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Integration of transitive Lie algebroids

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Preface

I would like to devote this section to thanking the nice people that support me in my pursuit of dream in mathematics.

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Summary

A Lie group is a natural object in differential geometry, and its infinitesimal counter part is called a Lie algebra, which is purely algebraic. In chapter 1, we start by introducing the basics of Lie groups and Lie algebras. Along the way, we exhibit various properties of Lie groups and Lie algebras, with an emphasis on their correspondences. As an application, we also introduce the notion of principal bundle, in an extensive manner.

Given any finite dimensional Lie algebra, there exists a unique simply connected Lie group whose Lie algebra is isomorphic to the given one. This is known as Lie's III theorem (Theorem 2.29). We give a detailed discussion of its proof in Chapter 2, following Duistermaat-Kolk's construction in 1999 [9].

In chapter 3, we turn to the discussion of Lie groupoids and Lie algebroids, which are natural generalizations of Lie group and Lie algebra. Surprisingly, on Lie groupoids and Lie algebroids, Lie's III theorem does not hold. Determining whether a Lie algebroid originates from a Lie groupoid is called the integration problem. We shall briefly discuss the Crainic and Fernandes [5] solution (Theorem 3.21) to the integration problem, which generalized the Duistermaat-Kolk construction. We end this chapter by discussing gauge groupoids and gauge algebroids, which are closely related to principal bundles.

Motivated by the example of gauge algebroids, we study a more general class, namely the transitive Lie algebroids in Chapter 4. The theory of transitive Lie algebroids was developed by Mackenzie [26], among which we discuss the local trivial property (Lemma 4.7) and gauge transformations on trivial algebroids (Proposition 4.13). The main theme of chapter is to demonstrate the relationship between principal bundles and transitive Lie algebroids, and we will prove that any integrable transitive Lie algebroid is a gauge algebroid (Lemma 4.6), which comes from a principal bundle. Inspired by the classification of principal bundles over the 2-sphere (Theorem 4.15, we classify (framed) transitive Lie algebroids over the 2-sphere (Theorem 4.16), expanding the sketch in Meinrenken's notes [28], and provide an alternative construction for the existing proof in [29]. We derive a corollary for the classification of (non framed) Lie algebroids over the 2-sphere (Corollary 4.17). Then we relate the above classification results by giving a topological interpretation of the gauge algebroid of a principal bundle (Proposition 4.19), following [28][29]. As an application, we state and prove the necessity of the integrability condition (Proposition 4.21), followed by a brief introduction on Mackenzie's non-constructive proof. Finally in Section 4.7, We use the example of prequantization algebroids $TM \times \mathbb{R}$ to illustrate the theory: we discuss the non-integrability of the Almeida-Molino example [1], and give an explicit construction of integration assuming the integrability condition.

In Chapter 5, we discuss a new and elementary construction of integration of transitive Lie algebroids, due to Meinrenken [29]. By choosing a splitting of a transitive Lie algebroid, one can describe the algebroid structure by a direct sum of the tangent bundle and

and the isotropy bundle $TM \oplus \mathfrak{h}$, which can be seen as a generalization of the case of prequantization algebroid. When the integrability condition is satisfied, the algebroid data on isotropy bundle can be transferred to groupoid data, namely the parallel transport and the holonomy. The groupoid is constructed by these data (Theorem 5.20). We finish the chapter by a computational comparison between Meinrenken's construction and the preceding constructions, on the integration of prequantization algebroids (Corollary 5.25).

We list our contributions here:

- The proof of Proposition 3.33 and Corollary 3.34, where we prove a general fact: the Lie algebroid of a gauge groupoid is a gauge algebroid.
- The proof of Lemma 4.6, showing gauge algebroids can be integrated to gauge groupoids only.
- The proof of well-definedness and injectivity in Theorem 4.16 on the classification of framed transitive Lie algebroids over $\mathbb{S}^2$.
- The statement and proof of Corollary 4.17, where we apply Theorem 4.16 to give a classification in the non framed case.
- The statement and proof of Proposition 4.26, and the following 3 corollaries, where we study a non trivial example by computations.
- The proof of Proposition 5.9 and Lemma 5.11, where we fill in some technical details omitted in Meinrenken's paper [29].
- The statement and proof of Proposition 5.24 and Corollary 5.25, where we apply Meinrenken's integration [29] to the example mentioned above, and compare it to the existing construction [6].

Contents

Chapter 1

Lie groups and Lie algebras

In the following two chapter, we introduce basics on Lie groups and Lie algebras, and prove the Lie's fundamental theorems (Theorem 1.15, Theorem 1.21, Theorem 2.29). The notion of principal bundles described in Section 1.5 will be useful throughout of this thesis.

1.1 Basic constructions

In Lie theory, we would like to do calculus on algebraic objects such as groups. In this section we introduce some basic definitions, following (Chapter 7 & 8, [21]). A Lie group is a group as well as a (finite-dimensional) smooth manifold such that the group structure is compatible with the smooth structure. To be more precise, we have the following definition.

Definition 1.1. A group G is a **Lie group** if G is a manifold such that the multiplication map $m : G \times G \to G$ and the inversion map $i : G \to G$ are both smooth. A **morphism** between Lie groups is a group morphism which is smooth, i.e. it is compatible with both group and smooth structures.

Example 1.2. Here are some examples for Lie groups:

- $\mathrm{GL}(n, \mathbb{R}) = \{A \in \mathrm{Mat}(n, \mathbb{R}) \,|\, \det(A) \neq 0\}$.
- $\mathrm{GL}(n, \mathbb{C}) = \{A \in \mathrm{Mat}(n, \mathbb{C}) \,|\, \det(A) \neq 0\}$.
- $\mathrm{SL}(n, \mathbb{R}) = \{A \in \mathrm{Mat}(n, \mathbb{R}) \,|\, \det(A) = 1\}$.
- $\mathrm{O}(n) = \{A \in \mathrm{Mat}(n, \mathbb{R}) | A^{-1} = A^T\}$.
- $\mathrm{U}(n) = \{A \in \mathrm{Mat}(n, \mathbb{C}) | A\overline{A}^T = I\}$.

Remark 1.3. Although most of the simple examples of Lie groups are 'matrix groups', it is not the case for all of them. This is a deep problem, and see [16] for further discussion.

Given a Lie group G, we are interested in the following maps which are called **translations**. For any $g \in G$, the map $L_g : G \to G : h \mapsto gh$ is called left translation, and $R_g : G \to G : h \mapsto hg$ is called right translation. It is easy to see that both maps are diffeomorphisms. Now we consider a family of vector fields that is invariant under the pushforward of the right translation map. These vector fields are called **left invariant**

vector fields and form a linear subspace of $\mathfrak{X}(M)$, denoted as $\mathfrak{X}^L(M)$. Note that the operation of the bracket of vector field preserves the left invariant vector fields, i.e. the bracket of left invariant vector fields is still left invariant. Moreover, we may observe that any left invariant vector field is determined by its value at the identity; and conversely, any tangent vector at the identity gives rise to a left invariant vector field by applying left transitions. Therefore, we get

Proposition 1.4. *There exists a linear isomorphism $T_eG \to \mathfrak{X}^L(G)$ mapping v to the left invariant vector field $\overleftarrow{v}$, where $\overleftarrow{v}|_g = TR_g v$, $\forall g \in G$.*

Now we can transport the bracket on the left invariant vector fields to T_eG, which makes it into a Lie algebra.

Definition 1.5. A **Lie algebra** $\mathfrak{g}$ is a vector space, together with a bilinear, skew symmetric map called the Lie bracket $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the Jacobi identity. A **morphism** of Lie algebras is a linear map between Lie algebras that preserves the brackets.

Example 1.6. Below are some examples of Lie algebras:

- $\mathfrak{gl}(n, \mathbb{R}) = \mathrm{Mat}(n, \mathbb{R})$. This is the Lie algebra of $\mathrm{GL}(n, \mathbb{R})$. The case for $\mathrm{GL}(n, \mathbb{C})$ is analogous.

- $\mathfrak{o}(n) = \{A \in \mathrm{Mat}(n, \mathbb{R}) | A + A^T = 0\}$. This is the Lie algebra of $\mathrm{O}(n)$.

- $\mathfrak{u}(n) = \{A \in \mathrm{Mat}(n, \mathbb{C}) | A + \overline{A}^T = 0\}$. This is the Lie algebra of $\mathrm{U}(n)$.

Remark 1.7. Unlike Lie groups, every (finite dimensional) Lie algebra come from $\mathfrak{gl}(n, \mathbb{C})$ or $\mathfrak{gl}(n, \mathbb{R})$ (as a Lie subalgebra, see Definition 1.12). This is Ado's theorem. However, the proof is quite involved, see ([18], Appendix B) for detailed discussions.

Remark 1.8. Given a mophism of Lie groups $\Phi : G \to H$, it is clear that $\Phi_* := (d\Phi)_e : T_eG \to T_eH$ is a morphism of Lie algebras. We shall prove a converse statement in Theorem 1.21.

Finally we introduce the exponential map. Let G be a Lie group, and $\mathfrak{g}$ be its Lie algebra. Given a vector $v \in T_eG$, we know that it determines a left invariant vector field on G, which is complete, i.e. its flow exists for any time t. Therefore, we may take the integral curve of the left invariant vector field starting at e, which is a map $\gamma_v : \mathbb{R} \to G$ and $\gamma_v'(0) = v$.

Proposition 1.9. *The integral curve γ_v is the unique morphism of Lie groups $(\mathbb{R}, +) \to G$ with $\gamma_v'(0) = v$. We define the* ***exponential map*** *of G to be $exp : \mathfrak{g} \to G, v \mapsto \gamma_v(1)$.*

For a proof of the above proposition, see ([21], Theorem 20.1). The exponential map possesses lots of interesting properties, for example, it is a diffeomorphism from a neighborhood of 0 in T_eG to a neighborhood of e in G. We shall use this in later sections.

1.2 Lie subgroups and Lie subalgebras

In this section, we discuss Lie subgroups and Lie subalgebras, following (Chapter 19, [21]). Recall that an immesed submanifold is a manifold, not necessarily with subspace topology, such that the inclusion map is an immersion. A Lie subgroup is a group version of immersed submanifolds.

Definition 1.10. Suppose H, G are Lie groups, and that $f : H \to G$ is a morphism of Lie groups, as well as an injective immersion, then we say H is a **Lie subgroup** of G.

Example 1.11. The Lie group $\mathrm{O}(n)$ is a Lie subgroup of $\mathrm{GL}(n,\mathbb{R})$; the Lie group $\mathrm{U}(n)$ is a Lie subgroup of $\mathrm{GL}(n,\mathbb{C})$.

Recall that bracketing of left invariant vector fields is still left invariant. We characterize this phenomenon as being a Lie subalgebra (although in our definition, the space of vector fields will normally be finite dimensional).

Definition 1.12. Given Lie algebras $\mathfrak{h} \subset \mathfrak{g}$, if $\forall \eta, \zeta \in \mathfrak{h}$, we have $[\eta, \zeta] \in \mathfrak{h}$ and $\mathfrak{h}$ is a linear subspace, then we say $\mathfrak{h}$ is a **Lie subalgebra** of $\mathfrak{g}$.

Example 1.13. The Lie algebra $\mathfrak{o}(n)$ is a Lie subalgebra of $\mathfrak{gl}(n,\mathbb{R})$; the Lie algebra $\mathfrak{u}(n)$ is a Lie subalgebra of $\mathfrak{gl}(n,\mathbb{C})$.

Example 1.14. If $f : G \to H$ is a Lie group morphism, then $\ker(f)$ is a Lie subgroup of G with Lie algebra $\ker(f_*)$.

The Lie subgroups and Lie subalgeras are corresponded via the following theorem, we will use the notion of foliations in Section 6.3.

Theorem 1.15. *Given a Lie group G, there exists a bijection between the set of connected Lie subgroups of G, and the set of Lie subalgebras of T_eG.*

Proof. [21] Suppose $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. Then $\mathfrak{h}$ induces a distribution D by left translations,

$$F_x = (L_x)_*\mathfrak{h}.$$

This distribution is involutive: take any basis for $\mathfrak{h}$. With the help of the left translation maps, we get a global frame for F, which we denote as $(X_1, \dots, X_k)$. Then by naturality of Lie brackets we have $[X_i, X_j] \in \mathfrak{h}$ for all $i, j \in \{1, \dots, k\}$. Then for $A = \sum_i A^i X_i$ and $B = \sum_i B^i X_i$, we have $[A, B] = \sum_{i,j} A^i B^j [X_i, X_j] + A^i(X_i B^j)X_j - B^j(X_j A^i)X_i$, which is clearly again a section for F. This shows F is an involutive distribution.

Let S be the foliation determined by D, by the Frobenius' theorem 6.2 since D is invariant under the left translations, so does S, i.e. $\forall g, g' \in G$, we have that $L_g(S_{g'}) = S_{gg'}$. Now let $H = S_e$, and we show that H is the Lie subgroup of G whose Lie algebra is $\mathfrak{h}$. First we observe that $T_eH = T_eS_e = D_e = \mathfrak{h}$, thus if H is a Lie group, then it has Lie algebra $\mathfrak{h}$. Next we show that H is a subgroup: for any $h, h' \in H$, we have $hh' = L_h(h') \in L_h(H) = S_h = H$, and $h^{-1} = h^{-1}e \in L_{h^{-1}}(H) = L_{h^{-1}}(S_h) = H$. Furthermore, the structure maps of H are smooth since they are just ones for G with restricted domains and codomains ([21], Theorem 19.17). Hence H is a Lie subgroup of G with desired Lie algebra.

We omit the proof of uniqueness, the interested readers can refer to ([21], Theorem 19.26) for a proof. ∎

1.3 Actions

Now we introduce Lie group and Lie algebra actions, which is useful in the proof of the correspondence of morphisms (Theorem 1.21), as well as in defining the notion of principal bundles in Section 1.4. The main reference for this section is (Chapter 20, [21]).

Definition 1.16. Given any Lie group G and a manifold M, a **smooth right action** of G on M is a smooth map $\theta : M \times G \to M$, and such that $p(gh) = (pg)h$, for any $p \in M, g, h \in G$.. Here, we write pg as $\theta(p, g)$. Similarly, we can define smooth left actions.

Example 1.17. Any Lie group G admits actions on itself. Let g be any element in G. The right translation $R_g : h \mapsto hg$ induces a right action $G \times G \to G$; similarly, the left translation $L_g : h \mapsto gh$ and the conjugation $C_g : h \mapsto ghg^{-1}$ induce left actions.

Definition 1.18. Given a smooth right action $\theta : M \times G \to M$, the **infinitesimal generator** $\hat{\theta} : T_eG \to \mathfrak{X}(M)$ is given by

$$X \in T_eG \mapsto \hat{X} \in \mathfrak{X}(M), \quad \hat{X}|_p := \frac{d}{dt}(p \cdot exp(tX))|_{t=0}.$$

More generally, we will call any Lie algebra morphism $\mathfrak{g} \to \mathfrak{X}(M)$ a Lie algebra action on M. This action is **complete** if for any $X \in T_eG$, the vector field $\hat{X}$ is complete, i.e. its flow is defined for any time t.

Remark 1.19. Let $\theta : M \times G \to G$ be a smooth right action. For any $p \in M$, define $\theta^p : G \to M, g \mapsto pg$. Then for any $X \in T_eG$, the left invariant vector field $\overleftarrow{X}$ and $\hat{X}$ are θ^p-related, i.e. $(\theta^p)_*(\overleftarrow{X}|_g) = \hat{X}|_{\theta^p(g)}$, for any $g \in G$..

The following result is called the fundamental theorem on Lie algebra actions. For a proof, see ([21], Theorem 20.16).

Theorem 1.20. *Let M be a manifold, G be a simply-connected Lie group, and $\mathfrak{g} = T_eG$. If $\hat{\theta} : \mathfrak{g} \to \mathfrak{X}(M)$ is complete, then there exists a unique right G-action on M whose infinitesimal generator is $\hat{\theta}$.*

Now we turn to the morphisms and prove the correspondence theorem.

Theorem 1.21. *Suppose G, H are Lie groups and G is simply connected. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G, H, respectively. For any Lie algebra morphism $\phi : \mathfrak{g} \to \mathfrak{h}$, there exists a unique Lie group morphism $\Phi : G \to H$ such that $\Phi_* = \phi$.*

Proof. [21] Note that we can identify the Lie algebra $\mathfrak{h}$ with the left-invariant vector fields $\mathfrak{X}^L(H)$. As a result, the morphism $\phi : \mathfrak{g} \to \mathfrak{h} \subset \mathfrak{X}(H)$ defines a Lie algebra action, which is complete. By Theorem 1.20, it corresponds to a unique right G-action $\theta : H \times G \to H$ on H. Then we get a smooth map $\theta^e : G \to H$.

By Remark 1.19, for each $h \in H, X \in \mathfrak{g}$, the vector fields $\overleftarrow{X}$ and $\hat{X}$ are θ^h-related. Thus by naturality, θ^h takes integral curve of $\overleftarrow{X}$, namely $exp(tX)$, to the integral curve of $\hat{X}$. Thus, we know that $\theta(h, exp(tX))$ is an integral curve of $\hat{X}$. Applying the above argument to L_h instead of θ^h, we get that $L_h(\theta(e, exp(tX)) = \theta(h, exp(tX))$. Now replacing h by $\theta(e, g)$, we get $\theta^e(g)\theta^e(exp(tX)) = \theta^e(g \cdot exp(tX))$. Since G is connected, it is generated by the any open neighborhood containing the identity element, thus in particular, it is generated by the image of exp. This shows that $\theta^e : G \to H$ is a morphism.

We need to see that $(\theta^e)_* = \phi$. For this, we use the fact that ϕ can be seen as the infinitesimal generator of θ. For any $X \in \mathfrak{g}$, we have

$$\phi(X) = \overleftarrow{\phi(X)}|_e = \frac{d}{dt}\theta(e, exp(tX))|_{t=0} = d(\theta^e)_e(X),$$

i.e. $(\theta^e)_* X = \phi(X)$. Thus we have constructed a desired Lie group morphism.

The uniqueness of such morphism follows from the following commutative diagram:

$$\begin{CD} \mathfrak{g} @>{\phi}>> \mathfrak{h} \\ @V{exp}VV @VV{exp}V \\ G @>{\theta^e}>> H \end{CD}$$

where $(\theta^e)_* = \phi$. If there is another morphism $\Phi : G \to H$ with $\Phi_* = \phi$, then using the diagram and the fact that the exponential is a diffeomorphism between an open neighborhood of 0 and an open neighborhood of e, we see that θ^e coincides with Φ on an open neighborhood of identity in G. This shows they coincide everywhere, since G is generated by the identity neighborhood. This shows uniqueness.

□

1.4 Adjoint representation

Now we turn to the adjoint representations, by which we can represent Lie group or Lie algebra elements as certain type of linear transformations. Let V be a vector space, denote $\mathrm{GL}(V)$ to be the set of linear automorphisms of the vector space V. It is clear that this is a Lie group. Its Lie algebra is $\mathfrak{gl}(V)$, the set of all linear endomorphisms of V. The main reference for this section is (Chapter 20, [21]).

Definition 1.22. A **representation** of a Lie group G is a morphism of Lie groups $\rho : G \to \mathrm{GL}(V)$. A **representation** of a Lie algebra $\mathfrak{g}$ is a morphism of Lie algebras $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$.

Remark 1.23. If a representation of a Lie group is faithful, i.e. $\rho : G \to \mathrm{GL}(V)$ is injective, then $G \cong \rho(G) \subset \mathrm{GL}(V)$. Thus, elements in G can be seen as matrices.

Example 1.24. Consider $\mathbb{S}^1 \cong \mathrm{U}(1) := \{e^{it} | t \in \mathbb{R}\}$. Below is a representation

$$\rho : \mathbb{S}^1 \to \mathrm{GL}(\mathbb{R}^2), \quad e^{ie} \mapsto \begin{pmatrix} cos(t) & -sin(t) \\ sin(t) & cos(t) \end{pmatrix}.$$

Moreover, ρ_* is a representation for the Lie algebra $\mathfrak{u}(1) = \mathbb{R}$.

Let $\mathfrak{g}$ be a Lie algebra (so in particular, a vector space), let $\mathrm{Aut}(\mathfrak{g})$ denote the set of isomorphisms of Lie algebras. It is easy to see that this is a Lie subgroup of $\mathrm{GL}(\mathfrak{g})$.

Definition 1.25. Given $g \in G$, we define $\mathrm{Ad}_g := (C_g)_* : \mathfrak{g} \to \mathfrak{g}$. This give rise to a map $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$, which is called the **adjoint representation** of G.

Similarly, let $\mathrm{Der}(\mathfrak{g})$ be the set of derivations, i.e. $\phi \in \mathfrak{gl}(\mathfrak{g})$ such that $\phi([X,Y]) = [\phi(X),Y] + [X,\phi(Y)]$. This is a Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$.

Definition 1.26. Given $X \in \mathfrak{g}$, we can define $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}, \mathrm{ad}_X(Y) = [X,Y]$. Again, ad can also be seen as a map $\mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$, called the **adjoint representation** of $\mathfrak{g}$.

Remark 1.27. The two representations are related by that $\mathrm{Ad}_* = \mathrm{ad}$, where Ad is for G, and ad is for $\mathfrak{g} = T_eG$. This provides us a 'bridge' between the group structure in G and the brackets of T_eG.

Now given a finite dimensional Lie algebra $\mathfrak{h}$, we denote the image of the map ad to be $\mathrm{ad}(\mathfrak{h})$, which is a Lie subalgebra of $\mathfrak{gl}(\mathfrak{h})$. By Theorem 1.15, it corresponds to a unique Lie subgroup of $\mathrm{GL}(\mathfrak{h})$. Note the existence of this Lie subgroup does not rely on the fact that the Lie algebra $\mathfrak{h}$ corresponds to a Lie group.

Definition 1.28. We denote the unique Lie subgroup of $\mathrm{GL}(\mathfrak{h})$, whose Lie algebra is $\mathrm{ad}(\mathfrak{h})$, to be $\mathrm{Ad}(\mathfrak{h})$.

Now we see some applications of the adjoint representation.

Definition 1.29. Given a Lie group G, the **center** of G is $\mathrm{Cent}(G) := \{g \in G : gh = hg, \forall h \in G\}$. Given a Lie algebra $\mathfrak{g}$, the **center** of $\mathfrak{g}$ is $\mathrm{Cent}(\mathfrak{g}) := \{X \in \mathfrak{g} : [X,Y] = 0, \forall Y \in \mathfrak{g}\}$.

Remark 1.30. Given a connected Lie group G with Lie algebra $\mathfrak{g}$, then $\mathrm{Cent}(G)$ is the kernel of the adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$. Similarly, $\mathrm{Cent}(\mathfrak{g})$ is the kernel of ad. Thus, as in Example 1.14, $\mathrm{Cent}(G)$ is a Lie subgroup of G with Lie algebra $\mathrm{Cent}(\mathfrak{g})$.

Definition 1.31. Let $\mathfrak{g}$ be a Lie algebra. A linear subspace $\mathfrak{h}$ of $\mathfrak{g}$ is called an **ideal** if for all $X \in \mathfrak{h}, Y \in \mathfrak{g}$, we have $[x,y] = 0$.

The proof of the following proposition can be found in ([21], Theorem 20.28).

Proposition 1.32. *Let H be a connected normal Lie subgroup of a connected Lie group G, with Lie algebras $\mathfrak{h}, \mathfrak{g}$, respectively. Then H is a normal subgroup of G if and only if $\mathfrak{h}$ is an ideal of $\mathfrak{g}$.*

1.5 Principal bundles

In this section we introduce the notion of principal bundles, which is a special class of fiber bundles, with a Lie group acting on each fiber. As we will see in the later chapters, the principal bundles play an important role in the study of transitive Lie algebroids. The main references are [17][37].

Definition 1.33. Let $\theta : M \times G \to M$ be a smooth right action. The action is **free** if for any $m \in M$, $g \in G$, we have: $mg = m \iff g = e$.

Let X, Y be topological spaces. A continuous map $f : X \to Y$ is **proper** if the preimages of compact sets are compact.

Definition 1.34. A smooth action $\theta : M \times G \to M$ is **proper** if the map $\overline{\theta} : G \times M \to M \times M : (m,g) \mapsto (mg,m)$ is proper.

Definition 1.35. [17] Let K be a Lie group. A **principal K-bundle** consists of a base M, which is a smooth manifold, and a smooth manifold P, called the total space of the bundle, and a smooth projection $\pi : P \to M$, with an action of K on P satisfying:

- The Lie group K acts freely on P.
- The manifold M is the quotient of P by the equivalence relation defined by the K-action, and $\pi : P \to M$ maps $q \in P$ to its equivalence class.
- The manifold P is locally trivial in the following sense: for each $x \in M$, there exist a neighborhood U of x and a diffeomorphism

$$\varphi : \pi^{-1}(U) \to U \times K$$

  of the form $\varphi(p) = (\pi(p), \psi(p))$ which is K-equivariant, i.e. $\varphi(pg) = (\pi(p), \psi(p)g)$ for all $g \in K$.

Remark 1.36. If we only require M to be a topological space instead of being a manifold, and all the maps above to be continuous instead of being smooth, then we call P a **continuous principal K-bundle**.

Remark 1.37. [37] Any smooth action of a Lie group K on a manifold M, which is free and proper, induces a principal G-bundle on M over M/K. To obtain the local trivializations for the principal bundle, one can apply a version of tubular neighborhood theorem ([35], Theorem 6.4.3). Conversely, any principal K-bundle P includes the data of a free K-action on P. One can show that this action is always proper.

Example 1.38. Here are some examples for principal bundles.

- Given any manifold M and Lie group K, we may consider the action of K on $M \times K$, given by right multiplications. This principal K-bundle, $M \times K$, is called the **trivial bundle**.
- Embed $\mathbb{S}^1$ in $\mathbb{C}$, and $\mathbb{S}^3$ in $\mathbb{C}^2$, and define an action of $\mathbb{S}^1$ on $\mathbb{S}^3$ by $\mathbb{S}^1 \times \mathbb{S}^3 \to \mathbb{S}^3, (e^{i\theta}, (z_1, z_2)) \mapsto (e^{i\theta} z_1, e^{i\theta} z_2)$. This is a principal $\mathbb{S}^1$-bundle over $\mathbb{S}^2$, called the **Hopf fibration**. It is easy to see that the quotient of $\mathbb{S}^3$ by this action is $\mathbb{C}P^1$. We shall continue our discussion on the Hopf fibration in Example 1.45.

