becomes minimal at the origin. From this it also immediately follows, by definition of the geometric resolution, that $H^{-k}(\mathcal{F}_\varphi,0) = \wedge^{k+1}\mathbb{C}^n$ and that for all $k \geq 2$ the k-ary brackets are the restrictions of the ones in example 3.2.1 in the following way

$$\{\partial_{I_1},\ldots,\partial_{I_k}\}_k = \sum_{i_1\in I_1,\ldots,i_k\in I_k} \epsilon(i_1,\ldots,i_k)\frac{\partial^k\varphi}{\partial x_{i_1}\cdots\partial x_{i_k}}(0)\partial_{I_1^{i_1}\bullet\cdots\bullet I_k^{i_k}}. \tag{4.13}$$

♦

4.3 Minimal Rank Lie algebroids Defining a Foliation

Above we saw that Lie algebroids account for a large class of examples of singular foliations. In this section we will exploit the isotropy L_∞-algebra to answer the following question: *does there always exist a Lie algebroid of minimal rank which locally induces the foliation $\mathcal{F}$?* Before proceeding with explaining these notions we do some preparatory work. The following proposition mentions Chevalley-Eilenberg cohomology, a very brief introduction to this formalism can be found in appendix C.

Proposition 4.3.1 (Proposition 4.27 in [LGLS20]). *Let $\mathcal{F}$ be a singular foliation that admits a geometric resolution of finite length in a neighborhood of $m \in M$. Equip $H^\bullet(\mathcal{F},m) = \oplus_{i\geq 1}H^{-i}(\mathcal{F},m)$ with the isotropy L_∞-algebra brackets $(\{\cdots\}_k)_{k\geq 2}$ constructed out of some universal Lie ∞-algebroid (E,Q) minimal at m. Then the following holds:*

1. *The restriction of $\{\cdot,\cdot\}_2$ in the following way*

$$\{\cdot,\cdot\}_2 : H^{-1}(\mathcal{F},m)\otimes H^{-2}(\mathcal{F},m) \to H^{-2}(\mathcal{F},m).$$

gives a Lie algebra representation of the Lie algebra $H^{-1}(\mathcal{F},m)$ on the vector space $H^{-2}(\mathcal{F},m)$ which does not depend on the choice of (E,Q).

2. *The restriction of the 3-ary bracket*

$$\{\cdot,\cdot,\cdot\}_3 : \wedge^3 H^{-1}(\mathcal{F},m) \to H^{-2}(\mathcal{F},m)$$

is a 3-cocycle for the Chevalley-Eilenberg complex of $H^{-1}(\mathcal{F},m)$ valued in the representation on $H^{-2}(\mathcal{F},m)$.

3. *The cohomology class of this cocycle does not depend on the choice of* (E, Q).

Proof. 1. This is an immediate consequence of proposition 4.2.7.

2. To show this we can do the following calculation. Let $\eta = \{\cdot,\cdot,\cdot\}_3$ then we want to show that

$$\begin{aligned}(d_{CE}l_3)(x_1,x_2,x_3,x_4) = &\sum_{i=1}^{4}\{x_i, \eta(x_1,\dots,\widehat{x}_i,\dots,x_4)\}_2 \\ &+ \sum_{1\le i,j\le 4}(-1)^{i+j}\eta(\{x_i,x_j\}_2, x_1,\dots,\widehat{x}_i,\dots,\widehat{x}_j,\dots,x_4) = 0,\end{aligned}$$

for arbitrary elements $x_1, x_2, x_3, x_4 \in H^{-1}(\mathcal{F}, m)$. Writing out this whole expression amounts to showing that

$$\begin{aligned}&-\{x_1,\{x_2,x_3,x_4\}_3\}_2 + \{x_2,\{x_1,x_3,x_4\}_3\}_2 - \{x_3,\{x_1,x_2,x_4\}_3\}_2 \\ &+ \{x_4,\{x_1,x_2,x_3\}_3\}_2 - \{\{x_1,x_2\}_2,x_3,x_4\}_3 + \{\{x_1,x_3\}_2,x_2,x_4\}_3 \\ &- \{\{x_1,x_4\}_2,x_2,x_3\}_3 - \{\{x_2,x_3\}_2,x_1,x_4\}_3 + \{\{x_2,x_4\}_2,x_1,x_3\}_3 \\ &- \{\{x_3,x_4\}_2,x_1,x_2\}_3 = 0.\end{aligned}$$

On the other hand, we have the $n = 4$ higher Jacobi identity which we can apply to the same elements i.e.

$$\sum_{i=1}^{4}\sum_{\sigma\in\mathcal{S}(i,4-i)}\epsilon(\sigma)\{\{x_{\sigma(1)},\dots,x_{\sigma(i)}\}_i, x_{\sigma(i+1)},\dots,x_{\sigma(n)}\}_{4-i+1} = 0.$$

Because $\{\cdot\}_1 = 0$ the only terms that remain from this equation are

$$\sum_{\sigma\in\mathcal{S}(2,2)}\epsilon(\sigma)\{\{x_{\sigma(1)},x_{\sigma(2)}\}_2,x_{\sigma(3)},x_{\sigma(4)}\}_2 + \sum_{\sigma\in\mathcal{S}(3,1)}\epsilon(\sigma)\{\{x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)}\}_3,x_{\sigma(4)}\}_2 = 0.$$

Now using that

$$\mathcal{S}(2,2) = \{\mathbb{1},(23),(243),(123),(1243),(13)(24)\},$$

and

$$\mathcal{S}(3,1) = \{\mathbb{1},(34),(234),(1234)\},$$

it is not too hard to see that this higher Jacobi identity exactly gives that $(d_{CE}\eta)(x_1,x_2,x_3,x_4) = 0$ for all $x_1,\dots,x_4 \in H^{-1}(\mathcal{F},m)$. This proves that l_3 indeed is a 3-cocycle for the Chevalley-Eilenberg differential with values in the representation $\{\cdot,\cdot\}_2 : H^{-1}(\mathcal{F},m)\otimes H^{-2}(\mathcal{F},m) \to H^{-2}(\mathcal{F},m)$.

