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Euler-Like Vector Fields

A New Method for Normal Forms

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Preface

This master's thesis is the result of work undertaken with the supervision of Professor Zambon and it would not have been possible without his advice and guidance. I am very thankful for all the patience and fruitful talks, for helping me hone my geometrical intuition, and for making the overall thesis experience a pleasure.

I show my gratitude also to all the people who supported me in my studies, including my family and friends.

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Abstract

Given a pair (M, N) of a manifold and a submanifold, the goal of this thesis is to discuss the bijection between germs of Euler-like vector fields and tubular neighborhood embeddings for (M, N). This connection allows us to simplify the search for normal forms for certain structures on this pair.
We will prove this relation when the submanifold is a point and in its most generality, and we will use it to prove classical and novel results in geometry.
We will present generalizations of this bijection to more advanced settings, and we will use them to simplify the proof of a recent result on gradient vector fields of Morse functions.

List of Main Symbols

$\hat{Y}$	Vector field on the deformation space constructed from Y
$\mathbb{N}$	Positive integers
$\mathbb{N}_0$	Positive integers including 0
$\mathcal{D}(\phi)$	Lift of a map ϕ to the deformation space
$\mathcal{D}(M,p)$, $\mathcal{D}(M,N)$	Deformation space for the pair (M,p) or (M,N)
$\mathcal{D}(Y)$	Vector field on the deformation space constructed from $\hat{Y}$
$\mathcal{L}_Y$	Lie derivative in direction of Y
νN, $\nu(M,N)$	Normal bundle for (M,N)
$\nu(\cdot)$	Linearization of a vector field, differential form, or singular foliation
π, κ, $\tilde{f}$	Maps defining the smooth structure of the deformation space
ψ^*	Pullback of a map ψ
ψ_*	Pushforward of a map ψ
$\mathbb{R}^\times$	Real numbers without the origin
θ	Canonical vector field on the deformation space
φ^Y	Flow of a vector field Y
E	Euler vector field
W	Vector field constructed from θ and $\mathcal{D}(X)$
X	Euler-like vector field

Contents

Chapter 1

Introduction

A natural question in Differential Geometry is whether a geometrical structure $\mathfrak{S}$ on a manifold M can be reduced to an equivalent, simplified structure around a submanifold N.
An example of this problem is the *simplification* of vector fields: given a vector field X on a smooth manifold, can we find local coordinates on which X takes a simple form, thereby facilitating the computation of its flow?
We refer to *normal form* theorems to indicate those results that allow a simplification of $\mathfrak{S}$. Such results often state the existence, under some conditions on $\mathfrak{S}$, of so-called *tubular neighborhood embeddings*. They are embeddings from (an open neighborhood of the zero section of) the *normal bundle* $\nu(M, N)$ of N in M to M itself that restrict to the identity map on N (seen as the zero section of the domain) and induce the natural identification $\nu(M, N) \simeq \nu(\nu(M, N), N)$.
In [BLM16] was proven that germs of *Euler-like* vector on M are in bijective correspondence with germs of tubular neighborhood embeddings. Hence, proving normal form theorems amounts to showing the existence of an explicit Euler-like vector field somehow compatible with the given structure $\mathfrak{S}$, which is a much simpler problem.
The following chapters will aim to prove this theorem in the case $N = point$ and for general submanifolds N of M, and to give some applications and generalizations of the theorem thereof.

The initial setting (chapter 2) is a particular case of *Sternberg's linearization theorem* (see for instance [Ste57, Ste58]). However, we will follow the strategy of [BBLM20] in both situations for two primary reasons. First, understanding the proof in the point-submanifold case provides a better insight into how the proof works and it will simplify the discussion of the general case (chapter 3). Moreover, the analytical point of view in [Ste57], and similarly, the concise proof in [BLM16], in contrast to [BBLM20], where the geometry is more explicit, do not always provide a clear method for extending this result to more advanced situations, as we will see at the end chapter 4.
In chapter 4, we will use the main theorem to prove some well-known results of differential and symplectic geometry, namely Morse's lemma, Darboux's theorem, and Morse-Bott and Weinstein Lagrangian neighborhood theorems in a simpler way, as well as a new *splitting* theorem for singular foliations presented in [BBLM20].
Chapter 5 will be devoted to generalizing the main theorem in the case of a point. We will talk about the linearization of *weighted* Euler-like vector fields, and how it can be generalized in the presence of *resonances* (appendix A). Our main contributions reside in the complemen-

tary *non-resonant* case, where we will prove a normal form for gradient vector fields of Morse functions. This result was proven in [Wan18], but our proof is much simplified and makes use of the above-mentioned Sternberg's linearization theorem.

According to our main reference, we will diverge from the discussion presented in [BLM16] and not talk about *germs* of Euler-like vector fields or tubular neighborhood embeddings. Instead, we will rephrase the result of [BLM16] into the following implications

Theorem 1.1 ($\Leftarrow$)**.** *If $\psi : \nu(M, N) \to M$ is a tubular neighborhood embedding, and E is the Euler vector field on $\nu(M, N)$, then*

$$X := \psi_* E$$

is an Euler-like vector field on the image of ψ.

Theorem 1.2 ($\Rightarrow$, Main Theorem)**.** *If X is an Euler-like vector field for the pair (M, N) and E is the Euler vector field on the normal bundle, then there exists a unique tubular neighborhood embedding*

$$\psi : \nu(M, N) \to M$$

such that $\psi_* E = X|_{\mathrm{Im}\,\psi}$.

To fix our notation, for any vector field X on a manifold M we will consider the Lie derivative of a vector field $Y \in \mathfrak{X}(M)$ on M with respect to X to be the vector field

$$\mathcal{L}_X Y = \frac{d}{dt}\Big|_{t=0} (\varphi^X_{-t})_* Y = [X, Y]$$

on M, where φ^X is the flow of X and, for any smooth map F between manifolds, we denote the pushforward of F with F_* and its pullback with F^*.
Similarly, we take the Lie derivative of a differential k-form $\omega \in \Omega^k(M)$ on M with respect to X to be the k-form $\mathcal{L}_X \omega$ on M with sign convention

$$\mathcal{L}_X \omega = \frac{d}{dt}\Big|_{t=0} (\varphi^X_t)^* \omega.$$

Chapter 2

Case $N = point$

This chapter will aim to prove theorems 1.1, 1.2 in the case of a point in a smooth manifold M, and will follow the work of main reference [BBLM20].
Let us give the statements adjusted to this setting

Theorem 2.1. *Let $p \in M$ be a point, $\psi : T_pM \to M$ a tubular neighborhood embedding, and E the Euler vector field on T_pM. Then, the pushforward*

$$X = \psi_* E$$

of E is an Euler-like vector field on the image of ψ.

Theorem 2.2. *An Euler-like vector field X for the pair (M, p) determines a unique tubular neighborhood embedding*

$$\psi : T_pM \to M$$

with $\psi_* E = X|_{\mathrm{Im}\,\psi}$.

2.1 Preliminaries

As mentioned earlier, this section is devoted to proving theorem 1.1 in the case of a point, namely theorem 2.1. We will do so after saying what it means for a vector field on M to be *Euler-like* and after defining the notion of *tubular neighborhood embedding*.
Let us start by introducing the objects in the statement and some of their properties.

Definition 2.3. Let V be a vector space. The *Euler* vector field on V is the unique vector field $E \in \mathfrak{X}(V)$ such that

$$E(f) = f \tag{2.1}$$

for every linear function $f \in V^*$.

Remark 2.4. The uniqueness in the definition above is given by the fact that a vector field on a vector space can be described uniquely by how the correspondence derivation acts on the linear functions on this vector space, in the same way as a real vector field is uniquely determined by how its derivation acts on the functions $x_1, \ldots, x_n$. The existence is guaranteed by the example below.

Example 2.5 (Euler vector field on V)**.** For a vector space V with linear coordinates $x_1, \dots, x_n$, the Euler vector field is

$$E = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}. \tag{2.2}$$

Indeed, if $f : V \to \mathbb{R}$ is linear, it takes the form

$$f(x) = \sum_{i=1}^{n} \lambda_i x_i$$

with $\lambda_i \in \mathbb{R}$. Then,

$$E(f)(x) = \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}(x) = \sum_{i=1}^{n} x_i \cdot \lambda_i = f(x).$$

Given a manifold M and a point $p \in M$, we know that its tangent space at p is a vector space on which we can take a set of linear coordinates $x_1, \dots, x_n$. The previous example shows that the Euler vector field on T_pM is defined in local coordinates by the linear functions x_i as its coefficients.
The next step will be to define *Euler-like* vector fields. Intuitively, we call X Euler-like if it is equal to the Euler vector field up to higher-order terms. This idea is made precise by the concept of *linearization*. Let us start by explaining what we mean by this.
Call $\mathcal{I}_p$ the ideal of smooth functions on M vanishing at p, and define the set[1]

$$\mathcal{I}_p^2 = \Big\{ \sum_{i=1}^{n} f_i g_i \, : \, n \in \mathbb{N}, \, f_i, \, g_i \in \mathcal{I}_p \Big\}.$$

Lemma 2.6. *We have a canonical isomorphism*

$$\Phi : \mathcal{I}_p / \mathcal{I}_p^2 \xrightarrow{\sim} T_p^*M$$

defined by

$$\Phi(f + \mathcal{I}_p^2) = d_p f.$$

Proof. Firstly, Φ is trivially linear and well-defined. This boils down to checking that if f is an element of $\mathcal{I}_p^2$, then its derivative at p vanishes.
So, let $f \in \mathcal{I}_p^2$. Then, $f = \sum_{i=1}^{n} f_i g_i$ with $f_i, \, g_i \in \mathcal{I}_p$. Taking its derivative gives

$$\begin{aligned} d_p f &= d_p\big(\sum_{i=1}^{n} f_i g_i \big) \\ &= \sum_{i=1}^{n} \big[d_p f_i \cdot g_i(p) + f_i(p) \cdot d_p g_i \big] \\ &= 0 \in T_p^*M \end{aligned}$$

because all f_i and g_i vanish at p. Hence, Φ is well-defined.
Now, considering a chart $\phi : U \to \mathbb{R}^n$ with $\phi(p) = 0$, without loss of generality, we can

[1] $\mathbb{N} := \mathbb{Z}_{>0} = \{1, 2, 3, 4 \dots\}$

work in $\mathbb{R}^n$.
Let $F \in T_0^*\mathbb{R}^n$. Since $d_0x_1, \ldots, d_0x_n$ form a basis for the cotangent space at the origin, there exist scalars $\lambda_1, \ldots, \lambda_n$ such that

$$F = \sum_{i=1}^{n} \lambda_i d_0 x_i \quad \text{and} \quad \Phi(x_i + \mathcal{I}_0^2) = d_0 x_i$$

for every i. But then,

$$\Phi\Big(\sum_{i=1}^{n} \lambda_i x_i + \mathcal{I}_0^2\Big) = \sum_{i=1}^{n} \lambda_i d_0 x_i = F.$$

So, Φ is surjective. To conclude the proof, we only have to show the injectivity.
Suppose $\Phi(f + \mathcal{I}_p^2) = d_p f = 0$. We have to prove that $f \in \mathcal{I}_p^2$. Take a smooth bump function $\phi : M \to [0,1]$ identically equal to 1 in a neighborhood U of p and zero outside a neighborhood of p containing U. Since $f = \phi f + (1-\phi)f$ and by construction $(1-\phi)f \in \mathcal{I}_p^2$, it suffices to prove that $\phi f \in \mathcal{I}_p^2$. Or equivalently, that $f \in \mathcal{I}_p^2$ in a neighborhood of p, since $f = \phi f$ in U. We can now take a chart of U to $\mathbb{R}^n$ and without loss of generality consider $p \in \mathbb{R}^n$ and f as a function $\mathbb{R}^n \to \mathbb{R}$ such that

$$f(p) = 0 \text{ and } \frac{\partial f}{\partial x_i}(p) = 0 \text{ for every } i, \tag{2.3}$$

because $d_p f = 0$.
Then, by Taylor's expansion, we know that there exist smooth functions g_j vanishing at p (i.e. $g_j \in \mathcal{I}_p$), for $j = 1, \ldots, n$, such that

$$f(x) = f(p) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p)(x_i - p_i) + \sum_{i,j=1}^{n} g_j(x)(x_i - p_i).$$

By condition (2.3), we have that

$$f(x) = \sum_{i,j=1}^{n} g_j(x)(x_i - p_i) \in \mathcal{I}_p^2.$$

This gives the injectivity and concludes the proof. □

Definition 2.7. Given $Y \in \mathfrak{X}(M)$ such that $Y_p = 0$, define the *linearization* of Y to be the vector field $\nu(Y) \in \mathfrak{X}(T_pM)$ acting on the linear functions on T_pM as

$$\nu(Y)(d_p f) := d_p(Y(f)) \tag{2.4}$$

or equivalently

$$\nu(Y)(f + \mathcal{I}_p^2) := Y(f) + \mathcal{I}_p^2.$$

Remark 2.8. 1. The operation that associates to a vector field its linearization as defined above is linear over $\mathbb{R}$. This follows from the linearity of the derivative. In local coordinates, the components of $\nu(X)$ will be the first-order terms of the Taylor expansion of the components of X at p. Hence the name.

2. Note that the property in the definition characterizes a unique vector field on T_pM because, as in remark 2.4, if V is a vector space, a vector field on V is determined by how the correspondence derivation acts on the linear functions of V, i.e. on $T_p^*M \simeq \mathcal{I}_p/\mathcal{I}_p^2$. Recall that the linear functions on T_pM are exactly the derivatives at p of smooth functions on M.

3. The definition is well-posed, i.e. $Y(f) + \mathcal{I}_p^2$ depends only on $f + \mathcal{I}_p^2$ not on f itself. Indeed, if $f + \mathcal{I}_p^2 = g + \mathcal{I}_p^2$, then $f = g + \sum_i f_i g_i$ and

$$Y(f) = Y(g) + \sum_i \big[Y(f_i)g_i + f_i Y(g_i)\big]. \tag{2.5}$$

But $Y(f_i)$ is a map $M \to \mathbb{R}$ and $(Y(f_i))(p) = Y_p(f_i) = 0$, because $Y_p = 0$. Then, $Y(f_i) \in \mathcal{I}_p$ and similarly $Y(g_i) \in \mathcal{I}_p$. This means that the second term on the right-hand side of equation (2.5) is an element of $\mathcal{I}_p^2$, thus

$$\nu(Y)(f + \mathcal{I}_p^2) = Y(f) + \mathcal{I}_p^2 = Y(g) + \mathcal{I}_p^2 = \nu(Y)(g + \mathcal{I}_p^2).$$

Definition 2.9. Fix $p \in M$. A vector field $X \in \mathfrak{X}(M)$ is *Euler-like* for the pair (M, p) if it is complete[2], $X_p = 0$, and the linearization $\nu(X) = E$, the Euler vector field on T_pM.

Example 2.10 (Euler-like vector fields)**.** Consider $M = \mathbb{R}^2$, $p = 0$, and

$$X = (x + y^2)\frac{\partial}{\partial x} + ye^x \frac{\partial}{\partial y}.$$

Then,

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + y^2\frac{\partial}{\partial x} + y(e^x - 1)\frac{\partial}{\partial y}$$

and if $f(x, y) = \lambda x + \mu y$ is linear, then

$$\begin{aligned}\nu(X)(f + \mathcal{I}_p^2) &= X(\lambda x + \mu y) + \mathcal{I}_p^2 \\ &= \lambda x + \mu y + \lambda y^2 + \mu y(e^x - 1) + \mathcal{I}_p^2 \\ &= \lambda x + \mu y + \mathcal{I}_p^2 \\ &= f + \mathcal{I}_p^2.\end{aligned}$$

Since $\nu(X)$ and E are characterized by the same action, we conclude that $\nu(X) = E$, so X is an Euler-like vector field.
In general, X is Euler-like for (M, p) if and only if in local coordinates around p it has the form

$$X = \sum_i (x_i + b_i(x))\frac{\partial}{\partial x_i}, \tag{2.6}$$

for smooth functions b_i that vanish to the second order at the origin.

Remark 2.11. In definition 2.9 we require that X be complete. This condition will be necessary in the proof of theorem 2.1, but we see that if $X_p = 0$ and $\nu(X) = E$, we can find a complete vector field that agrees with X in a neighborhood of p. Indeed, suppose

[2] i.e. its integral curves φ_t^X are defined for all $t \in \mathbb{R}$

X is a vector field satisfying these two requirements. Then, X has local form (2.6), and since the functions b_i have vanishing derivative at p, near this point the flow φ_s^X of X behaves like the flow φ_s^E of E. In particular, since

$$\lim_{s\to\infty} \varphi_{-s}^E(m) = p$$

for every $m \in M$, in a small enough neighborhood U of p we have

$$\lim_{s\to\infty} \varphi_{-s}^X(m) = p.$$

Hence, by multiplying X by a bump function that is zero outside U and 1 in a neighborhood of p contained in U, we can complete the vector field.
So, from now on we will consider X to be automatically complete whenever $X_p = 0$ and $\nu(X) = E$.

Lemma 2.12. *If $X \in \mathfrak{X}(M)$ is Euler-like and $f \in C^\infty(M)$ vanishes at p, then*

$$X(f) - f$$

vanishes to the second order at p. This means that $X(f) - f$ and its derivative vanish at p.

Proof. The proof is straightforward.

$$(X(f) - f)(p) = X_p(f) - f(p) = 0 - 0 = 0,$$

and

$$\begin{aligned} d_p(X(f) - f) &= d_p(X(f)) - d_pf \\ &= \nu(X)(d_pf) - d_pf \\ &= E(d_pf) - d_pf \\ &= d_pf - d_pf \\ &= 0 \end{aligned}$$

where the second equality is by definition of linearization. □

The second fundamental ingredient is the notion of *tubular neighborhood embeddings*. Recall that an *embedding* is defined to be a smooth map, which is a homeomorphism on its image and whose derivative at each point is injective.

Definition 2.13. Fix $p \in M$. A *tubular neighborhood embedding* for M at p is an embedding

$$\psi : T_pM \to M$$

such that $\psi(0) = p$ and

$$d_0\psi : T_0(T_pM) \to T_pM$$

is the identity map after taking the canonical identification $T_0(T_pM) \simeq T_pM$. With a light abuse of notation we will write $d_0\psi$ as a map from T_pM to itself, so that

$$d_0\psi = \mathrm{id}_{T_pM}\,.$$

Example 2.14 (Tubular neighborhood embedding for the circle at a point)**.** Let $M = S^1$ embedded in $\mathbb{R}^2$ with the standard (x, y)-coordinates, and let $p = (1, 0) \in S^1$, so that $T_pS^1 = \{1\} \times \mathbb{R} \simeq \mathbb{R}$. Then,

$$\psi : y \in T_pS^1 \mapsto (\cos(\arctan y), \sin(\arctan y)) \in S^1 \tag{2.7}$$

is a tubular neighborhood embedding, because $\psi(0) = (1, 0) = p$, and

$$\begin{aligned} d_0\psi &= \frac{d}{dy}\Big|_{y=0}\psi(y) \\ &= \Big(-\frac{\sin(\arctan y)}{1+y^2}, \frac{\cos(\arctan y)}{1+y^2}\Big)\Big|_{y=0} \\ &= (0, 1), \end{aligned}$$

which is the identity on T_pS^1 via the canonical identification $T_0(T_pS^1) \simeq T_pS^1$, $(0, y) \mapsto y$.

Recall the following

Lemma 2.15. *Given a vector field Z on M, $Z_p = 0$ if and only if $(Z(f))_p = 0$ for any function f on M vanishing at p.*

Proof. One direction is trivial because if $Z_p = 0$, then $(Z(f))_p = d_pf(Z_p) = 0$.
On the other hand, if $d_pf(Z_p) = 0$ for any f as in the statement, it holds in particular for any set of local coordinates $f = x_i$ around p.
Then, for every i

$$0 = d_px_i\Big(\sum_j z_j(p)\frac{\partial}{\partial x_j}\Big|_p\Big) = \sum_j z_j(p)\frac{\partial x_i}{\partial x_j}\Big|_p = z_i(p)$$

Thus, $Z_p = \sum_j z_j(p)\frac{\partial}{\partial x_j}\big|_p = 0$. □

In particular, this proves that the Euler vector field vanishes at the origin because any linear function does so, and $(E(f))(0) = f(0) = 0$. Equivalently, example 2.5 showed that in local $x_1, \ldots, x_n$ coordinates around p the Euler vector field takes form $E = \sum_i x_i\frac{\partial}{\partial x_i}$, which vanishes at zero.
As stated at the beginning of this chapter, we now have all the notions needed to prove theorem 2.1.

Theorem 2.1. *Let $p \in M$ be a point, $\psi : T_pM \to M$ a tubular neighborhood embedding, and E the Euler vector field on T_pM. Then, the pushforward*

$$X = \psi_*E$$

of E is an Euler-like vector field on the image of ψ.