Definition 1.39. Suppose P, Q are principal K-bundles. A diffeomorphism $f : P \to Q$ is an **isomorphism of principal bundles** if it is equivariant to the K-actions, i.e. $\forall g \in K, p \in P$, we have $f(pg) = f(p)g$.

Remark 1.40. It is easy to show that a principal bundle admits a global section if and only if is (isomorphic to) a trivial bundle.

We can establish the notion of connections on principal bundles, which is an Ehresmann connection (see Section 6.2), but compatible with the action.

Definition 1.41. Let $\pi : P \to M$ be a principal K-bundle, a **principal bundle connection** is an Ehresmann connection $\Gamma = \{H_u | u \in P\}$. such that the K-action on P preserves the horizontal subspaces, i.e. $H_{R_g(u)} = (R_g)_{*u} H_u, \forall g \in K, u \in M$.

Remark 1.42. On principal bundles, there is an equivalent way to describe the connections, namely by using the **connection 1-form** $\omega \in \Omega^1(P)$, which satisfies $(R_g)^*\omega = Ad(g^{-1}) \circ \omega$, and $w(\hat{X}) = X, \forall X \in \mathfrak{g}$. Here, $\hat{X}$ is the vector field on P given by $X^\#_u := \frac{d}{dt}(u \cdot exp(tX))|_{t=0}$, as in Definition 1.18.

We shall sketch the equivalence of the two different definitions: given a connection form, we can define the horizontal subspaces by $H_u := \ker(\omega_u)$. Conversely, with the horizontal subspace given, $\forall v \in T_uP$, we can take $\omega_u(v)$ to be the unique $X \in \mathfrak{g}$ such that $\hat{X}_u$ is the vertical component of v, and recover the connection form ω.

Definition 1.43. Given a principal K-bundle $\pi : P \to M$ with connection, let $c : [0,1] \to M$ be a smooth path with $c(0) = p \in M$, then for each choice of $u \in P_p$, there exists a unique path $\widetilde{c} : [0,1] \to P$ of c, such that $\widetilde{c}$ is tangent to the connection, $\pi \circ \widetilde{c} = c$ and $\widetilde{c}(0) = u$. We call $\widetilde{c}$ the **horizontal lift** of c.

The cocycle definition

The reference for this subsection is [37]. There is an alternative description for principal bundles, which is insightful, called the cocycle definition. Given a manifold M, and a principal K-bundle P over M. Pick a (locally finite) open cover $\{U_\alpha\}$ of M, such that for any index α, there exists a trivialization of P over $\{U_\alpha\}$ given by $f_\alpha : P_{U_\alpha} \to U_\alpha \times K$. Now suppose $U_\alpha \cap U_\beta \neq \emptyset$, then we get a transition map $f_\alpha \circ f_\beta^{-1}|_{U_\alpha \cap U_\beta} : U_\alpha \cap U_\beta \times K \to U_\alpha \cap U_\beta \times K$. Now the equivariance of the trivializations implies the equivariance of the transition maps, and in this case, it is equivalent to a map $g_{\alpha\beta} : U_\alpha \cap U_\beta \to K$.

By considering all α, β, such that $U_\alpha \cap U_\beta \neq \emptyset$, we get a family of the transition functions $\{g_{\alpha\beta}\}$, called the **clutching functions**. This family of functions satisfies the following three properties:

- $g_{\alpha\alpha} = e$.
- $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$.
- $g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}$, whenever $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$.

Conversely, the covering of M, as well as the family $\{g_{\alpha\beta}\}$ satisfying the above three properties always defines a principal K-bundle. We shall describe the converse procedure in details. Consider the space

$$E = \bigcup_\alpha U_\alpha \times K / \sim$$

where if $x \in U_\alpha \cap U_\beta$, then $(x,k) \in U_\alpha \times K$ is identified with $(x, g_{\alpha\beta}(x)k) \in U_\beta \times K$. We haven't shown that the construction above is a smooth bundle (e.g. E is a smooth manifold). However, it is clear that E is a continuous principal bundle as in Remark 1.36. According to [32], if K is a Lie group, then each continuous K-bundle over a manifold corresponds to a smooth principal bundle. And the smooth bundles are smoothly isomorphic if and only if they are continuously isomorphic. Thus we do get a smooth principal K-bundle.

If the original data of clutching functions came from locally trivializations of a bundle, then the construction recovers the bundle. Therefore we can describe principal bundles by using the clutching functions. This definition is called the **cocycle definition**, or the **clutching construction**. We will use this in Section 4.3.

Example 1.44. ([37], Section 10.3.1) Let $\pi : E \to M$ denote any vector bundle with rank n. Let $P_{\mathrm{GL}(E)}$ be the submanifold in $\bigoplus_n E$ consisting of all the n-tuples $(e_1, \cdots, e_n)$ such that these elements are in the same fiber, and they form a basis of the fiber, as a vector space. Given any element $(e_1, \cdots, e_n) \in P_{\mathrm{GL}(E)}$, we can apply a linear transformation in $\mathrm{GL}(n, \mathbb{R}^n)$ to get all the other possible basis. Thus $P_{\mathrm{GL}(E)}$ is a principal $\mathrm{GL}(n, \mathbb{R}^n)$-bundle, with the bundle projection $P_{\mathrm{GL}(E)} \to M$, mapping a basis to their common base point in M. This principal bundle is called the **frame bundle** of E.

Now we look at $P_{\mathrm{GL}(E)}$ from the cocycle point of view. First, pick a family of local trivializations of the vector bundle E that covers the base M. Similar to the cocycle definition above, by comparing the trivializations $\phi_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^n$ and $\phi_\beta : E|_{U_\beta} \to U_\beta \times \mathbb{R}^n$, we get transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{R}^n)$. Observe that each of these local trivializations of E identifies $E|_{U_\alpha}$ with $U_\alpha \times \mathbb{R}^n$, and thus identifies $(\bigoplus_n E)|_{U_\alpha}$ with $U_\alpha \times \bigoplus_n \mathbb{R}^n$. And thus identifies $P_{\mathrm{GL}(E)}$ with $U_\alpha \times \mathrm{GL}(n, \mathbb{R}^n) \subset U_\alpha \times \bigoplus_n \mathbb{R}^n$. Under this identification, one sees that the transitions functions of $P_{\mathrm{GL}(E)}$ is the same as those of the vector bundle E. To be more precise, when taking the same family of transition maps for $P_{\mathrm{GL}(E)}$ and E', then $E \cong E'$.

Example 1.45. We revisit Example 1.38 and describe the Hopf fibration by the cocycle definition. Again, embed $\mathbb{S}^3$ in $\mathbb{C}^2$. Let $V_0 = \{[x : y] \in \mathbb{C}^2 : x \neq 0\}$ and $V_1 = \{[x : y] \in \mathbb{C}^2 : y \neq 0\}$. Cover $\mathbb{C}P^1$ with two charts $\phi_0 : V_0 \to \mathbb{C}, [x : y] \mapsto \frac{y}{x}$, and $\phi_1 : V_1 \to \mathbb{C}, [x : y] \mapsto \frac{x}{y}$.

The bundle projection of the Hopf bundle is given by $p : \mathbb{S}^3 \to \mathbb{C}P^1, (x, y) \mapsto [x : y]$. We may write down the local trivializations for the principal bundle, namely

$$\psi_0 : p^{-1}(V_0) \to V_0 \times \mathbb{S}^1, \quad [x : y] \mapsto ([x : y], \frac{x}{|x|}),$$

and

$$\psi_1 : p^{-1}(V_1) \to V_1 \times \mathbb{S}^1, \quad [x : y] \mapsto ([x : y], \frac{y}{|y|}).$$

Compute the $\psi_0 \circ \psi_1^{-1}$ on the overlap, and identify $(V_0 \cap V_1)$ with $\phi_1((V_0 \cap V_1)) = \mathbb{C} \setminus \{0\}$ ($\phi_0((V_0 \cap V_1)$, respectively) on the left (right) hand side, the transition map is then given by

$$\Psi : \mathbb{C} \setminus \{0\} \times \mathbb{S}^1 \to \mathbb{C} \setminus \{0\} \times \mathbb{S}^1, \quad (x, y) \mapsto (\frac{1}{x}, y\frac{x}{|x|}).$$

We can identify $\mathbb{S}^2$ with $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$, as follows: recall that $\mathbb{C}$ is diffeomorphic to $U_+ := \{(z, d) \in \mathbb{C} \times \mathbb{R} : d > -1\}$, and $U_- := \{(z, d) \in \mathbb{S}^2 \subset \mathbb{C} \times \mathbb{R} : d < 1\}$, via the stereographic projections. Then we get natural diffeomorphisms $V_0 \cong U_+$ and $V_1 \cong U_-$, which give rise to a well-defined diffeomorphism $\mathbb{S}^2 \cong \mathbb{C}P^1$.

By viewing $\mathbb{S}^2$ as $\mathbb{C}P^1$, we can translate the transition data back to $\mathbb{S}^2 \subset \mathbb{C} \times \mathbb{R}$ and get the transition $\mathbb{S}^1$-valued function:

$$g : U_+ \cap U_- \to \mathbb{S}^1, \quad (z, d) \mapsto \frac{z}{|z|}.$$

For any $m \in \mathbb{Z}$, we may also take $g^m : U_+ \cap U_- \to \mathbb{S}^1$. Then each of these g^m also defines a principal $\mathbb{S}^1$-bundle, by the cocycle definition. In Section 4.3, we will see that these bundles are mutually non-isomorphic and, are the only principal $\mathbb{S}^1$-bundles over $\mathbb{S}^2$, up to isomorphism.

Chapter 2

Integration of Lie algebras

We have seen several results on correspondences in Lie theory, for example, Theorem 1.14 on subgroups and subalgebras, Theorem 1.20 on actions, Theorem 1.21 on morpisms, etc. There is another fundamental result, stating that every (finite-dimensional) Lie algebra integrates to a Lie group, i.e. it is the Lie algebra of some Lie group. This is usually referred to as Lie's III theorem. One way to show this is to use Ado's theorem mentioned in Remark 1.7. In this chapter, we follow a different approach given by Duistermaat and Kolk in 1999 ([9], Section 1.14), which make use of path spaces in the construction of integration. One significance of this construction is that it generalizes to the case of Lie groupoids and Lie algebroids, see Section 3.4.

2.1 Path space of manifolds

The main reference for the first four sections of this chapter is ([9], Section 1.13). Let M be a connected, smooth manifold and $x_0 \in M$. We denote $P(x_0, M)$ as the space of all continuous paths in M starting at x_0, and call it the **path space**. We assign the compact open topology (see Section 6.5) on $P(x_0, M)$. Usually, $P(x_0, M)$ will not be a finite dimensional manifold, but we can consider the quotient space under path-homotopy, i.e. homotopy fixing end points.

Definition 2.1. For any $\gamma \in P(x_0, M)$, we write $[\gamma]$ for the equivalence class under path-homotopy, and define $\widetilde{M}$ to be the collection of equivalence classes in the path space $P(x_0, M)$.

There is a natural projection map, and path-connectedness of M yields the surjectivity:

$$\pi : [\gamma] \to \gamma(1) : \widetilde{M} \to M.$$

Theorem 2.2. *The topological space $\widetilde{M}$ is a simply connected smooth manifold, with $\pi : \widetilde{M} \to M$ the covering map. We call the smooth manifold $\widetilde{M}$ the* ***universal covering manifold****.*

Proof. [9] We first show that $\widetilde{M}$ is a smooth manifold. To do this, we make use of the tubular neighborhood theorem (see for example Theorem 6.24 of [21]), which enable us to identify a neighborhood of the diagonal $\Delta := \{(x, x)|x \in M\} \subset M \times M$ with a neighborhood of the identity section in TM.

Let $x_1 \in M$, choose an open neighborhood V in M with $\{x_1\} \times V \subset \Delta$. Pick any $\gamma \in P(x_0, M)$ with $\gamma(1) = x_1$. For small δ, we have that $(\delta(t), x) \in \Delta$, for all $t \in [1-\delta, 1], x \in V$. Using tubular neighborhood theorem, we can get for any $x \in V$ a path γ_x, such that it coincides with γ for $t \in [0, 1-\delta]$, and $\gamma_x(1) = x$. This defines a map from V to $P(x_0, M)$, which is smooth in the sense of Banach manifold. And it descends to a map $s : V \to \widetilde{M}$, by taking the path-homotopy class. These maps can be taken as charts of $\widetilde{M}$, and one can show that this give rise to a manifold structure.

Note that $\pi \circ s = id_M$, so π is a local diffeomorphism and a smooth covering map. To show that $\widetilde{M}$ is simply connected, we consider $\widetilde{\widetilde{M}}$, the equivalence classes of paths in $\widetilde{M}$. The projection $\pi : \widetilde{M} \to M$ induces a map between $P(\widetilde{x}, \widetilde{M})$ and $P(x, M)$, where $\pi(\widetilde{x}) = x$. It descends to a map $\widetilde{\pi} : \widetilde{\widetilde{M}} \to \widetilde{M}$, which is a homeomorphism. Further, one can show that $\widetilde{\pi}$ coincides with the projection $\pi_{\widetilde{M}} : \widetilde{\widetilde{M}} \to \widetilde{M}$, which is obtained from replacing M by $\widetilde{M}$, and $\pi_{\widetilde{M}}$ by π. Then $\pi_{\widetilde{M}}$ being bijection implies that $\widetilde{M}$ is simply connected. □

Remark 2.3. Indeed, the universal covering $\widetilde{M}$ of M can be seen as a principal bundle over M. Let $\pi : \widetilde{M} \to M$ be the covering projection.

A diffeomorphism $h : \widetilde{M} \to \widetilde{M}$ is called a **covering transformation** if $\pi \circ h = \pi$. Denote the set of all covering transformation with respective to π to be $\mathrm{Aut}_\pi(\widetilde{M})$. It can be interpreted as a zero-dimensional Lie group, acting smoothly, freely and transitively on each fiber $\pi^{-1}(m), m \in M$ ([21], Proposition 7.23). It's easy to see that as groups, $\mathrm{Aut}_\pi(\widetilde{M}) \cong \pi_1(M, m)$.

We shall see that $\widetilde{M}$ is a principal $\mathrm{Aut}_\pi(\widetilde{M})$-bundle over M: it's clear that the group acts on $\widetilde{M}$ and the quotient space can be identified with M. For the local trivializations, note that locally, π is a diffeomorphism. This gives us the choice for a local zero section in $\widetilde{M}$, which is equivalent to a local trivialization. Thus, $\widetilde{M}$ is a principal bundle.

Definition 2.4. Let M be a connected manifold. For $l \in \mathbb{N}, x_0 \in M$, we will use $P(x_0, M) \cap C^l$ to denote the paths in $P(x_0, M)$ of class C^l.

Remark 2.5. We assign $P(x_0, M) \cap C^l$ with the C^l topology. The inclusion:

$$P(x_0, M) \cap C^l([0, 1], M) \to P(x_0, M),$$

induces a homeomorphism of the quotients. For more details, see ([9], Theorem 1.13.1).

2.2 Path space of Lie groups

Let G be a connected Lie group. We already saw in previous chapters that under path-homotopy, the space of paths starting at a fixed point can be considered as the universal covering manifold. Let $e \in G$ be the identity and $P(G) := \mathrm{P}\,(e, G)$.

Definition 2.6. Provided with the canonical pointwise multiplication $(\gamma \cdot \gamma')\,(t) = \gamma(t)\gamma'(t)$, $(t \in [0, 1])$, $P(G)$ becomes a group, called the **path group** of G.

Remark 2.7. The path component of $[e]$

$$P(G)_0 = \{\gamma \in P(1, G) \mid \gamma \sim e\}$$

is closed and normal in $P(G)$.

Further, $\gamma' \sim \gamma$ in $P(G)$ if and only if $\gamma' \in \gamma P(G)_0$, so that:

$$\widetilde{G} := P(G)/P(G)_0,$$

is the universal covering manifold of G

It is a classical result that when we are taking the universal covering of a Lie group, the covering is also a Lie group. For a detailed proof for the results below, see ([9], Proposition 1.13.4; [21], Theorem 7.7).

Proposition 2.8. *$\widetilde{G}$ is a Lie group, with the induced group structure from $P(G)$, and smooth structure described in Theorem 2.2. Moreover, the $\widetilde{G}$ covers G, and the covering map is a Lie group morphism.*

Example 2.9. Take $G = \mathbb{S}^1$, with multiplication defined by $e^{ix}e^{iy} = e^{i(x+y)}$. Then $P(G)/P(G)_0 \cong \mathbb{R}$, the isomorphism is given by $\mathbb{R} \to P(G)/P(G)_0, \quad t \mapsto [(s \mapsto e^{ist})]$. In general, the universal covering of $H = \mathbb{T}^n$ is $P(H)/P(H)_0 = \mathbb{R}^n$.

Remark 2.10. All statements remain valid if P(G), $\Lambda(G)$ are replaced by $P(G) \cap C^l$, $\Lambda(G) \cap C^l$, respectively, and the equivalence is replaced by smooth homotopy with fixed end points in $P(G) \cap C^l, l \in \mathbb{Z}_{\geq 0}$.

2.3 Path space of Lie algebras

Given any connected Lie group G, we know that the space of equivalence classes of paths can be seen as its universal covering. Now we shall see that the universal covering can also be described using paths in Lie algebras. This description will provide us the candidate of simply-connected integration of Lie algebras.

Definition 2.11. Suppose G has Lie algebra $\mathfrak{g}$, which can be identified with T_eG. Let $P(\mathfrak{g})$ denote the space of all continuous curves $\delta : [0,1] \to \mathfrak{g}$, and we call it the **path space** of Lie algebra $\mathfrak{g}$.

We can consider the following map:

$$D : P(G) \cap C^1 \to P(\mathfrak{g}) : \quad \gamma \mapsto D\gamma$$

where $D\gamma(t) = \frac{d}{ds}(\gamma(s)\gamma(t)^{-1})|_{s=t}$, i.e. applying D is the same as translating the tangent vector back to e.

Proposition 2.12. *The map D is a bijection.*

Proof. Given $\delta \in P(\mathfrak{g})$ denote by $F_\delta : G \times [0,1] \to TG$ the map

$$F_\delta(g,t) = (dR_g)\,\delta(t).$$

We may regard F_δ as a time dependent vector field on G. Note that $\delta = D\gamma$ if and only if

$$\dot{\gamma}(t) = F_\delta(\gamma(t),t), \quad \gamma(0) = e.$$

Thus injectivity of D follows from uniqueness of solutions to ODEs.

To show that D is surjective, we need to see that the solution of the above ODE can be defined on $[0,1]$. For fixed δ, let $\Phi^{s,t}(x)$ be the time-dependent flow of F_δ which is x at time s; then $\gamma(t) = \Phi^{0,t}(e)$ is a solution of $D\gamma = \delta$.

By the fundamental theorem on time-dependent vector fields ([21], Theorem 9.48), there exists an open subset $I \subset [0,1] \times [0,1]$, such that $I_s := \{t : (s,t) \in I\}$ is an open interval in $[0,1]$ containing s, and $\Phi^s(t,e) : I_s \to G$ is the unique maximal integral curve of F_δ. It is clear that $\Phi^{s,t}(e)$ is defined on I. Observe that for all $x \in G$, we may define $\Phi^{s,t}(x) := \Phi^{s,t}(e)x$. Therefore $\Phi^{s,t}(x)$ is defined for all $(s,t) \in I$. As a consequence, for all $(s,u),(u,t) \in I$, let $x = \Phi^{u,t}(e)$, then $\Phi^{s,y}(x)$ is an integral curve in the variable y. Again by the fundamental theorem on time-dependent vector fields, this implies that $(s,t) \in I$. It follows that if $(0,s),(s,t) \in I$ then $(0,t) \in I$. Hence $(0,t) \in I$ for every $t \in [0,1]$, i.e. $\gamma(t) = \Phi^{0,t}(e)$ is defined on I. Thus D is surjective. □

As for topology, it is shown in ([9], Theorem 1.13.4) that

Proposition 2.13. *The map D is a homeomorphism from $P(G) \cap C^1$ with C^1 topology onto $P(\mathfrak{g})$ with C^0 topology.*

Now we would like to see how do the product on $P(G)$ behaves when being transported to $P(\mathfrak{g})$.

Lemma 2.14. *$\forall \gamma, \gamma' \in P(G)$, we have*

$$D(\gamma\gamma')(t) = D(\gamma)(t) + Ad(\gamma(t))D(\gamma'(t)$$

Proof. The identity follows from

$$\begin{aligned} \frac{d}{dt}(\gamma(t)\gamma'(t)) &= \left(\mathrm{T}_{\gamma(t)}\mathrm{R}(\gamma'(t))\right)\frac{d\gamma}{dt}(t) + \left(\mathrm{T}_{\gamma'(t)}\mathrm{L}(\gamma(t))\right)\frac{d\gamma'}{dt}(t) \\ &= \left(\mathrm{T}_{\gamma(t)}\mathrm{R}(\gamma'(t)) \circ \mathrm{T}_1\mathrm{R}(\gamma(t))\right)\mathrm{D}\gamma(t) \\ &\quad + \left(\mathrm{T}_{\gamma'(t)}\mathrm{L}(\gamma(t)) \circ \mathrm{T}_1\mathrm{R}(\gamma'(t))\right)\mathrm{D}\gamma'(t) \\ &= \left(\mathrm{T}_1\mathrm{R}(\gamma(t)\gamma'(t))\right)\left(\mathrm{D}\gamma(t) + \mathrm{Ad}(\gamma(t))\mathrm{D}_{\gamma'(t)}\right). \end{aligned}$$

□

Now we denote $P(\mathfrak{g})_0$ as the image of $P(G)_0$ under the bijection D. Therefore, we can transport the topological and smooth structure on $\widetilde{G}$ to $P(\mathfrak{g})/P(\mathfrak{g})_0$. We will come back to this result in later sections.

2.4 The intrinstic description

In this section we follow [9] and give alternative descriptions for the group structure on $P(\mathfrak{g})$, as well as $P(\mathfrak{g})_0$, which is independent of the choice (or existence) of the Lie group G.

Proposition 2.15. *For any $\gamma \in P(1,G) \cap C^1$ and $t \in [0,1]$, we have $\mathrm{Ad}\,\gamma(t) = A_{D\gamma}(t)$.*

Here, given any $\delta \in P(\mathfrak{g})$, let $A_\delta \in C^1([0,1], \mathfrak{gl}(\mathfrak{g}))$ (defined in Section 1.4) be the solution A of the differential equation, with initial condition:

$$\frac{dA}{dt}(t) = \operatorname{ad}\delta(t)\cdot A(t), \quad A(0) = id_{\mathfrak{g}}.$$

Proof. We check that

$$\frac{d}{dt}\operatorname{Ad}\gamma(t) = \left.\frac{d}{dh}\right|_{h=0}\operatorname{Ad}\left(\gamma(t+h)\cdot\gamma(t)^{-1}\right)\circ\operatorname{Ad}\gamma(t) = \operatorname{adD}\gamma(t)\circ\operatorname{Ad}\gamma(t),$$
$$\operatorname{Ad}\gamma(0) = \operatorname{Ad}e = \mathrm{I}$$

and this shows that $A : t \mapsto \operatorname{Ad}\gamma(t)$ is the solution to the differential equation with initial conditions. □

Remark 2.16. We know that the equivalence relation in $P(G)\cap C^1$ is by definition the same as the existence of a continuous path-homotopy. We claim that: this is equivalent to the existence of a C^1-homotopy keeping the end points fixed. The idea of the proof is to make use of the tubular neighborhood theorem, then locally we can always construct a smooth homotopy by considering the neighborhood in the tangent bundle.

Remark 2.17. A similar argument shows: if there is a continuous homotopy between smooth maps, then there is a smooth homotopy. This is called Whitney's approximation theorem. For more discussion, see ([21], Theorem 6.21).

Now we would like to translate the description of the equivalence relation on $P(G)\cap C^1$ to $P(\mathfrak{g})$. The proof can be found in ([9], Proposition 1.13.4).

Lemma 2.18. *$P(\mathfrak{g})_0 \equiv D\left(P(G)_0\right)$ is the set of $\delta \in P(\mathfrak{g})$ such that there is a map $\epsilon \mapsto \delta_\epsilon$ such that $\delta_0 = \delta$ and $\delta_1 = 0$ and obeys the following equation:*

$$\int_0^1 A_{\delta_\epsilon}(t)^{-1}\frac{\partial\delta_\epsilon(t)}{\partial\epsilon}dt = 0.$$

Remark 2.19. As an application, we can give a new proof for Theorem 1.21 on the correspondence of morphisms, following [31]. Let G be a simply-connected Lie group and H a connected Lie group, with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Suppose $\phi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra morphism, then we want to show that there is a unique Lie group homomorphism $f : G \to H$ such that $\phi = (df)_e$.

Suppose first H is simply-connected, then we define $f =: P(\mathfrak{g}) \to P(\mathfrak{h})$ by $f(\delta) = \phi\circ\delta$. Since ϕ is a Lie algebra morphism we have that $\operatorname{ad} f(\delta)\circ\phi = \phi\circ\operatorname{ad}\delta$. Then $A_{f(\delta)}\circ\phi$ and $\phi\circ A_\delta$ satisfy the same differential equation in Proposition 2.15 with the same initial value, therefore they coincide. Write $\delta = D\gamma$ and $\delta' = D\gamma'$. By Lemma 2.14 and Proposition 2.15, f is a group homomorphism:

$$f(D(\gamma\gamma')) = f(D\gamma) + \phi\circ A_{D\gamma}D\gamma' = f(D\gamma) + A_{\phi\circ D\gamma}(\phi\circ D\gamma') = f(D(\gamma))f(D(\gamma')).$$

By Lemma 2.18, $\delta_0 \sim \delta_1$ implies $f(\delta_0) \sim f(\delta_1)$, and therefore f induces a map $\overline{f} : G = P(\mathfrak{g})/P(\mathfrak{g})_0 \to P(\mathfrak{h})/P(\mathfrak{h})_0 = H$ which is a homomorphism, here we used Proposition 2.8. One can further show f is a Lie group morphism lifting ϕ.

2.5 Integration of Lie algebras

We sketch a proof of Lie's III theorem (Theorem 2.29): given any Lie algebra, there exists a simply connected Lie group whose Lie algebra is the given Lie algebra. The main reference is ([9], Section 1.14) and notes by Moreira [31].

Recall that given a Lie group G with Lie algebra $\mathfrak{g}$, it admits a universal covering $\widetilde{G}$, which is isomorphic to $P(\mathfrak{g})/P(\mathfrak{g})_0$ as groups. Therefore, given any Lie algebra $\mathfrak{h}$, it is natural to guess that $P(\mathfrak{h})/P(\mathfrak{h})_0$ is the simply connected Lie group integrating $\mathfrak{h}$, with the correct topological and smooth structure on it.