3. Again, this requires some calculation. Note that an L_∞-algebra morphism $\Phi : (V,Q) \to (V',Q')$ consists of a collection of maps $\Phi_k : S^k(V) \to V'$. In particular the map $\Phi_2 : S^2(V) \to V$ has a component $\theta : S^2(H^{-1}(\mathcal{F},m)) \to H^{-2}(\mathcal{F},m)$. Applying the definition of an L_∞-algebra morphism applied to elements $x_1,x_2,x_3 \in H^{-1}(\mathcal{F},m)$ gives

us the following equation

$$\begin{aligned}\sum_{\sigma\in\mathcal{S}(1,2)} \epsilon(\sigma)\Phi_3\left(l_1\otimes\mathbb{1}^{\otimes 2}\right)(x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)})+&\sum_{\sigma\in\mathcal{S}(2,1)}\epsilon(\sigma)\Phi_2\left(l_2\otimes\mathbb{1}\right)(x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)})\\ &+\sum_{\sigma\in\mathcal{S}(3,0)}\epsilon(\sigma)\Phi_1\left(l_3\right)(x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)})\\ =\sum_{\sigma\in\mathcal{S}(3)}\epsilon(\sigma)l_1'\left(\Phi_3(x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)})\right)&+\sum_{\sigma\in\mathcal{S}(1,2)}\frac{\epsilon(\sigma)}{2}l_2'\left(\Phi_1\otimes\Phi_2(x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)})\right)\\ &+\sum_{\sigma\in\mathcal{S}(1,1,1)}\frac{\epsilon(\sigma)}{6}l_3'\left(\Phi_1^{\otimes 3}(x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)})\right)\end{aligned}$$

The different Koszul signs are calculated easily and one then obtains the following equation which holds for all homogeneous elements $x_1, x_2, x_3 \in H^{-1}(\mathcal{F}, m)$

$$\begin{aligned}&\Phi_3\left(l_1(x_1),x_2,x_3\right)-\Phi_3\left(l_1(x_2),x_1,x_3\right)+\Phi_3\left(l_1(x_3),x_1,x_2\right)\\&+\Phi_2\left(l_1(x_1,x_2),x_3\right)-\Phi_2\left(l_2(x_1,x_3),x_2\right)+\Phi_2\left(l_2(x_2,x_3),x_1\right)\\&+\Phi_1\left(l_3(x_1,x_2,x_3)\right)=l_1'\left(\Phi_1(x_1,x_2,x_3)\right)\\&+\frac{1}{2}\Big[l_2'\left(\Phi_1(x_1),\Phi_2(x_2,x_3)\right)-l_2'\left(\Phi_1(x_2),\Phi_2(x_1,x_3)\right)+l_2'\left(\Phi_1(x_3),\Phi_2(x_1,x_2)\right)\Big]\\&+\frac{1}{6}l_3'\left(\Phi_1(x_1),\Phi_1(x_2),\Phi_1(x_3)\right)\end{aligned}\tag{4.14}$$

Now remark that the map $\Phi_1 : H^{-1}(\mathcal{F}, m) \to H'^{-1}(\mathcal{F}, m)$ is the identity map by proposition 4.2.7 and this combined with lemma 4.2.8 gives that $l_2 = l_2'$ (i.e. the 2-ary brackets coincide). Lastly, we also have that the differential is zero so $l_1 = l_1' = 0$. Combined with equation (4.14) above this give an expression for the difference between the 3-ary brackets

$$\begin{aligned}l_3(x_1,x_2,x_3)-\frac{1}{6}l_3'(x_1,x_2,x_3)=&\frac{1}{2}\Big[l_2\left(x_1,\Phi_2(x_2,x_3)\right)-l_2\left(x_2,\Phi_2(x_1,x_3)\right)+l_2\left(x_3,\Phi_2(x_1,x_2)\right)\Big]\\&-\Phi_2\left(l_2(x_1,x_2),x_3\right)+\Phi_2\left(l_2(x_1,x_3),x_2\right)-\Phi_2\left(l_2(x_2,x_3),x_1\right).\end{aligned}\tag{4.15}$$

On the other hand, computing the Chevalley-Eilenberg differential of $\Phi_2 : S^2\left(H^{-1}(\mathcal{F}, m)\right) \to H^{-2}(\mathcal{F}, m)$ with values in the representation $l_2 = \{\cdot,\cdot\}_2 : H^{-1}(\mathcal{F}, m) \otimes H^{-2}(\mathcal{F}, m) \to H^{-2}(\mathcal{F}, m)$ gives us that

$$\begin{aligned}\left(d_{CE}\Phi_2\right)(x_1,x_2,x_3)=&l_2\left(x_1,\Phi_2(x_2,x_3)\right)-l_2\left(x_2,\Phi_2(x_1,x_3)\right)+l_2\left(x_3,\Phi_2(x_1,x_2)\right)\\&-\Phi_2\left(l_2(x_1,x_2),x_3\right)+\Phi_2\left(l_2(x_1,x_3),x_2\right)-\Phi_2\left(l_2(x_2,x_3),x_1\right).\end{aligned}\tag{4.16}$$

Comparing equations (4.15) and (4.16) it is not too hard to see that

$$l_3 - l_3' = \alpha \cdot d_{CE}\theta, \quad \alpha \in \mathbb{R}, \tag{4.17}$$

where we have used that on $S^2(H^{-1}(\mathcal{F}, m))$, the maps Φ_2 and θ coincide and we chose $x_1, x_2, x_3 \in H^{-1}(\mathcal{F}, m)$. Equation (4.17) now gives us exactly that the cohomology classes of l_3 and l_3' are the same (since they differ by an exact term).