Proof of theorem 2.1. By remark 2.11, we only have to prove that $X_p = 0$ and that the linearization $\nu(X)$ of X is the Euler vector field on the tangent space.
For the first goal, since ψ is a homeomorphism on the image, $\psi^{-1}(p) = 0$, and we have

$$X_p = (\psi_*E)_p = d\psi|_{\psi^{-1}(p)}(E_{\psi^{-1}(p)}) = d_0\psi(E_0) = E_0 = 0.$$

To prove that $\nu(X) = E$, it is enough to show that the linearization vector field acts on the space of linear functions over T_pM as the identity because, by remark 2.4, E is the only vector field over the tangent space with this property.
Recall that all linear functions over T_pM are derivatives at p of smooth functions over M. So, let $f : M \to \mathbb{R}$ be smooth and d_pf its derivative.
Then, by definition of linearization of a vector field,

$$\nu(X)(d_pf) = \nu(\psi_*E)(d_pf) = d_p(\psi_*E(f)). \tag{2.8}$$

Observe that for a general vector field V, we have

$$\begin{aligned}(\psi_*V(f))(q) &= (\psi_*V)_q(f) = d_qf((\psi_*V)_q) \\ &= d_qf(d_{\psi^{-1}(q)}\psi(V_{\psi^{-1}(q)})) \\ &= d_{\psi\circ\psi^{-1}(q)}f \circ d_{\psi^{-1}(q)}\psi(V_{\psi^{-1}(q)}) \\ &= d_{\psi^{-1}(q)}(f\circ\psi)(V_{\psi^{-1}(q)}) \\ &= (V(f\circ\psi)\circ\psi^{-1})(q),\end{aligned} \tag{2.9}$$

for any $q \in M$, using the chain rule in the fourth and fifth equalities. Then, equation (2.8) becomes

$$\begin{aligned}\nu(X)(d_pf) &= d_p(E(f\circ\psi)\circ\psi^{-1}) \\ &= d_0E(f\circ\psi)\circ d_p\psi^{-1} \\ &= d_0(\mathcal{L}_E(f\circ\psi))\circ d_p\psi^{-1}.\end{aligned} \tag{2.10}$$

Recall that the exterior derivative and the Lie derivative commute, i.e. for any function f and vector field X on M, $d(\mathcal{L}_Xf) = \mathcal{L}_X(df)$. Moreover, one has the well-known *Cartan's metric formula* that states

$$\mathcal{L}_X\omega = d(\iota_X\omega) + \iota_Xd\omega, \tag{2.11}$$

where ω is a differential m-form and $\iota_X\omega$ is the $(m-1)$-form defined as the *contraction* of ω with X, i.e. for vector fields $X_1, \dots, X_{m-1}$,

$$\iota_X\omega(X_1, \dots, X_{m-1}) = \omega(X, X_1, \dots, X_{m-1}).$$

Then, calling $\omega = d(f\circ\psi)$, we have

$$d(\mathcal{L}_E(f\circ\psi)) = \mathcal{L}_E(\omega) = d(\iota_E\omega) + \iota_Ed\omega = d(\iota_E\omega), \tag{2.12}$$

because ω is exact. Since ω is then closed, it locally takes the form

$$\omega = \sum_i u_idx_i,$$

and contracting with E gives

$$\iota_E\Big(\sum_i u_idx_i\Big) = \sum_i u_idx_i\Big(\sum_j x_j\frac{\partial}{\partial x_j}\Big) = \sum_i u_ix_i. \tag{2.13}$$

Hence,

$$\begin{aligned}d(\iota_E\omega)|_0 &= d\Big(\sum_i u_ix_i\Big)\Big|_0 = \sum_i \big(x_idu_i + u_idx_i\big)\big|_0 \\ &= \Big(\sum_i u_idx_i\Big)\Big|_0 = \omega|_0 = d_0(f\circ\psi) \\ &= d_pf\circ d_0\psi.\end{aligned} \tag{2.14}$$

Thus, equation (2.10) together with equations (2.12)-(2.14), and the fact that $(d_0\psi)^{-1} = d_p(\psi^{-1})$ gives

$$\nu(X)(d_pf) = d_pf.$$

As we said, this implies $\nu(X) = E$. □

Given an embedding ψ and a vector field X, none of them necessarily linear, the derivative $d_0\psi$ and the linearization of X are linear. Intuitively, the linearization should split ψ_*X into its linear parts, one associated to ψ, the other to X. The next proposition shows how this works.

Proposition 2.16. *If $\psi : T_pM \to M$ is an embedding with $\psi(0) = p$ and Y is a vector field on T_pM, we have*

$$\nu(\psi_*Y) = (d_0\psi)_*(\nu(Y)). \tag{2.15}$$

Proof. Let $f : M \to \mathbb{R}$ smooth. Then, we have

$$\begin{aligned}
(d_0\psi)_*(\nu(Y))(d_pf) &\overset{(2.9)}{=} \nu(Y)(d_pf \circ d_0\psi) \circ (d_0\psi)^{-1} \\
&= \nu(Y)(d_0(f \circ \psi)) \circ d_p\psi^{-1} \\
&= d_0(Y(f \circ \psi)) \circ d_p\psi^{-1} \\
&= d_p(Y(f \circ \psi) \circ \psi^{-1}) \\
&\overset{(2.9)}{=} d_p(\psi_*Y(f)) \\
&= \nu(\psi_*Y)(d_pf),
\end{aligned}$$

where in the third and sixth equality we used the definition of linearization of vector fields, and in the second and fourth the chain rule. □

We see then that with this result the proof of theorem 2.1 is trivial, because if $Y = E$ the Euler vector field, and ψ is a tubular neighborhood embedding, then $d_0\psi = \mathrm{id}_{T_pM}$ and, since E is already linear,

$$\nu(X) := \nu(\psi_*E) = \nu(E) = E.$$

The two proofs give two different ways to approach the problem, one more computational, the second more conceptual.

2.2 Deformation Space

In this section, we describe one of the most important objects necessary for the proof of theorem 2.2, the *deformation space*.

Definition 2.17. Fix a point $p \in M$. The *deformation space* $\mathcal{D}(M,p)$ for the pair (M,p) is the set[3]

$$\mathcal{D}(M,p) = (T_pM \times 0) \sqcup (M \times \mathbb{R}^\times) \tag{2.16}$$

equipped with the unique manifold structure determined by the following properties:

[3] To make the presentation more clear, with a little abuse of notation, we generally identify $(v,0) \in T_pM \times 0$ with $v \in T_pM$, and with $\mathbb{R}^\times := \mathbb{R}\backslash\{0\}$.

1. the map

$$\pi : \mathcal{D}(M,p) \to \mathbb{R}, \qquad \begin{cases} v \in T_pM \mapsto 0 \\ (m,t) \in M \times \mathbb{R}^\times \mapsto t \end{cases} \tag{2.17}$$

is a smooth submersion;

2. the map

$$\kappa : \mathcal{D}(M,p) \to M, \qquad \begin{cases} v \in T_pM \mapsto p \\ (m,t) \in M \times \mathbb{R}^\times \mapsto m \end{cases} \tag{2.18}$$

is smooth;

3. for every function $f \in C^\infty(M)$ vanishing at the point p, the map

$$\tilde{f} : \mathcal{D}(M,p) \to \mathbb{R}, \qquad \begin{cases} v \in T_pM \mapsto v(f) = d_pf(v) \\ (m,t) \in M \times \mathbb{R}^\times \mapsto \frac{1}{t}f(m) \end{cases} \tag{2.19}$$

is smooth.

To fix our notation, we will denote with v (or $(v,0)$) a vector in T_pM (or in $T_pM \times 0$), with (m,t) a point in $M \times \mathbb{R}^\times$, and with x a general point in $\mathcal{D}(M,p)$. Additionally, we will refer to $T_pM = \pi^{-1}(0)$ as the *zero fiber* (of π).
Since the definition is fairly involved and getting an idea of how the deformation space looks like is a non-trivial exercise, before giving an explicit example we show that $\mathcal{D}(M,p)$ is a smooth manifold and we introduce a set of *local coordinates* on it.
We equip $\mathcal{D}(M,p)$ with the smallest topology such that the maps π, κ, $\tilde{f}$ are continuous (for all smooth f), and with abuse of notation, we denote the map π simply by its image t. This topology is trivially Hausdorff and locally Euclidean.

Lemma 2.18 ([Hig10], Lemma 4.3). *Let U be an open subset of M and $(U, y := (y_1,\dots,y_n))$ be a local chart of M around p (i.e. y is an homeomorphism from U to an open subset of $\mathbb{R}^n$). Then,*

$$\tilde{y}_1,\dots,\tilde{y}_n,t \tag{2.20}$$

define an homeomorphism from an open subset $\mathcal{D}(U,p)$ of $\mathcal{D}(M,p)$ to an open in $\mathbb{R}^{n+1}$.

We will call the functions in (2.20) a set of *local coordinates* for the deformation space.

Proof. Let $y_1,\dots,y_n$ be defined on an open $U \subset M$. Then, $\mathcal{D}(U,p) = \kappa^{-1}(U)$ is open because κ is continuous. The map

$$u \in \mathcal{D}(U,p) \xrightarrow{\phi} (\tilde{y}(u),t) := (\tilde{y}_1(u),\dots,\tilde{y}_n(u),\pi(u)) \in \mathbb{R}^{n+1} \tag{2.21}$$

is continuous and bijective on its image, because its components are such.
Now, the map

$$\psi : \phi(\mathcal{D}(U,p)) \subset \mathbb{R}^{n+1} \to \mathcal{D}(U,p)$$

defined by

$$(x,t) \mapsto \begin{cases} \left(y^{-1}(tx),t\right) & \text{if } t \neq 0 \\ \left(d_0y^{-1}(x),0\right) & \text{if } t = 0 \end{cases} \tag{2.22}$$

is the inverse of ϕ. To prove its continuity, it is enough to show that its compositions with the maps in definition 2.17 are continuous. This check is trivial for π and κ, so consider a smooth function f vanishing at p. Then,

$$\tilde{f} \circ \psi(x,t) = \begin{cases} \frac{1}{t}(f \circ y^{-1})(tx) & \text{if } t \neq 0 \\ d_0(f \circ y^{-1})(x) & \text{if } t = 0 \end{cases}, \tag{2.23}$$

where we used the chain rule and $y^{-1}(0) = p$ for the case $t = 0$. This map is continuous because the two terms agree to the limit $t \to 0$, being f and y^{-1} continuous functions. This proves the claim or, equivalently, that the pair $(\mathcal{D}(U,p), \phi)$ is a local chart on the deformation space. □

Note that, if $y_1, \dots, y_n$ is a set of coordinates on an open subset U of M which does not contain p, then the set (2.20) allows us to cover only $U \times \mathbb{R}^\times$.
The following result shows the existence of a smooth atlas on the deformation space. For a further characterization of this structure, we refer to [Hig10], Proposition 4.5.

Proposition 2.19. *The deformation space has a smooth atlas consisting of charts $(\mathcal{D}(U,p), \phi)$ as above.*

Proof. The only thing left to prove is that the transition functions are smooth. So, take two charts $(\mathcal{D}(U,p), \phi)$, $(\mathcal{D}(V,p), \psi)$, with $\mathcal{D}(U,p) \cap \mathcal{D}(V,p) \neq \varnothing$ and

$$\phi = (\tilde{y}, t),\ \psi = (\tilde{z}, t),$$

as above. Then, by equations (2.21), (2.22),

$$\phi \circ \psi^{-1}(x,t) = \begin{cases} \left(\frac{1}{t}(y \circ z^{-1})(tx), t\right) & \text{if } t \neq 0 \\ \left(d_0(y \circ z^{-1})(x), 0\right) & \text{if } t = 0 \end{cases}, \tag{2.24}$$

where we used again the chain rule and $z^{-1}(0) = p$ for the case $t = 0$. Observe that this composition is smooth for both $t = 0$ and $t \neq 0$, because of the smoothness of the transition functions $y \circ z^{-1}$ on (an open subset of) M. Moreover, since $y \circ z^{-1}(0) = 0$, the two terms smoothly agree at the limit $t \to 0$, again by the smoothness of the transition functions. This shows the smoothness of $\phi \circ \psi^{-1}$ and concludes the proof. □

Now that we have given coordinates on the deformation space, it is time to consider a simple but effective example.

Example 2.20 (Deformation space for an interval)**.** Let $M = (-1,1) \subset \mathbb{R}$ be an interval and $p = 0 \in M$.
Then, $T_pM \simeq \mathbb{R}$ and

$$D := \mathcal{D}((-1,1), 0) = (\mathbb{R} \times 0) \sqcup ((-1,1) \times \mathbb{R}^\times).$$

Let $y : (-1,1) \to \mathbb{R}$ be the standard (global) coordinate on $(-1,1)$ that vanishes at 0. Then, D has (global) coordinates $\tilde{y}, t$, where on $(-1,1) \times \mathbb{R}^\times$

$$\tilde{y}(r,t) := \frac{1}{t} y(r) = \frac{r}{t}$$

and on $T_0(-1,1)$

$$\tilde{y}\Big(r\frac{\partial}{\partial y}\Big|_{r0}\Big) = d_0 y\Big(r\frac{\partial}{\partial y}\Big|_{r0}\Big) = r\frac{\partial y}{\partial y}\Big|_{r0} = r.$$

Hence, identifying $T_0(-1,1) \simeq \mathbb{R}$ via $r\frac{\partial}{\partial y}\Big|_{r0} \mapsto r$, we have $\tilde{y} = id_{\mathbb{R}\times 0}$. Then, D can be viewed in $(\tilde{y}, t)$ coordinates as in figure 2.1. These coordinates are simply a restriction of the Cartesian coordinates on the plane.

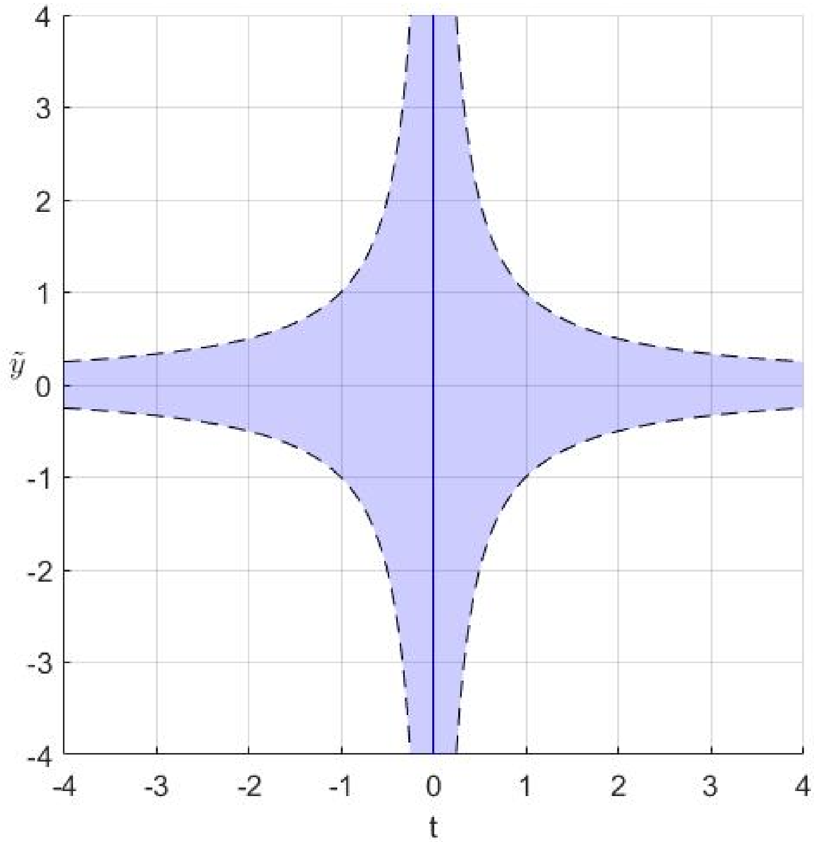


Figure 2.1: The deformation space in case M is an interval. The dashed lines are the loci $\tilde{y} = \pm 1/t$, the blue area is $(-1,1) \times \mathbb{R}^{\times}$, and the blue line is $T_0(-1,1) \simeq \mathbb{R}$

Below we list some basic properties of the deformation space.

(a) for any smooth map $\phi : M_1 \to M_2$, we can smoothly extend the map $\phi \times \mathrm{id} : M_1 \times \mathbb{R}^{\times} \to M_2 \times \mathbb{R}^{\times}$ to a map

$$\mathcal{D}(\phi) : \mathcal{D}(M_1, p) \to \mathcal{D}(M_2, \phi(p)) \tag{2.25}$$

whose restriction to the zero fibers is the usual derivative, $d_p\phi$.
This is simply because taking the derivative of $\phi \times \mathrm{id}$, we get

$$d_{(p,0)}(\phi \times \mathrm{id}) = d_p\phi \times d_0\,\mathrm{id} = d_p\phi \times \mathrm{id}$$

as a map $T_pM_1 \times 0 \to T_{\phi(p)}M_2 \times 0$.

(b) We have a canonical vector field θ on the deformation space that in local coordinates takes the form

$$\theta = t\frac{\partial}{\partial t} - \sum_i \tilde{y}_i \frac{\partial}{\partial \tilde{y}_i}. \tag{2.26}$$

θ satisfies the following

$$\theta(t) = t, \quad \theta(\kappa^* f) = 0, \quad \theta(\tilde{g}) = -\tilde{g} \tag{2.27}$$

for every smooth functions f and g on M, with g vanishing at p.
Indeed,

$$\theta(t) = t\frac{\partial t}{\partial t} - \sum_i \tilde{y}_i \frac{\partial t}{\partial \tilde{y}_i} = t.$$

For the other equations, note that $\theta = t\partial/\partial t$ in the $(y_1, \ldots, y_n, t)$ coordinates on $M \times \mathbb{R}^\times$. Indeed, if we call, to avoid confusion, $(\tilde{y}_1, \ldots, \tilde{y}_n, s) = (y_1/t, \ldots, y_n/t, t)$ the "new" coordinates on $M \times \mathbb{R}^\times$, by the chain rule we have

$$\begin{aligned} \frac{\partial}{\partial s} &= \frac{\partial t}{\partial s}\frac{\partial}{\partial t} + \sum_i \frac{\partial y_i}{\partial s}\frac{\partial}{\partial y_i} = \frac{\partial}{\partial t} + \sum_i \frac{1}{t} y_i \frac{\partial}{\partial y_i} \\ \frac{\partial}{\partial \tilde{y}_i} &= \frac{\partial t}{\partial \tilde{y}_i}\frac{\partial}{\partial t} + \sum_j \frac{\partial y_j}{\partial \tilde{y}_i}\frac{\partial}{\partial y_j} = t\frac{\partial}{\partial y_i}. \end{aligned} \tag{2.28}$$

Then, by equation (2.26)

$$\begin{aligned} \theta &= s\frac{\partial}{\partial s} - \sum_i \tilde{y}_i \frac{\partial}{\partial \tilde{y}_i} \\ &= t\frac{\partial}{\partial t} + \sum_i y_i \frac{\partial}{\partial y_i} - \sum_i \frac{y_i}{t} t \frac{\partial}{\partial y_i} = t\frac{\partial}{\partial t}. \end{aligned} \tag{2.29}$$

Hence, since $\kappa^* f$ is constant on T_pM and independent of t on $M \times \mathbb{R}^\times$,

$$\theta(\kappa^* f) = 0.$$

Now, on $M \times \mathbb{R}^\times$

$$\theta(\tilde{g}) = \theta(g/t) = gt\frac{\partial}{\partial t}(1/t) = -t/t^2 g = -\tilde{g},$$

and on $T_pM \simeq \mathbb{R}^n$, if $\tilde{g} = (\tilde{g}_1, \ldots, \tilde{g}_n)$,

$$\theta(\tilde{g}) = -\sum_i \tilde{y}_i \frac{\partial}{\partial \tilde{y}_i}(\tilde{g}_1, \ldots, \tilde{g}_n) = -(\tilde{g}_1, \ldots, \tilde{g}_n) = -\tilde{g}$$

and this concludes the proof of the system (2.27).
Since,

$$\theta|_{\pi^{-1}(0)} = -\sum_i \tilde{y}_i \frac{\partial}{\partial \tilde{y}_i}, \tag{2.30}$$

θ restricts to a vector field on the zero fiber. In this case, we say that θ is *tangent* to $\pi^{-1}(0) = T_pM$. In general,

Definition 2.21. Let $S \subset M$ be a submanifold. A vector field $X \in \mathfrak{X}(M)$ is called *tangent* to S if for all $p \in S$, the vector X_p lies in $T_pS \subset T_pM$. (Thus X restricts to a vector field $X|_S \in \mathfrak{X}(S)$).

Additionally, calling for $t \neq 0$

$$j_t : \pi^{-1}(t) = M \times \{t\} \hookrightarrow \mathcal{D}(M,p) \tag{2.31}$$

and for $t = 0$

$$j = j_0 : \pi^{-1}(0) = T_pM \hookrightarrow \mathcal{D}(M,p) \tag{2.32}$$

the inclusions, we have that $-j_*(E) = \theta$ and we write

$$-E \sim_j \theta. \tag{2.33}$$

In this case, we say that θ and $-E$ are *j-related*.

(c) Given a vector field $Y \in \mathfrak{X}(M)$, we have a vector field $\hat{Y}$ on the deformation space defined via

$$\hat{Y} = \begin{cases} Y_p & \text{on } T_pM \\ tY & \text{on } M \times \mathbb{R}^\times \end{cases}, \tag{2.34}$$

where on the tangent space $\hat{Y}$ is the constant vector field $v \mapsto Y_p \in T_pM \simeq T_0(T_pM)$ for every $v \in T_pM$.
$\hat{Y}$ satisfies a set of equations similar to (2.27), namely

$$\hat{Y}(t) = 0, \quad \hat{Y}(\kappa^* f) = t\kappa^*(Y(f)), \quad \hat{Y}(\tilde{g}) = \kappa^*(Y(g)). \tag{2.35}$$

To prove this, focus on $M \times \mathbb{R}^\times$ first. If Y is a vector field on M, by the inclusion, we can see it as a vector field on $M \times \mathbb{R}^\times$ that is tangent to $M \times \{t\}$ for all t, in particular $Y(t) = 0$. Since multiplying by t contributes only to the magnitude of the field, tY will still be tangent to $M \times \{t\}$, hence

$$\hat{Y}(t) = (tY)(t) = 0.$$

Now, the function $\kappa^* f = f \circ \kappa$ takes (m,t) to $f(m)$, so

$$\hat{Y}(\kappa^* f) = (tY)(f \circ \kappa) = tY(f) = t\kappa^*(Y(f)).$$

Finally,

$$\hat{Y}(\tilde{g}) = (tY)(g/t) = t/tY(g) = \kappa^*(Y(g)).$$

On T_pM instead, $t = 0$ and $\kappa^* f = f(p)$ constant. Hence, the first two equations of (2.35) hold trivially. For the last equation, observe that if we have coordinates $x_1, \ldots, x_n$ on T_pM, calling $e_j := dx_j$, we obtain

$$\begin{aligned}
\hat{Y}(\tilde{g}) = Y_p(d_p g) &= \sum_i v_i(p) \frac{\partial}{\partial e_i}\Big|_p \Big(\sum_j \frac{\partial g}{\partial x_j}(p)\, e_j|_p \Big) \\
&= \sum_i v_i(p) \frac{\partial g}{\partial x_j}(p) \frac{\partial e_j}{\partial e_i}\Big|_p \\
&= \Big(\sum_i v_i \frac{\partial g}{\partial x_j} \Big)(p) = \big(Y(g)\big)\big|_p \\
&= \kappa^*\big(Y(g)\big),
\end{aligned}$$

where the third equality holds because $\frac{\partial g}{\partial x_j}$ does not depend on e_i.
Since the system (2.35) tells us how $\hat{Y}$ smoothly acts on the smooth functions on the deformation space, we have that $\hat{Y}$ as defined above is a smooth vector field.