$P(\mathfrak{g})$ as a Banach Lie group

Although we consider the quotient space of the path space as a candidate for our simply connected integration, which we shall see that has a structure of finite dimensional manifold, it is not the case for the path space itself, since it is in general infinite dimensional. To solve this, we use the notion of Banach manifold, and show that the group structure is compatible with the smooth structure in the Banach sense.

Definition 2.20. A **Banach Lie group** is a group that also has a Banach manifold structure, and such that the multiplication and inversion of the group are smooth maps, in the sence of Section 6.4.

Now we describe the smooth structure on $P(\mathfrak{g})$. First, since $\mathfrak{g}$ is finite dimensional vector space, it is a Banach space with any norm $\| \ \|$. Then for a continuous path $\delta : [0,1] \to \mathfrak{g}$, we define its norm to be the maximum of the function $\|\delta\|$. This makes the vector space $P(\mathfrak{g})$ in to a Banach space. Note that the topology induced by this norm is the same as the C^0-topology we considered in Proposition 2.13.

Proposition 2.21. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. For $\delta, \delta' \in P(\mathfrak{g})$, define the product and inverse in $P(\mathfrak{g})$ by:*

$$(\delta \cdot \delta')\,(t) = \delta(t) + A_\delta(t)\delta'(t) \quad (t \in [0,1]),$$

$$\delta^{-1}(t) = -A_\delta(t)^{-1}\delta(t)$$

where A_δ is defined in Proposition 2.15. This makes $P(\mathfrak{g})$ a Banach Lie group.

Proof. First we show that the map $A : P(\mathfrak{g}) \to P(\mathfrak{gl}(\mathfrak{g}))$ is a smooth map, which will imply that the smoothness of the structure maps. Recall that A is the solution of the differential equation

$$\frac{dA}{dt}(t) = \operatorname{ad}\delta(t) \circ A(t), \quad A(0) = id_{\mathfrak{g}}.$$

Here $\operatorname{ad}\delta(t)$ takes value in $\mathfrak{gl}(\mathfrak{g})$, and $\circ$ means matrix multiplication. Since the differential equation depends smoothly on the coefficients, we know that the map A is smooth.

Moreover, the image of A_δ always lies in $\operatorname{ad}(\mathfrak{g}) \subset \mathrm{GL}(\mathfrak{g})$ (defined in Definition 1.28). Indeed, it is easy to see that $\operatorname{ad}\delta(t) = D' \circ A_\delta(t)$, where $D' : P(\mathrm{GL}(\mathfrak{g})) \to P(\mathfrak{gl}(\mathfrak{g}))$ is defined in the same way as D, but we replace G by $\mathrm{GL}(\mathfrak{g})$. Thus $A_\delta = (D')^{-1} \circ \operatorname{ad}$, so $A_\delta(t) \in \mathrm{Ad}(\mathfrak{g})$, for all $t \in [0,1]$.

Now we shall see that the product is also smooth. It suffices to show the smoothness of the second term in the product. Suppose the derivative of A at $\delta_1 \in P(\mathfrak{g})$ is the linear map $\lambda : P(\mathfrak{g}) \to P(\mathfrak{gl}(\mathfrak{g}))$. Then we have

$$\frac{\partial A_\delta(t)\delta'(t)}{\partial \delta}|_{(\delta_1,\delta_2)} = \lambda\_(\delta_2).$$

where $\lambda\_(\delta_2)$ is a linear map $P(\mathfrak{g}) \to P(\mathfrak{g})$. In view of the smooth condition defined in Section 6.4, the smoothness of the multiplication map is clear. Similarly, the inverse mapping is also smooth.

By the fact $A_\delta(t) \in \mathrm{Ad}(\mathfrak{g})$, one can verify by computation

$$A_{\delta\cdot\delta'}(t) = A_\delta(t) \circ A_{\delta'}(t) \quad (t \in [0,1]).$$

This further implies the associativity of the product in $P(g)$, which makes $(P(\mathfrak{g}), \cdot)$ a Banach Lie group, with $\delta(t) = \underline{0}(t) \equiv 0$ as the identity element. □

The proof of the following proposition can be found in ([9], Proposition 1.14.1).

Proposition 2.22. *The Lie algebra of $P(\mathfrak{g})$, denoted $P(\mathfrak{g})^{\mathrm{alg}}$, is the space $P(\mathfrak{g})$, provided with the bracket:*

$$[X,Y](t) = \frac{d}{dt}\left[\int_0^t X(s)ds, \int_0^t Y(s)ds\right] \quad (X, Y \in P(\mathfrak{g}), t \in [0,1])$$

The subgroup $P(\mathfrak{g})_0$

In this section we establish $P(\mathfrak{g})_0$ (defined in Lemma 2.18) as a Lie subgroup of $P(\mathfrak{g})$. Similar to what we did before, we want the definition of $P(\mathfrak{g})_0$ to be 'intrinstic', i.e. not depending on the existence of G.

Given a Banach Lie group G, a **Banach Lie subgroup** of G is a Banach Lie group H admitting an injective inclusion map $H \hookrightarrow G$ such that the induced map on the Lie algebras is an embedding.

As we shall expect, $P(\mathfrak{g})_0$ is a Banach Lie subgroup:

Proposition 2.23. *$P(\mathfrak{g})_0$ is a connected normal Banach Lie subgroup of $P(\mathfrak{g})$ with corresponding Lie algebra*

$$P(\mathfrak{g})_0^{\mathrm{alg}} = \left\{X \in P(\mathfrak{g})^{\mathrm{alg}} : \int_0^1 X(s)ds = 0\right\}.$$

Proof. ([9], 1.14.1) First we note that $P(\mathfrak{g})_0$ is connected by definition. Now denote $\mathrm{av} : P(\mathfrak{g})^{\mathrm{alg}} \to \mathfrak{g}$ to br the averaging map, given by

$$\mathrm{av}(X) = \int_0^1 X(t)dt.$$

By Proposition 2.22, it is clear that av is a Lie algebra homomorphism, hence

$$\ker(\mathrm{av}) = \left\{X \in P(\mathfrak{g})^{\mathrm{alg}} : \int_0^1 X(s)ds = 0\right\}$$

is an ideal (see Definition 1.31) of $P(\mathfrak{g})$. Consider the left-invariant distribution on $P(\mathfrak{g})$ given by $D_\delta = (dL_\delta)(\ker(\mathrm{av})) \subset T_\delta P(\mathfrak{g})$, for any $\delta \in P(\mathfrak{g})$. This distribution is involutive and the maximal integral submanifold through the zero path is a Lie subgroup of $P(\mathfrak{g})$. Here we use that Frobenius' theorem (Theorem 6.2) can be generalized to Banach manifolds (see [20], Chapter 6).

We show that $P(\mathfrak{g})_0$ is the maximal connected integral manifold. Note that $(dL_\delta)\,X(t) = A_\delta(t)X(t)$, for all $X(t) \in P(\mathfrak{g})^{alg}$: we may represent $X(t)$ as the derivative of a family of paths $\Gamma_s(t) \in P(P(\mathfrak{g}))$, with $\frac{d}{ds}\Gamma_s(t)|_{s=0} = X(t)$. Then $L_\delta(\Gamma_s) = \delta + A_\delta(\Gamma_s)$. It follows that

$$(dL_\delta)X(t) = \frac{d}{ds}(\delta + A_\delta(\Gamma_s))|_{s=0} = \frac{d}{ds}(A_\delta(\Gamma_s))|_{s=0} = A_\delta(t)X(t).$$

Hence, the definition of $P(\mathfrak{g})_0$ says that $\delta \in P(\mathfrak{g})_0$ if and only if there is a path δ_ϵ in $P(\mathfrak{g})$ from δ to 0 such that

$$\int_0^1 A_{\delta_\epsilon}(t)^{-1}\frac{\partial\delta_\epsilon(t)}{\partial\epsilon}dt = 0.$$

This is equivalent to $(dL_{\delta_c})^{-1}\frac{d\delta_s}{d\epsilon} \in \ker(\mathrm{av})$, that is, such that $\frac{d\delta_s}{d\epsilon} \in D$ for every ϵ. This shows that $P(\mathfrak{g})_0$ is the maximal integral submanifold tangent to D, where the maximality follows from the fact the $P(\mathfrak{g})$ is a Lie subgroup. In view the proof of Theorem 1.15, we have that $P(\mathfrak{g})_0$ is the connected Lie subgroup of $P(\mathfrak{g})$ with Lie algebra $P(\mathfrak{g})_0^{\mathrm{alg}} = \ker(\mathrm{av})$. Moreover, since the Lie algebra $\ker(\mathrm{av})$ of $P(\mathfrak{g})_0$ is an ideal, by Proposition 1.32, the Lie subgroup $P(\mathfrak{g})_0$ is a normal subgroup. □

The quotient group $P(\mathfrak{g})/P(\mathfrak{g})_0$

Since av is surjective, it induces an isomorphism of Lie algebras $P(\mathfrak{g})^{\mathrm{alg}}/P(\mathfrak{g})_0^{\mathrm{alg}} \cong \mathfrak{g}$. We expect that $P(\mathfrak{g})/P(\mathfrak{g})_0$ is a Banach Lie group with Lie algebra $P(\mathfrak{g})^{\mathrm{alg}}/P(\mathfrak{g})_0^{\mathrm{alg}}$, which implies that $P(\mathfrak{g})/P(\mathfrak{g})_0$ is a (finite-dimensional) Lie group integrating $\mathfrak{g}$.

Definition 2.24. A smooth map $F : M \to N$ between manifolds is an **embedding** if it is a homeomorphism onto its image, and an immersion.

Definition 2.25. Suppose H is a Banach Lie subgroup of G. If the inclusion $H \hookrightarrow G$ is an embedding we say that H is an **embedded (Banach) Lie subgroup**.

The following lemmas are taken from ([13], Lemma 2.17, Corollary 2.21). The proofs are quite involved.

Lemma 2.26. *If G is a Banach Lie group and N is a normal embedded Lie subgroup of G then G/N can be given a Banach Lie group structure compatible with the quotient topology. Moreover, the Lie algebra of G/N is identified with $\mathfrak{g}/\mathfrak{n}$ where $\mathfrak{g}$ and $\mathfrak{n}$ are the Lie algebras of G and N.*

Lemma 2.27. *If $f : G \to H$ is a smooth homomorphism of Banach Lie groups and $T \subseteq H$ is an embedded Lie subgroup of H, then $f^{-1}(T) \subseteq G$ is an embedded Lie subgroup of G.*

Proposition 2.28. *The Banach Lie group $P(\mathfrak{g})_0$ is an embedded Lie subgroup of $P(\mathfrak{g})$.*

Proof. We shall sketch the construction of the morphism in Lemma 2.26, which enable us to view $P(\mathfrak{g})_0$ as the kernel, following [31].

Consider the morphism of Banach Lie group $\pi : P(\mathfrak{g}) \to Ad(\mathfrak{g}), \delta \mapsto A_\delta(1)$, and define $P(\mathfrak{g})_1 = \ker(\pi)$. Then we know it is embedded Lie subgroup, and its Lie algebra is given by $\{X \in P(\mathfrak{g})^{alg} : \mathrm{av}(X) \in \mathfrak{z}\}$, where $\mathfrak{z}$ is the center of $\mathfrak{g}$. Since π is surjective, we get an isomorphism of Lie groups $Ad(\mathfrak{g}) \cong P(\mathfrak{g})/P(\mathfrak{g})_1$.

Using the fact that the second homotopy group of a Lie group is trivial ([9], Theorem 1.14.2), one can show that the identity component $(P(\mathfrak{g})_1)^\circ$ is simply-connected (i.e. the connected component of the identity element). We can integrate the morphism of Lie algebras $\mathrm{av} : P(\mathfrak{g})_1^{alg} \to \mathfrak{z}$ to a morphism of Lie groups $\phi : (P(\mathfrak{g})_1)^\circ \to Z$. Then the Lie algebra of $\ker(\phi)$ coincides with $P(\mathfrak{g})_0$. Since $P(\mathfrak{g})_0$ is a connected Lie subgroup of $P(\mathfrak{g})$, by Theorem 1.15, we conclude that $P(\mathfrak{g})_0$ coincides with the identity component of $\ker(\phi)$. This shows that $P(\mathfrak{g})_0$ is an embedded Banach Lie subgroup. □

Theorem 2.29. $P(\mathfrak{g})/P(\mathfrak{g})_0$ *is a simply connected Lie group and it integrates* $\mathfrak{g}$.

Proof. By the surjectivity of $\mathrm{av} : P(\mathfrak{g})^{alg} \to \mathfrak{g}$, we have that $P(\mathfrak{g})^{\mathrm{alg}} / P(\mathfrak{g})_0^{\mathrm{alg}} \cong \mathfrak{g}$. Since $P(\mathfrak{g})_0$ is an embedded Banach Lie subgroup of $P(\mathfrak{g})$, we can apply Lemma 2.26 to conclude that the quotient $P(\mathfrak{g})/P(\mathfrak{g})_0$ is the (Banach) Lie group integrates $\mathfrak{g}$.

For simply connectedness, let $G = P(\mathfrak{g})/P(\mathfrak{g})_0$, then we have

$$G = P(\mathfrak{g})/P(\mathfrak{g})_0 \cong P(G)/P(G)_0 = \widetilde{G}.$$

The second equivalence stands for homeomorphism induced by D, which is mentioned in Proposition 2.13 (defined in Section 2.3). The last equality follows from Remark 2.7, and $\widetilde{G}$ is the universal covering. Hence G is simply connected. □

Chapter 3

Lie groupoids and Lie algebroids

Lie groupoids and Lie algebroids are natural generalizations of Lie groups and Lie algebras. In this chapter, we discuss basic definitions and first examples of Lie groupoids and Lie algebroids. We discuss in Section 3.4 the Lie correspondence, which generalizes various results we have seen in the previous chapters, and motivates our detailed discussion for transitive Lie algebroids in later chapters. In the last section, we introduce the gauge groupoid and algebroid, which has a close relation with principal bundles, and play an important role in the later chapters.

3.1 Notion of groupoid

The reference for this section is [11]. The notion of groupoid occurs naturally in mathematics. Let us first consider an example from topology. Suppose we have a topological space X, then we look at the set of isomorphism classes of continuous paths under path-homotopy (by abuse of notation, we use 'path' to denote the path-homotopy class of paths). There is an interesting structure on this set: we can define the 'composition' of elements to be concatenation of paths; at each point of X, there is the constant path, whose composition with any path starting at this point is path-homotopic to the given path; and for each path there is a well-defined 'inverse', namely, its composition with the original path is path-homotopic to the constant path. This structure is not a group structure, since not any two paths are 'composable', however, one recovers the fundamental group by restricting our attention to a fixed point i.e. loops. This structure on the topological space, which is a generalization of the group structure, is called a groupoid structure. The notion of groupoid is first introduced by H. Brandt [2] in 1926.

Definition 3.1. [11] A **groupoid** consists of a set $\mathcal{G}$ (the set of arrows) and a set M (the set of objects), equipped with the following structure maps, satisfying the law of composition, associativity, units and inverses:

- the **source** and the **target** maps

$$\mathbf{s}, \mathbf{t} : \mathcal{G} \longrightarrow M,$$

  associating to each arrow g its source object $\mathbf{s}(g)$ and its target object $\mathbf{t}(g)$. Given $g \in \mathcal{G}$, we use $y \xleftarrow{g} x$ to indicate that g is an arrow from x to y.

- the **composition** map
$$m : \mathcal{G}_2 \longrightarrow \mathcal{G},$$
is defined on the set $\mathcal{G}_2$ of composable arrows:
$$\mathcal{G}_2 = \{(g, h) \in \mathcal{G} \times \mathcal{G} : \mathbf{s}(g) = \mathbf{t}(h)\}.$$
For a pair (g, h) of composable arrows, $m(g, h)$ is the composition $g \circ h$. We also use the notation $m(g, h) = gh$.

- the **unit** map
$$u : M \longrightarrow \mathcal{G},$$
which sends $x \in M_{\mathcal{G}}$ to the identity arrow $1_x \in \mathcal{G}$ at x. We will often identify 1_x and x, and call them the units.

- the **inverse** map
$$i : \mathcal{G} \longrightarrow \mathcal{G},$$
which sends an arrow g to its inverse g^{-1}.

The structure maps satisfies:

- law of composition: if $x \xleftarrow{g} y \xleftarrow{h} z$, then $x \xleftarrow{gh} z$.

- law of associativity: if $x \xleftarrow{g} y \xleftarrow{h} z \xleftarrow{k} u$, then $g(hk) = (gh)k$.

- law of units: $x \xleftarrow{1_x} x$ and, for all $x \xleftarrow{g} y$, $1_x g = g 1_y = g$.

- law of inverses: if $x \xleftarrow{g} y$, then $y \xleftarrow{g^{-1}} x$ and $gg^{-1} = 1_y, g^{-1}g = 1_x$.

We call $\mathcal{G}$ a groupoid over M.

Example 3.2 (The fundamental groupoid)**.** We can rephrase the example mentioned in the beginning using the language of Definition 4.1. Given a topological space X, we can consider the set $\Pi(X)$, consisting of the path-homotopy classes of continuous paths in X, relative to the same end points. We consider X to be the set of objects, and $\Pi(X)$ as the set of arrows joining the objects, along with the structure maps:

- $\mathbf{s}[\gamma] = \gamma(0)$, $\mathbf{t}[\gamma] = \gamma(1)$, where γ is any continuous path in X.

- Two homotopy classes of paths $[\gamma_1]$, $[\gamma_2]$ is composable if and only if $\gamma_1(0) = \gamma_2(1)$. In this case, $m([\gamma_1], [\gamma_2])$ is defined to be $[\gamma_1 * \gamma_2]$, here $*$ stands for concatenation of paths.

- $i([\gamma]) = [\overline{\gamma}]$, here γ is any continuous path in X, and $\overline{\gamma}$ is the inverse path.

One advantage of working with the fundamental groupoid is that one can get rid of the choice of base points. Notably, there is also a groupoid version of Serfert-van Kampen Theorem. For more discussion on this topic, see [3].

3.2 Lie groupoids

The reference for this section is [11]. In differential geometry, we introduce the notion of Lie groups in order to do calculus on groups: it have compatible group and smooth (manifold) structures in the sense that the composition and inverse map of the group are smooth maps. Analogously, we can define the Lie groupoid, as follows. It was first introduced by Ehresman in the 1950s [10].

Definition 3.3. A **Lie groupoid** is a groupoid $(\mathcal{G}, M)$ such that $\mathcal{G}$ is a possibly non-Hausdorff smooth manifold, M is smooth manifold, the structure maps $\mathbf{s}, \mathbf{t}$ are smooth submersions, and m, u, i are smooth maps.

Remark 3.4. Note that the smoothness is not defined for $m : \mathcal{G}_2 \longrightarrow \mathcal{G}$ unless $\mathcal{G}_2$ is a smooth manifold. This is guaranteed by $\mathbf{s}, \mathbf{t}$ being smooth submersions.

Remark 3.5. There is a good reason why we do not expect $\mathcal{G}$ to be Hausdorff: we would like to include the family of foliation groupoids, which is non-Hausdorff in general. In addition, similar to Lie groups, we can find a universal covering of a Lie groupoid (see Section 4.10 of [28]). However, this construction relies on foliation groupoids, and the universal covering groupoid can be non-Hausdorff even if the given Lie groupoid is Hausdorff.

Example 3.6. Here are some simple examples of Lie groupoids:

- A manifold M is a Lie groupoid over M. Just take $\mathbf{s} = \mathbf{t} = u = i = id_M$, and $m(x, y)$ is defined only when $x = y \in M$. In this case, $m(x, x) = x$.

- A Lie group G is a Lie groupoid. We set the set of objects to be a singleton, and define $\mathbf{s}, \mathbf{t}$ to be the same constant mapping from the Lie group to the single point. Note that any pair of elements in the Lie group is composable, so the map m is defined on $G \times G$. The map u maps the singleton to the identity of Lie group, and i is defined in the same way as the inversion map of G.

- Given a manifold M, the manifold $M \times M$ is a groupoid with the following structure maps. For $(m', m) \in M \times M$, we have $\mathbf{s}(m', m) = m$, and $\mathbf{t}(m', m) = m'$. The composition map, m is defined for two arrows $(m'_1, m_1), (m'_2, m_2) \in M \times M$, if and only if $m_1 = m'_2$. If so, $(m'_1, m_1)(m'_2, m_2) = (m'_1, m_2)$. The units are given by the diagonal embedding of M into $M \times M$, and $i(m', m) = (m, m')$. This groupoid is called the **pair groupoid**, denoted as $\mathrm{Pair}(M)$.

Definition 3.7. A **morphism** of Lie groupoids $F : \mathcal{H} \to \mathcal{G}$ is a smooth map such that

$$F(h_1 \circ h_2) = F(h_1) \circ F(h_2)$$

for all $(h_1, h_2) \in \mathcal{H}_2$. If F is an injective immersion, we say that $\mathcal{H}$ is a **Lie subgroupoid** of $\mathcal{G}$.

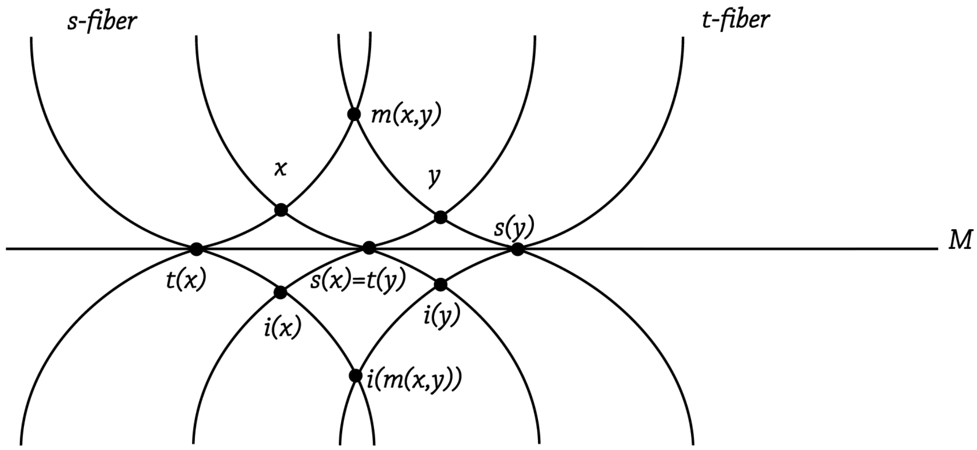


Figure 3.1: Visualisation of pair groupoid.

3.3 Lie algebroids

In this section we follow [11] to study the notion of Lie algebroids. The concept of a Lie algebroid was introduced by J. Pradines [34]. It is a vector bundle, with a bracket similar to that of the tangent bundle.

Definition 3.8. A **Lie algebroid** over a manifold M consists of a vector bundle A together with a bundle map $\rho_A : A \to TM$ and a Lie bracket $[\ ,\ ]_A$ on the space of sections $\Gamma(A)$, satisfying the Leibniz identity

$$[\alpha, f\beta]_A = f[\alpha, \beta]_A + \mathcal{L}_{\rho_A(\alpha)}(f)\beta,$$

for all $\alpha, \beta \in \Gamma(A)$ and all $f \in C^\infty(M)$. Note that here $\mathcal{L}_X f = X(f) = f_*X$, for any vector field X. We will also write ρ for the anchor, and $[\ ,\ ]$ for the bracket when it is clear that which algebroid we are referring to.

Remark 3.9. Here, 'Lie bracket' means that it is a map from $\Gamma(A) \times \Gamma(A)$ to $\Gamma(A)$, such that it is bilinear, skew symmetric, and satisfies the Jacobi identity. To be more precise, for any sections $\alpha, \beta, \gamma \in \Gamma(A)$ and any $a, b \in \mathbb{R}$,

- $[a\alpha + b\beta,\ \gamma] = a[\alpha,\ \gamma] + b[\beta,\ \gamma]$.
- $[\alpha,\ \beta] = -[\beta,\ \alpha]$.
- $[[\alpha,\ \beta],\ \gamma] + [[\beta,\ \gamma],\ \alpha] + [[\gamma,\ \alpha],\ \beta] = 0$.

Example 3.10. Here are some simple examples of Lie algebroids:

- Given a manifold M, we can consider it as a vector bundle of rank 0, so it becomes a Lie algebroid with the trivial bracket.
- A vector space can be seen as a vector bundle over a single point. Thus a Lie algebra can be seen as a Lie algebroid with the Lie bracket.
- Given a manifold M, the tangent bundle is a Lie algebroid. Take $\rho_{TM} = id_{TM}$, and let the bracket be the bracket of vector fields. It is easy to see that the Leibniz rule holds.

Next we define the notion of Lie subalgebroid, following Meinrenken [28]. We will make use of the following notation, for a vector bundle $V \to M$ with given subbundle $W \to N$, we define:

$$\Gamma(V,W) = \{\sigma \in \Gamma(V) | \quad \sigma|_N \in \Gamma(W)\}$$

The map $\Gamma(V,W) \to \Gamma(W)$ is surjective, with kernel $\Gamma(V, 0_N)$ consisting of sections of V whose restriction to N vanishes.

Definition 3.11. A subbundle $B \to N$ of a Lie algebroid $A \to M$ is called a **Lie subalgebroid** if it has the following two properties:

- $\rho_A(B) \subseteq TN$,
- $\Gamma(A,B) \subseteq \Gamma(A)$ is a Lie subalgebra.

Remark 3.12. A Lie subalgebroid B of A inherits a unique Lie algebroid structure, in such a way that the restriction map $\Gamma(A,B) \to \Gamma(B)$ preserves Lie brackets.

Now we define morpisms between Lie algebroids $A \to M$ and $B \to N$. Note that for the case that the vector bundle map, whose base map between the base manifolds is a diffeomorphism, the notion of **morphism** of Lie algebroids can be defined easily: we just need to make sure that it preserves the anchor and the brackets. However, similar approach does not apply to the general case since we cannot push forward sections of A to get sections of B. Therefore we need the following definition, which coincides with our intuitive definition when the base map is a diffeomorphism.

Definition 3.13. Let $A \to M$ and $B \to N$ be two Lie algebroids. A **morphism** of Lie algebroids $\phi : B \to A$ is a vector bundle morphism whose graph

$$\mathrm{Gr}(\phi) \subseteq A \times B$$

is a Lie subalgebroid of the direct product (the direct product of algebroids is defined naturally [28]).