□

We are now ready to use the developed theory to prove a result regarding the existence of a Lie algebroid inducing a given singular foliation $\mathcal{F}$. However, we require something more of the Lie algebroid: it must have minimal rank[2], i.e. the same rank as the foliation $\mathcal{F}$ at a point m (this is defined as the minimal number of generators for $\mathcal{F}$ in a neighborhood of m). In general, so without further assumptions, this is still an open question as remarked earlier.

The following definition is the main object of interest to determine if a minimal rank Lie algebroid inducing $\mathcal{F}$ exists and so is named appropriately.

Definition 4.3.2 (Definition 4.28 in [LGLS20])**.** The 3-cohomology class from proposition 4.3.1 is called the **No-Minimal-Rank-Lie-Algebroid class** or NMRLA class.

We will now show the following proposition.

Proposition 4.3.3 (Proposition 4.29 in [LGLS20])**.** *Let $\mathcal{F}$ be a singular foliation on a manifold M that admits a geometric resolution of finite length, let r be the rank of $\mathcal{F}$ at m. If the NMRLA 3-class does not vanish, it is not possible to find a Lie algebroid $(A, [\cdot,\cdot], \rho_A)$ defined in a neighborhood U_m of m satisfying the following two conditions:*

1. *the rank of the vector bundle A is r,*
2. $\rho(\Gamma(A)) = \mathcal{F}|_{U_m}$.

Before proceeding to the proof of proposition 4.3.3 we need some preparatory results.

Lemma 4.3.4 (Lemma 4.31 in [LGLS20])**.** *For every geometric resolution (E, d, ρ) of $\mathcal{F}$ which is minimal at m, the rank of the vector bundle E_{-1} is equal to the rank r of $\mathcal{F}$ at m.*

Proof. Let $r_E = \operatorname{rk} E_{-1}$ and r the rank of $\mathcal{F}$ at m. One can now choose a local trivialization of E_{-1} that we denote $e_1, \ldots, e_{r_E}$. By the definition of a geometric resolution we have that $\rho(\Gamma(E_{-1})) = \mathcal{F}$ and so the collection $(\rho(e_i))_{i=1,\ldots,r_E}$ generates $\mathcal{F}$ as a $C^\infty(M)$-module, clearly this implies that $r \leq r_E$. Now we show this inequality is, in fact, an equality. For this suppose $r < r_E$ then we must have that one of the generators $\rho(e_i)$ is a $C^\infty(M)$-linear combination of the other ones. After a possible renumbering we may assume that this is $\rho(e_1)$ so there exist smooth functions $f_2, \ldots, f_{r_E} \in C^\infty(M)$ such that

$$\rho(e_1) = \sum_{i=1}^{r_E} f_i \rho(e_i).$$

Completely similar to the proof of proposition 4.2.9 it now follows that this linear dependence implies the existence of a section $g \in \Gamma(E_{-2})$ such that

$$e_1 = \sum_{i=2}^{r_E} f_i e_i + d^{(2)} g.$$

Minimality of the geometric resolution at m now means that evaluating this equation at the point m implies

$$e_1(m) = \sum_{i=2}^{r_E} f_i(m) e_i(m).$$

Clearly this contradicts that $e_1, \ldots, e_{r_E}$ is a local trivialization of the bundle E_{-1} and hence we conclude $r = r_E$. $\square$

[2]Recall that the rank of a Lie algebroid is defined to be the rank of the underlying vector bundle.

We are now ready to start the proof of proposition 4.3.3, during the proof we will need one more lemma which is stated and proven after the following proof.

Proof of proposition 4.3.3. Assume that there exists a Lie algebroid $(A, [\cdot,\cdot], \rho_A)$ satisfying $\rho(\Gamma(A)) = \mathcal{F}$. Let (E, Q) be the universal Lie ∞-algebroid of $\mathcal{F}$ in a neighborhood of m that is constructed out of a geometric resolution minimal at m. By theorem 3.1.3 , combined with example 2.2.13 there exists a Lie ∞-algebroid morphism $\Phi : (A[1], Q_A) \to (E, Q)$. Recall remark 2.2.19 which gives a concrete explanation of what a Lie ∞-algebroid morphism looks like. It states that it is given as a collection of maps $(\Phi_k)_{k\geq 1}$ that give L_∞-algebra morphisms on the sections $\Gamma(A[1])$ and $\Gamma(E)$ and such that $\rho_A = \rho \circ \Phi_0$. In particular we have the linear component $\Phi_0 : A[1] \to E$. Remark that $A[1]$ is concentrated in degree -1 and that Φ is a degree 0 morphism so more specifically we have that $\Phi_0 : A[1] \to E_{-1}$.

Recall that r is the rank of $\mathcal{F}$ at m. Now assume the vector bundle A to have rank r, then by lemma 4.3.4 we have $\operatorname{rk} A = \operatorname{rk} E_{-1}$. By lemma 4.3.5 the restriction $\mathfrak{i}_m\Phi_0 : \mathfrak{i}_m A[1] \to \mathfrak{i}_m E_{-1}$ becomes a surjective map and combined with $\operatorname{rk} \mathfrak{i}_m A = \operatorname{rk} \mathfrak{i}_m A[1] = \operatorname{rk} \mathfrak{i}_m E_{-1}$ this means $\mathfrak{i}_m\Phi_0$ becomes a bijection. Note that we also have two other maps $\Phi_1 : S^2(A[1]) \to E_{-2}$ and $\Phi_2 : S^3(A[1]) \to E_{-3}$ (again the codomain can be seen by a simple degree count). Some general remarks:

- in example 2.2.13 we denoted the bracket on $\Gamma(A[1])$ as $\{\cdot,\cdot\}$. We shall do the same here being careful not to confuse this bracket with the 2-ary bracket $\{\cdot,\cdot\}_2$ on $\Gamma(E)$,
- the Lie ∞-algebroid $(A[1], Q_A)$ only has a binary bracket (the one from the previous point) and all the others are zero,
- recall that the 1-ary bracket on $\Gamma(E)$ corresponds to the differentials $(d^{(i)})_{i\geq 2}$.