(d) If $Y \in \mathfrak{X}(M)$ vanishes at p, then $\hat{Y}$ is zero along the zero fiber $\pi^{-1}(0) = T_pM$ and hence is divisible by t. Indeed, if $t \mapsto \hat{Y}(\cdot, t)$ vanishes at zero, then $\hat{Y}(\cdot, t) = tZ(\cdot, t)$ for some vector field Z on $\mathcal{D}(M, p)$, i.e. $\hat{Y}$ is divisible by t. Hence, we have a well-defined vector field $\mathcal{D}(Y) := t^{-1}\hat{Y}$ on the whole deformation space. By construction, $\mathcal{D}(Y)$ is simply $Y \times 0$ on $M \times \mathbb{R}^\times$, so it follows straightforwardly from (2.35) that on $M \times \mathbb{R}^\times$ we have

$$\mathcal{D}(Y)(t) = 0, \quad \mathcal{D}(Y)(\kappa^* f) = \kappa^*(Y(f)), \quad \mathcal{D}(Y)(\tilde{g}) = \widetilde{Y(g)}. \tag{2.36}$$

By continuity, they hold also on T_pM.
Observe that, since $Y_p = 0$, the last equation in (2.36) translates on T_pM to

$$\mathcal{D}(Y)(d_p g) = d_p(Y(g)),$$

which is the property definition of the linearization of Y. Thus,

$$\mathcal{D}(Y) = \begin{cases} \nu(Y) & \text{on } T_pM \\ Y \times 0 & \text{on } M \times \mathbb{R}^\times \end{cases}. \tag{2.37}$$

To get a better understanding of these vector fields let us study them in the frame of example 2.20.

Example 2.22 (Vector fields on the deformation space)**.** Let $M = (-1, 1)$, $p = 0$.
If we consider the standard (y, t) coordinates on $(-1, 1) \times \mathbb{R}^\times$, a natural vector field to consider in our context is $t\frac{\partial}{\partial t}$, being the Euler vector field on $\{x\} \times \mathbb{R}^\times$ for each $x \in (-1, 1)$, and similarly on $T_pM = \mathbb{R}$ we could consider $-y\frac{\partial}{\partial y}$. Now, intuitively, the way θ is formed is simply deform the coordinate y of $(-1, 1) \times \mathbb{R}^\times$ via $\frac{1}{t}y$ to get the coordinate $\tilde{y}$ and obtain the canonical vector field we defined in equation (2.26)

$$\theta = t\frac{\partial}{\partial t} - \tilde{y}\frac{\partial}{\partial \tilde{y}}.$$

Figure 2.2 shows a plot of the vector field θ on the first quadrant of the $(t, \tilde{y})$-plane. By the symmetries of θ, we can reconstruct its form on the other quadrants by simply reflecting on each axis. The length of the vectors in the plot is rescaled to avoid intersections between them. The next step is to visualize the vector fields $\hat{Y}$ and $\mathcal{D}(Y)$ for some vector field Y. So, let us first consider $Y = \frac{\partial}{\partial y}$, the constant vector field of length one on $(-1, 1)$. Then, on the zero fiber $\hat{Y} = \frac{\partial}{\partial \tilde{y}}$ as constant vector field of length one, while on $(-1, 1) \times \mathbb{R}^\times$,

$$\hat{Y} = tY = t\frac{\partial}{\partial y} = \frac{\partial}{\partial \tilde{y}}.$$

Hence, $\hat{Y}$ is the constant vector field of length one on the deformation space.
Observe that since Y does not vanish at the origin, $\mathcal{D}(Y)$ is not defined.

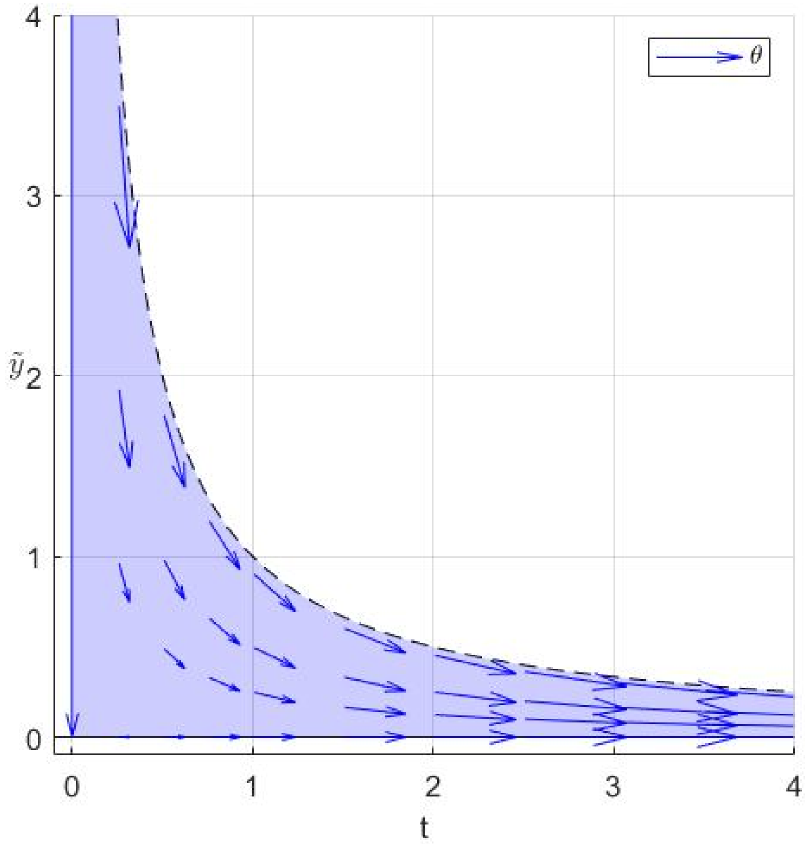


Figure 2.2: The vector field θ in the first quadrant after rescaling.

Another vector field worth considering is $Y = y\frac{\partial}{\partial y}$. In (y, t) coordinates on $(-1, 1) \times \mathbb{R}^{\times}$, $\hat{Y}$ takes the form $ty\frac{\partial}{\partial y}$, which in $(\tilde{y}, t)$ coordinates becomes

$$\hat{Y} = t(t\tilde{y})\frac{\partial \tilde{y}}{\partial y}\frac{\partial}{\partial \tilde{y}} = t\tilde{y}\frac{\partial}{\partial \tilde{y}}.$$

On the other hand, on the zero fiber $\hat{Y} = Y_0 = 0$, the zero vector field.
Now, since Y vanishes at zero, we can construct $\mathcal{D}(Y) = t^{-1}\hat{Y}$. Since

$$\nu(Y) = \nu\Big(y\frac{\partial}{\partial y}\Big) = y\frac{\partial}{\partial y},$$

which is $\tilde{y}\frac{\partial}{\partial \tilde{y}}$ in $(\tilde{y}, t)$ coordinates, by equation (2.37), we have

$$\mathcal{D}(Y) = \tilde{y}\frac{\partial}{\partial \tilde{y}}$$

on the whole deformation space.

2.3 Proof of Main Theorem

Our goal in this section is to prove the previously mentioned

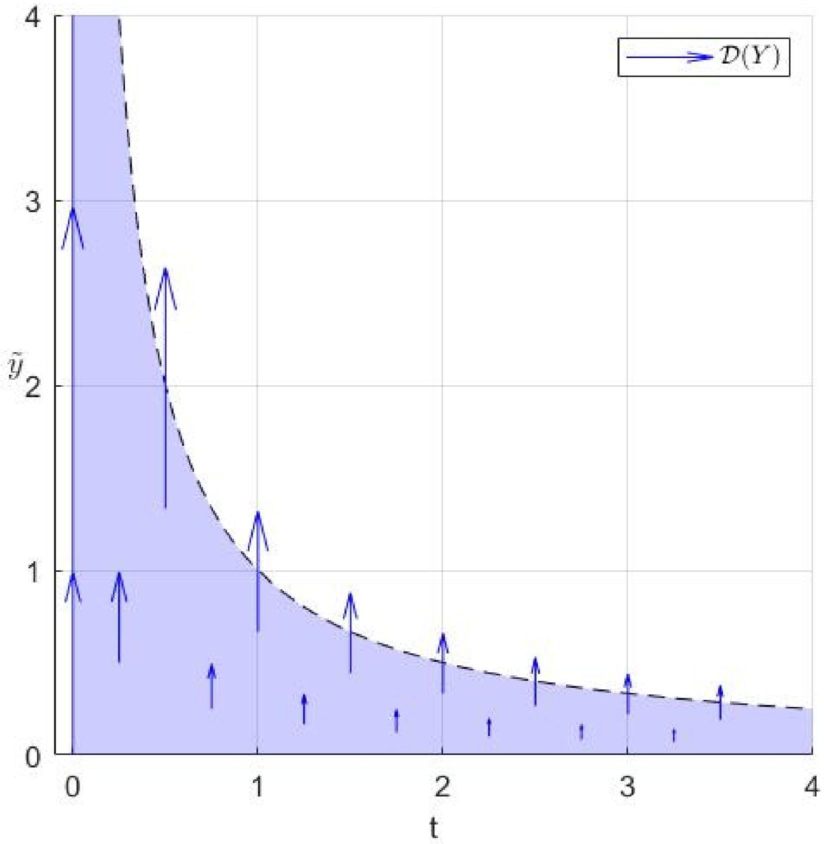


Figure 2.3: The vector field $\mathcal{D}(Y)$ in the first quadrant, when $Y = y\frac{\partial}{\partial y}$. Reflecting along each axis gives the full behavior.

Theorem 2.2. *An Euler-like vector field X for the pair (M, p) determines a unique tubular neighborhood embedding*

$$\psi : T_pM \to M$$

with $\psi_* E = X|_{\mathrm{Im}\,\psi}$.

As we said in remark 2.11, the condition of X being complete can be omitted, since we can multiply by a suitable bump function to get a complete vector field. On the other hand, if one was not to follow such a process and work only with non-complete vector fields, the result would be a tubular neighborhood embedding from an open neighborhood of the origin in T_pM to an open neighborhood of p in M.
We will need some results before getting into the proof itself.

Lemma 2.23. *If $X \in \mathfrak{X}(M)$ is Euler-like, then $\frac{\partial}{\partial t} + \frac{1}{t}X \in \mathfrak{X}(M \times \mathbb{R}^\times)$ extends to a vector field $W \in \mathfrak{X}(\mathcal{D}(M,p))$.*

Proof. If X is Euler-like, then $X_p = 0$ and we can construct $\mathcal{D}(X)$. By equation (2.37), $\mathcal{D}(X)|_{T_pM} = \nu(X) = E$ the Euler vector field on T_pM, i.e. $E \sim_j \mathcal{D}(X)$, where $j : T_pM \hookrightarrow \mathcal{D}(M,p)$ is the inclusion. Moreover, by construction $-E \sim_j \theta$.
Hence, $\theta + \mathcal{D}(X)$ vanishes on $T_pM = \pi^{-1}(0)$. This means that $\theta + \mathcal{D}(X) = tW$ for some vector field $W \in \mathfrak{X}(\mathcal{D}(M,p))$. Then, $W = \frac{1}{t}(\theta + \mathcal{D}(X))$ is the required extension because on $M \times \mathbb{R}^\times$ it restricts to $\frac{1}{t}(t\frac{\partial}{\partial t} + X \times 0) = \frac{\partial}{\partial t} + \frac{1}{t}X$. $\square$

Remark 2.24. Recall that if X is Euler-like, by equation (2.6) in local coordinates $X = \sum_i (y_i + b_i(y))\frac{\partial}{\partial y_i}$, for functions b_i vanishing to second order at $y = 0$. Then, on $M \times \mathbb{R}^\times$

$$\mathcal{D}(X) = \sum_i (t\tilde{y}_i + b_i(t\tilde{y})) \sum_j \frac{\partial \tilde{y}_i}{\partial y_i}\frac{\partial}{\partial \tilde{y}_j} = \sum_i \left(\tilde{y}_i + \frac{1}{t} b_i(t\tilde{y})\right)\frac{\partial}{\partial \tilde{y}_i}.$$

Note that for $t \to 0$ we have $\frac{1}{t}b_i(t\tilde{y}) \to 0$ because $b_i(t\tilde{y})$ vanishes to the second order at zero. So, we see that $\mathcal{D}(X)$ is defined also for $t = 0$ and equals $\nu(X) = E$.
Moreover,

$$\begin{aligned} W &= \frac{1}{t}(\theta + \mathcal{D}(X)) \\ &= \frac{1}{t}\big(t\frac{\partial}{\partial t} - \sum_i \tilde{y}_i \frac{\partial}{\partial \tilde{y}_i} + \sum_i \tilde{y}_i \frac{\partial}{\partial \tilde{y}_i} + \frac{1}{t}\sum_i b_i(t\tilde{y})\frac{\partial}{\partial \tilde{y}_i}\big) \\ &= \frac{\partial}{\partial t} + \sum_i \frac{1}{t^2} b_i(t\tilde{y})\frac{\partial}{\partial \tilde{y}_i}, \end{aligned}$$

which confirms that W is defined also for $t = 0$.

Lemma 2.25. *If X is an Euler-like vector field on M, then*

$$[\mathcal{D}(X), W] = 0 \tag{2.38}$$

on $\mathcal{D}(M, p)$.

Proof. On $M \times \mathbb{R}^\times$,

$$\begin{aligned} [\mathcal{D}(X), W] &= \left[X \times 0, \frac{1}{t}X + \frac{\partial}{\partial t}\right] \\ &= \left[X \times 0, \frac{1}{t}X\right] + \left[X \times 0, \frac{\partial}{\partial t}\right] \\ &= \frac{1}{t}[X, X] + (X \times 0)\left(\frac{1}{t}\right)X \\ &= 0, \end{aligned}$$

where in the third equality we used the fact that X independent of t, so the commutator of X and $\frac{\partial}{\partial t}$ vanishes everywhere, and the well-known formula

$$[X, fY] = f[X, Y] + X(f)Y. \tag{2.39}$$

By continuity, $[\mathcal{D}(X), W] = 0$ also on the zero fiber. □

From now on, we will denote the flow of W with φ^W.

Lemma 2.26. *For every $v \in T_pM \subset \mathcal{D}(M, p)$ the integral curve $\varphi^W_s(v)$ of W is defined for all $s \in \mathbb{R}$.*

Proof. We will show that for all $v \in T_pM$, $\varphi^W_s(v)$ is defined for all $s > 0$. The case $s < 0$ is similar and we know it is defined for $s = 0$ as $\varphi^W_0(v) = v$.
Since W and $\frac{\partial}{\partial t}$ are π-related, we have that

$$\pi \circ \varphi^W_s = \varphi^{\frac{\partial}{\partial t}}_s \circ \pi, \tag{2.40}$$

where $\varphi_s^{\frac{\partial}{\partial t}}(t) = t + s$.
Since $\pi(v) = 0$ by definition and $\pi(\varphi_s^W(v)) = \varphi_s^{\frac{\partial}{\partial t}}(\pi(v)) = \varphi_s^{\frac{\partial}{\partial t}}(0) = s \in \mathbb{R}_{>0}$ for all $s > 0$, by short time existence, we have that $\varphi_s^W(v) \in M \times \mathbb{R}_{>0}$ for all $s > 0$ and small enough. Hence, it suffices to prove that for any $(m, t) \in M \times \mathbb{R}_{>0} \subset \mathcal{D}(M, p)$, the integral curve $\varphi_s^W(m, t)$ is defined for all $s > 0$.
Now, on $M \times \mathbb{R}_{>0}$ we have $W = \frac{1}{t}X + \frac{\partial}{\partial t}$, hence

$$\varphi_s^W(m, t) = (\varphi^X_{\log(s/t+1)}(m), t + s) \tag{2.41}$$

for $-1 < s/t < +\infty$.
Indeed, on this interval,

$$\begin{aligned} W|_{(\varphi^X_{\log(s/t+1)}(m), t+s)} &= \Big(\frac{1}{t}X + \frac{\partial}{\partial t}\Big)|_{(\varphi^X_{\log(s/t+1)}(m), t+s)} \\ &= \frac{1}{t+s}X|_{\varphi^X_{\log(s/t+1)}(m)} + \frac{\partial}{\partial t}, \end{aligned}$$

and

$$\begin{aligned} \frac{d}{ds}(\varphi^X_{\log(s/t+1)}(m), t + s) &= X|_{\varphi^X_{\log(s/t+1)}(m)} \cdot \frac{d}{ds}(\log(s/t + 1)) + \frac{\partial}{\partial t} \\ &= \frac{1}{t+s}X|_{\varphi^X_{\log(s/t+1)}(m)} + \frac{\partial}{\partial t}. \end{aligned}$$

Now, since $t > 0$, φ_s^W is defined for all $s > 0$ because φ_r^X is defined over $\mathbb{R}$ (X is Euler-like, thus complete). □

We are now ready to prove the main theorem.

Proof of theorem 2.2. Existence.
Call $D_s \subset \mathcal{D}(M, p)$ the domain of the diffeomorphism φ_s^W, for fixed $s \in \mathbb{R}$. By lemma 2.26, D_s is an open neighborhood of T_pM in $\mathcal{D}(M, p)$ for all $s \in \mathbb{R}$. Hence, for any $s \neq 0$ we have a smooth map $\psi_s := \kappa \circ \varphi_s^W \circ j$

$$T_pM \xhookrightarrow{\ j\ } D_s \xrightarrow{\ \varphi_s^W\ } \mathcal{D}(M, p) \xrightarrow{\ \kappa\ } M. \tag{2.42}$$

Equivalently, ψ_s is the restriction of φ_s^W to a map from the submanifold $\pi^{-1}(0) = T_pM$ of its domain to the submanifold $\pi^{-1}(s) = M$ of the image. Thus, since φ_s^W is a diffeomorphism, ψ_s is a diffeomorphism on its image (because by the proof of lemma 2.26, $\psi_s(T_pM) \subseteq M$).
Recalling definition 2.13, we still need to prove that the derivative of ψ_s is everywhere injective (so ψ_s will be an embedding), that for some s not zero $d_0\psi_s$ is the identity map, that $\psi_s(0) = p$, and $(\psi_s)_*(E) = X$.
Let us show these results, starting with $(\psi_s)_*(E) = X$.
Observe that $[\mathcal{D}(X), W] = 0$, by lemma 2.25. Hence, the pushforward of φ_s^W preserves $\mathcal{D}(X)$, i.e. $(\varphi_s^W)_*\mathcal{D}(X) = \mathcal{D}(X)$, in the sense that

$$d_x\varphi_s^W(\mathcal{D}(X)_x) = \mathcal{D}(X)_{\varphi_s^W(x)}. \tag{2.43}$$

By consequence,

$$\begin{aligned}(\psi_s)_*(\mathcal{D}(X)|_{T_pM}) &= \kappa_* \circ (\varphi_s^W)_* \circ j_*(\mathcal{D}(X)|_{T_pM}) \\ &= \kappa_* \circ (\varphi_s^W)_*(\mathcal{D}(X)|_{T_pM\times 0}) \\ &= \kappa_*(\mathcal{D}(X)|_{\varphi_s^W(T_pM\times 0)}) \\ &= \kappa_*(X \times 0) \\ &= X,\end{aligned} \tag{2.44}$$

where we used the fact that $\varphi_s^W(T_pM \times 0) \subset M \times \mathbb{R}^\times$ for any $s \neq 0$, as shown in the proof of lemma 2.26.
But, $\mathcal{D}(X)|_{T_pM} = \nu(X) = E$. Thus,

$$(\psi_s)_*(E) = X. \tag{2.45}$$

Let now $0 \in T_pM$ be the zero vector. Then, $\varphi_s^W(0) = (p, s) \in \{p\} \times \mathbb{R}$ because W on $\{p\} \times \mathbb{R} = \kappa^{-1}(p)$ restricts to $\frac{\partial}{\partial t}$. Thus,

$$\psi_s(0) = (\kappa \circ \varphi_s^W \circ j)(0) = p. \tag{2.46}$$

Let us show that the derivative of ψ_s is the identity map for some value of s.
Let $v \in T_pM$. Then, we can find a vector field Y on M such that $Y_p = v$. But, on T_pM we have that $\hat{Y} = Y_p$ as a constant vector field, so $\hat{Y}_p = v$.
Now, similarly to the proof of lemma 2.25, we find that

$$[W, \hat{Y}] = \mathcal{D}\big(Y + [X, Y]\big). \tag{2.47}$$

But, for any smooth function f on M vanishing at p, we have

$$\begin{aligned}\big(Y + [X, Y]\big)(f)(p) &= Y(f)(p) + X(Y(f))(p) - Y(X(f))(p) \\ &= Y(f - X(f))(p) \\ &= 0,\end{aligned} \tag{2.48}$$

where in the second equality we used that by definition $X_p = 0$ and in the third that $f - X(f)$ vanishes to second order at p, by lemma 2.12.Then, by lemma 2.15, $Y + [X, Y]$ vanishes at p, hence on $\{p\} \times \mathbb{R}^\times$ the commutator of W and $\hat{Y}$ vanishes, because it coincides with $(Y + [X, Y]) \times 0$, and by continuity it does so also on $\{p\} \times \mathbb{R}$.
We claim that on $\{p\} \times \mathbb{R}$

$$(\varphi_s^W)_* \hat{Y} = \hat{Y}. \tag{2.49}$$

Indeed, by definition of Lie derivative, for any s and any vector fields X, Y, we have

$$\begin{aligned}(\varphi_s^X)_* \mathcal{L}_X Y &= (\varphi_s^X)_* \frac{d}{dt}\Big|_{t=0} (\varphi_{-t}^X)_* Y \\ &= \frac{d}{dt}\Big|_{t=0} (\varphi_{-t+s}^X)_* Y \\ &= -\frac{d}{dt}\Big|_{t=s} (\varphi_t^X)_* Y \\ &= \frac{d}{ds}\big((\varphi_{-s}^X)_* Y\big),\end{aligned} \tag{2.50}$$

where we used $t \leftrightarrow -t+s$ in the third equality and $s \leftrightarrow -s$ in the last. Hence, $\frac{d}{ds}\big((\varphi^X_{-s})_* Y\big)$ vanishes on $\{p\}\times\mathbb{R}$, i.e. $(\varphi^X_{-s})_* Y$ is constant in s on $\{p\}\times\mathbb{R}$. Observing that $(\varphi^X_0)_* Y = Y$, we have $(\varphi^X_s)_* Y = Y$ on $\{p\}\times\mathbb{R}$, proving our claim.
By consequence of equation (2.49),

$$\begin{aligned} d_{(p,0)}\varphi^W_s(v) &= d_{(p,0)}\varphi^W_s(\hat{Y}_p) \\ &\overset{(2.49)}{=} \hat{Y}_{\varphi^W_s(p,0)} = \hat{Y}_{(p,s)} \\ &= (tY)_{(p,s)} = sY_p \\ &= sv. \end{aligned} \tag{2.51}$$

Then,

$$d_0\psi_s = d_{(p,s)}\kappa \circ d_{(p,0)}\varphi^W_s \circ d_0 j = s\cdot \mathrm{id}_{T_pM}\,. \tag{2.52}$$

This tells us that for every $s \neq 0$ the derivative of ψ_s is injective, and in particular that ψ_s is an embedding for $s \neq 0$. Then, taking $s = 1$, ψ_1 is the desired tubular neighborhood embedding.