3.4 Lie theory of Lie groupoids and Lie algebroids

Since Lie groupoids and Lie algebroids are generalizations of the notion of Lie groups and Lie algebras, they share many common properties. It is well-known that in a Lie group, any open neighborhood of the identity generates the identity component, which is a connected Lie group. Similarly, according to Mackenzie [26], a neighborhood of the identity section of a Lie groupoid, which is symmetric and fibrewise open, also generates the identity component subgroupoid. Another property of a Lie group is that any subgroup which is also an embedded submanifold, is automatically a Lie subgroup. The groupoid version of this result is stated in Theorem 4.12 of [28].

However, one should not expect the theory of Lie groupoids and Lie algebroids to be completely parallel to that of Lie groups and Lie algebras. For example, unlike the case of Lie groups, a bijective morphism of Lie groupoid is not necessarily an isomorphism.

One thing that is particularly nice and important is the Lie group - Lie algebra correspondence. This is summarized as Lie's I, II, & III theorems. In the case of Lie groupoids, we can also get a unique Lie algebroid corresponds to a given Lie groupoid, and analogous of Lie's I and II theorems are still true. However, Lie's III theorem is not true for groupoids.

The Lie algebroid of a Lie groupoid

The main reference for this subsection is [27]. Let $\mathcal{G}$ be a Lie groupoid over M.

Proposition 3.14. *For each $x \in \mathcal{G}$, the maps*

$$y \mapsto L_x(y) = xy \quad \text{and} \quad z \mapsto R_x(z) = zx$$

are diffeomorphisms, from $\mathbf{t}^{-1}(\mathbf{s}(x))$ onto $\mathbf{t}^{-1}(\mathbf{t}(x))$ and from $\mathbf{s}^{-1}(\mathbf{t}(x))$ onto $\mathbf{s}^{-1}(\mathbf{s}(x))$, respectively. These maps are called the ***left translation*** *and the* ***right translation*** *by x, respectively.*

Proof. The smoothness of the groupoid composition law $m : (x, y) \mapsto xy$ implies the smoothness of L_x and R_x. These maps are diffeomorphisms whose inverses are

$$(L_x)^{-1} = L_{x^{-1}}, \quad (R_x)^{-1} = R_{x^{-1}}.$$

□

The invariant vector fields under right multiplication is called right invariant vector field, see the definition below. Similarly, one can define left invariant vector fields.

Definition 3.15. A vector field Z, defined on an open subset of $\mathcal{G}$, is said to be **right invariant** if they satisfy the two properties:

- the vector field Y is tangent to the $\mathbf{s}$-fibers, i.e. $T\mathbf{s}(Z) = 0$.
- for each z in the domain of definition of Z and each $x \in \mathbf{s}^{-1}(\mathbf{s}(z))$, one has that zx is in the domain of definition of Z and

$$Z(zx) = TR_x(Z(z)).$$

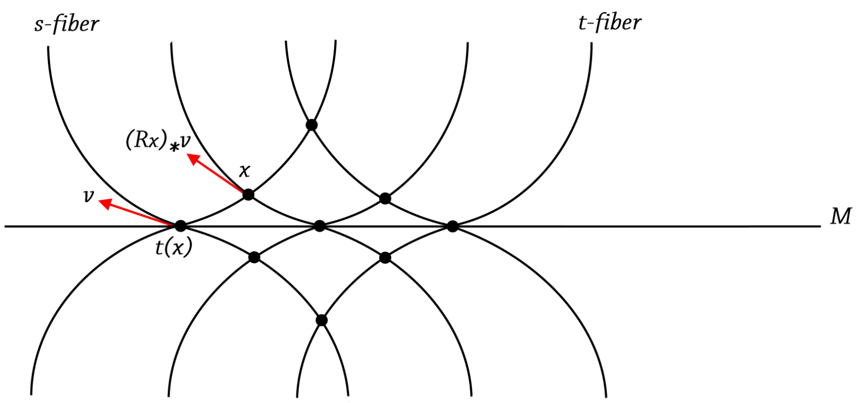


Figure 3.2: The right invariant property.

Now we can define the Lie algebroid of a Lie groupoid. Given a Lie groupoid $\mathcal{G}$, let A be the intersection of $\ker T\mathbf{s}$ and $T\mathcal{G}|_M$. We will establish A as a Lie algebroid. The anchor ρ is given by the map $T\mathbf{t}$ restricted to A. Its Lie bracket is defined as follows: let w_1 and w_2 be two smooth sections of that bundle over an open subset U of M. Let $\widehat{w}_1$ and $\widehat{w}_2$ be the two right-invariant vector fields, defined on $\mathbf{t}^{-1}(U)$, whose restrictions to U are equal to w_1 and w_2 respectively. For each $u \in U$, we define

$$[w_1, w_2]_A(u) := [\widehat{w}_1, \widehat{w}_2](u).$$

One can show that A with the above structure is a Lie algebroid. We will denote Lie algebroid A by $\mathrm{Lie}(\mathcal{G})$ and call it the **Lie algebroid of $\mathcal{G}$**.

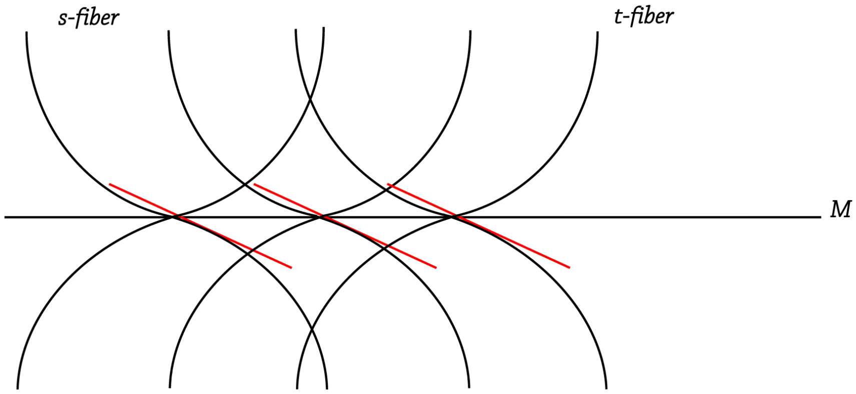


Figure 3.3: The Lie algebroid of a Lie groupoid.

Remark 3.16. Given a morphism of Lie groupoid $F : \mathcal{G} \to \mathcal{H}$, then the differential TF induces a map from $\mathrm{Lie}(\mathcal{G})$ to $\mathrm{Lie}(\mathcal{H})$. Readers could check easily that this is a morphism of Lie algebroids.

Example 3.17. A **bundle of Lie groups** is a Lie groupoid with the same source and target map. A **bundle of Lie algebras** is a Lie algebroid with zero anchor. It's easy to see that the Lie algebroid of any bundle of Lie groups must be a bundle of Lie algebras. The converse is also true, which is due to [8].

Example 3.18. Suppose $\theta : G \times M \to M$ is a smooth left action, we can define a Lie groupoid structure on $G \times M \times M$, as follows: for any $(g, m) \in G \times M$, the source is $m \in M$ and the target is $gm \in M$. Two arrows $(g_1, m_1), (g_2, m_2)$ are composable if and only if $m_1 = g_2 m_2$, and in this case, the product is $(g_1 g_2, m_2)$. We call this Lie groupoid $G \times M$ the **action groupoid**.

Now let $\rho : \mathfrak{g} \to \mathfrak{X}(M)$ be a Lie algebra action. Then we can define a Lie algebroid structure on the trivial vector bundle $M \times \mathfrak{g}$: the anchor ρ_A is given by the infinitesimal action, i.e. $\rho_A(x, v) = \rho(v)|_x$, where $x \in M, v \in \mathfrak{g}$. The bracket on constant sections is given by $[c_v, c_w] = c_{[v,w]}$, where $c_v, c_w \in \Gamma(M \times \mathfrak{g})$ denotes sections with constant value $v, w \in \mathfrak{g}$, respectively. Then the Lie bracket on $\Gamma(M \times \mathfrak{g})$ is uniquely determined by the Leibniz rule. This Lie algebroid $M \times \mathfrak{g}$ is called the **action algebroid**.

One can show that given an smooth action, the algebroid of the action groupoid is the action algebroid associated to its infinitesimal action. Hence by Theorem 1.20, action algebroids arose from complete Lie algebra actions are integrable to action groupoids. It is shown by Dazord [7] that any action algebroid is integrable.

Lie correspondence

The references for this subsection are [6][11]. While we can always get a Lie algebroid given a Lie groupoid, the converse is not true: only some of Lie algebroids come from Lie groupoids. In this case, we say the Lie algebroid **integrates** this Lie groupoid, i.e. Lie algerboid A integrates $\mathcal{G} \iff A = \text{Lie}(\mathcal{G})$.

We have the following version of Lie's I and II theorems, proven by [30], [25] respectively.

Theorem 3.19 (Lie I). *If A is an integrable Lie algebroid, then there exists a (unique) s-simply connected Lie groupoid integrating A.*

Theorem 3.20 (Lie II). *Let $\phi : A \to B$ be a morphism of integrable Lie algebroids, and let $\mathcal{G}$ and $\mathcal{H}$ be integrations of A and B. If $\mathcal{G}$ is s-simply connected, then there exists a (unique) morphism of Lie groupoids $\Phi : \mathcal{G} \to \mathcal{H}$ integrating ϕ.*

Now we look at the problem of integrability of general Lie algebroids. This problem goes back to Pradines [33], who claimed that every Lie algebroid is integrable. Almeida and Molino [1] gave the first example of a non-integrable Lie algebroid (discussed in Corollary 4.31). In 2003, Crainic and Fernandes [5] solved this problem by giving a necessary and sufficient condition for the integrability.

Theorem 3.21 (Lie III). *A Lie algebroid is integrable $\iff$ its monodromy groups are locally uniformly discrete.*

Remark 3.22. We will not discuss the full proof here. Instead, we will present in later chapters a more elementary and geometric construction given by Meinrenken [29], which deals with the special case of transitive Lie algebroids. In particular, in Section 4.6, we discuss the notion of monodromy, and the integrability conditions.

We briefly describe the construction of the integrated groupoid, often called the Weinstein groupoid, following [11]. The Weinstein groupoid, given any Lie algebroid, always exists (as a 'topological groupoid'), and it is a Lie groupoid if and only if the integrability condition in Theorem 3.21 holds. The construction is a generalization of Duistermaat-Kolk's construction for integration of Lie algebras, introduced in Chapter 2. Recall that any Lie group G defines a group structure on the path space $P(G)$, which can be transported to $P(\mathfrak{g})$, the path space of Lie algebra $\mathfrak{g}$ of G. We would like to generalize the construction to Lie groupoids and Lie algebroids. For this purpose, we introduce a new product in $P(G)$ in the remark below.

It turns out that given any finite dimensional Lie algebra $\mathfrak{h}$, we can describe the group structure on $P(\mathfrak{h})$ without referring to the Lie group it corresponds to.

Remark 3.23. Given a Lie group G, the product on $P(G)$ is defined by $\gamma \cdot \gamma'(t) := \gamma(t)\gamma'(t)$, in Definition 2.6. Alternatively, we can consider the concatenation of paths:

$$\gamma * \gamma'(t) = \begin{cases} \gamma'(2t) & 0 \leq t \leq \frac{1}{2}, \\ \gamma(2t-1)\gamma'(1) & \frac{1}{2} \leq t \leq L. \end{cases}$$

The concatenation is equivalent to the original one when we take the quotient $P(G)/P(G)_0$, i.e. given any two paths, the two products are path-homotopic, via the following map:

$$H_{\gamma,\gamma'}(s,t) = \begin{cases} \gamma'(2t) & 0 \le t \le \frac{s}{2}, \\ \gamma((1-s)t)\gamma'((1-s)t+s) & \frac{s}{2} \le t \le 1-\frac{s}{2}, \\ \gamma(2t-1)\gamma'(1) & 1-\frac{a}{2} \le t \le 1. \end{cases}$$

Now we turn to Lie groupoids and Lie algebroids. Let $\mathcal{G}$ be a Lie groupoid. A $\mathcal{G}$-path is a path of arrows, that lies in a $\mathbf{s}$-fiber of $\mathcal{G}$ and starts at the unit. To be more precise,

Definition 3.24. A **$\mathcal{G}$-path** $g : [0,1] \to \mathcal{G}$ is a path such that there exists $x \in M$, such that $\mathbf{s}(g(t)) = x$ for all t, and $g(0) = 1_x$. We denote by $P(\mathcal{G})$ the space of $\mathcal{G}$-paths, equipped with the C^2-topology.

Now let A be a Lie algebroid over M.

Definition 3.25. An **A-path** is a pair of paths (a, γ), where $a : [0,1] \to A$, and $\gamma : [0,1] \to M$ such that

- The path a covers γ, i.e. for any t, $a(t) \in A_{\gamma(t)}$,
- one has $\rho_A(a(t)) = \frac{d\gamma}{dt}(t), \forall t.$

We denote the space of A-paths to be $P(A)$, equipped with the C^1-topology.

Parallel to Proposition 2.13, we have the following result. See ([5], Proposition 1.1) for a proof.

Proposition 3.26. *Suppose $\mathcal{G}$ is a Lie groupoid, with Lie algebroid A, then there exists a homeomorphism $D^R : P(\mathcal{G}) \to P(A)$. For any $\mathcal{G}$-path g, D^R takes g to the A-path $(t \mapsto \frac{d}{ds}(dR_{g(t)^{-1}}g(s))|_{s=t})$.*

Following Remark 3.22, we can define the product in $P(\mathcal{G})$ to be the concatenation of paths. It turns out that the map D^R takes this product in $P(\mathcal{G})$ to the concatenation of A-paths in $P(A)$, i.e. with the products defined on both sides, D^R is an isomorphism of groups.

Now we define a equivalence relation on $P(A)$. A **variation of A-paths** is a map $a_\epsilon = a(\epsilon, t) : [0,1] \times [0,1] \to A$, C^2 on the variable ϵ, such that all $a_\epsilon(0)$ lies in the same fiber, and all $a_\epsilon(1)$ lies in the same fiber. Equivalently, this means when a_ϵ is projected to paths γ_ϵ in M, then they have the same ending points. Let a_ϵ be a variation of A-paths, and ξ_ϵ be a section of A, defined by $\xi_\epsilon(\gamma_\epsilon) = a_\epsilon(1)$. Let $\phi^s_{\xi_\epsilon}$ be the unique 1-parameter group of Lie algebroid ismorphisms $\phi^s_{\xi_\epsilon} : A \to A$, such that $\frac{d}{ds}\phi^s_{\xi_\epsilon}(\beta)|_{s=0} = [\xi_\epsilon, \beta]$.

Two A-paths a_0, a_1 are equivalent if and only if there exists a variation of A-paths a_ϵ, and such that with ξ_ϵ, $\phi^s_{\xi_\epsilon}$ defined as above, we have that

$$\int_0^1 \phi^s_{\xi_\epsilon} \frac{\partial \xi_\epsilon}{\partial \epsilon}(s, \gamma_\epsilon(s)) ds = 0.$$

One can show that in the special case of Lie algebras, this equivalence relation is the same as the condition in Lemma 2.18.

Definition 3.27. Given a Lie algebroid A, the **Weinstein groupoid**, $\mathcal{G}(A)$, is given by the quotient of $P(A)$ seen as a groupoid, by the equivalence relation defined above. The Weinstein groupoid is the unique (source simply connected) candidate for the integration.

3.5 Gauge groupoids and algebroids

We have seen in Example 3.6 that given any manifold P, we can take the pair groupoid $P\times P$. Let P be a principal K-bundle. The K-action on P enable us to study the quotient of $P \times P$. This is an important class of Lie groupoids, called gauge groupoids. In this section we study gauge groupoid and its algebroids.

We first review a result regarding to taking the quotients, see ([21], Theorem 21.10) for a proof.

Lemma 3.28. *Let K be a Lie group acting smooothly, freely, and properly on a smooth manifold M. Then the orbit space M/K is a topological manifold of dimension* $\dim M - \dim G$, *and has a unique smooth structure such that the quotient map $\pi : M \to M/K$ is a smooth submersion.*

By considering the diagonal K-action on $P\times P$, it is easy to see that $\mathrm{Pair}(P) = P\times P$ is a principal G-bundle, and thus $(P \times P)/K$ is a well-defined manifold. The groupoid structure on $(P \times P)/K$ is defined trivially: all the structure maps are induced from structure maps of $P \times P$, and then passing to the quotients ([21], Theorem 4.30).

Definition 3.29. Any principal K-bundle $\pi : P \to M$, with action map $P \times K \to P$, given by $(p, k) \mapsto p \cdot k$, defines a transitive Lie groupoid called the **gauge groupoid**

$$G(P) = \mathrm{Pair}(P)/K \rightrightarrows M.$$

Next we see the notion of gauge algebroid of a principal K-bundle P, which is a quotient of the tangent bundle TP, called gauge algebroids. Here, elements in K acting on TP by the tangent map of right translations. We will see later that this is the Lie algebroid of a gauge groupoid. Again, we need a lemma to ensure that there exists a vector bundle structure on the quotient TP/K.

Lemma 3.30. *Suppose P is a principal K-bundle over M, let $\pi : P \to M$ be the bundle projection. With the induced action of K on TP, the quotient TP/K is a smooth vector bundle over M.*

Proof. ([12], Proposition 1.3.3) We can describe the vector bundle structure of TP/K. For any element $u \in TP$, we denote $[u]$ as the equivalence class in TP/K. For each $m \in M$, and $[u], [v] \in (TP/K)|_m$, there exists unique $g \in K$ such that $\pi(vg) = \pi(u)$, i.e. they are both in $T_{\pi(u)}P$. Thus we can take the sum of u and vg in E, and then take the equivalence class, i.e. $[u] + [v] = [u + vg]$. The scalar multiplication is defined as $t[u] := [tu]$.

As for the local trivialization, we first require that P is trivial over U. This is equivalent to the existence of a local section $\sigma : U \to \mathcal{U} \subset P$. The local trivialization map is then given as follows:

$$\begin{array}{ccc} U \times \mathbb{R}^n & \longrightarrow & (TP/K)|_U \\ \big\downarrow{\scriptstyle (\sigma, id)} & & \big\uparrow \\ \mathcal{U} \times \mathbb{R}^n & \xrightarrow{\ f\ } & TP|_{\mathcal{U}} \end{array}$$

□

We would like to define a Lie algebroid structre on TP/K. For the anchor map, let π be the bundle projection. Then $\rho : TP/K \to TM, [X] \mapsto \pi_* X$ is a well-defined vector bundle morphism ([12], Proposition 1.3.3). This map ρ will be taken as the anchor map.

Now we study the sections of TP/K and define a Lie bracket on sections. Given any local section μ of TP, we can form a map $P \to TP \to TP/K$. A sufficient condition for this map to correspond to a section of TP/K is that $\mu_{qg} = \mu_q g, \forall q \in P, \forall g \in K$. We call sections of TP with this property **K-invariant**. Any K-invariant section μ in TP corresponds to a section $\widetilde{\mu} \in \Gamma(TP/K)$, with $\widetilde{\mu} := [\mu_q]$, where $\pi(q) = m$. One can show the well-definedness and smoothness. Conversely, given any section ν of TP/K, we would like to find the corresponding K-invariant section in TP. Let $\pi^*(TP/K)$ be the pullback bundle (see Proposition 3.32 below). Notice that $\pi : P \to M$ induces a vector bundle isomorphism

$$TP \to \pi^*(TP/K), v_p \mapsto (p, [v_p]).$$

Thus we get a section of $\pi^*(TP/K)$, and thus a section $\overline{\nu}$ of TP. Explicitly,

$$\overline{\nu}_q = \overline{\pi}_q^{-1}(\nu_{\pi(q)}) \in T_q P.$$

Here, $\overline{\pi}$ is the projection $TP \to TP/K$. Moreover, $\overline{\nu}$ is K-invariant.

Hence, we have seen that the sections of TP/K are in one-to-one correspondence to K-invariant sections of TP. Moreover, it is easy to show that with the bracket of vector fields in TP, the bracket of K-invariant sections is again K-invariant. This allows us to use this Lie bracket as our bracket on TP/K.

Definition 3.31. Given a principal K-bundle P, the vector bundle TP/K, with the anchor and brackets defined above, is a Lie algebroid, called the **gauge algebroid** of P. We denote this Lie algebroid as $A(P)$.

Finally, we would like to see that the algebroid of a gauge groupoid is a gauge algebroid. To do this, we introduce and prove a result on more general setting, namely on the pullbacks. Let $f : M \to N$ be a smooth map between manifolds.

Proposition 3.32. *Given a principal K-bundle P over N, with the bundle projection $\pi : P \to N$. Then*

$$f^*P := \{(m, p) \in M \times P : \quad f(m) = \pi(p)\}.$$

admits a principal K-bundle structure over M, with K acting on the second component of f^*P. We call f^*P the ***pullback principal bundle*** *of P by f.*

Proof. See ([37], Section 10.7). □

Given any Lie algebroid A over N, with anchor map ρ, we can consider the following set

$$f^!A := A \times_{TN} TM = \{(x, y) \in A \times TM : \quad \rho(x) = Tf(y)\}.$$

There is no guarantee that the above construction defines a Lie algebroid for arbitrary A. However, when we have that the Lie algebroid A is 'transitive' i.e. the anchor map ρ is surjective, $f^!A$ is indeed a Lie algebroid over M (see [28], Section 7.4). In this case, we call the Lie algebroid $f^!A$ the **pullback algebroid** of A by f.

We shall describe the Lie algebroid structure here. The anchor of $f^!A$ is induced by the natural projection $A \times TM \to TM$. The bracket of $f^!A$ is induced from $A \times TM$, since

$f^!A$ is the preimage of $\mathrm{Gr}(f) = \mathrm{Gr}(Tf)$ under the anchor $A \times TM \to TN \times TM$. More explicitly, given two sections of the pullback, i.e. let $\xi, \eta \in \Gamma(A)$, and $X, Y \in \Gamma(TM)$, such that $\rho(\xi) = Tf(X)$, and $\rho(\eta) = Tf(Y)$, we have

$$[(\xi, X), (\eta, Y)] = ([\xi, \eta], [X, Y]).$$

The following proposition is stated as a general fact in [29]. For convenience, we work out the proof and state it here.

Proposition 3.33. *The gauge algebroid of the pullback principal bundle coincides with the pullback of the gauge algebroid, i.e.* $A(f^*P) = f^!(A(P))$. *Here,* $f : M \to N$ *is a smooth map and* P *is a principal* K*-bundle over* N.

Proof. The vector bundle structure coincides, since taking the pullback is equivalent to taking the fiber product $M_f \times_\pi P$, and the tangent space equals to $TM_{Tf} \times_{T\pi} TP$. Since the group action is only on the second component TP, we have that

$$A(f^*P) = T(f^*P)/K = (TM_{Tf} \times_{T\pi} TP)/K = TM_{Tf} \times_{\widetilde{T\pi}} TP/K = f^!(A(P)),$$

as vector bundles. Here, $\widetilde{\pi}$ denotes the anchor of $A(P)$.

It's clear that the anchor maps coincide. As for the brackets, we may observe that the a section $(X, Z) \in \Gamma(TM_{Tf} \times_{T\pi} TP)$ is K-invariant if and only if $Z \in \Gamma(TP)$ is K-invariant, thus we can identify the K-invariant vector fields. Then we conclude that the bracket in $A(f^*P)$ and $f^!(A(P))$ coincide. □

In the theory of gauge algebroids, it is a fundamental fact that the Lie algebroid of the gauge groupoid is a gauge algebroid. One way to check this is by direct computations (see for example [12]). Thanks to the last proposition, we observe that there is a simple approach avoiding the computations.

Corollary 3.34. The algebroid of the gauge groupoid is the gauge algebroid, i.e. $\mathrm{Lie}(G(P)) = A(P)$, for any principal K-bundle P with bundle projection $\pi : P \to M$.

Proof. This follows from Proposition 3.33, and that

$$TP \cong \pi^!(TP/K),$$

given by $v_p \mapsto (\pi_* v_p, [v_p])$. Since we have $\pi^!(A(P)) = A(\pi_* P) = TP$, we know that $\pi^!(TP/K) \cong \pi^!(A(P))$. This isomorphism is in the form (id_{TM}, f), where f gives an isomorphism between TP/K and $A(P)$. □

We finish this section by a discussion on the Lie groupoid structures on the fundamental groupoids.

Example 3.35. [11] Given a smooth manifold M, we can assign the fundamental groupoid $\Pi(M)$ (defined in Example 3.2) with a Lie groupoid structure. Indeed, it can be seen as the gauge groupoid of the universal covering, viewed as a principal bundle over M. Let $p : \widetilde{M} \to M$ be the covering projection. Recall that $\widetilde{M}$ is a principal $\mathrm{Aut}_p(\widetilde{M})$-bundle over M, by Remark 2.3.

Now consider the pair groupoid $\widetilde{M} \times \widetilde{M}$, we will construct a map from it to $\Pi_1(M)$, as follows: given a pair (x, y), we pick any path in $\widetilde{M}$ joining x, y. The we project it to a path in M, and take the homotopy class in $\Pi_1(M)$. By simply-connectedness, the resulting equivalence class does not depend on the choice of path. Moreover, by the definition of covering transformation and the smooth version of lifting lemma [14], one can show $(x_1, y_1), (x_2, y_2)$ has the same image if and only if $\exists h \in \operatorname{Aut}_p(\widetilde{M})$ such that $x_2 = h(y_1), x_2 = h(y_1)$. Combing with the fact that this map preserves groupoid structures, we conclude that $(\widetilde{M} \times \widetilde{M}) / \operatorname{Aut}_p(\widetilde{M})$ is isomorphic to $\Pi_1(M)$ as groupoids. Therefore, $\Pi_1(M)$ also has a Lie groupoid structure.

Chapter 4

The Transitive Theory

In this chapter, we discuss the theory of transitive Lie algebroids. The first three sections introduce some basic properties and give the necessary background for the main result (Theorem 4.16) in Section 4.4, namely we classify the transitive Lie algebroids over the 2-sphere. We relate the classification results on principal bundles with transitive Lie algebroids by Proposition 4.19 in Section 4.5. We apply these results in Section 4.6, namely we can prove that the integrability condition is necessary. Finally, the last part is devoted to an example illustrating the theory. Throughout the rest of this thesis, we assume M is a connected manifold.

4.1 Transitive Lie groupoids and Lie algebroids

The main reference for this section is [26] [29]. In this section we introduce the definition of transitive Lie groupoids and transitive Lie algebroids. We will show that: all transitive Lie groupoids are gauge groupoids. Conversely, gauge algebroids is integrable to gauge groupoids only.