Using remark 2.2.19 we may now do a computation completely similar to the one done in the proof of proposition 4.3.1 to obtain that the components Φ_0, Φ_1 and Φ_2 satisfy the following equation for all $a, b, c \in \Gamma(A[1])$ (keeping in mind the list above)

$$\begin{aligned}
&-\frac{1}{2}\left[\{\Phi_0(a), \Phi_1(b,c)\}_2 - \{\Phi_0(b), \Phi_1(a,c)\}_2 + \{\Phi_0(c), \Phi_1(a,b)\}_2\right] \\
&+ \Phi_1(\{a,b\}, c) - \Phi_1(\{a,c\}, b) + \Phi_1(\{b,c\}, a) \\
&= d^{(2)}\Phi_2(a,b,c) + \frac{1}{6}\{\Phi_0(a), \Phi_0(b), \Phi_0(c)\}_3.
\end{aligned} \tag{4.18}$$

We now proceed by evaluating equation (4.18) at the point m and invoking a couple of earlier results. First note that by minimality the first term on the right-hand side of equation (4.18) vanishes by definition. Moreover, by the bijective correspondence between $\mathfrak{i}_m A[1]$ and $\mathfrak{i}_m E_{-1}$ we may assume that $\mathfrak{i}_m\Phi_0$ is the identity map. Now recall the construction of the isotropy L_∞-algebra at m: it was constructed as $\ker \mathfrak{i}_m\rho \oplus \bigoplus_{i\geq 2} \mathfrak{i}_m E_{-i}$ and so choosing $a, b, c \in \Gamma(A[1])$ such that $\mathfrak{i}_m a, \mathfrak{i}_m b, \mathfrak{i}_m c \in \ker \mathfrak{i}_m\rho$ (after an application of $\Phi_0 = \mathbb{1}$) we recover the 3-cohomology class from proposition 4.3.1 on the right hand side of equation (4.18). Also note that when working in the restriction to m, by the same reasons as in the proof of proposition 4.3.1 the binary brackets coincide and so the left hand side of equation (4.18) yields exactly the expression for the Chevalley-Eilenberg differential of

$\Phi_1(\cdot,\cdot)$ with values in the representation determined by $\{\cdot,\cdot\}$ (up to a constant multiple). Hence it is shown that the NMRLA 3-class is a Chevalley-Eilenberg coboundary.

□

The following lemma was used in the proof above.

Lemma 4.3.5 (Lemma 4.31 in [LGLS20])**.** *Restricting the vector bundle morphism $\Phi_0 : A[1] \to E_{-1}$ to the fiber at m yields a surjective linear map.*

Proof. Pick an arbitrary element $e \in \mathfrak{i}_m E_{-1}$ and let $\tilde{e} \in \Gamma(E_{-1})$ be an extension of e to a section. We assumed $\rho_A(\Gamma(A)) = \mathcal{F}$ and by the definition of a geometric resolution $\rho(\Gamma(E_{-1})) = \mathcal{F}$ so, modulo a degree shift, there exists an element $a \in \Gamma(A[1])$ such that $\rho_A(a) = \rho(\tilde{e})$. Hence by the Lie ∞-algebroid morphism property we have that $\rho(\tilde{e}) = \rho(\Phi_0(a))$ i.e. $\rho(\tilde{e} - \Phi_0(a)) = 0$. As we already saw a couple of times this implies that there exists a $g \in \Gamma(E_{-2})$ such that $\tilde{e} - \Phi_1(a) = d^{(2)}g$. Because (E, d, ρ) was chosen to be minimal at m this implies that $e = \mathfrak{i}_m\Phi_0(a(m))$ which means $\mathfrak{i}_m\Phi_0 : \mathfrak{i}_m A[1] \to \mathfrak{i}_m E_{-1}$ is surjective. □

We will now display two examples that have non-trivial NMRLA 3-class, thus showing the following corollary to proposition 4.3.3.

Corollary 4.3.6 (Proposition 4.33 in [LGLS20])**.** *There exist singular foliations of rank r that, even locally, cannot be induced by a Lie algebroid of rank r.*

As we already remarked we will now illustrate corollary 4.3.6 by two examples. The first one is taken from [LGLS20], the second one takes inspiration from the first one but is original.

Example 4.3.7 (Example 4.32 in [LGLS20])**.** This example looks at the foliation $\mathcal{F}_\varphi$ that we already saw in examples 1.2.36 and 1.2.60. It can be shown that this foliation has rank $n(n-1)/2$. In example 4.2.12 we displayed the k-ary brackets on the isotropy L_∞-algebra of $\mathcal{F}_\varphi$ at the origin 0. We now consider $n \geq 4$ and the homogeneous polynomial

$$\varphi : \mathbb{C}^n \to \mathbb{C} : (x_1, \ldots, x_n) \mapsto \sum_{i=1}^{n} x_i^3.$$

Remark that this polynomial certainly satisfies the conditions that we set earlier: it is homogeneous and so certainly weight-homogeneous and it has an isolated singularity at the origin. According to equation (4.13), which defines the brackets, the 2-ary bracket vanishes, indeed $\frac{\partial^2\varphi}{\partial x_i \partial x_j}(0) = 0$ for all $i, j = 1, \ldots, n$. Combining this with proposition 4.3.1 one sees that the representation of $H^{-1}(\mathcal{F}_\varphi, 0)$ on $H^{-2}(\mathcal{F}_\varphi, 0)$ is trivial. Also a small calculation, using (4.13) shows that

$$\left\{\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_4}\right\}_3 = \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4}.$$

This shows that the NMRLA 3-class does not vanish and so the foliation $\mathcal{F}_\varphi$ is not induced by a rank $n(n-1)/2$ Lie algebroid in a neighborhood of the origin.