Uniqueness.
Suppose that there exist two tubular neighborhood embeddings ϕ, $\psi : T_pM \to M$ such that $\phi_* E = X = \psi_* E$.
Then, $\chi := \phi^{-1}\circ\psi : T_pM \to T_pM$ satisfies

$$\chi(0) = 0, \quad d_0\chi = \mathrm{id}_{T_pM}, \quad \chi_*(E) = E. \tag{2.53}$$

By the last equation,

$$\chi\circ\varphi^E_{-s} = \varphi^E_{-s}\circ\chi,$$

and since $\varphi^E_{-s}(v) = e^{-s}v$, we have

$$\chi(e^{-s}v) = e^{-s}\chi(v) \tag{2.54}$$

for all $v \in T_pM$.
Now, since T_pM is a vector space, we can consider a norm (any) and by the Taylor expansion of χ at 0, for any $u \in T_pM$ close enough to 0 we have

$$\chi(u) = \chi(0) + d_0\chi(u) + H(u), \tag{2.55}$$

where $H : T_pM \to T_pM$ is such that $||H(u)|| = \mathcal{O}(||u||^2)$. Hence, recalling the system (2.53) we can find a constant a such that

$$||\chi(u) - u|| \leqslant a||u||^2. \tag{2.56}$$

In particular, we can take $u = e^{-t}v$ and the last inequality will hold for every $v \in T_pM$ and $t \in \mathbb{R}$ large enough. By equation (2.54),

$$e^{-t}||\chi(v) - v|| = ||\chi(u) - u|| \leqslant a||u||^2 = ae^{-2t}||v||^2$$

and

$$||\chi(v) - v|| = \lim_{t\to\infty} ||\chi(v) - v|| \leqslant \lim_{t\to\infty} e^{-t}a||v||^2 = 0$$

for any $v \in T_pM$. This implies that $||\chi(v) - v|| = 0$ and so $\chi(v) = v$ for every $v \in T_pM$. Thus, $\phi^{-1}\circ\psi = \chi = \mathrm{id}_{T_pM}$ and $\phi = \psi$. □

Remark 2.27. We say that a vector field X on M is *linearizable* at p if there exists a tubular neighborhood embedding $\psi : T_pM \to M$ such that

$$\psi_*(\nu(X)) = X.$$

In particular, theorem 2.2 tells us that Euler-like vector fields are linearizable.

Example 2.28 (Linearizable vector fields on $\mathbb{R}$)**.** An easy application of theorem 2.2 is the following characterization:
If $M = \mathbb{R}$ and $p = 0$, a non-zero vector field

$$X = f(x)\frac{\partial}{\partial x},$$

with $f(0) = 0$, is linearizable at the origin if and only if $f'(0) \neq 0$.
Indeed, suppose X is linearizable, then there exists a smooth map $\psi : \mathbb{R} \simeq T_0\mathbb{R} \to \mathbb{R}$, with $\psi(0) = 0$, $\psi'(0) = 1$ such that

$$\psi_*(\nu(X)) = X.$$

But,

$$\nu(X) = f'(0)x\frac{\partial}{\partial x}.$$

So, if $f'(0) = 0$, then $\nu(X) = 0$ and

$$0 = \psi_*(0) = X \neq 0$$

which leads to a contradiction.
On the other hand, if $f'(0) \neq 0$, the vector field $\frac{1}{f'(0)}X$ is Euler-like for $(\mathbb{R}, 0)$, hence linearizable by theorem 2.2.

Chapter 3

Case $N \subset M$ general submanifold

In this chapter, we generalize to submanifolds the notions and the results studied previously. For this reason, the discussion will be rather fast and we will dive into detail only when major differences with the previous case occur. We will follow our main reference [BBLM20].
The purpose will be to prove

Theorem 1.1. *If $\psi : \nu(M, N) \to M$ is a tubular neighborhood embedding and E is the Euler vector field on $\nu(M, N)$, then*

$$X := \psi_* E$$

is an Euler-like vector field on the image of ψ.

Theorem 1.2 (Main Theorem)**.** *If X is an Euler-like vector field on M and E is the Euler vector field on the normal bundle, then there exists a unique tubular neighborhood embedding*

$$\psi : \nu(M, N) \to M$$

such that $\psi_* E = X|_{\mathrm{Im}\,\psi}$.

3.1 Preliminaries

In the last chapter, we defined the notion of an Euler vector field on a vector space, in particular the tangent space. We can now generalize this notion to vector bundles over a smooth manifold.

Definition 3.1. Let $V \xrightarrow{\pi} M$ be a vector bundle over a smooth manifold. The *Euler* vector field E on V is the unique vector field on V that restricts to the Euler vector field on the fibers in the sense of definition 2.3.

Observe that the uniqueness of E follows from the uniqueness of the Euler vector field on the fibers.

Proposition 3.2. *The following are equivalent*

1. *E is the Euler vector field for V*
2. *$E(f) = f$ for any linear map f on V*

3. $E = \sum_i x_i \frac{\partial}{\partial x_i}$ *on each fiber of* V *with linear coordinates* $x_1, \ldots, x_n$.

Proof. Since E is defined fiber-wise and by example 2.5 the three properties are equivalent on each fiber, we have the conclusion. □

A natural generalization of the tangent space of M to a point is the notion of *normal bundle* for a submanifold in M. Here is its definition.

Definition 3.3. Let M be a smooth manifold and $N \subset M$ a submanifold. We define the *normal bundle* of N in M (or for the pair (M, N)) to be the vector bundle

$$\nu(M, N) := TM|_N / TN \tag{3.1}$$

over N. We will denote by $i : N \hookrightarrow M$ the inclusion and by $pr : \nu(M, N) \twoheadrightarrow N$ the projection map.

$$\begin{array}{ccc} \nu(M,N) & & \\ \Big\downarrow{\scriptstyle pr} & & \\ N & \overset{i}{\hookrightarrow} & M \end{array}$$

With a small abuse of notation, in equation (3.1) we wrote $TM|_N$ instead of $TM|_{i(N)}$. Additionally, whenever the ambient space is obvious, we will denote the normal bundle with νN. With this definition at hand, from now on we will refer to the Euler vector field as being an element of $\mathfrak{X}(\nu(M, N))$.

Example 3.4. Observe that if N is a point, then $TN = 0$, because the only paths on N are the constant paths, whose derivatives are zero.
Additionally, $TM|_N = TM|_{\{p\}} = T_pM$. Then,

$$\nu N = TM|_N / TN = T_pM/0 = T_pM.$$

I.e. in the case $N = \{p\}$, the normal bundle is simply the tangent space at the point and we recover the theory explained in the previous chapter.

Remark 3.5. Recall that if $f : (M, N) \to (M', N')$ is smooth (i.e. $f : M \to M'$ is smooth and such that $f(N) \subset N'$), then we have an induced map, its derivative, on the tangent spaces $f_* : TM \to TM'$ sending TN to TN'. Thus, we have a unique linear map $\nu(f)$ on the quotients defined by $\nu(f)(v + TN) = v(f) + TN'$, and making the following diagram with exact rows commute

$$\begin{array}{ccccccccc} 0 & \longrightarrow & TN & \xrightarrow{i_*} & TM|_N & \xrightarrow{p} & \nu N & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle (f|_N)_*} & & \Big\downarrow{\scriptstyle f_*|_N} & & \Big\downarrow{\scriptstyle \nu(f)} & & \\ 0 & \longrightarrow & TN' & \xrightarrow{(i')_*} & TM'|_{N'} & \xrightarrow{p'} & \nu N' & \longrightarrow & 0 \end{array}$$

where p and p' are the respective projections on the quotients.

Definition 3.6. With the notation above, we will call $\nu(f)$ the *linearization* of the map f.

In particular, if $(M', N') = (\mathbb{R}, 0)$ and $f \in C^\infty(M)$ such that $f|_N = 0$, then

$$\nu(f) : \nu N \to \mathbb{R}, \; v + TN \mapsto v(f). \tag{3.2}$$

Example 3.7. If $N = \{p\}$ and $f : M \to \mathbb{R}$ vanishes at p, then $\nu(f)$ is the usual derivative of f at p.

Remark 3.8. $\nu(M, N)$ is a quotient of the vector bundle $TM|_N$, on which the linear functions are all and only the derivatives $f_*|_N$ of some smooth function f over M vanishing on N. Hence, all and the only linear functions on the normal bundle are linearizations of functions over M vanishing at N. In particular, by proposition 3.2, the Euler vector field E is defined by the property

$$E(\nu(f)) = \nu(f) \tag{3.3}$$

for all f as above.

Definition 3.9. Let $X \in \mathfrak{X}(M)$ be tangent to N. We define the *linearization* of X to be the unique vector field $\nu(X) \in \mathfrak{X}(\nu N)$ acting on linear functions over the normal bundle via

$$\nu(X)(\nu(f)) = \nu(X(f)). \tag{3.4}$$

Definition 3.10. A vector field $X \in \mathfrak{X}(M)$ is *Euler-like* for the pair (M, N) if it is complete, $X|_N = 0$, and $\nu(X) = E$.

Remark 3.11. Observe that lemma 2.12 can be generalized to the following characterization in the submanifold case: X is Euler-like if and only if

$$X(f) - f$$

vanishes to the second order on N for any smooth $f : M \to \mathbb{R}$ that vanish on N.
Indeed, if X is Euler-like, then

$$(X(f) - f)|_N = X|_N(f) - f|_N = 0$$

and

$$\begin{aligned} \nu(X(f) - f) &= \nu(X(f)) - \nu(f) \\ &\overset{(3.4)}{=} \nu(X)(\nu(f)) - \nu(f) \\ &= E(\nu(f)) - \nu(f) \\ &\overset{(3.3)}{=} 0. \end{aligned}$$

Conversely, assume that $X(f) - f$ vanishes to second order on N for any f such that $f|_N = 0$. Then,

$$0 = (X(f) - f)|_N = X|_N(f).$$

So, by lemma 2.15, $X|_N = 0$. Moreover,

$$0 = \nu(X(f) - f) = \nu(X)(\nu(f)) - \nu(f),$$

showing that $\nu(X)$ acts on linear functions the same way that E does. Thus, $\nu(X) = E$.

Example 3.12 (Euler-like vector fields in local coordinates)**.** Consider $M = \mathbb{R}^2$ with the standard x, y coordinates and $N = \mathbb{R} = \{x \in \mathbb{R}\}$.
Then,

$$TM|_N/TN = \bigsqcup_{r\in\mathbb{R}} (T_r\mathbb{R}^2/T_r\mathbb{R}) \simeq \bigsqcup_{r\in\mathbb{R}} (\mathbb{R}^2/\mathbb{R}) \simeq \mathbb{R} \times \mathbb{R}. \tag{3.5}$$

For any $x \in \mathbb{R}$ the Euler vector field on the fiber $pr^{-1}(x) = \{(x, y) : y \in \mathbb{R}\} \simeq \mathbb{R}$ takes form $E_x = y\frac{\partial}{\partial y}$, independently on x. Hence, the Euler vector field on the normal bundle $\mathbb{R} \times \mathbb{R}$ is

$$E = y\frac{\partial}{\partial y},$$

which is significantly different from the Euler vector field $E = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ on $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ as a vector space! Special attention is, thus, needed when working with these objects.
Then, an Euler-like vector field for the pair $(\mathbb{R}^2, \mathbb{R})$ as above takes the form

$$X = a(x, y)\frac{\partial}{\partial x} + (y + b(x, y))\frac{\partial}{\partial y}, \tag{3.6}$$

with $a, b \in C^\infty(\mathbb{R}^2)$ vanishing for $y = 0$ and $\frac{\partial b}{\partial y}(x, 0) = 0$.
Analogously, for $M = \mathbb{R}^m$, $N = \mathbb{R}^n$ with $n < m$, and standard coordinates $x_1, \ldots, x_n, y_1, \ldots, y_{m-n}$ such that $x_1, \ldots, x_n$ are the standard coordinates on $\mathbb{R}^n$ and $y_i|_{\mathbb{R}^n} = 0$ for every $i = 1, \ldots, m - n$, an Euler-like vector field for (M, N) takes form

$$X = \sum_{i=1}^{n} a_i(x, y)\frac{\partial}{\partial x_i} + \sum_{j=1}^{m-n} (y_j + b_j(x, y))\frac{\partial}{\partial y_j}, \tag{3.7}$$

with functions a_i, b_j respectively satisfying the same conditions for a, b above for every i, j.
This extends to a general manifold M and submanifold N with local coordinates $x_1, \ldots, x_n, y_1, \ldots, y_{m-n}$ such that $x_1, \ldots, x_n$ restricts to local coordinates on N and $y_i|_N = 0$. In this setting, an Euler-like vector field will again take the form (3.7).

Definition 3.13. A *tubular neighborhood embedding* for the pair (M, N) is an embedding

$$\psi : \nu N \to M$$

that restricts to the identity on N, seen as the zero section of the normal bundle, and such that the induced map $\nu(\psi)$ is the natural identification

$$\nu(\nu(M, N), N) \simeq \nu(M, N).$$

Let us unroll this definition. Viewing $N \subset \nu(M, N)$ as the zero section, the first condition states to

$$\psi|_N = \mathrm{id}_N,$$

or, equivalently, to $\psi(0_p) = p$ for every $p \in N$, where 0_p is the zero vector of the fiber $\nu_p N = pr^{-1}(p)$. The second condition, instead, says that after canonically identifying $\nu(\nu(M, N), N) \simeq \nu(M, N)$[1], the linearization (i.e. the induced map on $\nu(M, N)$) of ψ is the identity on the normal bundle.

[1] In general, for any vector bundle $E \to N$, we have a natural isomorphism $\nu(E, N) \simeq E$

Remark 3.14. In the case $M = S^2$ and $N = S^1$ seen as the equator, the normal bundle will be $S^1 \times \mathbb{R}$, i.e. a "tube", or cylinder (see picture 3.1). In general, the normal bundle of a submanifold N in M will, intuitively, be a "tube" of base N. In this context, one usually refers to a *tubular neighborhood* of N in M to indicate an open neighborhood of N in M that "locally looks like a tube", i.e. that is diffeomorphic to (an open neighborhood of the zero section of) the normal bundle, where this diffeomorphism restricts to the identity on the zero section and induces the identity on the normal bundle. In particular, the image of a tubular neighborhood embedding is a tubular neighborhood, explaining the name.

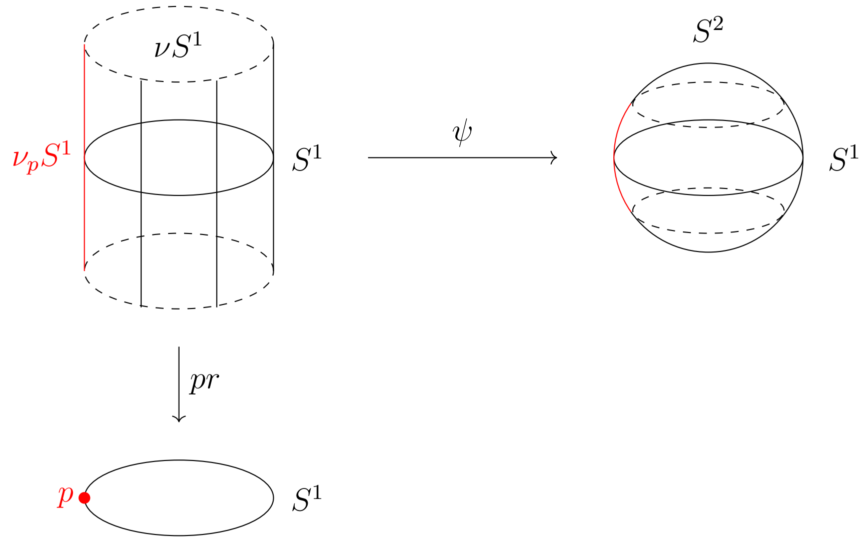


Figure 3.1: Qualitative example of a tubular neighborhood embedding ψ for $M = S^2$ and $N = S^1$. The open neighborhood of $S^1 \subset \nu S^2$ delimited by the dashed circles is diffeomorphically mapped to the tubular neighborhood of $S^1 \subset S^2$ delimited by the dashed circles. In red are shown the fiber $pr^{-1}(p) = \nu_p S^1$ at $p \in N$ and its image under ψ.

With these concepts at hand, we can now prove theorem 1.1.

Proof of theorem 1.1. The proof follows essentially the same logic of theorem 2.1 and proposition 2.16.
For every $p \in N$ we have

$$X_p = (\psi_* E)_p = d_{\psi^{-1}(p)}(E_{\psi^{-1}(p)}) = d_{0_p}\psi(E_{0_p}) = 0,$$

where the last equality holds because the Euler vector field on the fiber at p, as shown in the previous chapter, vanishes at zero.
Moreover, observe that

$$\nu(\psi_* E) = \nu(\psi)_*(\nu(E)) \tag{3.8}$$

for any smooth map ψ and vector field E. This is proven analogously to equation (2.15) because the linearization of a map is simply its derivative modulo TN, as in equation (3.2). Hence,

$$\nu(X) = \nu(\psi_* E) = \nu(\psi)_*(\nu(E)) = E.$$

□

3.2 Deformation Space

In the previous chapter, we saw that the deformation space is a construction that "enhances", for $t \to 0$, the tangent directions to a point in a manifold. We also saw in example 3.4, that the normal bundle for a point in a manifold is exactly the tangent space to this point. Hence, we can rephrase the intuition behind the deformation space by saying that it is a construction that "enhances" the normal directions to a given point. We will see now how this notion generalizes to general submanifolds.

Definition 3.15. Let $N \subset M$ be a submanifold. The *deformation space* for the pair (M, N) is the set[2]

$$\mathcal{D}(M,N) = (\nu(M,N) \times 0) \sqcup (M \times \mathbb{R}^\times),$$

equipped with the unique manifold structure determined by the following conditions:

1. the map
$$\pi : \mathcal{D}(M,N) \to \mathbb{R}, \qquad \begin{cases} v + TN \mapsto 0 \\ (m,t) \mapsto t \end{cases} \tag{3.9}$$
is a smooth submersion,

2. the map
$$\kappa : \mathcal{D}(M,N) \to M, \qquad \begin{cases} v + TN \mapsto i \circ pr(v + TN) \\ (m,t) \mapsto m \end{cases} \tag{3.10}$$
is smooth,

3. for any $f \in C^\infty(M)$ vanishing on N, the map
$$\tilde{f} : \mathcal{D}(M,N) \to \mathbb{R}, \qquad \begin{cases} v + TN \mapsto v(f) \\ (m,t) \mapsto \frac{1}{t} f(m) \end{cases} \tag{3.11}$$
is smooth.

Note that the restriction of $\tilde{f}$ to the zero fiber $\pi^{-1}(0) = \nu N$ is the linearization $\nu(f)$. We will denote again π as an element of $C^\infty(\mathcal{D}(M,N))$ by its image t.

Lemma 3.16. *Let $x_1, \dots, x_n, y_1, \dots, y_{m-n}$ be a set of local coordinates*[3] *on M such that the x_i's restrict to local coordinates on N and $y_j|_N = 0$, for every j*[4]*. Then,*

$$\kappa^* x_1, \dots, \kappa^* x_n, \tilde{y}_1, \dots \tilde{y}_{m-n}, t \tag{3.12}$$

is a set of local coordinates on $\mathcal{D}(M,N)$.

[2] We use the following convention for points in the deformation space: $v + TN \in \nu N$, $(m,t) \in M \times \mathbb{R}^\times$, $x \in \mathcal{D}(M,N)$, where $\mathbb{R}^\times := \mathbb{R}\backslash\{0\}$.

[3] In the sense that this set defines a homeomorphism from an open subset U of M to an open subset of $\mathbb{R}^n$.

[4] In the literature, such coordinates on M are called *adapted* to N.