Definition 4.1. A Lie groupoid $\mathcal{G} \rightrightarrows M$ is called **transitive** if it has a unique orbit: for any two elements $m, m' \in M$ there is an arrow g from $m = \mathbf{s}(g)$ to $m' = \mathbf{t}(g)$.

Example 4.2. An action groupoid is transitive when the action is transitive.

Proposition 4.3. *Every transitive Lie groupoid $\mathcal{G} \rightrightarrows M$ is isomorphic to a gauge groupoid (defined in Definition 3.29).*

Proof. ([22]; [30], Proposition 5.14; [28], Theorem 3.10) It is clear that gauge groupoids are transitive. For the converse, given any transitive Lie groupoid $\mathcal{G}$, pick any point $m \in M$, then the source fiber $\mathbf{s}^{-1}(m)$ is a principal $\mathbf{s}^{-1}(m) \cap \mathbf{t}^{-1}(m)$-bundle. Consider the map $\mathbf{s}^{-1}(m) \times \mathbf{s}^{-1}(m) \to \mathcal{G}$, mapping (g, g') to $g(g')^{-1} \in \mathcal{G}$. One can show that it descends to a isomorphism of Lie groupoids $(\mathbf{s}^{-1}(m) \times \mathbf{s}^{-1}(m))/(\mathbf{s}^{-1}(m) \cap \mathbf{t}^{-1}(m)) \to \mathcal{G}$. Hence, the gauge groupoid of $\mathbf{s}^{-1}(m)$ is isomorphic to $\mathcal{G}$. □

Definition 4.4. A Lie algebroid $A \Rightarrow M$ is **transitive** if its anchor map $\rho_A : A \to TM$ is surjective.

Example 4.5. Gauge algebroids are transitive Lie algebroids.

Lemma 4.6. *Given a transitive Lie algebroid A over a connected manifold M, which is integrable to a Lie groupoid $\mathcal{G}$, then $\mathcal{G}$ must be transitive.*

Proof. Suppose the Lie groupoid $\mathcal{G}$ has orbits $\mathcal{O}_1, \mathcal{O}_2, \cdots$, where each orbit is a subset of M such that for any two element in the same orbit, there is an arrow connecting them. Recall that A is isomorphic to $\mathrm{Lie}(\mathcal{G}) = \ker(T\mathbf{s})|_M$, and the anchor is given by $T\mathbf{t}$ restricted to A. So A being transitive implies that for any $m \in M$, $T_m\mathbf{t} : \ker(T_m\mathbf{s}) = T_m\mathbf{s}^{-1}(m) \to T_mM$ must be surjective. Then for $m \in \mathcal{O}_i$ we have $T_m\mathbf{t}(\ker T_m\mathbf{s}) \subset T_m\mathcal{O}_i$, and therefore $T_m\mathcal{O}_i = T_mM$ for each $m \in \mathcal{O}_i$. Thus $\mathbf{t}|_{\mathbf{s}^{-1}(m)} : \mathbf{s}^{-1}(m) \to M$ is a submersion. Hence $\mathbf{t}(\mathbf{s}^{-1}(m))$ is open in M for all $m \in M$, i.e. $\mathcal{O}_i$ is open in M. Since M is connected, there can be only one orbit which equals to M. Thus a transitive Lie algebroid can only integrates to a transitive Lie groupoid. □

The following lemma says that the local structure of transitive Lie algebroids is simple. We postpone the proof of it to Proposition 5.3.

Lemma 4.7. *[26] Given any Lie algebroid A over M, and contractible open set $U \subset M$, there exists an isomorphism of Lie algebroids*

$$A|_U \cong TU \times \mathfrak{k}.$$

where $\mathfrak{k}$ is a Lie algebra. The bracket on $TU \times \mathfrak{k}$ is given by bracketing sections on TU and bracketing on $\mathfrak{k}$, respectively.

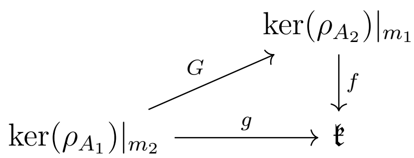
The isomorphism in the previous lemma is called a **trivialization** of A. It is easy to see that, when taking different trivializations of A over U, V, say $A|_U \cong TU \times \mathfrak{k}$, and $A|_V \cong TV \times \mathfrak{h}$, one has that $\mathfrak{k}$ and $\mathfrak{h}$ are isomorphic Lie algebras. We refer to any Lie algebra $\mathfrak{k}$ in this isomorphism class as the **structure Lie algebra** (or isotropy) of A.

Finally, we introduce the notion of framings, which stands for trivializations with fixed values at one point. We will use this in later sections.

Definition 4.8. Let $A \Rightarrow M$ be a transitive Lie algebroid, with anchor ρ and structure Lie algebra $\mathfrak{k}$, we define a **framing** of A at m_0 to be an isomorphism of Lie algebras $\phi_0 : \ker(a)|_{m_0} \to \mathfrak{k}$. We will call an algebroid with framing a **framed algebroid**. We denote by

$$\mathrm{Tran}_{\mathfrak{k}}(M, m_0)$$

the set of isomorphism classes of framed transitive Lie algebroids $A \Rightarrow M$ with structure Lie algebra $\mathfrak{k}$, whose framings are at m_0, modulo isomorphisms intertwining the framing. To be more precise, $G : A_1 \to A_2$ intertwines the framing if



where f, g denote the framings.

Remark 4.9. Given a transitive Lie groupoid A over M, and a contractible open set $U \subset M$, then by Lemma 4.7 we get a local trivialisation f of A. Pick any point $m \in U$, and choose any framing at m. The local trivialisation f restricted to $\ker(\rho_A)|_m$ may be different from the chosen framing by an automorphism of Lie algebra $\phi \in \mathrm{Aut}(\mathfrak{k})$, but $(id_{TU} \times \phi) \circ f$ is a trivialization respecting the framing. In short, given any framing, we can extend it to a trivialization over any given contractible open neighborhood.

4.2 Gauge transformations

Recall in Section 1.5, when studying the principal K-bundles, we can take local triviliztions and compare them. The resulting transition map can be seen as a K-valued function. Thanks to Lemma 4.7, similar approach also apply to transitive Lie algebroids. To compare the different local trivializations and hopefully, obtain a nice description of the transition data, we study certain isomorphisms between trivial algebroids of the form $TU \times \mathfrak{k}$, called the gauge transformations. The main reference for this section is [26][29].

We first introduce the Maurer-Cartan form. Let K be a Lie group with Lie algebra $\mathfrak{k}$.

Definition 4.10. The left-invariant **Maurer-Cartan form** on K is the $\mathfrak{k}$-valued 1-form $\theta^L_K \in \Omega^1(K, \mathfrak{k})$ given by

$$\theta^L_K(v) = (L_{k^{-1}})_* v \in T_e K, \quad v \in T_k K.$$

A fundamental property of the Maurer-Cartan form says that it satisfies the Maurer-Cartan equation: $d\theta^L_K = \frac{1}{2}[\theta^L_K, \theta^L_K]$. The following lemma is well-known in the theory of Cartan forms, for a proof see [15].

Lemma 4.11. *Let K be a Lie group with Lie algebra $\mathfrak{k}$ and θ^L_K the left-invariant Maurer-Cartan form on K. Let M be a simply connected manifold and θ be a $\mathfrak{k}$-valued 1-form on M. If θ satisfies the Maurer-Cartan equation $d\theta + \frac{1}{2}[\theta, \theta] = 0$, then there exists a unique smooth mapping $f : M \to K$ such that $f(m) = e$ and $f^*\theta^L_K = \theta$.*

Definition 4.12. Given a transitive Lie algebroid $A \Rightarrow M$ with structure Lie algebra $\mathfrak{k}$, a **gauge transformation** is a Lie algebroid automorphism covering the identity on TM, and we denote the group of gauge transformations to be $\mathrm{Gau}(A)$. To be more precise, $F \in \mathrm{Gau}(A)$ if and only if $F : A \to A$ is a Lie algebroid (auto)morphism, and such that $\rho_A = \rho_A \circ F$, where ρ_A is the anchor.

We are interested in the gauge transformations of a trivial transitive Lie algebroid $A = TM \times \mathfrak{k} \Rightarrow M.$, where $\rho_A = pr_1 : TM \times \mathfrak{k} \to TM$. Note that given two trivializations of A, $F_1 : A|_U \to TU \times \mathfrak{k}$ and $F_2 : A|_V \to TV \times \mathfrak{k}$, where $U \cap V \neq \emptyset$, then the difference $F_1 \circ F_2^{-1}|_{U \cap V} \in \mathrm{Gau}(T(U \cap V) \times \mathfrak{k})$.

To study $\mathrm{Gau}(TM \times \mathfrak{k})$, first observe that any element $F \in \mathrm{Gau}(TM \times \mathfrak{k})$ is in particular a vector bundle isomorphism covering id_M. Hence for any $p \in M$, on the fiber $T_pM \times \mathfrak{k}$, F is of the form:

$$\begin{pmatrix} I & 0 \\ \theta_p & \Phi_p \end{pmatrix}$$

where $\theta_p : T_pM \to \mathfrak{k}$, and $\Phi_p : \mathfrak{k} \to \mathfrak{k}$. Since F is an isomorphism, we know Φ_p is invertible, so $\theta \in \Omega^1(M, \mathfrak{k}), \Phi \in C^\infty(M, \mathrm{GL}(\mathfrak{k}))$(defined in Section 1.4). We can describe $F \in \mathrm{Gau}(A)$ as the pair $(\theta, \Phi) \in \Omega^1(M, \mathfrak{k}) \rtimes C^\infty(M, \mathrm{GL}(\mathfrak{k}))$, here the semi-direct product comes from the composition of matrices. Since F is a Lie algebroid isomorphism, we can do more:

Proposition 4.13. *If M is connected and simply connected, then there is a surjective map*

$$C^\infty(M,K) \rtimes \mathrm{Aut}(\mathfrak{k}) \to \mathrm{Gau}(TM \times \mathfrak{k}).$$

Its kernel consists of pairs (c^{-1}, Ad_c) with $c \in K$ (as a constant function). In particular, given a base point m_0, every gauge transformation is given by a unique pair (f, Ψ) such that $f(m_0) = e$.

Proof. ([29], Proposition 3.7) We have already seen that each gauge transformation can be seen as a pair $(\theta, \Phi) \in \Omega^1(M, \mathfrak{k}) \rtimes C^\infty(M, \mathrm{GL}(\mathfrak{k}))$, and act on an arbitrary section $X + \xi$, with $X \in \mathfrak{X}(M), \xi \in C^\infty(M, \mathfrak{k})$, by

$$(\theta, \Phi) \cdot (X + \xi) = X + \iota_X \theta + \Phi(\xi),$$

here $\iota_X \theta = \theta(X)$.

For any $X, Y \in \mathfrak{X}(M), \xi, \eta \in C^\infty(M, \mathfrak{k})$, the gauge transformation represented by (θ, Φ) preserving the Lie algebroid bracket

$$[X + \xi, Y + \eta] = [X, Y] + [\xi, \eta] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi$$

is equivalent to the following conditions

$$\begin{aligned} &\Phi \in C^\infty(M, \mathrm{Aut}(\mathfrak{k})), \\ &\mathrm{d}\theta + \frac{1}{2}[\theta, \theta] = 0, \\ &\mathrm{d}\Phi + \mathrm{ad}_\theta \circ \Phi = 0. \end{aligned}$$

The above description for gauge transformations of $TM \times \mathfrak{k} \Rightarrow M$ in terms of pairs (θ, Φ) is due to Mackenzie [26]. Now fix a base point $m_0 \in M$. Since θ satisfies the Maurer-Cartan equation and M is simply connected, by Lemma 4.11, there is a unique function $f \in C^\infty(M, K)$, with $f(m_0) = e$, such that

$$\theta = f^* \theta_K^L.$$

By the third condition above,

$$\mathrm{d}\Phi + \mathrm{ad}_\theta \circ \Phi = 0 \Rightarrow \mathrm{d}\left(\mathrm{Ad}_{f^{-1}} \circ \Phi\right) = 0 \Rightarrow \mathrm{Ad}_{f^{-1}} \circ \Phi = \Psi \Rightarrow \Phi = \mathrm{Ad}_f \circ \Psi$$

where $\Psi = \Phi(m_0) \in \mathrm{Aut}(\mathfrak{k})$. We hence see that the action of (Φ, θ) can be regarded as the action of Ψ followed by the action of f. That is, there exits a bijection

$$\begin{aligned} C^\infty(M, K) \rtimes \mathrm{Aut}(\mathfrak{k}) &\to \Omega^1(M, \mathfrak{k}) \rtimes C^\infty(M, \mathrm{GL}(\mathfrak{k})) \\ (f, \Psi) &\mapsto (f^* \theta_K^L, \mathrm{Ad}_f \circ \Psi). \end{aligned}$$

Since each element of $\Omega^1(M, \mathfrak{k}) \rtimes C^\infty(M, \mathrm{GL}(\mathfrak{k}))$ represents a gauge transformation, we obtain a surjective map $C^\infty(M, K) \rtimes \mathrm{Aut}(\mathfrak{k}) \to \mathrm{Gau}(TM \times \mathfrak{k})$. The pair (f, Ψ) represents the trivial transformation if and only if $f^* \theta_K^L = 0$ and $\mathrm{Ad}_f \circ \Psi = \mathrm{id}$, i.e. $f = c^{-1}$ is a constant function, and $\Psi = \mathrm{Ad}_c$. □

Remark 4.14. The readers can check that the composition law of gauge transformations is given by $(f, \Psi_1) \circ (g, \Psi_2) = (fg, \Psi_1 \circ \Psi_2)$.

4.3 Classification of principal bundles over $\mathbb{S}^2$

In this section we apply the cocycle (clutching) construction and give a classification of principal bundles over $\mathbb{S}^2$. The main reference for this section is [4] [36].

Denote by $\mathrm{Prin}_K(M)$ the set of (smooth) isomorphism classes of smooth principal K-bundles over M. Following the discussion on page 8, we will also use $\mathrm{Prin}_K(M)$ to represent isomorphism classes of continuous bundles. The result below also applies to the case when G is a topological group, as well as bundles over n-spheres.

Theorem 4.15. *There is a bijective correspondence between principal bundles and the fundamental group*

$$\mathrm{Prin}_K(\mathbb{S}^2) \cong \pi_1(K, e)$$

where π_1 denotes the fundamental group.

We only sketch the correspondence here, and refer the interested readers to ([4], Theorem 2.7). Let $p : E \to \mathbb{S}^2$ be a principal K-bundle. Write $\mathbb{S}^2$ as the union of two contractible open sets, U_+, U_-, such that U_+ covers the upper hemisphere and the equator, U_- covers the lower hemisphere and the equator, as in Example 1.45.

Since U_+ and U_- are both contractible, E restricted to each of these hemispheres is trivial ([4], Corollary 2.6). Moreover, if we fix a trivialization of the fiber of E at the base point $x_0 \in \mathbb{S}^1 \subset \mathbb{S}^2$, then we can extend this trivialization to U_+ and U_-. The readers can compare this to Definition 4.8 and Remark 4.9. Following Section 1.5, we may write

$$E = (U_+ \times K) \cup_\theta (U_- \times K) := ((U_+ \times K) \cup (U_- \times K))/\sim$$

where $\theta : U_+ \cap U_- \to K$ is a clutching function. Here, we identify $(x, g) \in (U_+ \times K)$ with $(x, \theta(x)g) \in (U_- \times K)$, for all $x \in U_+ \cap U_-$. Notice that since our original trivializations extended a common trivialization on the base point $x_0 \in \mathbb{S}^1$, the change of trivialization over the point x_0 is just the identity transformation, and therefore the clutching function θ maps the base point x_0 to the identity $e \in K$. Now θ can be represented by an element $\theta \circ i$ in $\pi_1(K)$, where $i : \mathbb{S}^1 \to U_+ \cap U_-$ is the inclusion. Since each principal bundle is associated to a clutching function θ, we obtain a correspondence

$$\Theta : \mathrm{Prin}_K\left(\mathbb{S}^2\right) \to \pi_1(K).$$

One can show that the isomorphism class of principal bundle is independent of the choice of the clutching functions, given that the clutching functions represents the same element in $\pi_1(K)$. Therefore, we may use a base point preserving map $\mathbb{S}^1 \to K$ to represent θ.

Conversely, suppose E_1 and E_2 have homotopic clutching functions, $\theta_1 \simeq \theta_2 : \mathbb{S}^1 \to K$. Let $H : \mathbb{S}^1 \times [-1, 1] \to K$ be a homotopy between θ_1 and θ_2. One can show that the principal bundle with the clutching function H is isomorphic to both E_1 and E_2.

4.4 Classification of transitve Lie algebroids over $\mathbb{S}^2$

In this section we look for a similar classification for the isomorphism classes of transitive Lie algebroids over $\mathbb{S}^2$, i.e. $\mathrm{Tran}_{\mathfrak{k}}(\mathbb{S}^2, m_0)$ (defined in Definition 4.8). Suppose we are given any transitive Lie algebroid A over M, with framing at $m_0 \in M$. By Remark 4.9, we can extend this framing to local trivializations over U_+, U_- (same as last section, U_+ is a contractible open set covering the upper hemisphere and the equator, U_- is defined similarly). Then on $U_+ \cap U_-$ we get an automorphism of Lie algebroids covering $id_{T(U_+\cap U_-)}$, since it must commutes with the anchors. Then it is a gauge transformation, and recall from Proposition 4.13 that we can write this gauge transformation as a pair (θ, Φ), where $\theta \in \Omega^1(U_+ \cap U_-, \mathfrak{k})$, $\Phi \in C^\infty(U_+ \cap U_-, GL(\mathfrak{k}))$. The fact that both trivializations extend the framing implies that $\Phi(m_0) = id_{\mathfrak{k}}$.

Conversely, given any gauge transformation (θ, Φ) with $\Phi(m_0) = id_{\mathfrak{k}}$, we can glue trivial algebroids $TU_+ \times \mathfrak{k}$, $TU_- \times \mathfrak{k}$. The gauge transformation is the identity map on the level of base manifolds, so the gluing of base manifold is trivial. Also, the gauge transformation is a vector bundle isomorphism, so these two, as vector bundles, glue into a smooth vector bundle, by the reconstruction theorem(see for example [21]). It suffices to construct the Lie bracket, but since the gauge transformation preserves the Lie bracket, it's easy to see that the induced Lie bracket over U_+, U_- agree on the overlap, thus we can glue them. This algebroid we construct can have any framing, and we have to assign it one at m_0. However, given two Lie algebroids having the fixed gluing map, regardless of framing, we can construct an isomorphism, as follows: it is obvious that over U_+, U_- there are isomorphisms. Then the same gluing map implies that the two isomorphisms glue together. Interestingly, we get a framing-free description for algebroids with framings. And a more general version of the above result is explained in (Proposition 8.2.8, [26]).

The following is the main theorem of this chapter, which is due to Meinrenken [29][28]. Although a relatively complete treatment has been done in [29], we would like to follow a different, but more natural approach. This approach we take, which is parallel to the last section, has been sketched in Meinrenken's lecture notes [28], with the proof for well-definedness and injectivity parts missing. We would like to complete his proof here.

Theorem 4.16. *Let $m_0 \in \mathbb{S}^1 \subset \mathbb{S}^2$ (embedded as equator). Isomorphism classes of framed transitive Lie algebroid $A \in \mathrm{Tran}_{\mathfrak{k}}(\mathbb{S}^2, m_0)$ are classified by elements of the center of $\widetilde{K}$, where $\widetilde{K}$ is the connected and simply connected Lie group integrating $\mathfrak{k}$.*

Proof. We first construct a natural map from $\mathrm{Tran}_{\mathfrak{k}}(\mathbb{S}^2, m_0)$ to $\mathrm{Cent}(\widetilde{K})$, then we show that it is well-defined and bijective.

Construction of the map:

Let $h_+ : A|_{U_+} \to TU_+ \times \mathfrak{k}$, $h_- : A|_{U_-} \to TU_-$ be trivializations of the Lie algebroids A extending the framing. Then $h_+ \circ h_-|^{-1}_{A|_{U_+\cap U_-}} : T(U_+ \cap U_-) \times \mathfrak{k} \to T(U_+ \cap U_-) \times \mathfrak{k}$ is a gauge transformation. Thus it can be written as (θ, Φ), where $\theta \in \Omega^1(U_+ \cap U_-, \mathfrak{k})$, $\Phi \in C^\infty(U_+ \cap U_-, GL(\mathfrak{k}))$ with $\Phi(m_0) = id_{\mathfrak{k}}$. Notes that we cannot find a pair (f, Ψ) as Proposition 4.13, since $U_+ \cap U_- \simeq \mathbb{S}^1$ is not simply connected. However, we can find a universal cover of $U_+ \cap U_- \cong \mathbb{S}^1 \times (-1, 1)$, then pullback the gauge transformation.

Let $C = \mathbb{R} \times (-1, 1)$ be the universal cover of $U_+ \cap U_-$, with the covering map $\pi : C \to U_+ \cap U_-$, where $\pi(x, y) = \pi(x + 1, y)$, for all x, y. Define $\widetilde{\theta} = \pi^*\theta$, $\widetilde{\Phi} = \pi^*\Phi$. It's easy to check that they satisfy the conditions in the proof of Proposition 4.13, thus there exists a unique map $f : C \to \widetilde{K}$, with $f(m_0) = e$ and $\widetilde{\theta} = f^*\theta^L$, ($\theta^L$ is the left invariant Maurer-Cartan form on $\widetilde{K}$). Also $\widetilde{\Phi} = \mathrm{Ad}_{f^{-1}} \circ \Psi$, but $\widetilde{\Phi}(m_0) = id_{\mathfrak{k}}$ implies $\Psi = id_{\mathfrak{k}}$ i.e. $\widetilde{\Phi} = \mathrm{Ad}_{f^{-1}}$. The above construction is mentioned in [28]. Now we would like to use f to define the correspondence. We shall prove the following.

Claim: $f(x+1,y)(f(x,y))^{-1}$ is constant and $f(x+1,y)(f(x,y))^{-1} \in \mathrm{Cent}(\widetilde{K})$.

Define the translation on C by $\gamma : C \to C : (x,y) \to (x+1,y)$. We have that $\pi \circ \gamma = \pi$. Now by $\widetilde{\theta} = \pi^*\theta = f^*\theta^L$, we have for any $v = v_{(x,y)} \in T_{(x,y)}C$,

$$\widetilde{\theta}(v) = \theta(\pi_* v) = \theta^L(f_* v) = (L_{f(x,y)^{-1}})_* f_* v.$$
$$\widetilde{\theta}(\gamma_* v) = \theta^L(f_* \gamma_* v) = (L_{f(\gamma(x,y))^{-1}})_* f_* \gamma_* v.$$

On the other hand,

$$\widetilde{\theta}(v) = \theta(\pi_* \gamma_* v) = \widetilde{\theta}(\gamma_* v)$$

Thus we have

$$\begin{aligned} & (L_{f(x,y)^{-1}})_* f_* v = (L_{f(\gamma(x,y))^{-1}})_* f_* \gamma_* v \\ \iff & f_* \gamma_* v = (L_{f(\gamma(x,y)) f(x,y)^{-1}})_* f_* v \\ \iff & (R_{f(x,y)^{-1}})_* f_* \gamma_* v = (R_{f(x,y)^{-1}})_* (L_{f(\gamma(x,y)) f(x,y)^{-1}})_* f_* v \\ \iff & d((f \circ \gamma) f^{-1}) = 0. \end{aligned}$$

thus $f(x+1,y)(f(x,y))^{-1}$ is constant for all $(x,y) \in C$.

Now we look at $\widetilde{\Phi}$,

$$\begin{aligned} \widetilde{\Phi} = \pi^*\Phi \Rightarrow\ & \widetilde{\Phi} \circ \gamma = \widetilde{\Phi} \\ \Rightarrow\ & \mathrm{Ad}_{(f\circ\gamma)^{-1}} = \mathrm{Ad}_{f^{-1}} \\ \Rightarrow\ & (f \circ \gamma) f^{-1} \in \mathrm{Cent}(\widetilde{K}) \end{aligned}$$

this proves the claim. Thus we have construct a map

$$\mathrm{Tran}_{\mathfrak{k}}\left(\mathbb{S}^2, m_0\right) \to \mathrm{Cent}(\widetilde{K}), \quad A \mapsto c(A) := f(1,0)(f(0,0))^{-1}.$$

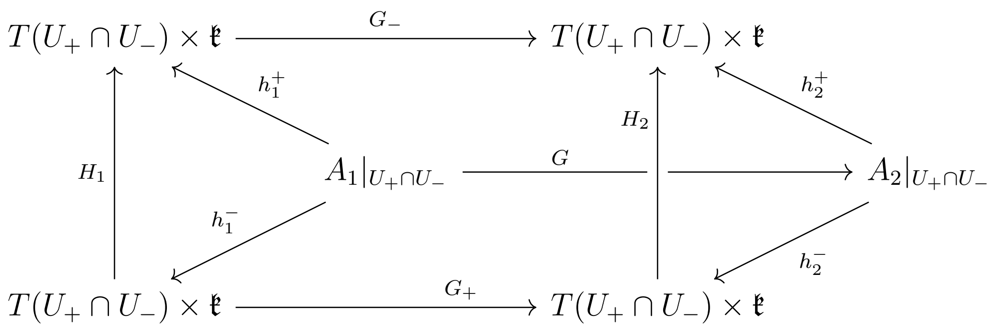
Next we show that this map is well-defined and bijective, and therefore defines a one-to-one correspondence.