♦

The following example was found by using inspiration from example 4.3.7.

Example 4.3.8. All the things we discussed earlier about $\mathcal{F}_\varphi$ were under the assumption that the polynomial φ is a *weight-homogeneous polynomial with an isolated singularity at the origin*. Up until now all explicit descriptions of φ where *homogeneous polynomials*. In this example we give an example where φ is weight-homogeneous but not homogeneous. Because this polynomial is quite specific, we let $n = 4$. Consider the following polynomial

$$\varphi : \mathbb{C}^4 \to \mathbb{C} : (x_1, x_2, x_3, x_4) \mapsto x_1^3 + x_2^5 + x_3^7 + x_4^{11}.$$

This is a weight homogeneous polynomial with weights $w_1 = 385, w_2 = 231, w_3 = 165$ and $w_4 = 105$. It also clearly has a singularity at the origin since

$$(3x_1^2, 5x_2^4, 7x_3^6, 11x_4^{10}) = (0, 0, 0, 0) \Leftrightarrow (x_1, x_2, x_3, x_4) = (0, 0, 0, 0).$$

It is also verified very easily that the representation $\{\cdot, \cdot\}_2 : H^{-1}(\mathcal{F}_\varphi, 0) \otimes H^{-2}(\mathcal{F}_\varphi, 0) \to H^{-2}(\mathcal{F}_\varphi, 0)$ is the trivial one and that in exactly the same way as the previous example we have, by equation (4.13)

$$\left\{\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_4}\right\}_3 = \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4}. \tag{4.19}$$

So again, we conclude that there does not exist a rank 6 Lie algebroid inducing the foliation $\mathcal{F}_\varphi$ in a neighborhood of the origin.

Actually, the procedure used for finding this polynomial with these specific properties can be replicated for $n > 4$. One can take a polynomial of the following form

$$\varphi : \mathbb{C}^n \to \mathbb{C} : (x_1, \dots, x_n) \mapsto \sum_{k=1}^{n} x_k^{i_k}, \quad \forall k = 1, ..., n : i_k \in \mathbb{N}.$$

We now want to choose the weights $w_1, \dots, w_n \neq 1$ (we exclude this case because this yields ordinary homogeneous polynomials) such that we have the following equations for some $w \in \mathbb{N}$

$$\begin{cases} w = w_1 i_1 \\ \quad \vdots \\ w = w_n i_n \end{cases}.$$

It is not hard to see that we want at least one $k = 1, \dots, n$ for which $i_k = 3$ (for otherwise the 3-ary bracket and thus the NMRLA 3-class vanishes), without loss of generality we may assume this is for $k = 1$. From $w = 3w_1$ we now also see that w must be a multiple of 3. We can now form w in the following way

$$w = 3p_2 \cdots p_n,$$

for primes $3 < p_2 < \cdots < p_n$. If we now let $i_k = p_k$ and $w_k = 3p_2 \cdots \widehat{p_k} \cdots p_n$ for $k \geq 2$ the system of equations from above is satisfied (here the hat denotes we left out the prime p_k). In this way we made sure, by construction, that φ is a weight-homogeneous polynomial and that it has an isolated singularity at the origin. Furthermore, we also have that the representation of the isotropy Lie algebra on $H^{-2}(\mathcal{F}_\varphi, 0)$ is trivial and that equation (4.19) still holds and so everything we said earlier applies to $\mathcal{F}_\varphi$.

♦

Conclusion

The aim of this thesis was to introduce the necessary concepts to understand the paper [LGLS20] and use them to study singular foliations as was done in [LGLS20].

In chapter 1 we introduced some preliminary material on regular foliations. After this we introduced singular foliations in the two ways they appear in the literature. In particular we introduced the sheaf point of view as this is the way singular foliations are defined in [LGLS20]. We illustrated these definitions by giving several examples. We also illustrated that one cannot expect to get a Lie algebroid from a singular foliation, but one does get an almost-Lie algebroid structure. This provided incentive to look for 'higher structures'. In the last two subsections we introduced and illustrated geometric resolutions. These are one of the main objects used in this thesis as they are the first building block for the universal Lie ∞-algebroid. In this context we provided the details to a proof concerning the existence of a geometric resolution for an algebraic singular foliation of a Zariski open set.

In chapter 2 we introduced L_∞-algebras and Lie ∞-algebroids. These are necessary to understand the main results of [LGLS20] that we introduced in the next chapter. We also explained the duality between Lie ∞-algebroids and NQ-manifolds. We did this to introduce morphisms and homotopies between Lie ∞-algebroids as these are more concrete in the category of NQ-manifolds.

In chapter 3 we introduced the main results of [LGLS20]: the existence of a universal Lie ∞-algebroid for a singular foliation admitting a geometric resolution, and a uniqueness result. We provided the main steps and some calculations in the proof of the existence result. Finally, we illustrated the universal Lie ∞-algebroid by displaying explicit examples following [LGLS20].

Chapter 4 contains the material developed in section 4 of [LGLS20]. It studies the geometry of singular foliations through their universal Lie ∞-algebroid. In this chapter we also provided more detailed calculations and proofs than the ones in the original publication. We ended with answering the question '*can all rank r singular foliations can be induced, locally, by a rank r Lie algebroid?*' and illustrated this by giving two examples. The first of these examples comes from [LGLS20] while the second example is heavily inspired by the first one but is original.