Proof. The proof is similar to the case $N = \{p\}$ (lemma 2.18).
Let $x_1, \dots, x_n, y_1, \dots, y_{m-n}$ be defined on an open $U \subset M$. Then, the set $\mathcal{D}(U, N \cap U) = \kappa^{-1}(U)$ is open because κ is continuous, and the map

$$\phi : \mathcal{D}(U, N \cap U) \to \mathbb{R}^n \times \mathbb{R}^{m-n} \times \mathbb{R}, \tag{3.13}$$

defined by (3.12), is smooth and bijective on its image. Now, viewing $U \times \mathbb{R}^\times$ as an open in $\mathbb{R}^n \times \mathbb{R}^{m-n} \times \mathbb{R}^\times$ via the coordinates on M, and $\nu(U, N \cap U)$ as $(N \cap U) \times \mathbb{R}^{m-n}$ via the local trivialization of the normal bundle, the map

$$(u, v, t) \in \mathbb{R}^n \times \mathbb{R}^{m-n} \times \mathbb{R}^\times \mapsto \begin{cases} (u, v, 0) \in (N \cap U) \times \mathbb{R}^{m-n} \times 0 \\ (u, tv, t) \in \mathbb{R}^n \times \mathbb{R}^{m-n} \times \mathbb{R}^\times \end{cases} \tag{3.14}$$

is the continuous inverse of ϕ.
The smoothness of the transition functions is proven analogously to the $N = \textit{point}$ case (see proposition 2.19). □

The algebra of smooth functions on the deformation space is generated by t (seen as the function π), and by functions of the form $\kappa^* f$ and $\tilde{g}$ for $f, g \in C^\infty(M)$ such that $g|_N = 0$. This follows by the previous lemma because, if $F \in C^\infty(\mathcal{D}(M, N))$ and $\kappa^* x_1, \dots, \kappa^* x_n, \tilde{y}_1, \dots \tilde{y}_{m-n}, t$ is a set of local coordinates for $\mathcal{D}(M, N)$, we can see F as a composition of such maps with a function $\mathbb{R}^{n+1} \to \mathbb{R}$.
Hence, to define a vector field on the deformation space is sufficient to define its action on functions $t, \kappa^* f, \tilde{g}$ as above.
With abuse of notation, we will denote the coordinate $\kappa^* x_i$ by x_i, and the inclusions of the fibers $\pi^{-1}(t)$ in $\mathcal{D}(M, N)$ by j_t, with $j := j_0$.
The following properties are proven in the same way as in the previous chapter. For any smooth functions f, g on M with $g|_N = 0$:

(a) For any morphism $\varphi : (M, N) \to (M', N')$, we can extend the map $\varphi \times \mathrm{id}_{\mathbb{R}^\times}$ to a map

$$\mathcal{D}(\varphi) : \mathcal{D}(M, N) \to \mathcal{D}(M', N')$$

that restricts to $\nu(\varphi)$ on $\pi^{-1}(0) = \nu N$.

(b) We have a canonical vector field $\theta \in \mathfrak{X}(\mathcal{D}(M, N))$ satisfying

$$\theta(t) = t, \quad \theta(\kappa^* f) = 0, \quad \theta(\tilde{g}) = -\tilde{g}. \tag{3.15}$$

In local $x_1, \dots, x_n, \tilde{y}_1, \dots, \tilde{y}_{m-n}, t$ coordinates,

$$\theta = t\frac{\partial}{\partial t} - \sum_j \tilde{y}_j \frac{\partial}{\partial \tilde{y}_j}, \tag{3.16}$$

from which we see that θ is tangent to the zero fiber $\pi^{-1}(0) = \nu N$, where

$$-E \sim_j \theta. \tag{3.17}$$

(c) For any vector field Y on M, we can define a vector field $\hat{Y} \in \mathfrak{X}(\mathcal{D}(M,N))$ by

$$\hat{Y} = \begin{cases} Y|_N + TN & \text{on } \nu N \\ tY & \text{on } M \times \mathbb{R}^\times \end{cases}, \tag{3.18}$$

where $\hat{Y}$ is seen on νN as the fiber-wise constant vector field of value $Y|_N + TN$. $\hat{Y}$ satisfies

$$\hat{Y}(t) = 0, \quad \hat{Y}(\kappa^* f) = t\kappa^*(Y(f)), \quad \hat{Y}(\tilde{g}) = \kappa^*(Y(g)). \tag{3.19}$$

(d) If $Y \in \mathfrak{X}(M)$ is tangent to N (in the sense of definition 2.21), then $Y|_N + TN = 0_{\nu N}$, i.e. $\hat{Y}$ vanishes on $\pi^{-1}(0) = \nu N$. Thus, $\hat{Y}$ is divisible by t and we can define a vector field $\mathcal{D}(Y)$ on $\mathcal{D}(M,N)$ by $\mathcal{D}(Y) = t^{-1}\hat{Y}$, which by the system (3.19) satisfies

$$\mathcal{D}(Y)(t) = 0, \quad \mathcal{D}(Y)(\kappa^* f) = \kappa^*(Y(f)), \quad \mathcal{D}(Y)(\tilde{g}) = \widetilde{Y(g)}. \tag{3.20}$$

Recalling that $\tilde{g}|_{\pi^{-1}(0)} = \nu(g)$, by the last equation of (3.20), we have that $\mathcal{D}(Y)|_{\pi^{-1}(0)} = \nu(Y)$. Hence,

$$\mathcal{D}(Y) = \begin{cases} \nu(Y) & \text{on } \nu N \\ Y \times 0 & \text{on } M \times \mathbb{R}^\times \end{cases}. \tag{3.21}$$

Remark 3.17. Consider a set of local coordinates for (M,N) as in lemma 3.16. A general vector field on M takes local form

$$Y = \sum_i a_i(x,y)\frac{\partial}{\partial x_i} + \sum_j b_j(x,y)\frac{\partial}{\partial y_j}.$$

Then, on $M \times \mathbb{R}^\times$ with coordinates x_i, $\tilde{y}_j := \frac{1}{t}y_j$, t, we have that

$$Y \times 0 = \sum_i a_i(x,t\tilde{y})\frac{\partial}{\partial x_i} + \frac{1}{t}\sum_j b_j(x,t\tilde{y})\frac{\partial}{\partial \tilde{y}_j}.$$

So, we see that $tY = t(Y \times 0)$ can be extended to a vector field $\hat{Y}$ on the whole deformation space, while $Y \times 0$ can be extended to a vector field $\mathcal{D}(Y)$ if and only if $b_j(x,0) = 0$ for every j. Since N is defined by the vanishing of the $y-$coordinates, this is equivalent to asking that

$$Y|_N = \sum_i a_i(x,0)\frac{\partial}{\partial x_i},$$

i.e. that Y be tangent to N.

3.3 Proof of Main Theorem

As stated in the introduction, the goal of this section is to prove the main theorem 1.2. To provide some context, let us report the statement here.

Theorem 1.2. *If X is an Euler-like vector field on M and E is the Euler vector field on the normal bundle, then there exists a unique tubular neighborhood embedding*

$$\psi : \nu(M,N) \to M$$

such that $\psi_* E = X|_{\mathrm{Im}\,\psi}$.

Remark 3.18. Analogously to remark 2.27, we say that a vector field X on M is *linearizable* around a submanifold N if there exists a tubular neighborhood embedding $\psi : \nu(M,N) \to M$ such that

$$\psi_*(\nu(X)) = X,$$

In particular, the previous theorem states that Euler-like vector fields for (M,N) are linearizable around N.

We will need several preparatory results for the actual proof of the theorem. One of the main tools will be a particular vector field W. The following lemma shows its construction and is proven in the same way as lemma 2.23, so the proof will be omitted.

Lemma 3.19. *If $X \in \mathfrak{X}(M)$ is Euler-like, the vector field $\frac{1}{t}X + \frac{\partial}{\partial t}$ on $M \times \mathbb{R}^\times$ can be extended to the vector field*

$$W := \frac{1}{t}(\mathcal{D}(X) + \theta)$$

on $\mathcal{D}(M,N)$.

Observe that, given a general vector field X on M, taking into account the three properties in definition 3.15, $\frac{1}{t}X + \frac{\partial}{\partial t}$ is a natural vector field to consider on $M \times \mathbb{R}^\times$.

Remark 3.20. Consider the usual local coordinates on the pair (M,N) and the corresponding ones on $\mathcal{D}(M,N)$ as in lemma 3.16. Then, we know that

$$\theta = t\frac{\partial}{\partial t} - \sum_j \tilde{y}_j \frac{\partial}{\partial \tilde{y}_j}$$
$$X = \sum_i a_i(x,y)\frac{\partial}{\partial x_i} + \sum_j \big(y_j + b_j(x,y)\big)\frac{\partial}{\partial y_j},$$

with a_i, b_j smooth functions on M vanishing at $y = 0$, such that $\frac{\partial b_j}{\partial y_k}(x,0) = 0$ for any i,j,k. Since $X|_N = 0$, we can construct $\mathcal{D}(X)$, which on $M \times \mathbb{R}^\times$ takes form

$$\mathcal{D}(X) = \sum_i a_i(x,t\tilde{y})\frac{\partial}{\partial x_i} + \sum_j \big(\tilde{y}_j + \frac{1}{t}b_j(x,t\tilde{y})\big)\frac{\partial}{\partial \tilde{y}_j}. \tag{3.22}$$

Then,

$$\frac{1}{t}\big(\mathcal{D}(X) + \theta\big) = \frac{\partial}{\partial t} + \frac{1}{t}\sum_i a_i(x,t\tilde{y})\frac{\partial}{\partial x_i} + \frac{1}{t^2}\sum_j b_j(x,t\tilde{y})\frac{\partial}{\partial \tilde{y}_j}. \tag{3.23}$$

Since $a_i(x,0) = 0$ for every i, the limit for $t \to 0$ of $a_i(x,t\tilde{y})/t$ is well-defined, and similarly is that of $b_j(x,t\tilde{y})/t^2$, because the functions b_j vanish to the second order for $t = 0$. This shows in a coordinate-dependent way that $\frac{1}{t}X + \frac{\partial}{\partial t}$ extends to the whole deformation space.

Lemma 3.21. *For any vector field $Y \in \mathfrak{X}(M)$ and any Euler-like vector field X, the vector field $Y + [X,Y]$ is tangent to N, and*

$$[W, \hat{Y}] = \mathcal{D}(Y + [X,Y]). \tag{3.24}$$

Moreover, if Y is tangent to N,

$$[W, \mathcal{D}(Y)] = \frac{1}{t}\mathcal{D}([X,Y]). \tag{3.25}$$

Before starting the proof of lemma 3.21, recall the following

Proposition 3.22 ([Lee13], Proposition 8.22). *A vector field Z on M is tangent to N if and only if $X(f)|_N = 0$ for every smooth function f over M vanishing on N.*

Proof of lemma 3.21. We want to use the last proposition to show that $Y + [X, Y]$ is tangent to N. Take $f \in C^\infty(M)$ such that $f|_N = 0$. Then,

$$\begin{aligned}(Y + [X, Y])(f)|_N &= Y(f)|_N + (X \circ Y)(f)|_N - (Y \circ X)(f)|_N \\ &= Y(f - X(f))|_N.\end{aligned}$$

because $X|_N = 0$. Seeing Y as a derivation, since $X(f) - f$ vanishes to second order by remark 3.11, $Y(f - X(f))|_N = 0$. Hence, $Y + [X, Y]$ is tangent to N, by proposition 3.22.
To prove equation (3.24), recall the well-known formulas

$$\begin{aligned}[X, fY] &= f[X, Y] + X(f)Y \\ [fX, Y] &= f[X, Y] - Y(f)X.\end{aligned} \tag{3.26}$$

Then, on $M \times \mathbb{R}^\times$ we have

$$\begin{aligned}[W, \hat{Y}] &= \Big[\frac{1}{t}X + \frac{\partial}{\partial t}, tY\Big] \\ &= t\Big[\frac{1}{t}X, Y\Big] + \Big(\frac{1}{t}X\Big)(t)Y + t\Big[\frac{\partial}{\partial t}, Y\Big] + \frac{\partial t}{\partial t}Y \\ &= t\Big[\frac{1}{t}X, Y\Big] + Y \\ &= t\frac{1}{t}[X, Y] - tY(1/t)X + Y \\ &= [X, Y] + Y,\end{aligned}$$

where in the third and fourth equality we used the fact X and Y are independent of t, so when applied to functions of t they vanish. For the same reason, the Lie derivative of Y in the direction of $\frac{\partial}{\partial t}$ is zero, i.e. their commutator vanishes.
Since all the vector fields above can be extended to the whole deformation space and they agree on an open dense subset ($M \times \mathbb{R}^\times$), by continuity, equation (3.24) holds on the whole deformation space.
Equation (3.25) follows from equation (3.24). By continuity, it is again sufficient to prove it only on $M \times \mathbb{R}^\times$.

$$\begin{aligned}[W, \mathcal{D}(Y)] &= \Big[W, \frac{1}{t}\hat{Y}\Big] = W\Big(\frac{1}{t}\Big)\hat{Y} + \frac{1}{t}[W, \hat{Y}] \\ &\overset{(3.24)}{=} \Big(\frac{\partial}{\partial t} + \frac{1}{t}X\Big)\Big(\frac{1}{t}\Big)tY + \frac{1}{t}\mathcal{D}(Y + [X, Y]) \\ &= -\frac{1}{t^2}tY + \frac{1}{t}X\Big(\frac{1}{t}\Big)tY + \frac{1}{t}\big(Y + [X, Y]\big) \\ &= -\frac{1}{t}Y + \frac{1}{t}Y + \frac{1}{t}[X, Y] \\ &= \frac{1}{t}[X, Y] \\ &= \frac{1}{t}\mathcal{D}([X, Y]).\end{aligned} \tag{3.27}$$

This concludes the proof. ☐

The proof of the following result is similar to the previous case, so it will be omitted.

Lemma 3.23. *Denote by φ^W the flow of W. Then, for any $v + TN \in \pi^{-1}(0) = \nu N$, the integral curve $\varphi^W_s(v + TN)$ is defined for all $s \in \mathbb{R}$.*

Let us now discuss the proof of our main theorem in the most general case. It will follow the logic of the previous case with some adjustments.

Proof of theorem 1.2. For any $s \in \mathbb{R}$, call $D_s \subset \mathcal{D}(M, N)$ the domain of φ^W_s, which by lemma 3.23 is an open neighborhood of νN in $\mathcal{D}(M, N)$. Then, for any $s \neq 0$, we have a well-defined smooth map $\psi_s := \kappa \circ \varphi^W_s \circ j$

$$\nu(M, N) \xhookrightarrow{\ j\ } D_s \xrightarrow{\ \varphi^W_s\ } \mathcal{D}(M, N) \xrightarrow{\ \kappa\ } M, \tag{3.28}$$

that is a diffeomorphism on its image.
By equation (3.25), $[W, \mathcal{D}(X)] = \frac{1}{t}\mathcal{D}([X, X]) = 0$. Thus, $(\varphi^W_s)_*\mathcal{D}(X) = \mathcal{D}(X)$ and consequently, as in equation (2.44),

$$(\psi_s)_*E = X.$$

To conclude the proof, we still have to show that ψ restricts to the identity on N and its linearization is the identity on the normal bundle.
For the first claim, since $W|_{N\times\mathbb{R}} = \frac{\partial}{\partial t}$, we have $\varphi^W_s|_{N\times\mathbb{R}} = \varphi^{\frac{\partial}{\partial t}}_s|_{N\times\mathbb{R}}$, and

$$\begin{aligned} \psi_s|_N &= \kappa \circ \varphi^W_s \circ j|_N \\ &= \kappa \circ \varphi^W_s|_{N\times\mathbb{R}} \\ &= \kappa \circ \varphi^{\frac{\partial}{\partial t}}_s|_{N\times\mathbb{R}} \\ &= \kappa|_{N\times\mathbb{R}} \\ &= \mathrm{id}_N, \end{aligned}$$

where the second to last equality holds because the flow of the translation vector field leaves N unchanged.
We are left to show that $\nu(\psi_s) = \mathrm{id}_{\nu N}$ for some $s \in \mathbb{R}$. So, take a general vector $v + TN$ on the fiber by a point $q \in N$. We can find a vector field Y on M such that

$$\hat{Y}_{(q,0)} = Y_q + TN = v + TN.$$

Note that, since by lemma 3.21 $(Y + [X, Y]) \times 0$ is tangent to $N \times \{t\}$ for all $t \neq 0$, we have that $\mathcal{D}(Y + [X, Y])$ is tangent to $N \times \{t\}$ for all t by continuity. Hence, by

$$\mathcal{L}_W\hat{Y} = [W, \hat{Y}] \overset{(3.24)}{=} \mathcal{D}(Y + [X, Y]), \tag{3.29}$$

we have that

$$(\varphi^W_s)_*\hat{Y} - \hat{Y} = \int_0^s \frac{d}{dr}((\varphi^W_r)_*\hat{Y})\, dr \overset{(2.50)}{=} \int_0^s (\varphi^W_{-r})_*(\mathcal{L}_W\hat{Y})\, dr \tag{3.30}$$

is tangent to $N \times \{t\}$ for all t.
This means that the flow of W preserves the vector field $\hat{Y}$ up to a vector field tangent

to $N \times \{t\}$, say U.
But then,

$$\begin{aligned} \big((\varphi_s^W)_* \hat{Y}\big)_{(q,0)} &= \hat{Y}_{\varphi_s^W(q,0)} + U_{\varphi_s^W(q,0)} \\ &= \hat{Y}_{(q,s)} + U_{(q,s)} \\ &= sY_q + U_{(q,s)}. \end{aligned} \tag{3.31}$$

On the other hand,

$$\big((\varphi_s^W)_* \hat{Y}\big)_{(q,0)} = d_{(q,0)}\varphi_s^W(\hat{Y}_{(q,0)}) = d_{(q,0)}\varphi_s^W(v + TN). \tag{3.32}$$

Combining equations (3.31) and (3.32) modulo TN, we then have

$$\nu(\varphi_s^W)(v + TN) = s(v + TN), \tag{3.33}$$

Thus,

$$\begin{aligned} \nu(\psi_s)(v + TN) &= \nu(\kappa) \circ \nu(\varphi_s^W) \circ \nu(j)(v + TN) \\ &= \nu(\kappa) \circ \nu(\varphi_s^W)(v + TN) \\ &= \nu(\kappa)\big(s(v + TN)\big) \\ &= s(v + TN). \end{aligned} \tag{3.34}$$

Then, ψ_s is an embedding, and $\nu(\psi_s) = \mathrm{id}_{\nu N}$ if and only if $s = 1$. So, ψ_1 is the desired tubular neighborhood embedding.
The uniqueness is analogous to the proof of theorem 2.2. □

Chapter 4

Applications

In this chapter, we will discuss some applications of our main theorem, both in the point-submanifold and general cases. In primis, we will prove Morse lemma and Darboux theorem via theorem 2.2 (see [Mei21], section 2.2). In the second and third sections, we will show the generalizations of these two results to submanifolds aided by theorem 1.2 (see [Mei21], sections 4.1-2). Finally, in the last section, we will present and prove a splitting theorem for singular foliations (see [BBLM20], section 2.4).

4.1 Morse Lemma and Darboux Theorem

We saw that if E is the Euler vector field on a vector space V, then $E(f) = f$ for any linear function f on V. There is a similar property for k-homogeneous polynomials, namely

Lemma 4.1. *If $E \in \mathfrak{X}(V)$ is the Euler vector field on V and f is a homogeneous polynomial on V of degree k, then*

$$E(f) = kf.$$

Proof. Without loss of generality by the linearity of $f \mapsto E(f)$, consider $f = f_1 \cdots f_k$, where all f_i are linear maps of V. Then,

$$E(f) = E(f_1)f_2 \cdots f_k + \ldots + f_1 \cdots f_{k-1}E(f_k) = kf_1 \cdots f_k.$$

□

The last fact turns out to be useful in proving our first application of theorem 2.2; the Morse lemma, a normal form result for *Morse functions*.
Recall that if f is a smooth function on a manifold M and $p \in M$ is such that $d_pf = 0$, then we can define the *Hessian*[1] of f at p to be the symmetric bilinear form $\operatorname{Hess} f(p)$ on T_pM defined by

$$\operatorname{Hess} f(p)(u, v) := X(Y(f))(p), \tag{4.1}$$

for any pair of vector fields X, Y on M such that $X_p = u$, $Y_p = v \in T_pM$.
Observe that this definition does not depend on the choice of the two vector fields, and the symmetricity follows from $d_pf = 0$, indeed

$$X(Y(f))(p) - Y(X(f))(p) = [X, Y](f)(p) = d_pf([X, Y]_p) = 0,$$

for any vector fields X, Y on M.

[1] See for instance [Pla13]

Definition 4.2. A smooth function f on a manifold M is called *Morse* if all its critical points are non-degenerate, i.e. for all $p \in M$ such that $d_p f = 0$ we have that $\operatorname{Hess} f(p)$ is non-degenerate.

Since the Morse lemma is a local condition around a critical point, we can reduce to the case $M = \mathbb{R}^n$ and $p = 0$, with $f(0) = 0$.

Lemma 4.3 (Morse)**.** *Let $f \in C^\infty(\mathbb{R}^n)$ be a Morse function having a critical point at the origin, with $f(0) = 0$. Then, there exists a tubular neighborhood embedding ψ defined on a neighborhood of the origin such that*

$$\psi^* f(x) = \sum_{i=1}^{n} \pm x_i^2. \tag{4.2}$$

.

Proof. By expanding f at the origin, we have

$$f(x) = f(0) + \sum_i \frac{\partial f}{\partial x_i}(0) x_i + \frac{1}{2} \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(0) x_i x_j + R(x),$$

with $R(x)$ vanishing to third order. Since $f(0) = 0$ and $\frac{\partial f}{\partial x_i}(0) = 0$ for all i, we can rewrite f as

$$f(x) = \frac{1}{2} \sum_{i,j} H_{ij}(x) x_i x_j, \tag{4.3}$$

where $x \mapsto H(x)$ is a smooth matrix-valued function with H symmetric and $H(0) = \operatorname{Hess} f(0)$.
The derivative of f then satisfies

$$\frac{\partial f}{\partial x_j}(x) = \sum_k G_{jk}(x) x_k, \tag{4.4}$$

with

$$G_{jk}(x) = H_{jk}(x) + \frac{1}{2} \sum_l \frac{\partial H_{kl}}{\partial x_j}(x) x_l. \tag{4.5}$$

Since $G(0) = H(0) = \operatorname{Hess} f(0)$, H and G are non-degenerate (and hence invertible) in a neighborhood of the origin.
Then, we can define a smooth vector field in a neighborhood of 0 by

$$X = \sum_{i,j} (H(x) G(x)^{-1})_{ij} x_i \frac{\partial}{\partial x_j}. \tag{4.6}$$

X is then Euler-like since HG^{-1} is the identity up to higher-order terms (simply observe that $HG^{-1} = (I + (G - H)H^{-1})^{-1}$, which is the geometric series close enough to the origin). Then, by theorem 2.2, there exists a tubular neighborhood embedding ψ on a neighborhood of the origin such that $\psi_* E = X$.