Well-defined:

Suppose we have A_1, A_2, both are Lie algebroids over M with isotropy Lie algebra $\mathfrak{k}$, representing the same isomorphism class in $\mathrm{Tran}_{\mathfrak{k}}(\mathbb{S}^2, m_0)$ i.e. there is a isomorphism $G : A_1 \to A_2$ preserving the framing. By choosing trivializations preserving the framings, we get the following commutative diagrams.

$$\begin{array}{ccc} A_1|_{U_+} & \xrightarrow{G} & A_2|_{U_+} \\ {\scriptstyle h_1^+}\downarrow & & \downarrow{\scriptstyle h_2^+} \\ TU_+ \times \mathfrak{k} & \xrightarrow{G_+} & TU_+ \times \mathfrak{k} \end{array} \qquad \begin{array}{ccc} A_1|_{U_-} & \xrightarrow{G} & A_2|_{U_-} \\ {\scriptstyle h_1^-}\downarrow & & \downarrow{\scriptstyle h_2^-} \\ TU_- \times \mathfrak{k} & \xrightarrow{G_-} & TU_- \times \mathfrak{k} \end{array}$$

Here, G_+, G_- are gauge transformations over contractible open sets, thus can be represented as (g_+, Ψ_+), (g_-, Ψ_-), respectively. Recall that we have $g_+(m_0) = g_-(m_0) = e \in \widetilde{K}$, and G preserve framings implies $\Psi_+ = \Psi_- = id_{\mathfrak{k}}$. Restricting all of these maps to $U_+ \cap U_-$, we get



here $H_1 = h_1^+ \circ (h_1^-|_{A_1|_{U_+\cap U_-}})^{-1}$, $H_2 = h_2^+ \circ (h_2^-|_{A_2|_{U_+\cap U_-}})^{-1}$ are gauge transformations. From the diagram we have $G_- \circ H_1 = H_2 \circ G_+$. We can pull the maps G_-, G_+, H_1, H_2 back to C and get $\widetilde{G_-} \circ \widetilde{H_1} = \widetilde{H_2} \circ \widetilde{G_+}$. Note that now $\widetilde{G_-}$, $\widetilde{G_+}$ corresponds to $\pi^* g_-$, $\pi^* g_+$, respectively. Now suppose $\widetilde{H_1}$, $\widetilde{H_2}$ corresponds to f_1, f_2, then we have $(\pi^* g_-) f_1 = f_2(\pi^* g_+)$. Since $\pi^* g_-(0,0) = \pi^* g_-(1,0) = g_-(m_0) = e$, $\pi^* g_+(0,0) = \pi^* g_+(1,0) = e$, we have $f_1(1,0) = f_2(1,0)$, $f_1(0,0) = f_2(0,0)$. Hence we conclude that A_1, A_2 must corresponds to the same element in $\mathrm{Cent}(\widetilde{K})$.

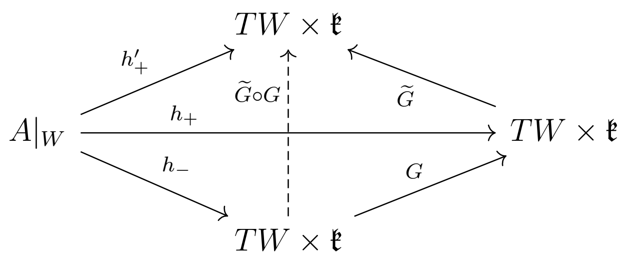
Surjectivity:

We follow [28] for this part. Given any element $k \in \mathrm{Cent}(\widetilde{K})$, we can choose a path inside $\widetilde{K}$ with sitting instances from e to k, i.e. it is a smooth path which is constant near $t = 0$ and $t = 1$. Then we can extend this path to get a smooth map $\tilde{g} : \mathbb{R} \to \widetilde{K}$, by letting $\tilde{g}(t+1) = \tilde{g}(1)\tilde{g}(t)$. Then we define $g : C \to \widetilde{K}$, $g(x, y) = \tilde{g}(x)$. Now $(g, id_{\mathfrak{k}})$ defines a gauge transformation on $TC \times \mathfrak{k}$, and this gauge transformation is unchanged under a translation γ, therefore it descends to a gauge transformation on $T(U_+ \cap U_-) \times \mathfrak{k}$, which is our desired gluing data. Thus we get a framed transitive Lie algebroid over $\mathbb{S}^2$. By applying the 'construction' part of the proof to this Lie algebroid, we can recover the map g as the gluing data, and therefore the element $k \in \mathrm{Cent}(\widetilde{K})$.

Injectivity:

Suppose we are given framed Lie algebroids $A, A' \in \mathrm{Tran}_{\mathfrak{k}}(S^2, m_0)$, which are mapped to the same element k, we want to see that they are isomorphic as framed algebroids. Suppose h_+, h_- are trivializations of A over U_+, U_-, respectively, both extending the framing, and thus A corresponds to the gauge transformation G and thus $(g, id_{\mathfrak{k}})$, where $g : C \to \widetilde{K}$. We can do the same on A', which corresponds to $(g', id_{\mathfrak{k}})$. Let $f = g' g^{-1} : C \to \widetilde{K}$. Then we know that $f(x+1, y) = f(x, y) = e$, for all x, y.

We know that in this case, the map $f|_{[0,1]\times\{y\}}$ is a loop, for any fixed y, thus f descends to $\overline{f} : U_+ \cap U_- \to \widetilde{K}$. Since $\widetilde{K}$ is simply-connected, each of the loops is contractible, and we can get a continuous map $U_+ \to \widetilde{K}$, which agree with $\overline{f}$ in a neighborhood $\overline{W}$ of the equator. By Corollary 6.29 of [21], there is also a smooth extension $\widetilde{f} : U_+ \to \widetilde{K}$ agreeing with $\overline{f}$ on $\overline{W}$. Now $\widetilde{f}$ determines a gauge transformation $\widetilde{G}$ on $TU_+ \times \mathfrak{k}$, i.e. a change of trivialization over U_+, which agrees with G over W. Take this new trivialization h'_+ on U_+ and the old one h_- on U_-, we get the gluing gauge transformation $\widetilde{G} \circ G$ on W, which corresponds to the map $g'g^{-1}g = g'$, as the following diagram illustrates.



This shows A and A' are isomorphic. $\square$

We have adapted the viewpoint of seeing isomorphism class of framed Lie algebroids as gluing two trivial pieces by the gauge transformations, and they are equivalent if and only if they correspond to the same element in $\mathrm{Cent}(\widetilde{K})$. Now we look at them from another point of view: we fix a transitive Lie algebroid with given framing, and see what happens in $\mathrm{Cent}(\widetilde{K})$ if we look at the same algebroid, but with a different framing. In this way, we can obtain a new result, i.e. giving a classification of algebroids without framing. This result will be useful in the study of the example in Section 4.7.

Corollary 4.17. There is a one-to-one correspondence between the isomorphism classes of transitive Lie algebroids over $\mathbb{S}^2$ with structure algebra $\mathfrak{k}$, and orbits of the action $\mathrm{Aut}(\widetilde{K})$ on $\mathrm{Cent}(\widetilde{K})$. Here, $\widetilde{K}$ is the connected and simply-connected Lie group integrating $\mathfrak{k}$.

Proof. Given $A \in \mathrm{Tran}_{\mathfrak{k}}(\mathbb{S}^2, m_0)$, with any framing. By Remark 4.9, we can extend this framing to trivializations of A over U_+ and U_-. And these trivializations are corresponded by a gauge transformation over $U_+ \cap U_-$. Now for A with any other framing different to the original one by an element $\tau \in \mathrm{Aut}(\mathfrak{k})$, we get new trivializations of A extending the new framing.

$$\begin{array}{ccc}
A|_{U_+\cap U_-} & & A|_{U_+\cap U_-} \\
\downarrow{\scriptstyle h_+} & & \downarrow{\scriptstyle h_-} \\
T(U_+\cap U_-)\times\mathfrak{k} & \xrightarrow{G_1} & T(U_+\cap U_-)\times\mathfrak{k} \\
\downarrow{\scriptstyle id_{T(U_+\cap U_-)}\times\tau} & & \downarrow{\scriptstyle id_{T(U_+\cap U_-)}\times\tau} \\
T(U_+\cap U_-)\times\mathfrak{k} & \xrightarrow{G_2} & T(U_+\cap U_-)\times\mathfrak{k}
\end{array}$$

Here, $h_+ : A|_{U_+} \to TU_+ \times \mathfrak{k}$ and $h_- : A|_{U_-} \to TU_- \times \mathfrak{k}$ are trivializations extending the old framing. Suppose the gauge transformation G_1 is given by (θ, Φ), where $\theta \in \Omega^1(U_+ \cap U_-, \mathfrak{k})$, $\Phi \in C^\infty(U_+ \cap U_-, GL(\mathfrak{k}))$ with $\Phi(m_0) = id_{\mathfrak{k}}$. Then we can pullback θ by $\pi : C \to U_+ \cap U_-$, and get $f : C \to \widetilde{K}$ with $\pi^*\theta = f^*\theta^L_{\widetilde{K}}$, and $f(0,0) = e$. We also know that (θ, Φ) maps any section (X, ξ) to $X + \iota_X\theta + \Phi(\xi)$. By the commutative diagram above, G_2 maps (X, ξ) to $X + \iota_X(\tau \circ \theta) + \Phi(\xi)$ i.e. G_2 corresponds to the pair $(\tau \circ \theta, \Phi)$.

We denote by g the automorphism of $\widetilde{K}$ integrating to τ. Then $\pi^*(\tau \circ \theta) = g_*(f^*\theta^L_{\widetilde{K}}) = (g \circ f)_*\theta^L_{\widetilde{K}}$. Thus G_2 corresponds to the element $g(f(1,0)) \in \mathrm{Cent}(\widetilde{K})$ (here we used $f((0,0)) = e$). This shows that a fixed algebroid with different framings lies in the same orbit of the action of $\mathrm{Aut}(\widetilde{K})$.

Now consider isomorphism of Lie algebroids $B \to A$, as non-framed algebroids. Then it is an isomorphism of framed algebroids for some framings of B and A. But then they must lie in the same orbit. The converse is trivial because an isomorphism of framed Lie algebroids must be an isomorphism of non-framed Lie algebroids. This finishes our proof. □

4.5 Mapping principal bundles to transitive Lie algebroids

It is clear that given a principal bundle, by considering the gauge algebroid of the principal bundle (see Section 3.5), we get a transitive Lie algebroid. However, for our purpose, we want the algebroids we get to have a framing. Therefore, we define $\mathrm{Prin}_K(M, m_0)$ to be the set of principal K-bundles P with a choice of $p \in P|_{m_0}$, and we call P a **principal bundle with framing**.

Given a principal K-bundle with framing $p \in P_m$, we get framing of the Lie algebroid $A(P)$ with framing at m, as follows: we can extend this framing to a local trivialisation of principal bundles $P|_U \to U \times K$, thus $TP|_U \to TU \times TK$ (we can compute this by simply taking the differential, note that this is equivalent to first take a morpism of pair groupoids and then differentiate to Lie algebroids). This isomorphism descends to $A(P|_U) = A(P)|_U = (TP/K)|_U \cong TU \times \mathfrak{k}$. Thus we get a framing on $A(P)$, which is independent of the extension. Denote $[A(P)]$ as the (framed) isomorphism class of gauge algebroid $A(P)$ with the induced framing mentioned above. We get

Proposition 4.18. *There is a natural map*

$$\mathrm{Prin}_K(M, m_0) \to \mathrm{Tran}_{\mathfrak{k}}(M, m_0), \quad P \mapsto [A(P)].$$

We recall a basic property of the covering space: let $p : \widetilde{M} \to M$ be a universal covering of the topological space M, then given any path $\gamma : [0,1] \to M$, and a fixed element $m \in \widetilde{M}$ with $p(m) = \gamma(0)$. Then there exists a unique path $\widetilde{\gamma} : [0,1] \to \widetilde{M}$ such that $p \circ \widetilde{\gamma} = \gamma$, and $\widetilde{\gamma}(0) = m$. Now for any loop λ in M based at $m' := p(m)$, there exists a unique path $\widetilde{\lambda}$ in $\widetilde{M}$ starting from m. Fixing the point m, the map $\pi_1(M, m') \to \widetilde{M}$ given by mapping each loop λ to $\widetilde{\lambda}(1)$ is well-defined, and called the **lifting correspondence**.

The following proposition shows that there is a nice relation between the two classifications we discussed in Section 4.3 and Section 4.4. Also, the natural map we constructed in Theorem 4.16 induces the lifting correspondence map on the level of groups classifying principal bundles and transitive Lie algebroids. The proposition and the proof is sketched by Meinrenken in [28], and we added a few detailed computations.

Proposition 4.19. *There is a commutative diagram*

$$\begin{array}{ccc} \mathrm{Prin}_K(\mathbb{S}^2, m_0) & \longrightarrow & \mathrm{Tran}_{\mathfrak{k}}(\mathbb{S}^2, m_0) \\ \downarrow & & \downarrow \\ \pi_1(K, e) & \longrightarrow & \mathrm{Cent}(\widetilde{K}) \end{array}$$

where the map $\pi_1(K,e) \to \mathrm{Cent}(\widetilde{K})$ *is the lifting correspondence map.*

Proof. Given a framed principal K-bundle $P \in \mathrm{Prin}_K(\mathbb{S}^2, m_0)$, we can find local trivializations of principal bundles, extending the framing

$$\begin{aligned}\Phi_+ &: P|_{U_+} \to U_+ \times K\\ \Phi_- &: P|_{U_-} \to U_- \times K.\end{aligned}$$

Therefore we get a map $g : U_+ \cap U_- \to K$ representing the gluing map. The trivializations extends the framing at m_0 implies that $g(m_0) = e$. Now that $[g] \in \pi_1(K,e)$ is our desired element under the left vertical map in the diagram.

On the other hand, we can take the differentials of Φ_+, Φ_- to get

$$\begin{aligned}T\Phi_+ &: TP|_{U_+} \to TU_+ \times TK\\ T\Phi_- &: TP|_{U_-} \to TU_- \times TK.\end{aligned}$$

Then we can compute the gauge transformation by considering

$$(id_{T(U_+\cap U_-} \times \theta^L_K) \circ (\Phi_+ \circ \Phi_-^{-1})_*|_{T(U_+\cap U_-)\times T_eK} : T(U_+ \cap U_-) \times \mathfrak{k} \to T(U_+ \cap U_-) \times \mathfrak{k}.$$

here θ^L_K is the left-invariant Maurer-Cartan form of K. To compute θ corresponding to this gauge transformation, we can pick any section $(X, 0)$ (recall the gauge transformation acts by $(\theta, \Phi)(X, 0) = (X, \theta(X))$, see Proposition 4.13). Thus $(X, \theta(X)) = (id \times \theta^L_K) \circ (\Phi_+ \circ \Phi_-^{-1})_*(X, 0)$, and $\theta(X) = \theta^L_K \circ g_*(X)$. Also we denote the covering map of $\widetilde{K}$ by $\widetilde{\pi}$, and the covering map of the annulus C by $\pi : C \to U_+ \cap U_-$. Define $\overline{g} = g \circ \pi : C \to K$, and let

$$\widetilde{g} : C \to \widetilde{K}$$

be the lifting of g. Now for any vector field $\widetilde{X} \in \mathfrak{X}(C)$,

$$\begin{aligned}\widetilde{\theta}(\widetilde{X}) &= \pi^*\theta(\widetilde{X})\\ &= \theta^L_K(g_*\pi_*\widetilde{X})\\ &= \theta^L_K(\overline{g}_*\widetilde{X})\\ &= \theta^L_K((\widetilde{\pi} \circ \widetilde{g})_*\widetilde{X})\\ &= \widetilde{g}^*\widetilde{\pi}^*\theta^L_K(\widetilde{X})\\ &= \widetilde{g}^*\theta^L_{\widetilde{K}}(\widetilde{X}).\end{aligned}$$

i.e. $\widetilde{g}$ is the map we shall use to compute the element in $\mathrm{Cent}(\widetilde{K})$, and $\widetilde{g}(1,0)(\widetilde{g}(0,0))^{-1}$ is our desired element under the right vertical map. Now we see that the horizontal map is given by $\pi_1(K,e) \to \mathrm{Cent}(\widetilde{K}) : [g] \mapsto \widetilde{g}(1,0)(\widetilde{g}(0,0))^{-1}$. This is exactly the lifting correspondence.

□

4.6 The integrability condition

We have seen in Theorem 3.21 the integrability condition for a general Lie algebroid. In this section we discuss the integrability condition in more details, for the case of transitive Lie algebroids. Throughout this section, we assume that A is a transitive Lie algebroid over M, with structure Lie algebra $\mathfrak{k}$. Let $\widetilde{K}$ be the (unique) simply connected Lie group whose Lie algebra is $\mathfrak{k}$, and let $\mathrm{Cent}(\widetilde{K})$ denotes its center (see Definition 1.29).

We first introduce the notion of monodromy, which involves the pullback construction (see Section 3.5). Given any transitive Lie algeboid A over M with framing at m, let $f : N \to M$ be a smooth map. Then $f^!A$ is a transitive Lie algebroid over $\mathbb{S}^2$, with $\ker(\rho_{f^!A}) = (0, \ker(\rho_A))$. It is easy to see that there is a natural choice of framing on $f^!A$ induced by that of A.

Let $\pi_2(M, m)$ be the second homotopy group. Its elements consists of homotopy classes of base-point preserving maps $\mathbb{S}^2 \to M$.

Definition 4.20. The **monodromy map** of framed transitive Lie algebroid A at $m \in M$ is the group homomorphism

$$\delta_A : \pi_2(M, m) \to \mathrm{Cent}(\widetilde{K}) \subset \widetilde{K}, \quad [f] \mapsto c\left(f^!A\right).$$

Here, c is the natural map defined in Theorem 4.16, and $f^!A$ is equipped with the natural choice of framing. The image $\Lambda \subset \mathrm{Cent}(\widetilde{K})$ of the monodromy map is called the **monodromy group** at m.

Now we can apply the result in the last section to get a necessary condition for integrability.

Proposition 4.21. *A necessary condition for the integrability of a transitive Lie algebroid $A \Rightarrow M$ is that the monodromy group Λ is discrete in $\widetilde{K}$.*

Proof. [29] Suppose $A \Rightarrow M$ is integrable to a Lie groupoid $\mathcal{G} \Rightarrow M$, and fix $m \in M$. According to Lemma 4.6, $\mathcal{G}$ is necessarily transitive. Suppose the source and target map of $\mathcal{G}$ are given by $\mathbf{s}, \mathbf{t}$, respectively. Pick any point $m \in M$. Then the transitive Lie groupoid $\mathcal{G}$ is the gauge groupoid $G(P)$ of the principal K-bundle $P = \mathbf{s}^{-1}(m)$, where $K = \mathbf{s}^{-1}(m) \cap \mathbf{t}^{-1}(m)$, as in Proposition 4.3. We know that $A \cong A(P)$, where $A(P)$ is the gauge algebroid of P. Given a smooth base-point preserving map $f : \mathbb{S}^2 \to M$, it follow from Proposition 3.33 that $f^!A = A\left(f^*P\right)$. By the commutative diagram in Proposition 4.19, the element $\delta_A([f]) = c\left(f^!A\right)$ must lie in $\pi_1\left(K, e\right) \subseteq \widetilde{K}$. This shows that

$$\Lambda \subseteq \pi_1\left(K, e\right) \subseteq \widetilde{K}.$$

Since $\pi_1(K, e)$ is discrete in $\widetilde{K}$, we know that Λ must be discrete. □

The necessary condition mentioned above turns out to be sufficient:

Theorem 4.22 (Integrability condition[26][29])**.** *A transitive Lie algebroid is integrable if and only if the monodromy group is discrete.*

Remark 4.23. The definition of the monodromy in Definition 4.20 is the same as that of Crainic-Fernandes [5]. As a consequence, Theorem 4.21 is a direct corollary of Theorem 3.21.

Remark 4.24. Historically, the integration of transitive Lie algebroid was first solved by Mackenzie, in [23]. According to [11], the integrability condition derived by Mackenzie is equivalent to the condition in Theorem 4.21. We shall briefly mention the idea of Mackenzie: we already know that each principal bundle can be seen as several trivial pieces glued together, by the clutching functions. In light of Proposition 4.13 and ([26], Proposition 8.2.8), any transitive Lie algebroid can also be seen as trivial pieces glued together, by the transition data (θ, Φ). Moreover, when the integrability condition is satisfied and the base manifold M is simply connected, then the transition datas for transitive Lie algebroids can be translated to those for principal bundles ([26], Section 8.3). When the base manifold is not simply connected, one can consider the pullback algebroid by the universal covering map. The Lie algebroid is integrable if and only if the pullback algebroid is integrable ([26], Theorem 8.3.4).

4.7 Example: the prequantization algebroid

In (Example 4.2, [29]), the author mentioned that it is possible to apply the classification results in Section 4.4 to study the prequantization algebroids, which is a trivial line bundle over $\mathbb{S}^2$, with nontrivial Lie brackets. In this section we study this example in full detail: we compute the monodromy, and discuss its integrability. Let $\mathfrak{k} = \mathbb{R}$, we have $\mathrm{Cent}(\widetilde{K}) = \mathbb{R}$, so $\mathrm{Tran}_{\mathbb{R}}(\mathbb{S}^2, m_0) \cong \mathbb{R}$.

Definition 4.25. Letting ω be the standard symplectic form on $\mathbb{S}^2$, and $\lambda \in \mathbb{R}$. We can define a Lie algebroid structure on the vector bundle $T\mathbb{S}^2 \times \mathbb{R}$, with the bracket $[\ ,\ ]_A$ given by

$$[(X, f), (Y, g)]_A = [X, Y] + \mathcal{L}_X g - \mathcal{L}_Y f + \lambda\omega(X, Y),$$

where $[\ ,\ ]$ denotes the bracket of $T\mathbb{S}^2$. This Lie algebroid is called the **prequantization algebroid over** $\mathbb{S}^2$ of $\lambda\omega$, denoted $A^{(\lambda)}$.

To apply the results of Section 4.4, it is convenient to assign $A^{(\lambda)}$ with the trivial framing $id_{\mathbb{R}}$, and think $A^{(\lambda)}$ as a framed Lie algebroid. Indeed, $A^{(\lambda)}$ with different framing is not isomorphic to that with the trivial framing, as framed Lie algebroids. Instead, it corresponds to another Lie algebroid $A^{(\lambda')}$ with the trivial framing, for some $\lambda \neq \lambda'$. By abuse of notation, we use $A^{(\lambda)}$ to represent $A^{(\lambda)}$ with trivial framing.

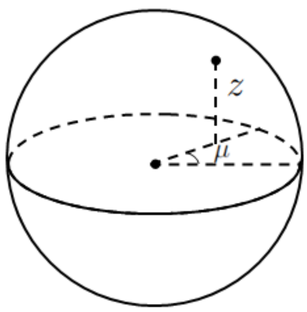


Figure 4.1: The cylindrical coordinates on $\mathbb{S}^2$

We embed $\mathbb{S}^2$ in $\mathbb{C} \times \mathbb{R}$, for each point $(y, z) \in \mathbb{S}^2$, we use (μ, z) as the coordinates of $\mathbb{S}^2$, where $\mu = \arg(y)$. Under this coordinate, $\omega = d\mu \wedge dz$. Take the universal covering of $\mathbb{S}^1$ by $\mathbb{R}$ to be the identification of $\mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1$.

Proposition 4.26. *Under the correspondence described in Theorem 4.16, the prequantization algebroid $A^{(\lambda)}$ corresponds to $-4\pi\lambda \in \mathbb{R}$.*

Proof. We extend the trivial framing to local trivializations, and then compute the gauge transformation of $A^{(\lambda)}$. On $\mathbb{S}^2 \setminus \{s\}$, we have that $\omega = d\beta$, where $\beta = (-z + 1)d\mu$. And we have trivialisation

$$h_+ : A|_{U_+} \cong TU_+ \times \mathbb{R}$$
$$(v, \xi) \mapsto (v, \xi - \lambda\iota_v\beta).$$

We check that it is indeed a Lie algebroid isomorphism. It suffices to see that it preserves Lie brackets. We use $[\ ,\ ]_0$ to denote the bracket of the trivial Lie algebroid. For any sections (X, f), (Y, g) of $A|_{U_+}$, we have

$$\begin{aligned}[h_+(X, f), h_+(Y, g)]_0 &= [(X, \xi - \lambda\iota_X\beta, (Y, \xi - \lambda\iota_Y\beta]_0 \\ &= ([X, Y], \mathcal{L}_X g - \mathcal{L}_Y f + \lambda\mathcal{L}_X\iota_Y\beta - \lambda\mathcal{L}_Y\iota_X\beta).\end{aligned}$$

On the other hand,

$$\omega(X, Y) = d\beta(X, Y) = \mathcal{L}_X\beta(Y) - \mathcal{L}_Y\beta(X) - \beta([X, Y]).$$

It follows that

$$\begin{aligned}h_+([(X, f), (Y, g)]_A) &= h_+([X, Y], \mathcal{L}_X g - \mathcal{L}_Y f + \lambda\omega(X, Y)) \\ &= ([X, Y], \mathcal{L}_X g - \mathcal{L}_Y f + \lambda d\beta(X, Y) + \lambda\iota_{[X,Y]}\beta) \\ &= ([X, Y], \mathcal{L}_X g - \mathcal{L}_Y f + \lambda\mathcal{L}_X\beta(Y) - \lambda\mathcal{L}_Y\beta(X) \\ &= [h_+(X, f), h_+(Y, g)]_0.\end{aligned}$$

Thus h_+ is indeed a Lie algebroid isomorphism.

Similarly, we have trivialisation $h_- : A|_{U_-} \cong TU_- \times \mathbb{R}$, sending section (X, ξ) to $(X, \xi - \lambda\iota_X\alpha)$, where $\alpha = -(z+1)d\mu$. Since $h_-(\frac{\partial}{\partial\mu}, \lambda(z-1)) = (\frac{\partial}{\partial\mu}, 0)$, $h_+(\frac{\partial}{\partial\mu}, \lambda(z-1)) = (\frac{\partial}{\partial\mu}, -2\lambda)$, thus the gauge transformation sends the section $\frac{\partial}{\partial\mu}$ to $\frac{\partial}{\partial\mu} - 2\lambda$. On the other hand, the gauge transformation over $C = (-1, 1) \times \mathbb{R}$ corresponding to $c \in \mathrm{Cent}(\mathbb{R})$ is $\widetilde{\theta} = cdx$, $\widetilde{\Phi} = id_{\mathbb{R}}$, where x denotes the coordinate of $\mathbb{R}$ in $(-1, 1) \times \mathbb{R}$. It descends to $(\theta = \frac{c}{2\pi}d\mu, \Phi = id_{\mathbb{R}})$ over $U_+ \cap U_-$, under the identification $\mathbb{R}/\mathbb{Z} \cong \mathbb{R}/2\pi\mathbb{Z}$. One can compute easily that this transformation sends $\frac{\partial}{\partial\mu}$ to $\frac{\partial}{\partial\mu} + \frac{c}{2\pi}$. In this case, we have $\frac{c}{2\pi} = -2\lambda$. Therefore, we conclude that $A^{(\lambda)}$ corresponds to $-4\pi\lambda \in \mathbb{R}$, under the bijection defined in Theorem 4.16.

□

In particular, this shows:

Corollary 4.27. Every transitive Lie algebroids $A \in \mathrm{Tran}_{\mathbb{R}}(\mathbb{S}^2, m_0)$ is isomorphic as framed algebroid to some $A^{(\lambda)}$; and $A^{(\lambda)}$ is not isomorphic to $A^{(\lambda')}$ as long as $\lambda' \neq \lambda$, as framed algebroids.