Chapter 5

Appendices

5.1 Appendix A

In this appendix we will show that claim used in chapter 3, namely that $\rho : E_{-1} \to TM$ is a morphism of the brackets if and only if

$$\left[Q^{(1)}_{E_{-1}}, Q^{(1)}_{E_{-1}}\right] \in \mathfrak{X}_{\text{vert}}(E_{-1}).$$

To do the calculations we will make use of lemma 2.11 in [ZZ12]. This lemma describes a correspondence between Lie algebroids and NQ-manifolds concentrated in degree 1, based on the derived bracket construction from [KS04]. Given an NQ-manifold[1] $\mathcal{M} = (E_{-1}[1], Q = Q^{(1)}_{E_{-1}})$ we have that the degree -1 vector fields on $\mathcal{M}$ are $\Gamma(E_{-1})$ (see lemma 2.6 [ZZ12]). Furthermore, we have the following expressions for the anchor and bracket

$$[a,b]_{E_{-1}} = [[Q,a],b], \qquad \rho(a)f = [[Q,a],f], \tag{5.1}$$

for $a, b \in \Gamma(E_{-1})$ and $f \in C^\infty(M)$. We will now apply this to get our desired result.

Let (x_i) be a set of coordinates on the base M and ξ_j a set of degree $+1$ coordinates on the vector bundle E_{-1}. Note that $\left[Q^{(1)}_{E_{-1}}, Q^{(1)}_{E_{-1}}\right]$ is a degree $+2$ vector field and so it can be written as

$$\left[Q^{(1)}_{E_{-1}}, Q^{(1)}_{E_{-1}}\right] = \underbrace{\sum_{i,j,k} a^k_{ij}(x)\xi_i\xi_j \frac{\partial}{\partial x_k}}_{\text{horizontal part}} + \underbrace{\sum_{i,j,k,l} b^l_{ijk}(x)\xi_i\xi_j\xi_k \frac{\partial}{\partial \xi_l}}_{\text{vertical part}}, \tag{5.2}$$

for $a^k_{ij}(x), b^l_{ijk}(x) \in C^\infty(M)$. Now let $e_1 = \frac{\partial}{\partial \xi_i}$ and $e_2 = \frac{\partial}{\partial \xi_j}$ then these are sections of the vector bundle E_{-1} and so they can be viewed as degree -1 vector fields on the graded manifold $(E_{-1}, Q = Q^{(1)}_{E_{-1}})$. Furthermore, they are part of the canonical local frame and so it suffices to work with this type of sections .This allows us to compute the following

$$[[[Q,Q],e_1],e_2] = \sum_k a^k_{ij}(x)\frac{\partial}{\partial x_k} + \sum_l b^l_{ijk}(x)\xi_k \frac{\partial}{\partial \xi_l}.$$

[1]Here $E_{-1} \to M$ is a vector bundle.

Now pick a function $f \in C^\infty(M)$ then using the above it is easy to see that

$$\begin{aligned}\left[\left[\left[\left[Q,Q\right],e_1\right],e_2\right],f\right] &= \sum_k a_{ij}^k(x)\frac{\partial f}{\partial x_k} + \sum_{k,l} b_{ijk}^l(x)\xi_k\frac{\partial f}{\partial \xi_l} \\ &= \sum_k a_{ij}^k(x)\frac{\partial f}{\partial x_k}.\end{aligned}$$

So when we choose $f = x_k$ we have

$$\left[\left[\left[\left[Q,Q\right],e_1\right],e_2\right],x_k\right] = a_{ij}^k(x).$$

On the other hand we can do the following calculation for general $f \in C^\infty(M)$ and $e_1, e_2 \in \Gamma(E_{-1})$. From the graded Jacobi identity, it is not hard to see that

$$\left[\left[Q,Q\right],e_1\right] = 2\left[Q,\left[Q,e_1\right]\right]. \tag{5.3}$$

Hence we may write

$$\left[\left[\left[\left[Q,Q\right],e_1\right],e_2\right],f\right] = 2\left[\left[\left[Q,\left[Q,e_1\right]\right],e_2\right],f\right].$$

Applying the graded Jacobi identity once more we have that

$$\left[\left[\left[Q,Q\right],e_1\right],e_2\right] = 2\left(\left[Q,\left[\left[Q,e_1\right],e_2\right]\right] - \left[\left[Q,e_1\right],\left[Q,e_2\right]\right]\right).$$

Now we apply the equations from (5.1) to conclude that

$$\begin{aligned}\left[Q,e_1\right] &= \rho(e_1) + \text{vertical part}, \\ \left[Q,e_2\right] &= \rho(e_2) + \text{vertical part}, \\ \left[Q,\left[\left[Q,e_1\right],e_2\right]\right] &= \rho\left(\left[e_1,e_2\right]_{E_{-1}}\right) + \text{vertical part}.\end{aligned}$$

Letting $f \in C^\infty(M)$ as above we see that

$$\left[\left[\left[\left[Q,Q\right],e_1\right],e_2\right],f\right] = 2\left(\rho\left(\left[e_1,e_2\right]_{E_{-1}}\right)f - \left[\rho(e_1),\rho(e_2)\right]_{\mathfrak{X}(M)}f\right). \tag{5.4}$$

Indeed, writing a bit informally, we have (vertical part)$(f) = 0$ because f does not depend on the coordinates ξ_i. Now note that the right-hand side of equation (5.4) is zero if and only if for all sections e_1 and e_2 of the form $\frac{\partial}{\partial \xi_k}$ and all $f \in C^\infty(M)$ the anchor ρ preserves the brackets. On the other hand we have shown above that for this particular choice of sections and $f = x_k$ we have that the left-hand side of equation (5.4) is zero if and only if $a_{ij}^k(x) = 0$ for all i, j and k. By the Leibniz property of the bracket the choice of coordinate functions suffices and so we conclude the following: the anchor preserves brackets if and only if $a_{ij}^k(x) = 0$ for all i, j and k. Going back to equation (5.2) this exactly means that the horizontal part vanishes and so $[Q,Q] = \left[Q_{E_{-1}}^{(1)}, Q_{E_{-1}}^{(1)}\right]$ is vertical. This is exactly what we wanted to show.