Now, by equations (4.4) and (4.6),

$$\begin{aligned} X(f) &\overset{(4.6)}{=} \sum_{i,j} (H(x)G(x)^{-1})_{ij} x_i \frac{\partial f}{\partial x_j} \\ &\overset{(4.4)}{=} \sum_{i,j,k} (H(x)G(x)^{-1})_{ij} x_i G_{jk}(x) x_k \\ &= \sum_{i,k} H_{ik}(x) x_i x_k \\ &= 2f, \end{aligned}$$

and recalling equation (2.9),

$$E(\psi^* f) \overset{(2.9)}{=} \psi^*(X(f)) = 2\psi^* f. \tag{4.7}$$

By lemma 4.1, $\psi^* f$ is a homogeneous quadratic polynomial, which then defines a bilinear form that is symmetric and thus diagonalizable.
Hence, up to a change of coordinates where this bilinear form is diagonal, we have $\psi^* f = \sum_i \lambda_i x_i^2$, for some real numbers λ_i, which by a coordinate rescaling takes the form (4.2). □

In the same fashion, we can prove the Darboux theorem for 2-forms. It relies on the following

Lemma 4.4. *If E is the Euler vector field on a vector space V and $\omega \in \Omega^k(V)$, then $\mathcal{L}_E \omega = k\omega$ if and only if ω has constant coefficients.*

Proof. Without loss of generality by linearity of the Lie derivative, consider $\omega = f\, dx_1 \wedge \ldots \wedge dx_k$. Then,

$$\mathcal{L}_E \omega = \mathcal{L}_E f\, dx_1 \wedge \ldots \wedge dx_k + f \mathcal{L}_E (dx_1 \wedge \ldots \wedge dx_k).$$

The second term on the right-hand side, using Cartan's magic formula, becomes

$$f \iota_E d(dx_1 \wedge \ldots \wedge dx_k) + f d\iota_E (dx_1 \wedge \ldots \wedge dx_k).$$

Since $dx_1 \wedge \ldots \wedge dx_k$ is a closed form and its contraction with E is

$$\sum_i (-1)^{i-1} x_i dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_k,$$

where the symbol $\widehat{dx_i}$ means that we omit dx_i, and the exterior derivative of this contraction is $k dx_1 \wedge \ldots \wedge dx_k$, we have that

$$\mathcal{L}_E \omega = \mathcal{L}_E f\, dx_1 \wedge \ldots \wedge dx_k + k f dx_1 \wedge \ldots \wedge dx_k.$$

Hence, $\mathcal{L}_E \omega = k\omega$ if and only if $\mathcal{L}_E f = 0$, i.e. f is constant along the integral curves of E, which are rays through the origin. Since f is continuous, the value of this constant on each ray is $f(0)$, i.e. f is constant on V. □

Theorem 4.5 (Darboux)**.** *Let $\omega \in \Omega^2(\mathbb{R}^{2n})$ be closed and non-degenerate (i.e. symplectic). Then, there exists a tubular neighborhood embedding ψ defined on a neighborhood of the origin such that $\psi^* \omega$ is constant.*

Remark 4.6. The classical statement of Darboux theorem actually states the existence of a coordinate system $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ on a neighborhood U of the origin such that

$$\omega|_U = \sum_{i=1}^{n} dq_i \wedge dp_1. \tag{4.8}$$

This is equivalent to asking that there is a tubular neighborhood embedding ψ on a neighborhood of the origin such that $\psi^*\omega$ takes form (4.8). Now, if $\psi^*\omega$ is constant, since on a vector space for a form being closed is equivalent to being skew-symmetric, by a classical result of linear algebra (see for instance [Con], theorem 5.4), there exists a coordinate system on which $\psi^*\omega$ takes the form (4.8). This shows the equivalence between of the two statements of Darboux theorem.

Proof. Since ω is a closed 2-form on $\mathbb{R}^{2n}$, by the Poincaré lemma, there exists a 1-form α on $\mathbb{R}^{2n}$ such that $d\alpha = \omega$ and we can take the coordinate expressions

$$\begin{aligned} \omega &= \sum_{i<j} (\omega_{ij} + \mathcal{O}(|x|)) dx_i \wedge dx_j \\ \alpha &= \frac{1}{2} \sum_{i,j} (\omega_{ij} x_i + \mathcal{O}(|x|^2)) dx_j \end{aligned} \tag{4.9}$$

in a neighborhood of zero, where $\omega_{ij} = -\omega_{ji}$.
Since ω is non-degenerate at the origin, and by continuity also in a neighborhood of 0, the equation

$$\iota_X \omega = 2\alpha$$

has a solution $X \in \mathfrak{X}(U)$ in a neighborhood U of 0.
By equation (4.9),

$$X = \sum_i (x_i + \mathcal{O}(|x|^2)) \frac{\partial}{\partial x_i},$$

i.e. X is an Euler-like vector field for the pair $(U, 0)$. Hence, there exists a tubular neighborhood embedding $\psi : (U, 0) \to (\mathbb{R}^{2n}, 0)$ such that $\psi_* E = X$. Since ω is closed, by Cartan's magic formula, we have

$$\mathcal{L}_X \omega = d\iota_X \omega + \iota_X d\omega = d(2\alpha) = 2\omega. \tag{4.10}$$

Then,

$$\mathcal{L}_E(\psi^*\omega) \overset{(2.9)}{=} \psi^*(\mathcal{L}_{\psi_* E}\omega) = \psi^*(\mathcal{L}_X \omega) = 2\psi^*\omega. \tag{4.11}$$

By the previous lemma, $\psi^*\omega$ has constant coefficients. □

4.2 Morse-Bott Functions

In this section, we prove the analogue of Morse lemma for submanifolds, the Morse-Bott lemma.
Let $N \subset M$ be a submanifold and $f \in C^\infty(M)$. If f vanishes to second order along N, then we can define its *quadratic approximation* to be the function $f_{[2]} \in C^\infty(\nu(M, N))$ made on each fiber $\nu_x(M, N)$ of the second-order terms of the Taylor expansion of f at $x \in N$. Then,

for every $x \in N$, its corresponding symmetric bilinear form on $\nu_x(M,N)$ is called the *normal hessian* of f

$$\text{Hess}(f)_x : \nu_x(M,N) \times \nu_x(M,N) \to \mathbb{R}.$$

Definition 4.7. A function $f \in C^\infty(M)$ that vanishes to second order along N is called *Morse-Bott* for the pair (M,N) if $\text{Hess}(f)_x$ is non-degenerate for every $x \in N$.

Lemma 4.8 (Morse-Bott)**.** *If f is a Morse-Bott function for the pair (M,N), there exists a tubular neighborhood embedding*

$$\psi : O \subset \nu(M,N) \to M,$$

with O being an open neighborhood of N in the normal bundle, such that

$$\psi^* f = f_{[2]}.$$

Proof. By taking an eventual initial tubular neighborhood embedding, we can consider M to be an open neighborhood of N in $\nu(M,N)$. By hypothesis, we have a Morse function on (a neighborhood of the origin of) each fiber of the normal bundle, so we can construct an Euler-like vector field on each $\nu_x(M,N)$ as in the proof of Morse lemma (equation (4.6)). Analogously to the proof of lemma 4.3, we then have an Euler-like vector field X on a neighborhood O of N in the normal bundle such that

$$X(f) = 2f.$$

By theorem 1.2, there exists a tubular neighborhood embedding ψ as in the statement satisfying $\psi_* E = X$, so that

$$E(\psi^* f) = 2\psi^* f,$$

i.e. by lemma 4.1, $\psi^* f$ is a homogeneous polynomial of degree 2. Hence, since $\nu(\psi)$ is the identity, we have the conclusion. □

4.3 Weinstein Lagrangian Neighborhood Theorem

The next application of theorem 1.2 we want to discuss is the well-known *Weinstein Lagrangian neighborhood theorem* in symplectic geometry, the analogous of Darboux theorem 4.5 for Lagrangian submanifolds.

Start considering a manifold M and a differential k-form ω on M, together with a submanifold $i : N \hookrightarrow M$ such that

$$i^*\omega = 0.$$

Then, $\kappa^*\omega$ is a differential k-form on the deformation space $\mathcal{D}(M,N)$ that vanishes on the zero fiber, because on $\nu(M,N)$ it is simply $pr^* i^* \omega = pr^* 0 = 0$, by definition of κ (see definition 3.15).

This means that $\kappa^*\omega$ is divisible by t on the deformation space, or equivalently, that $\frac{1}{t}(\omega \times 0) \in \Omega^k(M \times \mathbb{R}^\times)$ extends to a form

$$\mathcal{D}(\omega) := \frac{1}{t}\kappa^*\omega \in \Omega^k(\mathcal{D}(M,N)).$$

Definition 4.9. We define the *linearization* of ω to be the pullback

$$\nu(\omega) := j^*(\mathcal{D}(\omega)) \in \Omega^k(\nu(M,N)), \tag{4.12}$$

where $j : \nu(M,N) \hookrightarrow \mathcal{D}(M,N)$ is the inclusion.

In particular, if (M,ω) is a symplectic manifold with Lagrangian submanifold $i : N \hookrightarrow M$ (so, $i^*\omega = 0$), the linearization of ω is a well defined 2-form

$$\nu(\omega) \in \Omega^2(\nu(M,N)),$$

which, since N is Lagrangian, is symplectic along the zero section N of the normal bundle. To see this, note that since $\frac{1}{t}(\omega \times 0)$ is closed on each $\pi^{-1}(t) = M \times \{t\}$, by the smoothness of its extension, the form $\mathcal{D}(\omega)$ is closed also on the zero fiber, where it coincides with $\nu(\omega)$. The non-degeneracy can easily be checked using local coordinates.
Now, since $\nu(\omega)$ is symplectic along the zero section of the normal bundle, it is so also in a neighborhood of $N \subset \nu N$. We claim that $\nu(\omega)$ is symplectic on the whole νN. To check this, first observe the following

Lemma 4.10. *$\nu(\omega)$ defined above is a linear 2-form on νN, in the sense that it is linear on the fibers or, equivalently, that*

$$\mathcal{L}_E \nu(\omega) = \nu(\omega), \tag{4.13}$$

where E is the Euler vector field on the normal bundle.

Proof. Recall that we can construct a canonical vector field θ on $\mathcal{D}(M,N)$ that takes the form $t\frac{\partial}{\partial t}$ on $M \times \mathbb{R}^\times$, whereas $\mathcal{D}(\omega) = \frac{1}{t}(\omega \times 0)$. But then, on $M \times \mathbb{R}^\times$, taking the Lie derivative in direction of $-\theta$,

$$\mathcal{L}_{-\theta}\mathcal{D}(\omega)\big|_{M\times\mathbb{R}^\times} = \mathcal{L}_{-t\frac{\partial}{\partial t}} \frac{1}{t}(\omega \times 0) = \frac{1}{t}(\omega \times 0) = \mathcal{D}(\omega)|_{M\times\mathbb{R}^\times}.$$

Hence, by continuity, we have $\mathcal{L}_{-\theta}\mathcal{D}(\omega) = \mathcal{D}(\omega)$ on the totality of $\mathcal{D}(M,N)$. In particular, the equality holds also on the zero fiber νN, where $-\theta$ is E and $\mathcal{D}(\omega)$ is $\nu(\omega)$, i.e.

$$\mathcal{L}_E \nu(\omega) = \nu(\omega),$$

concluding the proof. □

In particular, equation (4.13) tells us that the flow of E preserves our linear form $\nu(\omega)$. Hence, by eventually pulling back $\nu(\omega)$ to a neighborhood where it is symplectic, we find that $\nu(\omega)$ is symplectic on the entire normal bundle.
We then proved that νN has a natural symplectic structure, given by $\nu(\omega)$.
The following normal form states that if we are interested in the behavior of ω around N, we might as well view it as its linearization on νN.

Theorem 4.11 (Weinstein Lagrangian neighborhood)**.** *If (M,ω) is a symplectic manifold and N is a Lagrangian submanifold, then there exists a tubular neighborhood embedding*

$$\psi : O \subset \nu(M,N) \to M,$$

with O being an open neighborhood of N in $\nu(M,N)$, such that

$$\psi^*\omega = \nu(\omega).$$

Remark 4.12 ([Mei21], Remark 4.1). Consider a vector bundle $V \to N$ with a linear symplectic form ω on V, in the sense of lemma 4.10. We know that $TV|_N = V \oplus TN$, so restricting ω to $TV|_N$ gives a pairing between V and TN via

$$\langle v, X\rangle := \omega|_{TV|_N}\big((v,0),(0,X)\big),$$

for all $v \in V$, $X \in TN$. Since this pairing is non-degenerate, we obtain an isomorphism

$$V \to T^*N, \quad v \mapsto \langle v, \cdot\rangle. \tag{4.14}$$

Furthermore, this map pushes the form ω on V to the standard symplectic form Ω_{T^*N} on T^*N, i.e. is a symplectomorphism.
Hence, taking $V = \nu(M,N)$ with symplectic form $\nu(\omega)$, gives us a diffeomorphism between $\nu(M,N)$ and T^*N preserving the symplectic structures.

This consideration allows us to recover the Weinstein neighborhood Lagrangian theorem in its most-known form

Theorem 4.13. *Let N be a Lagrangian submanifold of (M,ω). Then, there exist a neighborhood U of N in M, a neighborhood V of N in T^*N, and a symplectomorphism*

$$\psi : (U,\omega) \to (V, \Omega_{T^*N})$$

that restricts to the identity map on N.

Let us prove theorem 4.11 by means of our main theorem 1.2.

Proof. By the *tubular neighborhood theorem*, without loss of generality, consider M to be a fiber-wise convex, open neighborhood of N in $\nu(M,N)$. Since, ω pulls back to zero (via the inclusion) on N, by the *relative Poincaré lemma*, there exists a primitive α of ω that vanishes on N, and is constructed via

$$\alpha = \int_0^1 \frac{1}{t}\kappa_t^*(\iota_E\omega)\,dt, \tag{4.15}$$

where κ_t is the multiplication by t. Then, taking its linearization gives

$$\begin{aligned}\nu(\alpha) &= \int_0^1 \frac{1}{t}\kappa_t^*\nu(\iota_E\omega)\,dt\\ &= \int_0^1 \frac{1}{t}\kappa_t^*(\iota_{\nu(E)}\nu(\omega))\,dt\\ &= \int_0^1 \iota_E\frac{1}{t}\kappa_t^*\nu(\omega)\,dt\\ &= \iota_E\nu(\omega)\int_0^1 dt\\ &= \iota_E\nu(\omega),\end{aligned} \tag{4.16}$$

where in the third equality we used the fact that $\nu(E) = E$ and in the second to last that, since $\nu(\omega)$ is linear, $\frac{1}{t}\kappa_t^*\nu(\omega) = \nu(\omega)$, which does not depend on t. Now, define

$X \in \mathfrak{X}(M)$ via the equation $\iota_X\omega = \alpha$ (which has solutions because ω is non-degenerate). Then, taking the linearization on both sides,

$$\nu(\alpha) = \nu(\iota_X\omega) = \iota_{\nu(X)}\nu(\omega).$$

Hence, again by non-degeneracy of ω, we have $\nu(X) = E$, i.e. X is Euler-like (it might not be complete). Then, by theorem 1.2 and remark 2.11, there exists a tubular neighborhood embedding as in the statement such that $\psi_* E = X$. But,

$$\mathcal{L}_E(\psi^*\omega) = \psi^*(\mathcal{L}_X\omega) = \psi^*(\iota_X d\omega + d\iota_X\omega) = \psi^*\omega, \tag{4.17}$$

where in the second equality we used Cartan's magic formula and in the last that ω is closed and $d\iota_X\omega = d\alpha = \omega$. This means that $\psi^*\omega$ is linear, and since $\nu(\psi)$ is the identity, it coincides with $\nu(\omega)$. □

4.4 Splitting Theorem for Singular Foliations

Recall that a foliation on a manifold M can be seen as an involutive distribution, i.e. a subbundle of TM closed with respect to the Lie bracket of sections. In this context, all the leaves have the same dimension. Singular foliations extend this notion by allowing the dimension of the leaves to vary.
Following the flow of [BBLM20], our goal is to prove a so-called *splitting theorem* for these objects. The definition of singular foliations first appeared in [AS07], where the authors were inspired by earlier works of Stephan [Ste74] and Sussmann [Sus73]. One can consult [AS07, AZ16] for general discussions about the topic.

Definition 4.14. A *singular foliation* $\mathcal{F}$ on M is a $C^\infty(M)$-submodule of $\mathfrak{X}(M)$ such that

a. $\mathcal{F}$ is local: if $X \in \mathfrak{X}(M)$ such that for any $m \in M$ there exists $Y \in \mathcal{F}$ and a neighborhood $U \subset M$ of m such that $X|_U = Y|_U$, then $X \in \mathcal{F}$

b. $\mathcal{F}$ is locally finitely generated: for any $m \in M$ there exists a neighborhood U of m such that $\mathcal{F}|_U$ is spanned by finitely many $Y_1, \dots, Y_k \in \mathcal{F}$

c. $\mathcal{F}$ is involutive: $[\mathcal{F}, \mathcal{F}] \subset \mathcal{F}$.

Remark 4.15. One can show (see [AZ16], theorem 5.1) that $C^\infty(M)$-submodules of $\mathfrak{X}(M)$ that are local in the sense of the previous definition are in bijective correspondence with $C^\infty(M)$-submodules of the compactly supported vector fields $\mathfrak{X}_c(M)$. Hence, we can define singular foliations to be $C^\infty(M)$-submodules of $\mathfrak{X}_c(M)$ that are involutive and locally finitely generated.

Definition 4.16. Define $\exp \mathcal{F}$ to be the group generated by the time-1 flow $\exp X$ of vector fields X of $\mathcal{F}$, and $\mathrm{Aut}(M, \mathcal{F})$ to be the group made of diffeomorphisms $\phi : M \to M$ that preserve the foliation, i.e. $\phi_*\mathcal{F} = \mathcal{F}$.

Proposition 4.17 ([AS07], Proposition 1.6). *$\exp \mathcal{F}$ is a normal subgroup of* $\mathrm{Aut}(M, \mathcal{F})$.

We propose here a sketch of the proof by Androulidakis-Skandalis. For another equivalent proof, see [GY18].

Sketch of the proof. Let us first show that $\exp \mathcal{F}$ is a subgroup of $\mathrm{Aut}(M, \mathcal{F})$. We want to show that if $X \in \mathcal{F}$, then $\exp X \in \mathrm{Aut}(M, \mathcal{F})$.
We claim that it is enough to prove that

$$(\exp X)_* \mathcal{F} \subset \mathcal{F}. \tag{4.18}$$

Indeed, any $Y \in \mathcal{F}$ can be written as

$$Y = (\exp X)_* \big(\exp{(-X)}_* Y \big)$$

and, since $-X \in \mathcal{F}$, if equation (4.18) holds, $\exp{(-X)}_* Y \in \mathcal{F}$. This means that every $Y \in \mathcal{F}$ can be written as $Y = (\exp X)_* Z$ for some $Z \in \mathcal{F}$, i.e. $Y \in (\exp X)_* \mathcal{F}$, proving that $(\exp X)_* \mathcal{F} = \mathcal{F}$, so that $\exp X \in \mathrm{Aut}(M, \mathcal{F})$.
So, let us now prove the inclusion (4.18). Replacing eventually M with a neighborhood of the support of X, we can assume that $\mathcal{F}$ be generated by a finite number of vector fields $Y_1, \ldots . Y_n \in \mathcal{F}$.
By involutivity of the foliation, $[X, Y_i] \in \mathcal{F}$ for any i. Hence, there exist smooth functions α_{ij} on M such that

$$[X, Y_i] = \sum_j \alpha_{ji} Y_j. \tag{4.19}$$

Call L the linear mapping

$$L : C^\infty(N)^n \to C^\infty(N)^n, \quad (f_1, \ldots, f_n) \mapsto (g_1, \ldots, g_n), \tag{4.20}$$

with $g_i := X(f_i) + \sum_j \alpha_{ij} f_j$, and define a surjective map

$$S : C^\infty(N)^n \to \mathcal{F}, \quad (f_1, \ldots, f_n) \mapsto \sum_i f_i Y_i. \tag{4.21}$$

Then, for any $f_1, \ldots, f_n \in \mathcal{F}$, we have

$$\begin{aligned}
(\mathcal{L}_X \circ S)(f_1, \ldots . f_n) &= \mathcal{L}_X \Big(\sum_i f_i Y_i \Big) \\
&= \sum_i X(f_i) Y_i + [X, Y_i] \\
&= \sum_i X(f_i) Y_i + \sum_{i,j} \alpha_{ji} f_i Y_j \\
&= \sum_i X(f_i) Y_i + \sum_{i,j} \alpha_{ij} f_j Y_i \\
&= \sum_i \big(X(f_i) + \sum_j \alpha_{ij} f_j \big) Y_i \\
&= (S \circ L)(f_1, \ldots, f_n),
\end{aligned}$$

where in the third equality we used equation (4.19) and in the fourth we renamed the indices $i \leftrightarrow j$.
This shows that $\mathcal{L}_X \circ S = S \circ L$. One can then prove that

$$(\exp X)_* \circ S = S \circ \exp L.^2 \tag{4.22}$$

[2] Here $\exp L = \sum_{i=0}^\infty L^i / i!$, where L^k is the composition of L k times and L^0 is the identity map.

Since S is surjective,

$$(\exp X)_*\mathcal{F} = \big((\exp X)_* \circ S\big)\big(C^\infty(N)^n\big) \overset{(4.22)}{\subseteq} S\big(C^\infty(N)^n\big) = \mathcal{F}.$$

This proves equation (4.18) and hence that $\exp X \in \mathrm{Aut}(M, \mathcal{F})$.
To show that $\exp \mathcal{F}$ is a normal subgroup, take $g \in \mathrm{Aut}(M, \mathcal{F})$. Then, since X and $g_*X \in \mathcal{F}$ are g-related by definition, $g \circ \exp X = \exp(g_*X) \circ g$, i.e.

$$g \circ \exp X \circ g^{-1} = \exp(g_*X) \in \mathrm{Aut}(M, \mathcal{F}),$$

proving our claim. □

Definition 4.18. Given a singular foliation $\mathcal{F}$ on M, define its *leaves* as the orbits of the action of the group $\exp \mathcal{F}$ on M, that on the generators is defined via

$$\exp \mathcal{F} \times M \to M, \quad (\exp X, m) \mapsto \exp X(m).$$

Example 4.19 (A singular foliation on $\mathbb{R}$)**.** Let $M = \mathbb{R}$. Then, $\mathcal{F} := \mathrm{Span}_{C^\infty(\mathbb{R})}\{x\frac{\partial}{\partial x}\}$ is a singular foliation. Indeed, $x\frac{\partial}{\partial x}$ is the Euler vector field on $\mathbb{R}$, so its flow is $\phi_t(y) = e^t y$, thus its time-1 flow is the multiplication by e. The leaves of $\mathcal{F}$ are then

$$(-\infty, 0),\ \{0\},\ (0, +\infty),$$

so, in contrast with regular foliations, the dimension of the leaves can vary. This explains why such objects are called *singular*.