Since automorphisms of Lie group $\mathbb{R}$ acts by scalar multiplication, we know that the orbits of the $\mathrm{Aut}(\mathbb{R})$ action on $\mathbb{R}$ are $\{0\}$ and $\{x \in \mathbb{R} : x \neq 0\}$. By Corollary 4.17, we get

Corollary 4.28. Up to isomorphism (of non-framed Lie algebroids), there is only one nontrivial transitive Lie algebroid over $\mathbb{S}^2$ with structure Lie algebra $\mathbb{R}$. That is, if we forget about the framing, $A^{(\lambda)}$ are isomorphic Lie algebroids for all $\lambda \neq 0$.

Next we compute the monodromy group of $A^{(\lambda)}$ with trivial framing, following section 4.6, and conclude the following:

Corollary 4.29. The monodromy group associated to $A^{(\lambda)}$ is $4\pi\lambda\mathbb{Z} \subset \mathbb{R}$. In particular, the monodromy group is discrete.

Proof. Given $f : \mathbb{S}^2 \to \mathbb{S}^2$, the pullback of $A^{(\lambda)}$ is $f^! A^{(\lambda)} = T\mathbb{S}^2 \times_{\mathbb{S}^2} A^{(\lambda)}$, with elements of the form $(u, (f_* u, \xi))$. The bracket is given by

$$[(X, (f_* X, \xi)), (Y, (f_* Y, \eta))] = ([X, Y], f_*[X, Y] + \mathcal{L}_{f_* X}\eta - \mathcal{L}_{f_* Y}\xi + \lambda\omega(f_* X, f_* Y)).$$

We claim that $f^! A$ is isomorphic to $A^{(\lambda')}$ with $\lambda' = \deg(f)\lambda$, via the map $(u, (f_* u, \xi)) \mapsto (u, \xi)$. By comparing the brackets, it suffices to show

$$\lambda\omega(f_* X, f_* Y) = \lambda(f^*\omega)(X, Y) = \lambda'\omega(X, Y)$$

i.e. $\lambda'\omega = \lambda f^*\omega$. Integrate them on both sides over $\mathbb{S}^2$, we get $\lambda' \int_{\mathbb{S}^2} \omega = \lambda \int_{\mathbb{S}^2} f^*\omega = \lambda \deg(f) \int_{\mathbb{S}^2} \omega$. The claim follows.

□

We know by the integrability condition (Theorem 4.22) that $A^{(\lambda)}$ must be integrable. Then one can easily recover the (framed) principal bundle whose gauge algebroid is $A^{(\lambda)}$, with the help of the commutative diagram in Proposition 4.19, and Example 1.45. Recall that the Hopf bundle corresponds to the clutching function

$$g : U_+ \cap U_- \to \mathbb{S}^1, \quad (z, d) \mapsto \frac{z}{|z|},$$

whose pullback to $\mathbb{S}^1$ is just the identity map $id_{\mathbb{S}^1}$. Thus the Hopf bundle corresponds to $1 \in \mathbb{R}$, and therefore its (framed) gauge algebroid is $A^{(-4\pi)}$. This Lie algebroid, as a non-framed Lie algebroid, is the only nontrivial transitive Lie algebroid mentioned in Corollary 4.28.

More generally, we may consider any connected manifold M, and a closed form $\sigma \in \Omega^2(M)$. From now on, we follow [6]. The **prequantization algebroid over** M of σ is given by the Lie algebroid $A^\sigma = TM \times \mathbb{R}$, with the Lie bracket defined in the same way as the prequantization algebroid over $\mathbb{S}^2$, described in Definition 4.25.

Now we study the integrability of A^σ. To compute the monodromy, we need to consider all the pullback algebroids $\gamma^! A^\sigma$, where $\gamma : \mathbb{S}^2 \to M$. By generalizing the arguments in Corollary 4.29, one sees that $\gamma^! A^\sigma \cong A^{\gamma^*\sigma}$, and then corresponds to the element $\int_{\mathbb{S}^2} \gamma^*\sigma \in \mathbb{R}$. Therefore,

Proposition 4.30. *The monodromy group of* A^σ *is* $\{\int_{\mathbb{S}^2} \gamma^*\sigma | \quad \gamma : \mathbb{S}^2 \to M\} \subset \mathbb{R}$.

Using the result above, it is not hard to give examples of non-integrable Lie algebroids, among which the example mentioned below is the first one, due to Almeida and Molino.

Corollary 4.31. [1] Take $M = \mathbb{S}^2 \times \mathbb{S}^2$. Let $\omega \in \Omega^2(\mathbb{S}^2)$ be the symplectic form, and $\widetilde{\omega} = \mathrm{pr}_1^* \omega + \sqrt{2}\, \mathrm{pr}_2^* \omega$. The monodromy group of $A^{\widetilde{\omega}}$ is $\mathbb{Z} + \sqrt{2}\mathbb{Z} \subset \mathbb{R}$, therefore not integrable.

Assuming the integrability condition, we can now give a description for the integration of the prequantization algebroid A^σ, following [6]. For simplicity, assume M is simply connected. Pick $m \in M$, let $P(M) := P(M, m)$ be the space of C^2-paths in M ending at m. Recall that we may see this as a Banach manifold. Consider the quotient space

$$P_\sigma := P(M) \times \mathbb{R}/\sim,$$

with the equivalence relation:

$$(\gamma_0, r_0) \sim (\gamma_1, r_1) \iff \gamma_0 \simeq \gamma_1, r_0 - r_1 = \int_H \sigma.$$

Here, H is any homotopy $H : I^2 \to M$ between γ_0 and γ_1 keeping the end points fixed.

Lemma 4.32. *[6] If the integrability condition holds, then P_σ is a principal $\mathbb{S}^1$-bundle over M. Here, $\mathbb{S}^1$ is identified with $\mathbb{R}/\Lambda$, where $\Lambda = \{\int_{\mathbb{S}^2} \gamma^*\sigma | \quad \gamma : \mathbb{S}^2 \to M\}$. The $\mathbb{S}^1$-action on P_σ is given by the usual $\mathbb{R}$-action on $P(M) \times \mathbb{R}$ passing to the quotients.*

In this case, we denote $\mathcal{G}_\sigma$ to be the gauge groupoid of P_σ. Moreover, its Lie algebroid is A^σ.

Remark 4.33. We omit the proof of smoothness of principal bundle (and therefore the groupoid) here. One can apply Proposition 5.20 for a proof. Now we sketch the idea of proving the Lie algebroid of $\mathcal{G}_\sigma$ is A^σ : by assuming the integrability condition, one can find a (multiplicative) 1-form $\widetilde{\theta}$ on $\mathcal{G}_\sigma$, which serve as a connection form (see Remark 1.42) of the principal $\mathbb{S}^1$-bundle $\pi : \mathcal{G}_\sigma \to M \times M$. We also require that $d\widetilde{\theta} = \pi^*\widetilde{\sigma}$, where $\widetilde{\sigma} = \mathrm{pr}_1^* \sigma - \mathrm{pr}_2^* \sigma$. Indeed, finding such 1-form is called the prequantization problem, and see [6] for more details.

The algebroid of $\mathcal{G}_\sigma$ is $\ker T\mathbf{s}|_M \subset T\mathcal{G}_\sigma$. Observe the 1-form $\widetilde{\theta}$ can be seen as a map $T\mathcal{G}_\sigma \to \mathbb{R}$. By restricting it to $\ker T\mathbf{s}|_M$, we obtain a map $l : \ker T\mathbf{s}|_M \to \mathbb{R}$. Now the vector bundle isomorphism $\ker T\mathbf{s}|_M \to A^\sigma = TM \times \mathbb{R} : \alpha \mapsto (T\mathbf{t}(\alpha), l(\alpha))$ is a Lie algebroid isomorphism: it preserves the Lie brackets, by the quantization condition $d\widetilde{\theta} = \pi^*\widetilde{\sigma}$.

Chapter 5

A Construction for Integration

In Section 4.6 and 4.7, we discussed the integrability condition for arbitrary transitive Lie algebroids, and constructed an integration, in the special case of the prequantization algebroids. This motivates us to find a construction for the integration of transitive Lie algebroids. While Crainic and Ferenandes [5] gave a construction for general Lie algebroids in 2004, recently, Meinrenken [28] [29] introduced a new elementary construction for transitive Lie algebroids. The main goal of this chapter is to explain Meinrenken's new construction.

5.1 The isotropy bundle

To understand Meinrenken's construction, we first introduce the notion of isotropy and connections, which allow us to describe the Lie algebroid structure of a transitive Lie algebroid in a way similar to that of the prequantization algebroid (see Section 4.7). The reference for this section is Mackenzie's book [26], although all of the results also appear in [29].

Definition 5.1. Given a Lie algebroid A over M, the kernel of the anchor map, $\mathfrak{h} := \ker(\rho_A)$, is a bundle of Lie algebra (mentioned in Example 3.17), with isomorphic fibers. We call this Lie algebroid $\mathfrak{h}$ the **isotropy bundle** (adjoint bundle) of A.

The following result is due to Mackenzie and Xu [24], the proof is quite involved.

Lemma 5.2. *Let $A \rightrightarrows M$ be a transitive Lie groupoid, and Q another manifold. If*

$$\varphi : \mathbb{R} \times Q \to M, (t, q) \mapsto \varphi_t(q)$$

is a smooth map, then the pullback Lie algebroids $\varphi_t^! A \Rightarrow Q$ are all isomorphic.

One consequence of this result is Lemma 4.7. We restate it here:

Proposition 5.3. *[26] Let $U \subseteq M$ be a contractible open subset, then any choice of a smooth deformation retract of U onto a base point $m_0 \in U$ determines an isomorphism of Lie algebroids $A|_U \cong TU \times \mathfrak{h}_{m_0}$.*

Proof. [29] Let $i : \{m_0\} \to U$ be the inclusion, and $p : U \to \{m_0\}$ be the retraction. Let $\varphi : \mathbb{R} \times U \to U, (t, m) \mapsto \varphi_t(m)$ be a smooth map such that $\varphi_0 = i \circ p$ and $\varphi_1 = \mathrm{id}_U$. By

Lemma 5.2, φ determines a Lie algebroid isomorphism

$$A|_U = \varphi_1^! A \cong \varphi_0^! A = p^! i^! A = p^! \mathfrak{h}_{m_0} = TU \times \mathfrak{h}_{m_0}.$$

□

In particular, we know that the isotropy bundle $\mathfrak{h}$ is not only an algebroid, but admits local trivializations compatible with the fiberwise Lie algebra structure. We call any bundle of Lie algebras with this property to be a **Lie algebra bundle**.

Note that the isotropy bundle $\mathfrak{h}$ fits into an exact sequence of Lie algebroids

$$0 \to \mathfrak{h} \to A \to TM \to 0.$$

Definition 5.4. We call a vector bundle morphism $j : TM \to A$ a **splitting** (or connection) of A if it is a vector bundle splitting of the above sequence, i.e. $\rho_A \circ j = id_{TM}$.

Let $\mathrm{Cent}(\mathfrak{h}) \subset \mathfrak{h}$ be the subalgebroid, given by $\mathrm{Cent}(\mathfrak{h})_m := \mathrm{Cent}(\mathfrak{h}_m)$, for all $m \in M$. The following lemma explains why we call j in the previous definition a connection. The proof is straightforward, see [5].

Lemma 5.5. *A splitting $j : TM \to A$ of a transitive Lie algebroid $A \Rightarrow M$ induces a linear connection (called the adjoint connection) ∇ on the Lie algebra bundle $\mathfrak{h} = \ker(\rho_A)$, by*

$$\nabla_X \sigma = [j(X), \sigma], \quad X \in \mathfrak{X}(M), \sigma \in \Gamma(\mathfrak{h})$$

This connection preserves $\mathrm{Cent}(\mathfrak{h})$*, and restricts to a flat connection on* $\mathrm{Cent}(\mathfrak{h})$ *independent of the choice of j.*

Remark 5.6. Given any transitive Lie algebroid A over M, the choice of splitting j enable us to see $A|_V$ as a direct sum $TV \oplus \mathfrak{h}$, where $\mathfrak{h}$ is the isotropy Lie algebra bundle, and V is some contractible open set. To see this, $\Omega \in \Omega^2(M; \mathfrak{h})$ be the 2-form given by $\Omega(X,Y) := [j(X), j(Y)] - j([X,Y])$. The bracket on $TU \oplus \mathfrak{h}$ is defined by

$$[(X,\xi),(Y,\eta)] = ([X,Y],\ [\xi,\eta] + \nabla_X \eta - \nabla_Y \xi + \Omega(X,Y)).$$

Consider $f : TV \oplus \mathfrak{h} \to A$, given by $(X,\xi) \mapsto j(X) + \xi$. We show that f preserves the brackets:

$$\begin{aligned} f([(X,\xi),(Y,\eta)]) &= j([X,Y]) + [\xi,\eta] + \nabla_X \eta - \nabla_Y \xi + \Omega(X,Y) \\ &= j([X,Y]) + [j(X),\eta] + [\xi, j(Y)] + [j(X), j(Y)] - j([X,Y]) + [\xi,\eta] \\ &= [j(X) + \xi, j(Y) + \eta]. \end{aligned}$$

Thus the vector bundle isomorphism f is an isomorphism of Lie algebroids. We conclude that: the Lie algebroid structure on A is determined by $\mathfrak{h}$, ∇ and Ω.

Similarly, a **Lie group bundle** is a fiber bundle such that each fibre is a Lie group, and that there exists an atlas among which each chart is a Lie group isomorphism on each fiber.

For a Lie groupoid $\mathcal{G}$ over M, the **isotropy Lie group bundle** is a Lie group bundle H with fibers $H_m = \mathbf{s}^{-1}(m) \cap \mathbf{t}^{-1}(m)$. We have the following exact sequence of Lie groupoids

$$1 \to H \to \mathcal{G} \to \mathrm{Pair}(M) \to 1.$$

Remark 5.7. Given a transitive Lie algebroid A, then $ker(\rho_A)$ seen as a Lie algebroid, integrates to a Lie groupoid $\widetilde{H}$. If A is transitive, then we can take $\widetilde{H}$ to be a Lie group bundle with simply connected fibers.

5.2 Isotropy modulo monodromy

Remark 6.6 in the last section tells us that the Lie algebroid structure of a transitive Lie algebroid A is completely determined by the isotropy bundle $\mathfrak{h}$, the adjoint connection ∇, and the curvature 2-form Ω. In the following sections, we would like to transfer these 'algebroid data' to 'groupoid data'. The philosophy of constructing a new group bundle, which we will call 'isotropy modulo monodromy', is due to Meinrenken [29]. In complement, we prove some technical results in Proposition 5.9 and Lemma 5.11.

Given a transitive Lie algebroid $A \Rightarrow M$, let $\mathfrak{h} = \ker(\rho_A)$ be the isotropy bundle. Then $\mathfrak{h}$ integrates to a bundle of Lie groups $\widetilde{H}$ with simply connected fibers. The local trivializations for $\mathfrak{h}$ integrates to trivializations of $\widetilde{H}$. Denote by $Z = \mathrm{Cent}(\widetilde{H})$ the Lie group bundle with fibers $Z_m = \mathrm{Cent}(\widetilde{H}_m)$.

We have seen in Definition 4.20 the notion of monodromy. According to Meinrenken [29], the monodromy maps can be seen as a map $\cup_{m\in M}\pi_2(M,m) \to Z$. A small problem regarding to this definition is the following: we need to fix a transitive Lie algebroid A with framing, say at m', to get a corresponding element in the center of some Lie group. When we pullback A by an element in $\pi_2(M,m')$, there is no problem (there is a natural choice of framing, see page46). However, we also need to pullback A with elements in $\cup_{m\in M}\pi_2(M,m)$, so the map may not be defined for base points other than m'.

In the paper [29], the author omitted the discussion for this problem. We shall fill in the missing details. We will show the following: fix a base point $m \in M$, an element $f \in \pi_2(M,m)$, and a transitive Lie algebroid over M with any framing at M. Then the resulting element in the Lie group corresponds to the pullback Lie algebroid is independent of the choice of the framing. It is easy to see that this result will solve the above problem.

Definition 5.8. Given any transitive Lie algebroid A with framing at a fixed point $m \in M$, given by $g : \ker(\rho_A)|_m \cong \mathfrak{k}$. Then by Theorem 1.21, the isomorphism of Lie algebras g integrates to an isomorphism of Lie groups $\widetilde{g} : \widetilde{H}_m \cong \widetilde{K}$.

We define the map

$$\delta_{A,m} : \pi_2(M,m) \to Z_m \subset \widetilde{H}_m, \quad f \mapsto \widetilde{g}^{-1}(c(f^!A)).$$

Here, c is the natural map defined in the proof of 4.16. We define Λ_m to be the image of $\delta_{A,m}$ in Z_m.

Proposition 5.9. *Under the definitions above, given any transitive Lie algebroid A with framing at m, then $\Lambda_m \subset Z_m$ is independent of the choice of framings of A.*

Proof. First consider any transitive Lie algebroid B over $\mathbb{S}^2$. Suppose we have two framings differ by an automorphism $\tau \in \mathrm{Aut}(\mathfrak{k})$, and it integrates to $g \in \mathrm{Aut}(\widetilde{K})$. By the proof of Corollary 4.17, the resulting element in $\widetilde{K}$ of B (with different framings) differ by g. But then by the commutative diagram below, they corresponds to the same element in

$\widetilde{H}_m$.

$$\begin{array}{ccc} \ker(\rho_A)|_m & \longrightarrow & \mathfrak{k} \\ \downarrow & \overset{\tau}{\nearrow} & \\ \mathfrak{k} & & \end{array} \qquad \begin{array}{ccc} \widetilde{H}_m & \longrightarrow & \widetilde{K} \\ \downarrow & \overset{g}{\nearrow} & \\ \widetilde{K} & & \end{array}$$

One can show that the above argument works for the general case, where we consider an arbitrary transitive Lie algebroid pulled back to $\mathbb{S}^2$, with different framings. ☐

Corollary 5.10. Let A be a transitive Lie algebroid without framing. By taking at each $m \in M$ the map $\delta_{A,m}$, we get a map $\cup_{m \in M} \pi_2(M, m) \to Z$, as claimed in [29].

From now on, we assume that the **integrability condition** holds: Λ_m is discrete for some base point, and thus for any $m \in M$. One immediate consequence is that $U_m := \widetilde{H}_m/\Lambda_m$ is a well-defined Lie group. Indeed, we will establish the union of $U := \cup_{m \in M} U_m$ as a locally trivial bundle of Lie groups, and we will call U the **isotropy modulo monodromy**. This will be our groupoid counter part for $\mathfrak{h}$, and is of great importance in the construction of integration.

By the previous discussions, any choice of splitting of the transitive Lie algebroid $A \Rightarrow M$ induces a adjoint connection on its isotropy $\mathfrak{h} = ker(\rho_A)$. Given any path with sitting instances (see section 6.5) γ in M, we denote the parallel transport by $\gamma_* : \mathfrak{h}_{\gamma(0)} \to \mathfrak{h}_{\gamma(1)}$. The parallel transports exponentiate to $\gamma_* : \widetilde{H}_{\gamma(0)} \to \widetilde{H}_{\gamma(1)}$. We call it the parallel transport on $\widetilde{H}$. To further define similar notion on the quotient U_m we need the following lemma, which is stated in (Remark 5.3, [29]) without mentioning the proof. We give a proof here.

Lemma 5.11. *Given a path γ from m to m', we have the following commutative diagram.*

$$\begin{array}{ccc} \pi_2(M, m) & \xrightarrow{\delta_A} & Z_m \\ \downarrow{\scriptstyle \gamma_*} & & \downarrow{\scriptstyle \gamma_*} \\ \pi_2(M, m') & \xrightarrow{\delta_A} & Z_{m'} \end{array}$$

In particular, we know that $\gamma_*(\Lambda_m) = \Lambda_{m'}$.

Proof. Given $f \in \pi_2(M, m)$, we know that $f^! A$ is a transitive Lie algebroid over $\mathbb{S}^2$, and $\delta_A(f) = c(f^! A) \in \mathrm{Cent}(\widetilde{H}) = Z$. By going through the construction in Theorem 4.16, one sees that $(\gamma_* f)^! A$ corresponds to the same element $\delta_A(f) \in Z$. Here, the framing of $(\gamma_* f)^! A$ at m' is obtained from the framing of A at m, namely: $\mathfrak{h} \to \mathfrak{h}_m$, compositing with the parallel transport along γ: $\mathfrak{h}_m \to \mathfrak{h}_{m'}$. The change of framing induces the following diagram

$$\begin{array}{ccc} Z & \longrightarrow & Z_{m'} \\ \downarrow & \overset{\gamma_*}{\nearrow} & \\ Z_m & & \end{array}$$

Thus the lemma follows.

☐

Thus the parallel transport $\gamma_* : \widetilde{H}_{\gamma(0)} \to \widetilde{H}_{\gamma(1)}$ passes to the quotient $\mathcal{P}_\gamma : U_{\gamma(0)} \to U_{\gamma(1)}$. We call $\mathcal{P}_\gamma$ the **parallel transport** along γ in U. The local trivializations of U is defined using those of $\widetilde{H}$, and we use the fact that the monodromy is preserved by parallel transport in $\widetilde{H}$.

5.3 Holonomy

The parallel transport on U determines the parallel transport on $\mathfrak{h}$, and therefore the (adjoint) connection ∇ on $\mathfrak{h}$. However, This does not necessarily determine the Lie algebroid structure, since the curvature form Ω may vary. Therefore, we introduce in this section the holonomy on U, which will contain more information than the parallel transport. The reference for this section is [29].

Given any transitive Lie algebroid $A \Rightarrow M$, we fix a choice of a splitting $j : TM \to A$. Let $\mathfrak{h}$ be the isotropy (Lie algebra) bundle. Let $Loop^0(M)$ denotes the set of thin homotopy classes (See Section 6.6) of contractible loops (in the sense of smooth path homotopy) in M.

Definition 5.12. [29] The **holonomy** is a map

$$Hol : Loop^0(M) \to U.$$

Given a contractible loop $(\zeta : [0,1] \to M) \in Loop^0(M)$, $Hol(\zeta)$ can be computed by the following steps:

- Pick a local trivialization of the pullback Lie algebroid $\zeta^! A$, say $\phi : \zeta^! A \cong T[0,1] \times \mathfrak{h}_m$. Moreover, we shall assume that ϕ extends over a deformation retract pf ζ.
- Transport the splitting j to a splitting of $\zeta^! A$, and therefore a splitting of $T[0,1] \times \mathfrak{h}_m$.
- Note that $T[0,1] \times \mathfrak{h}_m$ is the gauge algebroid of the trivial principal bundle $[0,1] \times U_m$, therefore a splitting $T[0,1] \to T[0,1] \times \mathfrak{h}_m$ defines a principal connection (see Definition 1.31) on $[0,1] \times U_m$.
- Compute the horizontal lift $k : [0,1] \to [0,1] \times \mathfrak{h}_m$ of the identity map on $[0,1]$ (seen as a path), and such that $k(0) = (0, e)$.

Then $(1, Hol(\zeta)) = k(1) \in \{1\} \times U_m$.

Remark 5.13. The definition presented here is slightly different from the definition in [29]: in the paper, the Lie algebroid A has a framing at some point m. In this case, we face with the same problem as the last section, namely the holonomy map is only defined for loops based at m. Again, one way to solve this is to prove that the holonomy map does not depend on the framings at a fixed point.

Remark 5.14. The local trivialization in the previous definition always exists. We may obtain it as follows: first we pick a local trivialization of A covering the image of ζ, then we pull it back by ζ.

The definition for holonomy involves a choice of the local trivialization. Therefore we need the following lemma. We will only sketch a proof, and refer the interested readers to (Proposition 5.4, [29]).

Lemma 5.15. *[29] The value $Hol(\zeta)$ does not depend on our choice of local trivialization ϕ.*

Proof. We know that the local trivialization ϕ extends over a deformation retract H, which can be seen as a map $D_+ \to M$, and such that $H|_{\partial D_+}$ can be identified with ζ. Here, D_+ denotes the closed upper hemisphere. We can repeat the procedure for D_-, and then glue the two homotopies to a map $\sigma : \mathbb{S}^2 \to M$.

Since $U_m = \widetilde{H}_m/\Lambda_m$, with Λ_m discrete, we have that the map $\pi_2(U_m) \to \Lambda_m$ defined in Definition 5.8 is an isomorphism. Thus there exists a principal U_m-bundle P over $\mathbb{S}^2$ whose gauge Lie algebroid is $\sigma^! A$. It follows that the pullback splitting $\sigma^! j : T\mathbb{S}^2 \to \delta^! A$ defines a connection on P. One can show that $Hol(\zeta)$ can be computed using this connection, which is independent of the choice of the trivialization ϕ. □

Now we study the relationship between the holonomy and the parallel transport. For the proof of the following lemma, see ([29], Proposition 5.4).

Lemma 5.16. *The map $Hol : Loop^0(M) \to U$ is equivariant for the parallel transports on both sides, i.e.*

$$\begin{array}{ccc} Loop^0(M) & \xrightarrow{Hol} & U \\ \downarrow{\gamma_*} & & \downarrow{\mathcal{P}_\gamma} \\ Loop^0(M) & \xrightarrow{Hol} & U \end{array}$$

We observe that let $\zeta, \zeta' \in Loop^0(M)_m$, then $Hol(\zeta' * \zeta) = Hol(\zeta')Hol(\zeta)$. The following corollary is immediate.

Corollary 5.17. [29] Let ζ be a contractible loop, then $\mathcal{P}_\zeta = C_{Hol(\zeta)}$. Here, C denotes the conjugation by a Lie group element.

5.4 Integration

Now we are ready to use the bundle U and the data on U to construct the Lie groupoid for integration. Let U be given as before, and $j : TM \to A$ be a spitting. The reference for this section is [29].

Definition 5.18. Let $\mathrm{Path}(M)$ be the groupoid consisting of thin homotopy classes of paths (see Section 6.5), with product given by the usual concatenation of paths. We call $\mathrm{Path}(M)$ the **path groupoid** of M.

Now we define the semidirect product of groupoids $\mathrm{Path}(M) \ltimes U$. As a set, it coincides with the direct product of $\mathrm{Path}(M)$ and U. The multiplication on $\mathrm{Path}(M) \ltimes U$ is given by

$$([\gamma'], u') \circ ([\gamma], u) = ([\gamma' * \gamma], ((\mathcal{P}_{\gamma^{-1}}(u')u).$$

The multiplication is defined whenever $[\gamma' * \gamma]$ is defined in $\mathrm{Path}(M)$.