5.2 Appendix B

Lemma 5.2.1. *Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with $\operatorname{rk}(A) = k$. Then any small perturbation B of A satisfies that $\operatorname{rk}(B) \geq k$.*

Proof. Since the determinant det : $M_{n\times m}(\mathbb{R}) \to \mathbb{R}$ can be expressed as a polynomial it is a continuous function. The linear map A having rank k is the same as saying its representative matrix, also denoted A, has rank k. A matrix has rank k if and only if there exists a $k \times k$-submatrix with nonzero determinant. Now since det is continuous it is easy to see that any small perturbation of the entries in A (and thus also the $k \times k$-submatrix) does not change the value of the determinant. Hence, we conclude the lemma. □

5.3 Appendix C

Here we will expose very briefly the theory of Chevalley-Eilenberg cohomology or Lie algebra cohomology. From standard sources on differential geometry like [Lee12], we see that the de Rham cohomology is constructed on the space of differential forms

$$\Omega^\bullet(M) = \bigoplus_{k=0}^{\infty} \Gamma\left(\wedge^k T^* M\right).$$

Taking inspiration from this we now define the Chevalley-Eilenberg differential on the space

$$\wedge^\bullet \mathfrak{g}^* = \bigoplus_{k=0}^{\infty} \wedge^k \mathfrak{g}^*.$$

Recall here that $\wedge^k \mathfrak{g}^* = \{\text{multilinear and antisymmetric maps } \mathfrak{g} \times \cdots \times \mathfrak{g} \to \mathbb{R}\}$. We now define the Chevalley-Eilenberg differential to be the degree $+1$ map

$$d_{CE} : \wedge^\bullet \mathfrak{g}^* \to \wedge^\bullet \mathfrak{g}^*,$$

defined for all $\eta \in \wedge^\bullet \mathfrak{g}^*$ as

$$(d_{CE}\eta)(v_1, \ldots, v_{k+1}) := \sum_{1 \le i < j \le k+1} (-1)^{i+j} \eta\left([v_i, v_j], v_1, \ldots, \widehat{v_i}, \ldots, \widehat{v_j}, \ldots, v_{k+1}\right).$$

Note that with some small calculations one can show that for all $\eta \in \wedge^1 \mathfrak{g}^*$ we have $d_{CE}(d_{CE}(\eta)) = 0$ and that for all $\zeta \in \wedge^k \mathfrak{g}^*, \xi \in \wedge^l \mathfrak{g}^*$

$$d_{CE}(\zeta \wedge \xi) = d_{CE}(\zeta) \wedge \xi + (-1)^k \zeta \wedge d_{CE}(\xi).$$

From this it can be seen that $d_{CE} \circ d_{CE} = 0$ and so d_{CE} indeed is a differential. We now define the Chevalley-Eilenberg cohomology groups as

$$H^n_{CE}(\mathfrak{g}) := \frac{\ker(d_{CE} : \wedge^n \mathfrak{g}^* \to \wedge^{n+1} \mathfrak{g}^*)}{\operatorname{im}(d_{CE} : \wedge^{n-1} \mathfrak{g}^* \to \wedge^n \mathfrak{g}^*)}.$$

We now continue with Chevalley-Eilenberg cohomology with values in a Lie algebra representation. For this let V denote some vector space and $\mathfrak{gl}(V)$ the linear endomorphisms of V. A Lie algebra representation is now a Lie algebra homomorphism

$$\rho : \mathfrak{g} \to \mathfrak{gl}(V).$$

Now define the following space

$$\wedge^k \mathfrak{g}^* \otimes V = \{\text{antisymmetric maps } \underbrace{\mathfrak{g} \times \cdots \times \mathfrak{g}}_{k \text{ times}} \to V\}.$$

We now define the Chevalley-Eilenberg differential with values in a representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ as follows for all $\eta \in \wedge^k \mathfrak{g}^* \otimes V$

$$\begin{aligned}(d_{\mathfrak{g}}\eta)(v_1, \ldots, v_{k+1}) :=& \sum_{i=1}^{k+1} (-1)^i \rho(v_i)\eta(v_1, \ldots, \widehat{v}_i, \ldots, v_{k+1}) \\ &+ \sum_{1 \le i < j \le k+1} (-1)^{i+j} \eta\left([v_i, v_j], v_1, \ldots, \widehat{v}_i, \ldots, \widehat{v}_j, \ldots, v_{k+1}\right).\end{aligned}$$

This differential also has a resulting cohomology that we also denote by the Chevalley-Eilenberg cohomology with values in the representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$.

Bibliography

[Alu09] Paolo Aluffi. *Algebra: Chapter 0*, volume 104 of *Graduate studies in mathematics*. American Mathematical Society, 2009.

[AS09] Iakovos Androulidakis and Georgens Skandalis. The holonomy groupoid of a singular foliation. *Journal für die reine und angewandte Mathematik*, 626:1–37, 2009.

[AZ13] Iakovos Androulidakis and Marco Zambon. Smoothness of holonomy covers for singular foliations and essential isotropy. *Mathematische Zeitschrift*, 275(3):921–951, 2013.