Definition 4.20. We say that a smooth map $\phi : N \to M$ is *transverse* to $\mathcal{F}$ if for every $n \in N$

$$T_{\phi(n)}M = \mathrm{Im}(d_n\phi) + \{Y_{\phi(n)} : Y \in \mathcal{F}\}. \tag{4.23}$$

Clearly, submersions are transverse to any given singular foliation.
Recall that if we have a vector bundle $\pi : E \to M$ and a smooth map $\phi : N \to M$, we can define the *pullback bundle* $pr_1 : \phi^*(E) \to N$, where

$$\phi^*(E) = \{(n, e) \in N \times E : \phi(n) = \pi(e)\}^{3} \subset N \times E \tag{4.24}$$

and pr_1 is the projection onto the first component. The projection onto the second component, pr_2, makes the following diagram commute

$$\begin{array}{ccc} \phi^*(E) & \xrightarrow{\ pr_2\ } & E \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ N & \xrightarrow[\ \phi\ ]{} & M \end{array}$$

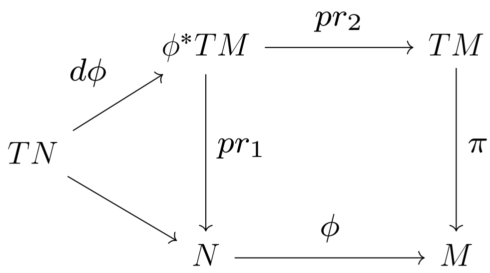
If $\mathcal{E}$ is a submodule of $\Gamma_c(E)$, define the *pullback module* $\phi^*(\mathcal{E})$ to be the submodule of $\Gamma_c(\phi^*(E))$ generated by elements $f \cdot (\xi \circ \phi)$, with $f \in C^\infty(N)$, $\xi \in \mathcal{E}$.

[3] In the literature, one often says that $\phi^*(E)$ is the *fibered product* of N and E over M, denoted with $\phi^*(E) := N \times_M E$

Definition 4.21. Given a smooth map $\phi : N \to M$ transverse to a singular foliation $\mathcal{F}$ on M, define the *pullback foliation* $\phi^!(\mathcal{F})$ to be the $C^\infty(N)$-submodule of $\mathfrak{X}_c(N)$[4] defined by

$$\begin{aligned}\phi^!(\mathcal{F}) &:= \{X \in \mathfrak{X}_c(N) : d\phi(X) \in \phi^*(\mathcal{F})\} \\ &= \{X \in \mathfrak{X}_c(N) : d\phi(X) = \sum_{i=1}^n f_i \cdot (Y_i \circ \phi),\ f_i \in C^\infty(N),\ Y_i \in \mathcal{F}\},\end{aligned} \tag{4.25}$$

seeing $d\phi(X)$ as a section of $\phi^*(TM)$.



Remark 4.22. Observe that the definition is well-posed. Indeed, $\phi^!(\mathcal{F})$ is always involutive and, if ϕ is transverse to $\mathcal{F}$, it is locally finitely generated (see [AS07], proposition 1.10).

Remark 4.23. Observe that if ϕ is a submersion, then the pullback foliation can be defined by

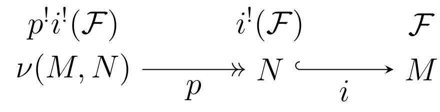
$$\phi^!\mathcal{F} := \mathrm{Span}_{C_c^\infty(N)}\left\{X \in \mathfrak{X}(N)\ \phi\text{-related to an element of } \mathcal{F}\right\}$$

Consider a submanifold N of M. If the inclusion $i : N \hookrightarrow M$ is transverse to $\mathcal{F}$, then we have a singular foliation

$$i^!(\mathcal{F}) = \{X|_N \in \mathfrak{X}(N) : X \in \mathcal{F} \text{ is tangent to } N\}. \tag{4.26}$$

Moreover, calling $p : \nu(M, N) \to N$ the projection (instead of pr as in definition 3.3, for convenience of notation), we have a natural singular foliation on the normal bundle, which we will call *linear approximation* of $\mathcal{F}$ around N, defined by

$$\nu(\mathcal{F}) := p^!i^!(\mathcal{F}). \tag{4.27}$$



We now have all the tools to prove the following[5]

Theorem 4.24 (Splitting Theorem for Singular Foliation)**.** *Let $\mathcal{F}$ be a singular foliation on M, N be a submanifold transverse (i.e. the inclusion is transverse) to $\mathcal{F}$. Then, there exists a tubular neighborhood embedding $\psi : \nu(M, N) \to M$ such that*

$$\psi^!(\mathcal{F}) = \nu(\mathcal{F}).$$

[4]Here we use the equivalence between the two definitions of singular foliations, explained in remark 4.15

[5]see [BBLM20], Theorem 2.8.

The proof of this result is not a direct implication of theorem 1.2, instead it follows the logic of its proof due to [BBLM20], showing an instance of how we can generalize its geometry to more advanced situations. This approach has not been possible yet, at least to our knowledge, by means of the *Moser trick*, used in the proofs of theorem 1.2 by [BLM16], or theorem 5.5 by [Mei21].

Proof. First of all, let us show that $\mathcal{F}$ contains an Euler-like vector field.
Denote by $r = \dim M - \dim N$. Since the inclusion $N \hookrightarrow M$ is transverse to $\mathcal{F}$, for any point $n \in N$ there exist a neighborhood V of n and vector fields $Y_1, \dots, Y_r \in \mathcal{F}|_V$ spanning a subbundle $K \subset TM|_V$ such that

$$TM|_{N\cap V} = K|_{N\cap V} \oplus TN|_{N\cap V}. \tag{4.28}$$

By lemma 3.9 of [BLM16], there exists a section $\sigma \in \Gamma(K)$ such that $\sigma|_N = 0$ and is Euler-like[6]. Since K is finitely generated, $\sigma = \sum_i f_i Y_i$ for some $f_i \in C^\infty(V)$, hence $\sigma \in \mathcal{F}|_V$. Using a partition of unity, by the locality of $\mathcal{F}$, we can construct a vector field $X \in \mathcal{F}$ which is Euler-like for (M, N).
Recall that we have a canonical vector field θ on the deformation space $\mathcal{D}(M, N)$ that by the system (3.15) satisfies $\theta \sim_\kappa 0$, with κ as in definition 3.15. Additionally, if X is Euler-like, we can construct a vector field $\mathcal{D}(X)$ on $\mathcal{D}(M, N)$ such that $\mathcal{D}(X) \sim_\kappa X$ (see equation (3.21)). This means

$$d\kappa(\theta) = 0, \quad d\kappa(\mathcal{D}(X)) = X.$$

But then, since a vector field Y on $\mathcal{D}(M, N)$ is an element of $\kappa^!(\mathcal{F})$ if it is of the form

$$d\kappa(Y) = \sum_i f_i(Y_i \circ \kappa),$$

with $f_i \in C^\infty(N)$ and $Y_i \in \mathcal{F}$ for all i, we have that θ and $\mathcal{D}(X)$ both lie in $\kappa^!(\mathcal{F})$.
Hence, also the vector field $W := \frac{1}{t}\big(\theta + \mathcal{D}(X)\big) \in \kappa^!(\mathcal{F})$. By proposition 4.17, its integral curves $\varphi_s = \exp sW$ are automorphisms of $\mathcal{D}(M, N)$ that preserve $\kappa^!(\mathcal{F})$. But then,

$$(\varphi_s)^!(\kappa^!(\mathcal{F})) = \kappa^!(\mathcal{F}).$$

Since the inclusion $j : \nu(M, N) \hookrightarrow \mathcal{D}(M, N)$ and $\kappa|_{\pi^{-1}(s)} : \pi^{-1}(s) \to M$ for $s \neq 0$ are foliation preserving in a trivial way, denoting $\psi_s = \kappa \circ \varphi_s \circ j$, we have

$$\begin{aligned}
\psi_s^! \mathcal{F} &= (\kappa \circ \varphi_s \circ j)^! \mathcal{F} \\
&= j^!(\varphi_s)^! \kappa^! \mathcal{F} \\
&= j^! \kappa^! \mathcal{F} \\
&= p^! i^! \mathcal{F} \\
&= \nu(\mathcal{F}),
\end{aligned} \tag{4.29}$$

where the third equality holds because, by definition, $\kappa \circ j := i \circ p$.
By the proof of theorem 1.2, we find that $\psi := \psi_1$ is the desired tubular neighborhood embedding. □

[6] The lemma applies to *anchored vector bundles*, but one can show that it holds also with our hypothesis in two ways; either by observing that anchored vector bundles define a singular foliation as the image of their *anchor*, or by mimicking the proof of the lemma.

Remark 4.25. The name *splitting theorem* comes from the fact that if $\hat{N}$ is a "small enough" *slice* at a point x in M, i.e. it is an embedded submanifold transverse to the leaf of $\mathcal{F}$ at x, then there exists a neighborhood W of x in M such that $\mathcal{F}$ restricted to W is diffeomorphic to the product of a trivial foliation with $i^!\mathcal{F}$, with $i : N = \hat{N} \cap W \hookrightarrow W$ is the inclusion. For more context, see [BBLM20] or [AZ11].

Chapter 5

Generalizations

In this chapter, we generalize theorem 2.2 by introducing a notion of *weights* on the local coordinates of a manifold, and of *eigenvalues* of vector fields on this manifold. We will first consider positive integer weights (see [Mei21]), which we will further generalize whenever we are in the presence of *resonances*.
In section 5.2 we will present a result on *non-resonant* eigenvalues (*Sternberg linearization theorem*, see [Ste57, Ste58]) and use it to prove a normal form for *gradient vector fields* of *Morse functions*, first shown (to our knowledge) in [Wan18] via analytic techniques. Our main contribution was to simplify the proof of this result by applying Sternberg's theorem to proposition 2.6 of [Wan18], which we demonstrated in a similar way, after having eliminated some redundancies.

5.1 Weighted Setting

Let us fix a point $p \in M$ and a set of local coordinates $x_1, \ldots, x_n$ around p, so that without loss of generality we can present the theory on $M = \mathbb{R}^n$.
We will follow the point of view of [Mei21] and, in contrast with the previous chapters, we will prove the analogue of theorem 2.2 employing germs of vector fields and functions.

Definition 5.1. We refer to an n-tuple of positive integers $w = (w_1, \ldots, w_n) \in \mathbb{N}^n$ as a *weight sequence* (or simply *weight*) and we call *weighted scalar multiplication* associated to w the smooth map

$$\kappa_t : \mathbb{R}^n \to \mathbb{R}^n, \quad x = (x_1, \ldots, x_n) \mapsto (t^{w_1} x_1, \ldots, t^{w_n} x_n). \tag{5.1}$$

Define the *weighted Euler vector field* associated to the weight w to be the unique vector field whose flow is $\phi_s = \kappa_{\exp s}$.

Example 5.2. A straightforward computation shows that the weighted Euler vector field associated to w takes form

$$E = \sum_i w_i x_i \frac{\partial}{\partial x_i}. \tag{5.2}$$

Recall that, if ϕ is a local diffeomorphism of $\mathbb{R}^n$ around the origin and X is a vector field on $\mathbb{R}^n$, we can define the *pullback* $\phi^* X$ to be the pushforward $(\phi^{-1})_* X$.

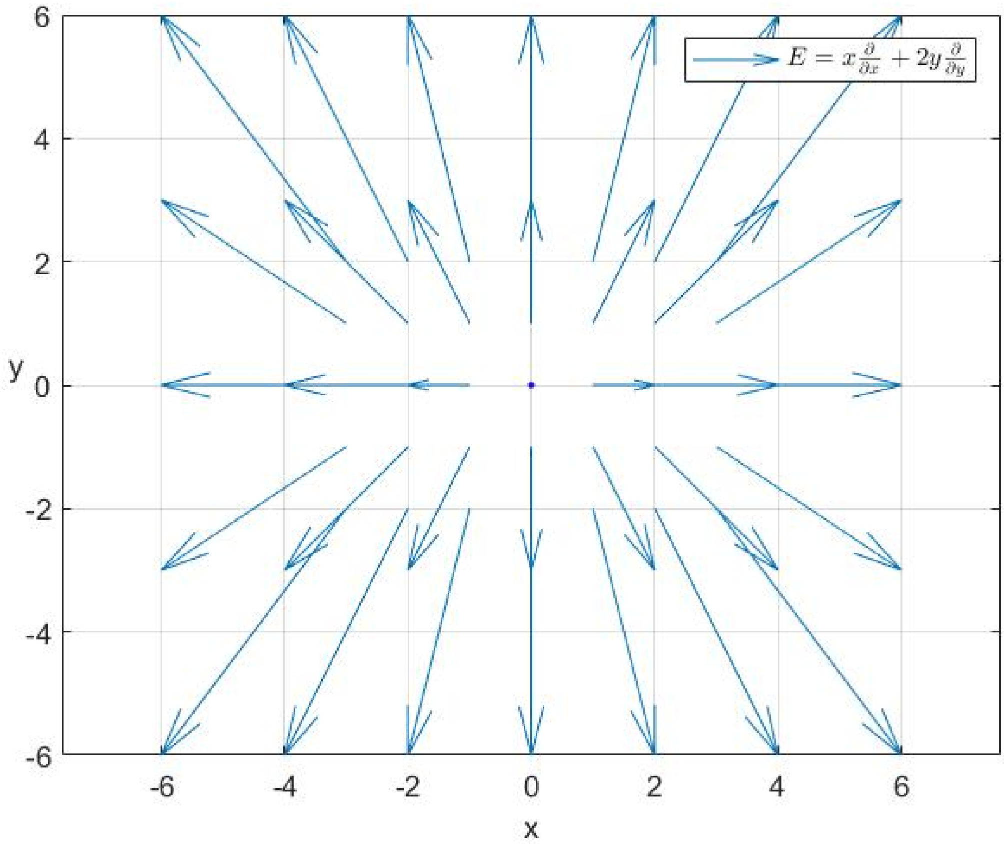


Figure 5.1: Portion of the weighted Euler vector field on $\mathbb{R}^2$ associated to the weight $w = (1, 2)$

Definition 5.3. A vector field $X = \sum_i a_i(x)\frac{\partial}{\partial x_i} \in \mathfrak{X}(\mathbb{R}^n)$ is called *weighted Euler-like* for $(\mathbb{R}^n, 0)$ if $X_0 = 0$ and

$$\lim_{t\to 0} \kappa_t^* X = E, \tag{5.3}$$

or equivalently, if

$$\lim_{t\to 0} t^{-w_i} a_i(\kappa_t(x)) = w_i x_i \tag{5.4}$$

for all $i = 1, \dots, n$.

Observe that for $w = (1, \dots, 1)$, this notion restricts to that of Euler-like vector field as in definition 2.9 whenever X is complete. As we introduced, we can drop this assumption since we are interested in germs of X.

Example 5.4 (Weighted Euler-like vector fields on $\mathbb{R}^2$)**.** Consider $\mathbb{R}^2$ with weight $w = (1, 2)$, $m \in \mathbb{N}$, and the vector fields

$$X = (x + y^m)\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}$$
$$Y = x\frac{\partial}{\partial x} + (2y + x^m)\frac{\partial}{\partial y}$$

vanishing at the origin.
Then, X is weighted Euler-like for all $m \in \mathbb{N}$. Indeed,

$$\begin{aligned}\lim_{t\to 0} t^{-1} a_1(tx, t^2y) &= \lim_{t\to 0} t^{-1}(tx + t^{2m}y^{2m}) \\ &= x + \lim_{t\to 0} t^{2m-1}y^{2m} \\ &= x,\end{aligned}$$

because $2m - 1 > 0$ for any value of m (the check for a_2 is trivial. This is the case whenever such a coefficient has already the form of the corresponding coefficient of the Euler vector field).
On the other hand, Y is weighted Euler-like if and only if $m \geqslant 3$ because, following the same logic,

$$\lim_{t\to 0} t^{-2}(2t^2y + t^my^m) = 2y + \lim_{t\to 0} t^{m-2}x^m = 2y$$

if and only if $m \geqslant 3$.
Intuitively, this example tells us that a vector field is weighted Euler-like if its 0th order terms (with respect to the weights) agree with the terms of the weighted Euler vector field. All the higher order terms do not contribute at all. It is useful to remark the following case: if $m = 2$, then Y as above takes the form

$$Y = x\frac{\partial}{\partial x} + (2y + x^2)\frac{\partial}{\partial y}.$$

So, up to the "usual" higher-order terms, Y coincides with the weighted Euler vector field, i.e. $\nu(Y) = E$, but we just saw that it is not weighted Euler-like as in definition 5.3 because, taking into account the weights, the term $x^2\frac{\partial}{\partial y}$ has order 0.

The following is the analogue of theorem 2.2 in the weighted setting. For sake of simplicity, we prove it accordingly to [Mei21], via a *Moser*-type argument, without passing through the deformation space, but losing the interesting geometrical picture related to it. For this version of the proof in the submanifold case, we refer to [Mei21], section 5.

Theorem 5.5. *If X is a weighted Euler-like vector field on $(\mathbb{R}^n, 0)$, then there exist a germ ψ at 0 of tubular neighborhood embeddings*[1] *$\mathbb{R}^n \to \mathbb{R}^n$ such that*

$$\psi^* X = E,$$

with E the weighted Euler vector field on $(\mathbb{R}^n, 0)$ as in definition 5.1.

Proof. Consider the time-dependent family of vector fields

$$X_t = \kappa_t^* X = \sum_i \frac{a_i(\kappa_t(x))}{t^{w_i}} \frac{\partial}{\partial x_i} \tag{5.5}$$

for $t \neq 0$. Since X is weighted Euler-like, by definition,

$$\lim_{t\to 0} X_t = E, \tag{5.6}$$

[1]In fact, of diffeomorphisms such that $\psi(0) = 0$ and $d_0\phi = \mathrm{id}_{\mathbb{R}^n}$

i.e. X_t extends to a smooth vector field at time 0 with $X_0 = E$.[2]
Now,

$$\frac{dX_t}{dt} = \frac{1}{t}[E, X_t] \tag{5.7}$$

because for any vector fields Z, Y with the flow of Z being denoted by φ_r, we have

$$\begin{aligned}\frac{d}{dr}(\varphi_{-r})_*Y &= (\varphi_{-r})_*\frac{d}{ds}\Big|_0(\varphi_{-s})_*Y\\ &= \frac{d}{ds}\Big|_0(\varphi_{-s})_*\big((\varphi_{-r})_*Y\big)\\ &= \mathcal{L}_Z\big((\varphi_{-r})_*Y\big)\\ &= [Z, \varphi_r^*Y]\end{aligned}$$

so that, if $Z = E$ and $Y = X$, we have $\varphi_r = \kappa_{\exp r}$, and substituting $r = \ln t$,

$$[E, \kappa_t^* X] = \Big(\frac{d(\ln t)}{dt}\Big)^{-1}\frac{d}{dt}(\kappa_{-t})_*X = t\frac{d}{dt}\kappa_t^*X.$$

Additionally, we can define a new time-dependent family of vector fields $\{W_t\}_{t\neq 0}$ by

$$W_t = \frac{1}{t}(E - X_t) \tag{5.8}$$

for $t \neq 0$, which extends at $t = 0$ because of equation (5.6). Call $\{\phi_t\}_t$ the germ of flow of the time-dependent vector field $\{W_t\}_t$, with $\phi_0 = \mathrm{id}_{\mathbb{R}^n}$.
Then,

$$\begin{aligned}\frac{d}{dt}\phi_t^*X_t &= \phi_t^*\big(\mathcal{L}_{W_t}X_t\big) + \phi_t^*\frac{d}{dt}X_t\\ &= \phi_t^*\Big([W_t, X_t] + \frac{dX_t}{dt}\Big)\\ &\overset{(5.7)}{=} \phi_t^*\Big(\Big[W_t + \frac{1}{t}E, X_t\Big]\Big)\\ &\overset{(5.8)}{=} \phi_t^*\Big(\Big[\frac{1}{t}X_t, X_t\Big]\Big)\\ &\overset{(3.26)}{=} \phi_t^*\Big(\frac{1}{t}[X_t, X_t] + X_t(1/t)X_t\Big)\\ &= 0.\end{aligned}$$

This shows that $\phi_t^*X_t$ is constant. Hence, calling $\psi = \phi_1$, we have

$$\psi^*X = \phi_1^*X_1 = \phi_0^*X_0 = E.$$

Moreover, for any given t, the time-dependent vector field $\{W_t\}_t$ vanishes to second order at the origin (to see this, combine equations (5.5) and (5.8)). Hence, its flow satisfies $d_0\phi_t = \mathrm{id}_{\mathbb{R}^n}$ for any t. In particular, $d_0\psi = \mathrm{id}_{\mathbb{R}^n}$. □

[2] This is why, as explained at the end of the previous example, in definition 5.3 we required the 0th order terms with respect to the weights to agree with the corresponding terms of the Euler vector field.