We first describe the groupoid structure on the semidirect product $\mathrm{Path}(M) \ltimes U$. The source and target maps are defined to be the same as the source and target of $\gamma \in \mathrm{Path}(M)$. The unit elements are of the form (m, e), where m represents the constant path at m, and $e \in U_m$. Given $(\gamma, u) \in \mathrm{Path}(M) \ltimes U$, the inversion is given by $(\gamma^{-1}, \gamma u^{-1})$.

Similar to Section 2.1, we do not expect the path groupoid (and therefore the semidirect product) to be a Lie groupoid. Instead, we will consider a quotient of the semidirect product as the candidate for the integration:

Definition 5.19. We denote $\mathcal{G}$ to be $(\mathrm{Path}(M) \ltimes U)/\sim$. Explicitly, the equivalence relation is given by

$$(\gamma_1, u_1) \sim (\gamma_2, u_2) \iff \gamma_1 \simeq \gamma_2, \quad u_1 = Hol(\gamma_2^{-1} * \gamma_1)u_2.$$

On the quotient $\mathcal{G}$, we shall see that the multiplication map inherited from $\mathrm{Path} \ltimes U$ is well-defined. This is guaranteed by the fact that, for any $\gamma \in \mathrm{Path}(M)$, $\zeta \in Loop^0(M)$, and $\gamma(0) = \zeta(0)$, we have by Lemma 5.16 that

$$(\gamma, u)(\zeta, Hol(\zeta^{-1}))(\gamma, u)^{-1} = (\widetilde{\zeta}, Hol(\widetilde{\zeta}^{-1})).$$

Here, $\widetilde{\zeta} = \gamma * \zeta * \gamma^{-1}$. Similarly, other structure maps on $\mathcal{G}$ are well-defined, and therefore $\mathcal{G}$ is a groupoid. As for the smooth structure, we follow ([29], Theorem 5.6).

Proposition 5.20. *When the integrability condition holds, i.e. the monodromy is discrete, the groupoid $\mathcal{G}$ is a Lie groupoid.*

Proof. ([29], Theorem 5.6) Recall from Example 3.35 that the fundamental groupoid $\Pi(M)$ is the gauge groupoid of the universal covering, and thus a Lie groupoid. We show that $\mathcal{G}$ is a fiber bundle over $\Pi(M)$, with typical fiber U_m for any $m \in M$. For the bundle projection, we use the fact that if we take $U = \emptyset$, then $\mathcal{G} = \mathrm{Path}(M)/\sim$ can be identified naturally to $\Pi(M)$. For the general case when $U \neq \emptyset$, we take the bundle projection to be the projection onto the first term $\mathrm{Path}(M)/\sim$. Here, the equivalence relation descends to the path homotopy in M.

Let $\widetilde{M}$ be the universal covering, then the projection $\widetilde{M} \to M$ is a local diffeomorphism. Thus $\Pi(M) = (\widetilde{M} \times \widetilde{M})/\mathrm{Aut}_\pi(M) \to \mathrm{Pair}(M)$ is a local diffeomorpism. Pick an open cover $\{O_v\}$ of $\Pi(M)$ such that the image of $\widetilde{O_v}$ under the projection is an open subset $O_v \subset \mathrm{Pair}(M)$, and such that each O_v is covered by a chart. The map $O_v \to \widetilde{O_v}$ can be seen as smooth map $\sigma_v : I \times O_v \to M$, such that $(t \mapsto \sigma_v(t, m', m))$ is a path in M, and its equivalence classes lies in $\widetilde{O_v}$, for any $(m', m) \in O_v$. Thus we get a map $s_v : O_v \to \mathrm{Path}(M), (m', m) \mapsto \sigma_v(\cdot, m', m)$. Using this, we can define local trivializations of $\mathcal{G}$:

$$F_v : O_v \times_M U \to G_{\widetilde{O_v}}, (m', m, u) \mapsto (\sigma_v(\cdot, m', m), u).$$

Consider trivializations F_{v_1}, F_{v_2} as above, defined on O_{v_1}, O_{v_2}, respectively. Let $\sigma_{v_2}^{-1} * \sigma_{v_1} : I \times (O_{v_1} \cap O_{v_2}) \to M$ be given by concatenation of paths. Each of these loops are contractible, since σ_{v_1} and σ_{v_2} are in the same equivalence class at each point (m', m). Let $f = Hol(\sigma_{v_2}^{-1} * \sigma_{v_1})$, we can compute the transition map

$$F_{v_2}^{-1} \circ F_{v_1} : (O_{v_1} \cap O_{v_2}) \times_M U \to (O_{v_1} \cap O_{v_2}) \times_M U, \quad (m', m, u) \mapsto (m', m, f(m', m)u),$$

which is smooth. □

Proposition 5.21. *The Lie groupoid $\mathcal{G}$ is source simply connected, i.e. for any $m \in M$, $\mathbf{s}^{-1}(m)$ is simply connected.*

Proof. This follows from ([29], Proposition 5.5 and Theorem 5.6). □

Example 5.22. When the Lie algebroid A coincides with the manifold M, the integration of A is given by $\Pi(M)$. To be more precise, we get $\mathrm{Path}(M)/Loop^0(M)$, i.e. the Lie groupoid is given by the set of thin homotopy classes of smooth paths in M, quotient by the set of thin homotopy classes of contractible loops (in the sense of the usual path-homotopy) in M.

Remark 5.23. In Meinrenken's paper [29], he did not prove that the Lie algebroid of $\mathcal{G}$ is the Lie algebroid A.

5.5 Revisit of the prequantization algebroid

In this section we revisit the prequantization algebroid over $\mathbb{S}^2$, construct an integration following the previous sections, and compare it to the preceding chapters.

Recall that the prequantization algebroids is the vector bundle $A^{(\lambda)} = T\mathbb{S}^2 \times \mathbb{R}$ with the bracket,

$$[(X, f), (Y, g)]_A = [X, Y] + \mathcal{L}_X g - \mathcal{L}_Y f + \lambda\omega(X, Y),$$

where ω is the standard symplectic form on $\mathbb{S}^2$.

Recall from 4.7 that on $U_+ = \mathbb{S}^2 \setminus \{s\}$ we have $\omega = d(-z+1)d\mu$, for $\beta = (-z+1)d\mu$. Moreover, there is a trivialization

$$h_+ : A|_{U_+} \cong TU_+ \times \mathbb{R}$$

$$(v, \xi) \mapsto (v, \xi - \lambda\iota_v\beta).$$

As a nontrivial example for Meinrenken's construction [29], the statement and proof of the following proposition is original.

Proposition 5.24. *Suppose $\zeta : [0, 1] \to \mathbb{S}^2$ is a smooth contractible loop based at m_0, then*

$$Hol(\zeta) = -\lambda \int_{[0,1]^2} F^*\omega \in \mathbb{R}/(4\pi\lambda\mathbb{Z}),$$

where $F : [0, 1]^2 \to \mathbb{S}^2$ denotes any path-homotopy contracting ζ to the constant loop.

Proof. Without loss of generality, assume the image of ζ lies in U_+. Fixing $A \in \mathrm{Tran}_{\mathbb{R}}(\mathbb{S}^2, m_0)$, we get the pullback $\zeta^! A$ along with a trivialization

$$\zeta^! A \to T[0, 1] \times \mathbb{R}$$

$$(v, (\zeta_* v, a)) \mapsto (v, a - \iota_{\zeta_* v}\beta) = (v, a - \lambda\iota_v(\zeta^*\beta)).$$

Here $v \in T[0, 1]$ and $a \in \mathbb{R}$. By taking the splitting $v \mapsto (v, (\beta_* v, 0))$ of $\zeta^! A$, we get a splitting of $T[0, 1] \times \mathbb{R}$

$$l : T[0, 1] \to T[0, 1] \times \mathbb{R}$$

$$v \mapsto (v, -\lambda\iota_v(\zeta^*\beta)).$$

Note that $T[0, 1] \times \mathbb{R}$ is the gauge algebroid of the trivial principal bundle $[0, 1] \times \mathbb{R}$. Thus, we get an Ehresmann connection on $[0, 1] \times \mathbb{R}$, as follows: first let $\pi : [0, 1] \times \mathbb{S}^1 \to$

$[0,1]$ be the projection. The splitting induces a vector bundle morphism

$$\pi^*(T[0,1]) \to T[0,1] \times T\mathbb{S}^1$$

$$((i,g),y_i) \mapsto (L_g)_* l(y_i),$$

where $i \in [0,1], g \in \mathbb{R}, y_i \in T_i[0,1]$, and $(L_g)_*$ acts on the second component. Then the image of this map gives us an Ehresmann connection.

Let $\widetilde{\gamma} : [0,1] \to [0,1] \times \mathbb{S}^1 : t \mapsto (t, \gamma(t))$ be a smooth path in the trivial principal bundle $[0,1] \times \mathbb{S}^1$. Then $\widetilde{\gamma}$ being parallel with respect to the connection is equivalent to

$$(\frac{\partial}{\partial t}|_i, \gamma'(i)) = (\frac{\partial}{\partial t}|_i, -\lambda(L_i)_*(\iota_{\frac{\partial}{\partial t}}(\zeta^*\beta)|_0)).$$

for all $i \in [0,1]$. Under the natural identification of $T_i\mathbb{S}^1$ and $\mathbb{R}$, we have that $(L_i)_*(\iota_{\frac{\partial}{\partial t}}(\zeta^*\beta)|_0) = (\zeta^*\beta)|_i$. i.e. $\gamma'(i) = (\zeta^*\beta)|_i$.

Suppose $F : [0,1]^2 \to \mathbb{S}^2$ is a map contracting ζ to a constant loop, i.e. $F|_{[0,1]\times\{0\}} = \zeta$.

Then by Stokes' theorem,

$$-\lambda \int_{[0,1]^2} F^*\omega = -\lambda \int_{[0,1]^2} dF^*\beta = -\lambda \int_{\partial([0,1]^2)} \zeta^*\beta = \gamma(1) - \gamma(0).$$

Here, we used the formula $\gamma'(i) = (\zeta^*\beta)|_i$ in the last equality.

□

Corollary 5.25. Following Section 4.7, the integration of prequantization algebroid over $\mathbb{S}^2$ by Meinrenken [29] coincides with the construction by Crainic [6] in Lemma 4.32.

Proof. Let $A^{(\lambda)}$ be the prequantization algebroid as above, and $\mathcal{G}^{(\lambda)}$ be its integration as in Proposition 5.20. By Proposition 4.3, it corresponds to the principal bundle

$$(\mathrm{Path}(M,m) \times \mathbb{R})/\sim$$

where $\mathrm{Path}(M,m)$ is the set of thin homotopy classes of smooth paths in M with ending point m.

Note that each C^2-path in M can be represented uniquely by a C^∞-path, and therefore uniquely represented by a thin homotopy class of smooth path. This gives rise to a natural identification between $\mathrm{Path}(M,m)$ and $P(M)$ (defined before Lemma 4.32). Using this identification, as well as Proposition 5.24, two elements $(\gamma_1, r_1), (\gamma_2, r_2)$ in $\mathcal{G}$ are equivalent if and only if the two paths are path-homotopic, and that $\gamma_2^{-1} * \gamma_1$, as a contractible loop, has holonomy

$$-\lambda \int_{[0,1]^2} F^*\omega,$$

where F is any path-homotopy contracting $\gamma_2^{-1} * \gamma_1$ to the constant path. Up to $4\pi\mathbb{Z}$ (and a sign change), this value is the same as $\lambda \int_H \omega$, where H is any path-homotopy between γ_1 and γ_2.

Then it is clear that the $\mathbb{S}^1$-actions on both principal bundles coincide, since they are both induced by the usual $\mathbb{R}$-action, then pass to the quotients. □

Chapter 6

Appendices

In this chapter we introduces some basic terminologies in geometry and topology. The readers could refer to them when needed.

6.1 Manifolds

In this section we review basics of the theory of manifolds, following [21].

Let $\mathbb{R}$ be the set of real numbers. A real vector space is a set of vectors such that we can add the vectors and multiply them with real numbers. In this section, we consider the finite dimensional vector spaces, which are of the form $\mathbb{R}^n$ for some natural number n. The subject of studying calculus $\mathbb{R}^n$ is called multi-variable calculus, which should be familiar to the readers.

Given a set, a topology of this set is a family of subsets that are considered to be open. A set with a topology is called a topological space. Taking $\mathbb{R}^n$ with the topology consisting open sets in the usual sense, then we get a topological space, which we call the Euclidean space. In general, arbitrary topology spaces can be quite different from $\mathbb{R}^n$ with the usual topology. We can use different properties to distinguish them. For example, a topological space is Hausdorff if any distinct points are separated by disjoint open neighborhoods. Given a topology, a basis is a subfamily such that each open set can be written as a union of member of the basis. A topological space is called second-countable, if it has a countable basis.

We would like to do calculus on more general objects than the Euclidean space. First, consider a topological space M which is locally Euclidean, i.e. near each point x there is a neighborhood U homeomorphic to some open subset of $\mathbb{R}^n$, via a map $\phi : U \to \mathbb{R}^n$. The pair (ϕ, U) is called a chart. If the natural number n is constant near any points, then we say that the topological space is a **topological manifold**. In practice, we will assume further that a topological manifolds satisfies the following technical properties: it has to be Hausdorff and second countable. With these, we obtain useful tricks like partition of unity and uniqueness of flow, when dealing with manifolds. Given any two charts, the change of charts map is a homeomorphism between opens of Euclidean spaces. If each of these maps is moreover a diffeomorphism in the Euclidean space, then we call the topological manifold M a **smooth manifold**.

With the notion of smooth manifolds, and the family of charts, we can use the charts to interpret the maps between manifolds as maps between opens of Euclidean spaces.

6.2 Bundles

Bundles are a family of interesting objects in geometry: we will introduce fibre bundles and vector bundles in this section. Later, we will study principal bundles, which has a deep connection with transitive Lie algebroids.

Suppose F, E, B are manifolds. A **fiber bundle** with typical fiber F is a surjective map $\pi : E \to B$, such that E is locally trivial in the following sense: near any $x \in B$, there exists an open neighborhood U of x and a diffeomorphism $\psi : \pi^{-1}(U) \to U \times F$, satisfying $\pi_1 \circ \psi = \pi$. A fiber bundles of the form $B \times F$ is called a trivial bundle. Given a fiber bundle $\pi : E \to B$, a **section** is a smooth map $f : B \to E$ such that $\pi \circ f = Id_B$. We denote $\Gamma(E)$ to be the space of sections of the fiber bundle.

Given a fiber bundle $\pi : E \to M$ with fiber F, for each $u \in E$, we define the vertical space $Vu := (E_{\pi(u)}) = ker(d\pi_u)$. The vertical space consists of the vectors tangent to the fiber $F_{\pi(u)}$. An **Ehresmann connection** is a complement of the vertical space, at the each point of E. To be more precise, it is a collection of vector subspaces of $\Gamma := \{H_u \subset T_uE | u \in E\}$, such that the assignment $u \mapsto H_u$ depends smoothly on $u \in E$, and H_u is horizontal, i.e. $\forall u \in E, T_uE = H_u \oplus V_u$. Now fix a fiber bundle and an Ehresmann connection. For any $u \in E$, $d\pi|_{H_u} : H_u \to T_{\pi(u)}M$ is an isomorphism. Hence under this isomorphism, each vector in $T_{\pi(u)}M$ corresponds to a unique element in T_uE, and we call it the **horizontal lift**.

As an example, consider the trivial bundle $E = M \times F$. Then the projection map induces an identification $TE \cong TM \oplus TF$, and we may take TF as a connection. This is called the trivial connection. For a general fibre bundal with a connection, if its local trivializations always take horizontal spaces of the connection to the trivial connection on the trivial bundle, then we say it is a **flat connection**.

Now we introduce **vector bundles**, which are fiber bundles whose fibers are linear spaces, and such that the local trivializations preserve linear structures. To be more precise, it is a surjective map $\pi : E \to B$ satisfying the following two properties: first, for any $P \in B$, $E_p := \pi^{-1}(p)$ is an n-dimensional vector space; second, near each point, there is a local trivialzation $\psi : E_U := \pi^{-1}(U) \to U \times \mathbb{R}^n$ in the sense of fiber bundles, and such that ψ restricts to linear isomorphisms on each fibers.

Let $\pi : E \to B$ be a vector bundle. A vector subbundle is a submanifold of E such that $\pi|_D : D \to B$ is a vector bundle, with fibers $D_p := D \cap E_p$ being linear subspace of E_p, for any $p \in B$. Suppose $\pi_1 : E_1 \to B_1$ and $\pi_2 : E_2 \to B_2$ are vector bundles, and $f : B_1 \to B_2$ is a smooth map. A smooth map $F : E_1 \to E_2$ is called a vector bundle morphism if $\pi_2 \circ F = f \circ \pi_1$, and F restricts to linear maps on each fiber of E_1.

6.3 Foliations

Let N be a manifold of dimension n. A subset $M \subset N$, is called a submanifold of dimension m, if we can find charts covering N such that its projection to the first m entries is exactly a chart for M. To be more precise, $\forall p \in M$, there exists a chart (U, ϕ) of N such that $\phi(U \cap M) = \phi(U) \cap (\mathbb{R}^n \times \{0\}$. The notion of widely used in the study of manifolds. For example, the image of the map $\mathbb{R}^n \times \{0\}$. The notion of widely used in the study of manifolds. For example, the image of the map $\mathbb{R} \to \mathbb{S}^1 \times \mathbb{S}^1 : t \mapsto (e^{z\pi it}, e^{2\pi i\lambda t}$, is a submanifold of the torus as long as $\lambda \in \mathbb{Q}$. The main reason that it has the structure as submanifold is that, the image is discrete. However, if λ is irrational, in any neighborhood there would be infinitely many slice of one-dimensional lines, and it is not a submanifold.

To describe objects similar to the previous example, we introduce the notion of **immersed submanifold**. It is a subset $H \subset M$ with an arbitrary topology, making H into a smooth manifold, and such that the inclusion of H into M is an immersion, i.e. its differential is injective. It is clear that the example described previously with λ irrational is an immersed submanifold.

The notion of immersed submanifold enable us to define foliations.

Definition 6.1. A rank k **foliation** is a collection of connected immersed submanifolds $\{L_\alpha\}$ of M, such that

- they are disjoint and covering M;
- given any $p \in M$, there exists a chart (U, ϕ) such that

$$\phi(U \cap L_\alpha) = \cup_{n\in\mathbb{N}}\{x_{k+1}, \cdots, x_m = \underline{c_n}\}.$$

  Here $\underline{c_n} \in \mathbb{R}^{m-k}$ is a constant vector.

Given a manifold M, there is a deep relation between foliations and certain type of subbundles of TM. We call a subbundle D over M of TM a **distribution**. We say that it is an **involutive distribution** if and only if $\Gamma(D)$ is closed under the Lie brackets. The Frobenius theorem states that the foliations on M corresponds to the involutive distributions on M, see ([21], 19.21) for a proof.

Theorem 6.2 (Frobenius)**.** *Let $\mathcal{F}$ be a foliation on a manifold M, the collection of tangent spaces to the leaves of $\mathcal{F}$ forms an involutive distribution on M.*

Conversely, given an involutive distribution D on M, the collection of all maximal connected integral manifolds of D forms a foliation of M.

Moreover, this describes a one-to-one correspondence.

Remark 6.3. Let M be a smooth manifold, $\phi : M \to M$ be a diffeomorphism. A distribution D on M is ϕ-invariant if $d\phi(D) = D$, i.e. $\forall x \in M$, we have $d\phi_x(D_x) = D_{\phi(x)}$. A foliation F on M is ϕ-invariant if for each leaf L of F, the submanifold $\phi(L)$ is also a leaf of F. Suppose D is an ϕ-invariant involutive distribution, we show that the foliation that it corresponds to also need to be ϕ-invariant. Let L be a leaf of F, then $TL = D|_L$. Then $T\phi(L) = d\phi(TL) = D_\phi\phi(L)$, so $\phi(L)$ is also an integral manifold. It's obvious that $\phi(L)$ is also maximal, thus F is $\phi(L)$-invariant. Similar argument also shows the converse, hence we conclude: invariant involutive distributions corresponds to invariant foliations.

6.4 Banach manifold

In the previous sections, we introduced smooth manifolds, which are locally equivalent to Euclidean spaces, i.e. finite dimensional real vector spaces. For our purpose, we will also need the generalized notion of manifolds, which are locally equivalent to Banach spaces. The main reference for this subsection is [20].

Given a real vector space X, and a real-valued function $\| \ \|$ on X, $(X, \| \ \|)$ is called a normed space if the function is nonnegative and vanishes only on the zero vector, compatible with scalar product, and the triangle inequality holds. The map $\| \|$ is called the norm, and induces a metric on X. A metric space is complete if all the Cauchy sequence converges. A **Banach space** is a complete normed space. We remark that finite dimensional Banach spaces are exactly the Euclidean spaces.

We can define smoothness of maps between Banach spaces, as follows: suppose E, F are Banach spaces, $U \subset E$ is open in the metric topology, a continuous map $f : U \to F$ is **differentiable** at $x \in U$ if and only if it can be written as $f(x+y) = f(x) + \lambda y + \phi(y)$ for small y, where $\lambda : E \to F$ is linear, and $\phi(y)$ is $o(y)$. If f is differentiable at x, the we call λ the derivative of f, denoted $f'(x)$. If f is differentiable everywhere in the domain, then we call it differentiable. A map is **smooth** if its n-th derivative exists and continuous for any natural number n.

Now with the notion of smooth map between Banach spaces, we can extend our definition of smooth manifolds, to infinite-dimensional, namely Banach manifolds.

Definition 6.4. A **Banach manifold** is a Hausdorff and second-countable topological space such that

- it is locally homeomorphic to a fixed Banach space, via the local charts;
- the change of charts maps are diffeomorphisms between opens of Banach spaces.

Many results on finite-dimensional manifolds, including the implicit function theorem,existence and uniqueness of ODE and Frobenius' theorem generalize naturally to Banach manifolds. Moreover, given a Banach space, its path space can also be endowed with a Banach space structure. Thus, it is not hard to believe that Banach manifolds will be useful in the study of path space of manifolds, as we will see in the later chapters.

6.5 Topology on the mapping space

The main reference of this section is ([19], Chapter 9). Let E, F be normed spaces, we denote $L^k_{sym}(E, F)$ to be the set of bounded symmetric multilinear mappings. A curve $c : \mathbb{R} \to E$ is called **locally Lipschitzian** if every point $r \in \mathbb{R}$ has a neighborhood U such that the Lipschitz condition is satisfied, i.e. the set $\{\frac{1}{t-s}(c(t) - c(s)) : t \neq s; t, s \in U\}$ is bounded. A curve $c : \mathbb{R} \to E$ is called $\mathcal{L}ip^k$ if all derivatives up to order k exist and are locally Lipshitzian. Let E be locally convex, a subset $U \subset E$ is said to be **open in the C^∞-topology** if there exists some $k \in \mathbb{N} \cup \{\infty\}$ such that, the preimages of U under all $\mathcal{L}ip^k$-curves are open in $\mathbb{R}$.

Now let $U \subset E$ and $V \subset F$ be C^∞-open. For any $k \in \mathbb{N} \cup \{\infty\}$, the space of **$k$-jets** from U to V is defined by

$$J^k(U, V) := U \times V \times L^l_{sym}(E, F) \times \cdots \times L^k_{sym}(E, F).$$

For a c^k mapping $f : U \to V$, the**k-jet extension** is given by

$$j^k f(x) = j^k_x f := (x, f(x), df(x), \frac{1}{2!}d^2 f(x), \cdots, \frac{1}{k!}d^k f(x)).$$

Thus $j^k f$ lies in $J^k(U, V)$.

Let A, B be Hausdorff topological spaces, let $C(A, B)$ denote the set of all continuous mappings.

Definition 6.5. The **compact-open topology** on $C(A, B)$ is generated by the family of all sets of the form $\{f \in C(A, B) : f(K) \subset U\} \subset C(A, B)$, where K runs through all compact subsets in A, and U through all open subsets of B.

Now consider M, N to be (possibly Banach) smooth manifolds.

Definition 6.6. For any $k \in \mathbb{N} \cup \{\infty\}$, the C^k**-topology** on $C^k(M, N)$ is the coarsest topology making the map $j^k : C^k(M, N) \to C(M, J^k(M, N))$ continuous, where the set $C(M, J^k(M, N))$ is assigned the compact-open topology.

It's easy to see that the C^0-topology on $C(M, N)$ coincides with the compact-open topology. Moreover, the following lemma will be useful when we want to construct manifolds out of the mapping space (e.g. taking the quotients). See ([19], Corollary 41.12) for a proof.

Lemma 6.7. *Suppose M, N are finite dimensional manifolds, then $C^k(M, N)$, equipped with the C^k-topology, is a Hausdorff, second-countable topological space.*

6.6 Thin homotopy

Given a smooth manifold M, the set of smooth paths in M does not behave well under concatenation of paths: the composition path could be not smooth. In this section, we introduce one way to overcome this problem.

Recall that there exists a smooth increasing function from $I = [0, 1]$ to itself: its value is identically 0 near a small neighborhood of 0, and is 1 near a neighborhood of 1. Given any smooth path, we can reparametrize it using the above function. The resulting path has the same image as the original path, but with the extra property that it stays at the starting point(ending point) near time 0 (time 1). We call a path having such property a **path with sitting instances**. It's easy to see that composition of paths with sitting instances is again smooth.

The reparametrized path is not the same as the original one, however, ther are equivalent in the sense of path-homotopy. Moreover, this homotopy, as a map from $I \times I$ to M, does not enclose any area. Equivalently, the differential of the homotopy map has rank at most one at each point. We call this special homotopy **thin homotopy**. To sum up, we can concatenate thin homotopy classes of smooth maps.

We can generalize this reparametrization trick to 2-dimensional case. Give a smooth homotopy between two smooth paths, say $F : I \times I \to M$, with $F(0, t) = \gamma_0(t)$, $F(1, t) = \gamma_1(t)$. We can reparametrize it horizontally to make it a smooth homotopy between paths with sitting instances, and such that for all fixed $s \in I$, $F(s, t)$ is also a smooth path with sitting instances. Moreover, we can reparametrize it vertically, and get a new homotopy $\widetilde{F}$, such that $\widetilde{F}(s, t) = \gamma_0(t)$ in a neighborhood of $\{0\} \times I$, and $\widetilde{F}(s, t) = \gamma_1(t)$ in a neighborhood of $\{1\} \times I$.

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