[AZ16] Iakovos Androulidakis and Marco Zambon. Stefan Sussmann singular foliations, singular subalgebroids and their associated sheaves. *International Journal of Geometric Methods in Modern Physics*, 13, 2016.

[BP13] Giuseppe Bonavolontá and Norbert Poncin. On the category of Lie n-algebroids. *Journal of Geometry and Physics*, 73:70–90, 2013.

[DLPR10] Lance D. Drager, Jeffrey M. Lee, Efton Park, and Ken Richardson. Smooth Distributions are not finitely generated. https://arxiv.org/abs/1012.5641v1, December 2010.

[DN05] Jean-Paul Dufour and Tien Zung Nguyen. *Poisson Structures and Their Normal Forms*, volume 242 of *Progress in Mathematics*. Birkhauser, 2005.

[dSW99] Ana Cannas da Silva and Alan Weinstein. *Geometric Models for Noncommutative Algebras*, volume 10 of *Berkeley mathematics lecture notes*. American Mathematical Society, 1999.

[Eis95] D. Eisenbud. *Commutative Algebra: With a View Toward Algebraic Geometry*. Graduate Texts in Mathematics. Springer, 1995.

[Fai17] Maxime Fairon. Introduction to graded geometry. *European Journal of Mathematics*, (3):208–222, 2017.

[FM07] Domenico Fiorenza and Marco Manetti. L-infinity structures on mapping cones. *Algebra & Number Theory*, 1(3):301–330, 2007.

[Gar19] Alfonso Gadir Garmendia. *Groupoids and singular foliations*. PhD thesis, KULeuven, July 2019.

[Har97] Robin Hartshorne. *Algebraic geometry*. Graduate texts in mathematics 52. Springer, New York (N.Y.), corr. 8th print. edition, 1997.

[Her63] Robert Hermann. On the Accessibility Problem in Control Theory. In *International Symposium on Nonlinear Differential Equations and Nonlinear Mechanics*, pages 325–332. Academic Press, 1963.

[Hue05] Johannes Huebschmann. *Higher homotopies and Maurer-Cartan algebras: Quasi-Lie-Rinehart, Gerstenhaber, and Batalin-Vilkovisky algebras*, pages 237–302. Birkhäuser Boston, Boston, MA, 2005.

[KS04] Yvette Kosmann-Schwarzbach. Derived Brackets. *Letters in Mathematical Physics*, 69:61–87, 2004.

[KS06] Hiroshige Kajiura and Jim Stasheff. Homotopy Algebras Inspired by Classical Open-Closed String Field Theory. *Communications in mathematical physics*, 263(3):553–581, 2006.

[Lan05] Serge Lang. *Algebra*. Graduate Texts in Mathematics. Springer, third edition, 2005.

[Lav16] Sylvain Lavau. *Lie ∞-algébroïdes et Feuilletages Singuliers*. PhD thesis, Université Claude Bernard Lyon 1, 2016.

[Lav18] Sylvain Lavau. A short guide through integration theorems of generalized distributions. *Differential Geometry and its Applications*, 61:42–58, 2018.

[Lee12] John Lee. *Introduction to Smooth Manifolds*, volume 218 of *Graduate texts in mathematics*. Springer New York, New York, NY, 2012.

[LGLS20] Camille Laurent-Gengoux, Sylvain Lavau, and Thomas Strobl. The Universal Lie infinity-Algebroid of a Singular Foliation. *Documenta Mathematica*, 25:1571–1652, 2020.

[LGPV13] Camille Laurent-Gengoux, Anne Pichereau, and Pol Vanhaecke. *Poisson Structures*, volume 347 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag Berlin Heidelberg, 1 edition, 2013.

[LM95] Tom Lada and Martin Markl. Strongly homotopy Lie algebras. *Communications in Algebra*, 23(6):2147–2161, 1995.

[LS93] Tom Lada and Jim Stasheff. Introduction to SH Lie algebras for physicists. *International Journal of Theoretical Physics*, 32(7):1087–1103, 1993.

[Mat87] H. Matsumura. *Commutative Ring Theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1987.

[Nes20] Jet Nestruev. *Smooth Manifolds and Observables*, volume 220 of *Graduate Texts in Mathematics*. Springer Nature Switzerland, 2 edition, 2020.

[nLa19] nLab. https://ncatlab.org/nlab/show/L-infinity-algebra#History, November 2019.

[Rot08] J.J. Rotman. *An Introduction to Homological Algebra*. Universitext. Springer New York, 2008.

[Sta92] Jim Stasheff. Differential graded Lie algebras, quasi-Hopf algebras and higher homotopy algebras. In Petr P. Kulish, editor, *Quantum Groups*, pages 120–137, Berlin, Heidelberg, 1992. Springer Berlin Heidelberg.

[Vak17] Ravi Vakil. The Rising Sea: foundations of algebraic geometry. draft, November 2017.

[Vor10] Th. Voronov. Q manifolds and Higher Analogs of Lie Algebroids. *AIP Conference Proceedings*, 1307(1):191–202, 2010.

[ZA16] M. Zambon and I. Androulidakis. Stefan-Sussman singular foliations, singular subalgebroids and their associated sheaves. *International Journal of Geometric Methods in Modern Physics*, 2016.

[Zwi93] Barton Zwiebach. Closed string field theory: Quantum action and the Batalin-Vilkovisky master equation. *Nuclear Physics B*, 390(1):33–152, 1993.

[ZZ12] Marco Zambon and Chenchang Zhu. Distributions and qoutients on degree 1 NQ-manifolds and Lie algebroids. *Journal of Geometric Mechanics*, 4(4):469–485, December 2012.

Department of Mathematics
Celestijnenlaan 200B
3001 HEVERLEE, BELGIË
tel. + 32 16 32 70 06
fax + 32 16 32 79 98
www.kuleuven.be