Remark 5.6. Opposite to the non-weighted case, the germ of tubular neighborhood embeddings we find is not unique. For instance, in the setting of example 5.4 the diffeomorphism

$$\phi : \mathbb{R}^2 \to \mathbb{R}^2,\ (x, y) \mapsto (x, y + x^2) \tag{5.9}$$

commutes with the weighted scalar multiplication, thus it preserves the weighted Euler vector field. Moreover, $d_0\phi = \mathrm{id}_{\mathbb{R}^2}$. Hence, if ψ is a germ of tubular neighborhood embeddings as in the theorem, $\phi \circ \psi$ is one as well, and

$$(\phi \circ \psi)_* X = \phi_* E = E. \tag{5.10}$$

Remark 5.7. The previous theorem shows in particular that the vector fields X and Y in example 5.4 can be linearized to the weighted Euler vector field for all m and for $m \geqslant 3$, respectively.
A closer look at this example tells us something more. Consider a vector field

$$X = E + \sum_i a_i(x) \frac{\partial}{\partial x_i}, \tag{5.11}$$

where E is the weighted Euler vector field associated to a weight w, and the non-linear parts a_i are monomials in $x_1, \ldots, x_n$ of degree at least 2 (we can extend this discussion to polynomials and formal polynomials). For instance, take

$$a_i(x) = x_1^{k_1} \cdots x_n^{k_n},$$

with $k_j \in \mathbb{N}$ for all j and $k_1 + \ldots + k_n \geqslant 2$. Let us define the *total weight* of a_i to be its degree as a monomial taking into account the weight on each x_i, i.e.

$$w_1 k_1 + \ldots + w_n k_n,$$

and the *total weight* of $\frac{\partial}{\partial x_i}$ to be $-w_i$.
Hence, the computations in example 5.4 tell us that the vector field X is weighted Euler-like, hence linearizable (to E), if and only if the total weight of each component $a_i(x)\frac{\partial}{\partial x_i}$, i.e.

$$w_1 k_1 + \ldots + w_n k_n - w_i$$

is positive.
Given that X is linearizable if and only if $-X$ is linearizable, and $-X$ will have negative total weights whenever X has positive total weights, we can conclude that a vector field of the form (5.11) is linearizable to the weighted Euler vector field whenever the total weights of its non-linear parts are either all positive or all negative.
As we will see in appendix A, it turns out that this statement is true (at least in the formal case) not only for positive integer weights but also when $w \in \mathbb{Q}^n$. Actually, this statement can be relaxed, since linearizability is granted by theorem 5.9 in the next section if the "weights" are *non-resonant*. In fact, appendix A will discuss the complementary *resonant* case.

5.2 Non-Resonant Eigenvalues

The aim of this section is to prove theorem 5.11 (see [Wan18], Theorem 1.2). It gives conditions under which the *gradient vector field* of a *Morse function* on a Riemannian manifold is linearizable. Our proof will be similar, but simpler and more geometric, and will rely on Sternberg's linearization theorem 5.9.
Fix a manifold M and a point $p \in M$, and consider a vector field X on M vanishing at p, whose linearization $\nu(X)$, in local coordinates around p, takes form

$$\nu(X) = \sum_{i,j=1}^{n} a_{ij} x_j \frac{\partial}{\partial x_i}, \tag{5.12}$$

where $a_{ij} \in \mathbb{R}$ and the matrix $A = [a_{ij}]_{i,j}$ has non-zero (possibly complex) eigenvalues $\lambda_1, \dots, \lambda_n$. We define the *eigenvalues* of $\nu(X)$ to be the eigenvalues of A.

Definition 5.8. We say that a set of (possibly complex) numbers $\lambda_1, \dots, \lambda_n$ is *non-resonant* (or *satisfies the* $\mathbb{N}_0$[3]*-linearity condition*) if for any $(k_1, \dots, k_n) \in \mathbb{N}_0^n$ with $\sum_{j=1}^{n} k_j \geqslant 2$

$$\sum_{j=i}^{n} k_j \lambda_j - \lambda_i \neq 0 \tag{5.13}$$

for all $i = 1, \dots, n$.
Equivalently, if every sum (with possible repetitions) of at least two elements in $\{\lambda_1, \dots, \lambda_n\}$ does not belong to $\{\lambda_1, \dots, \lambda_n\}$.

The following is a classical result due to Sternberg (see [Ste58], theorem 2).

Theorem 5.9 (Sternberg's Linearization)**.** *Let X be a vector field on M vanishing at p. If the eigenvalues of its linearization $\nu(X)$ are non-resonant, then there exists a local diffeomorphism*

$$\psi : U \subset T_pM \to U' \subset M,$$

with $0 \in U$, $p \in U'$, such that $\psi(0) = p$ and

$$\psi^* X = \nu(X),$$

i.e. X is linearizable.

5.2.1 Gradient Vector Fields of Morse Functions

Let (M, g) an n-dimensional Riemannian manifold with metric $g \in \Gamma(T^*M \otimes T^*M)$ and Levi-Civita connection ∇. Recall that for a smooth map $f : M \to \mathbb{R}$, we can define its *gradient (vector field)* ∇f (or $\operatorname{Grad} f$) as the unique vector field on M such that

$$g(\nabla f, X) = df(X), \tag{5.14}$$

and the *Hessian* of f to be the section $\operatorname{Hess} f \in \Gamma(T^*M \otimes T^*M)$ given by

$$\operatorname{Hess} f(X, Y) = g(\nabla_X \operatorname{Grad} f, Y) \tag{5.15}$$

[3] $\mathbb{N}_0 := \mathbb{Z}_{\geqslant 0} = \{0, 1, 2, 3, \dots\}$

for every $X, Y \in \mathfrak{X}(M)$.
We call a point $p \in M$ a *critical point* for f if

$$\mathrm{Grad}\, f|_p = 0. \tag{5.16}$$

If additionally $\mathrm{Hess}\, f|_p$ is non-degenerate (as a bilinear form on T_pM), p is said to be a *non-degenerate* critical point.
Now, for a bilinear form B on T_pM, we can consider its associated linear map

$$\widetilde{B} : T_pM \to T_p^*M, \quad v \mapsto B(v, \cdot),$$

which is non-degenerate (equivalently, invertible) if and only if B is.
Hence, fixing a point p, we can think of $(g^{-1} \circ \mathrm{Hess}\, f)|_p$ as its associated linear map

$$\left(\widetilde{g}^{-1} \circ \widetilde{\mathrm{Hess}\, f}\right)\Big|_p : T_pM \to T_pM.$$

Definition 5.10. A smooth function $f : M \to \mathbb{R}$ is a *Morse function* if all its critical points are non-degenerate.
So, if f is a Morse function and p is a critical point, $(g^{-1} \circ \mathrm{Hess}\, f)|_p$ has n non-zero eigenvalues counted with multiplicity, called *Morse eigenvalues* of f at p.

It turns out that the Morse eigenvalues are all real. In fact, in the proof of proposition 5.12 we will show that $(g^{-1} \circ \mathrm{Hess}\, f)|_p$ is self-adjoint.
In Morse lemma 4.3, we stated that there exists a local change of coordinates on $\mathbb{R}^n$ under which a Morse function takes the following normal form

$$f(x) = \sum_i \pm x_i^2. \tag{5.17}$$

Hence, in Euclidean metric on $\mathbb{R}^n$, its gradient vector field is

$$\nabla f = \sum_i \pm x_i \frac{\partial}{\partial x_i},$$

i.e. it is a linear vector field.
For a general metric g, there is no reason why, in the above-mentioned coordinates, should both f take the form (5.17) and g be the Euclidean metric (or even a general flat metric). It is then natural to ask when the gradient vector field of a Morse function on a Riemannian manifold is linearizable.
With the next theorem we show that, under some conditions on f and g, the gradient vector field of f admits a normal form, even though the metric alone does not.

Theorem 5.11. *If $f : M \to \mathbb{R}$ is a Morse function on (M, g) whose Morse eigenvalues at a critical point p are non-resonant, then there exists a local diffeomorphism*

$$\psi : U \subset T_pM \to U' \subset M,$$

with $0 \in U$, $p \in U'$, such that $\psi(0) = p$, that linearizes ∇f.

The proof of the previous theorem is a straightforward application of Sternberg linearization theorem 5.9 to the following result. Its proof is taken from [Wan18], but with some redundancies here simplified.

Proposition 5.12. *If f is a Morse function on (M, g), there exists a local chart $(U, (x_1, \dots, x_n))$ around every critical point p such that*

$$\nabla f(x) = \sum_i (\lambda_i x_i + a_i(x)) \frac{\partial}{\partial x_i}, \tag{5.18}$$

where the a_i's are functions vanishing to second order at $x = 0$ and the λ_i's are the Morse eigenvalues of f at p.

Proof. By the proof of Morse lemma 4.3, we can find a chart around p where f takes form (4.3)

$$f(x) = \frac{1}{2} \sum_{i,j} H_{ij}(x) x_i x_j,$$

with $H(x)$ symmetric and $H(0) = \operatorname{Hess} f|_0$.
We claim that the linear map associated to $h := (g^{-1}\operatorname{Hess} f)|_{x=0} = g^{-1}(0)H(0)$ is self-adjoint, i.e. that for any $v,\, w \in T_pM$

$$g(\tilde{h}v, w) = g(v, \tilde{h}w). \tag{5.19}$$

Indeed,

$$g\big((\tilde{g}^{-1}|_0 \widetilde{H}|_0)(v), w\big) = \widetilde{H}|_0(v)(w) = \operatorname{Hess} f|_0(v, w),$$

and since g is symmetric

$$g\big(v, (\tilde{g}^{-1}|_0 \widetilde{H}|_0)w\big) = g\big((\tilde{g}^{-1}|_0 \widetilde{H}|_0)w, v\big) = \widetilde{H}|_0(w)(v) = \operatorname{Hess} f|_0(w, v).$$

So, we have our claim by the symmetricity of the Hessian.
Since $(g^{-1}\operatorname{Hess} f)|_{x=0}$ is self-adjoint, we can find an orthogonal matrix (i.e. a coordinate change $x \mapsto y$) that diagonalizes $(g^{-1}\operatorname{Hess} f)|_{x=0}$ into the diagonal matrix of (Morse) eigenvalues $\lambda_1, \dots, \lambda_n$, i.e.

$$g^{-1}\operatorname{Hess} f|_{y=0} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}. \tag{5.20}$$

Hence, expanding at $y = 0$ by Hadamard theorem, we find

$$\sum_k (g^{-1})_{ik}(y) H_{kj}(y) = (g^{-1}(y)H(y))_{ij} = \lambda_i \delta_{ij} + \mathcal{O}(||y||). \tag{5.21}$$

Recalling that by equations (4.4) and (4.5),

$$\frac{\partial f}{\partial y_j}(y) = \sum_k H_{jk}(y) y_k + \frac{1}{2} \sum_{k,l} \frac{\partial H_{kl}}{\partial y_j}(y) y_k y_l, \tag{5.22}$$

and that the gradient has local expression

$$\nabla f(y) = \sum_{i,j} (g^{-1})_{ij}(y) \frac{\partial f}{\partial y_j}(y) \frac{\partial}{\partial y_i}, \tag{5.23}$$

plugging equation (5.22) into (5.23) and using (5.21), we obtain

$$\begin{aligned}
\nabla f(y) &= \sum_{i,j} \big((g_{ij}^{-1}(y) \sum_k H_{jk}(y) y_k + \mathcal{O}(||y||^2)\big) \frac{\partial}{\partial y_i} \\
&= \sum_{i,k} \big(y_k \sum_j (g_{ij}^{-1}(y) H_{jk}(y) + \mathcal{O}(||y||^2)\big) \frac{\partial}{\partial y_i} \\
&= \sum_i (\lambda_i y_i + \mathcal{O}(||y||^2)) \frac{\partial}{\partial y_i}
\end{aligned}$$

concluding the proof. □

Remark 5.13. Observe that the coefficients $\lambda_1, \ldots, \lambda_n$ do not depend on the local chart. Indeed, in the previous proof, they are constructed as the eigenvalues of $g^{-1} \operatorname{Hess} f|_0$, and we know that the eigenvalues of a matrix do not depend on the choice of coordinates.

Proof of theorem 5.11. By the last proposition, under a suitable local chart, the gradient vector field of f takes the form (5.18). Hence, its linearization is

$$\nu(\nabla f)(y) = \sum_i \lambda_i y_i \frac{\partial}{\partial y_i},$$

which by hypothesis has non-resonant eigenvalues $\lambda_1, \ldots, \lambda_n$. The conclusion holds by applying Sternberg's linearization theorem 5.9. □

Appendix A

Resonant Eigenvalues

For the sake of completeness, we explain the problem of linearization of vector fields in the presence of resonances. This discussion generalizes the weighted setting of section 5.1 when the weights are resonant (in the non-resonant case, we know linearizability holds by Sternberg theorem) and is inspired by [BG10]. An interested reader is invited to check this reference for a richer analysis of the topic.
Without loss of generality, take $M = \mathbb{R}^n$ with coordinates $x_1, \ldots, x_n$ and $p = 0$. Consider a formal vector field X on $\mathbb{R}^n$ (i.e. a derivation on the algebra of formal power series of $x_1, \ldots, x_n$ over $\mathbb{R}$. E.g the expansion around 0 of a vector field on $\mathbb{R}^n$) vanishing at the origin, whose linearization takes the form

$$\nu(X) = \sum_{i=1}^{n} \lambda_i x_i \frac{\partial}{\partial x_i},$$

with $\lambda_1, \ldots, \lambda_n$ resonant and none of them zero. So that

$$X = \nu(X) + A,$$

with A being a vector field with no linear components

$$A = \sum_{i=1}^{n} \sum_{|K| \geqslant 2} a_i^K x^K \frac{\partial}{\partial x_i},$$

for $a_i^K \in \mathbb{R}$, where $K = (k_1, \ldots, k_n) \in \mathbb{N}_0^n$ multi-index, $|K| = k_1 + \ldots + k_n$, and $x^K := x_1^{k_1} \cdots x_n^{k_n}$.
We say that the vector field A is *admissible* if all linear combinations with non-negative integers (not all zero) of its *total weights*

$$Q_i^K := k_1\lambda_1 + \ldots k_n\lambda_n - \lambda_i,$$

such that $a_i^K \neq 0$ are non-zero.

Theorem A.1 ([BG10], Theorem 3.4)**.** *Let X be a (formal) vector field as above. If its non-linear part A is admissible, then there exists a formal change of coordinates linearizing X.*

Remark A.2. Note that if all the λ_i are positive integers such that each of the Q_i^K is positive, then any linear combination with non-negative integers of the total weights will

again be positive. This means that weighted Euler-like vector fields, as in section 5.1, have admissible non-linear parts (by remark 5.7) and are thus (formally) linearizable, as we anticipated at the end of the section. Notice that in the case of non-resonant weights, we already know by Sternberg that they can be smoothly linearized.
Similarly, if the $\lambda_i \in \mathbb{Z}$ are such that the total weights Q_i^K are all positive or all negative, then we have admissible non-linearity and the same conclusion holds. Analogously for $\lambda_i \in \mathbb{Q}$, multiplying eventually by the product of the denominators of the λ_i's.

Appendix B

Functoriality of Linearization

As we anticipated, the linearization of a vector field can be defined in a functorial way. We explain this process following [BLM16], Section 2.2.
Consider a vector bundle $pr : E \to M$ and a vector subbundle $F \to N$, with $F \subset E$ and $N \subset M$ submanifolds. Then, by remark 3.5, pr induces a projection

$$\nu(pr) : \nu(E, F) \to \nu(M, N),$$

turning $\nu(E, F)$ into a vector bundle over $\nu(M, N)$.
By definition of the normal bundle, we also have vector bundle structures

$$p : \ \nu(M, N) \to N,$$

and

$$\nu(E, F) \to F.$$

Hence, we obtain a so-called *double vector bundle*

$$\begin{array}{ccc} \nu(E,F) & \longrightarrow & F \\ {\scriptstyle \nu(pr)}\big\downarrow & & \big\downarrow \\ \nu(M,N) & \xrightarrow{\ p\ } & N \end{array}$$

where all the horizontal and vertical maps are projections.
In particular, considering the tangent bundles $E = TM$ and $F = TN$ gives rise to a double vector bundle

$$\begin{array}{ccc} \nu(TM,TN) & \longrightarrow & TN \\ {\scriptstyle \nu(pr)}\big\downarrow & & \big\downarrow \\ \nu(M,N) & \xrightarrow{\ p\ } & N \end{array} \tag{B.1}$$

On the other hand, we have the tangent bundle

$$T\nu(M, N) \to \nu(M, N)$$

and a vector bundle

$$Tp : T\nu(M, N) \to TN$$

constructed as in remark 3.5. This gives rise to another double vector bundle

$$\begin{array}{ccc} T\nu(M,N) & \xrightarrow{Tp} & TN \\ \downarrow & & \downarrow \\ \nu(M,N) & \xrightarrow{p} & N \end{array} \tag{B.2}$$

By lemma B.1 below, we have an isomorphism between the double vector bundles (B.1) and (B.2), i.e. $\nu(TM,TN) \simeq T\nu(M,N)$.
By remark 3.5, a vector field X on M tangent to a submanifold N, i.e. a smooth map of pairs $X : (M,N) \to (TM,TN)$, gives rise to a map

$$\nu(X) : \nu(M,N) \to \nu(TM,TN),$$

which by the lemma below can be seen as a map

$$\nu(X) : \nu(M,N) \to T\nu(M,N),$$

i.e. a vector field $\nu(X)$ on $\nu(M,N)$.
This gives a functorial way to construct the *linearization* of X, which we defined via its action in definition 3.9.
Let us now show the following

Lemma B.1. *There exists a vector bundle isomorphism*

$$\nu(TM,TN) \xrightarrow{\sim} T\nu(M,N)$$

with respect to the vector bundle structures over TN and $\nu(M,N)$ that restricts to the identity on each base.

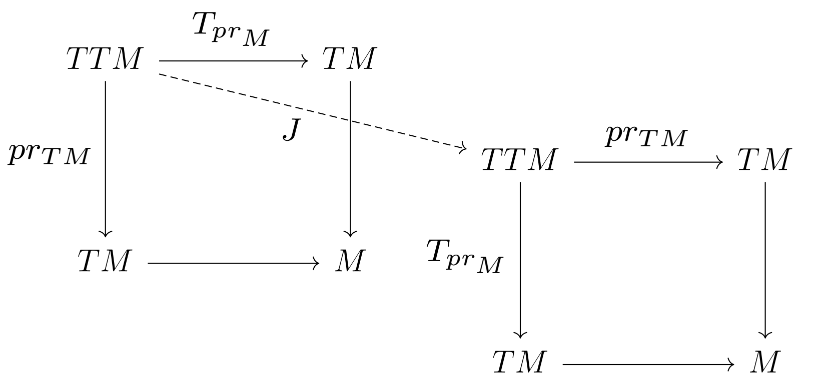
Proof. Consider the tangent bundle $pr_M : TM \to M$ and the double tangent bundle

$$\begin{array}{ccc} TTM & \xrightarrow{T_{pr_M}} & TM \\ {\scriptstyle pr_{TM}}\downarrow & & \downarrow{\scriptstyle pr_M} \\ TM & \xrightarrow{pr_M} & M \end{array}$$

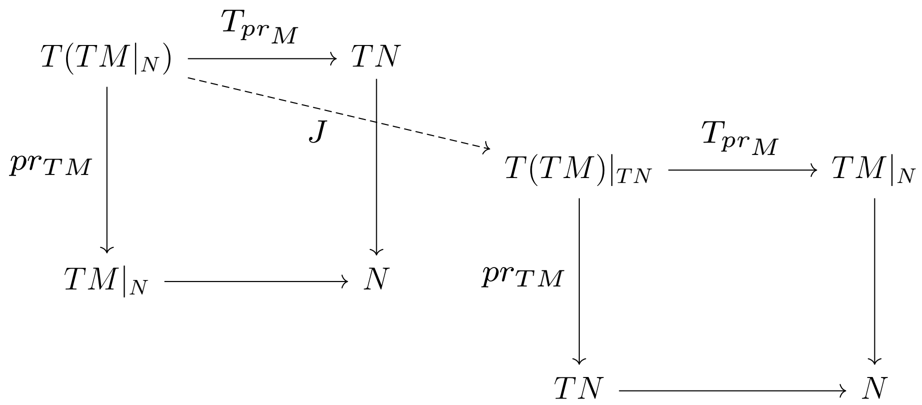
If we view an element of TTM as a double derivation at the origin of TTM, we find an isomorphism $J : TTM \to TTM$, referred as to the *canonical involution*, defined by the switching of the order of derivation

$$J\Big(\frac{\partial^2 m}{\partial t \partial s}\Big|_{(0,0)}\Big) = \frac{\partial^2 m}{\partial s \partial t}\Big|_{(0,0)},$$

so that $T_{pr_M} \circ J = pr_{TM}$ (see [Mac05], chapter 9.6).



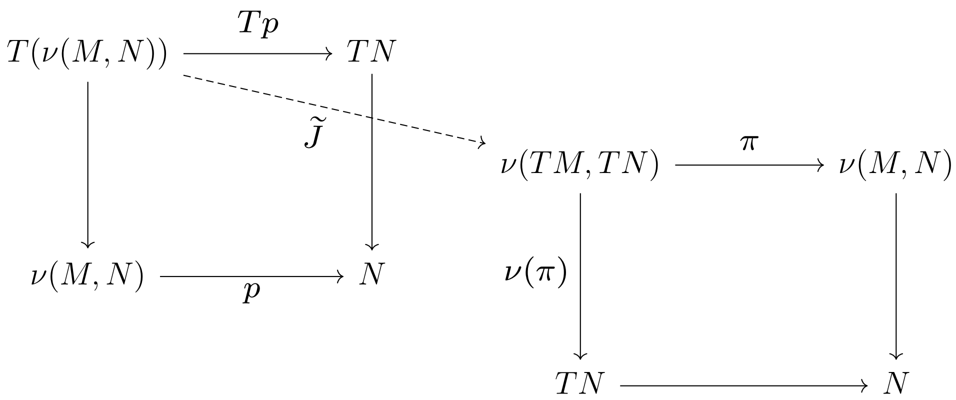
Now, for any submanifold $N \subset M$, we can consider the submanifolds $T(TM|_N)$ and $T(TM)|_{TN}$ of TTM. These are both double vector subbundles on which J restricts to an isomorphism (that we call again J).



By construction J restricts to the canonical involution $J : TTN \subset T(TM|_N) \to TTN \subset T(TM)|_{TN}$. Hence, we have an isomorphism on the quotients

$$\widetilde{J} : \frac{T(TM|_N)}{TTN} \to \frac{T(TM)|_{TN}}{TTN},$$

i.e. an isomorphism of the double vector bundle



□

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