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On Submanifolds and Deformations in Poisson Geometry

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Dissertation presented in partial fulfillment of the requirements for the degree of Doctor of Science (PhD): Mathematics

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Abstract

This thesis concerns specific classes of submanifolds in Poisson geometry. The emphasis lies on normal form statements, and we present an application in deformation theory. The results are divided into three themes.

We first study coisotropic submanifolds in log-symplectic manifolds. We provide a normal form around coisotropic submanifolds transverse to the degeneracy locus, and we prove a reduction statement for coisotropic submanifolds transverse to the symplectic leaves.

Next, we address Lagrangian submanifolds contained in the singular locus of a log-symplectic manifold. We establish a normal form around such Lagrangians, which we use to study their deformations. On the algebraic side, we show that the deformations correspond with Maurer-Cartan elements of a suitable DGLA. On the geometric side, we discuss when small deformations of the Lagrangian are constrained to the singular locus, and we find criteria for unobstructedness of first order deformations. We also address equivalences of deformations and we prove a rigidity result.

At last, we consider a class of submanifolds in arbitrary Poisson manifolds, which are defined by imposing a suitable constant rank condition. We show that their local Poisson saturation is smooth, and we give a normal form for the induced Poisson structure. This result extends some normal form theorems around distinguished types of submanifolds in symplectic and Poisson geometry. As an application, we prove a uniqueness statement concerning coisotropic embeddings of Dirac manifolds into Poisson manifolds.

Beknopte samenvatting

Deze thesis behandelt bepaalde klassen van deelvariëteiten in Poissonmeetkunde. De nadruk ligt op normaalvormstellingen, en we geven een toepassing ervan in deformatietheorie. De resultaten zijn onderverdeeld in drie thema's.

Eerst bestuderen we coisotrope deelvariëteiten in log-symplectische variëteiten. We vinden een normaalvorm rond coisotrope deelvariëteiten die de singuliere locus transversaal snijden, en we bewijzen een reductiestelling voor coisotrope deelvariëteiten die de symplectische bladeren transversaal snijden.

Vervolgens beschouwen we Lagrangiaanse deelvariëteiten die bevat zijn in de singuliere locus van een log-symplectische variëteit. We maken gebruik van een normaalvorm rond zulke Lagrangianen om hun deformaties te bestuderen. Op algebraïsch vlak tonen we aan dat deformaties overeenkomen met Maurer-Cartan elementen van een bepaalde DGLA. Op meetkundig vlak onderzoeken we wanneer deformaties niet uit de singuliere locus kunnen ontsnappen, en we vinden voorwaarden onder dewelke eerste-orde deformaties rakend zijn aan een pad van deformaties. Tevens behandelen we equivalentierelaties op de ruimte van deformaties en bewijzen we een rigiditeitsstelling.

Het laatste deel van de thesis gaat over bepaalde deelvariëteiten van willekeurige Poissonvariëteiten, die gedefinieerd zijn in termen van een regulariteitsvoorwaarde. We tonen aan dat de lokale Poissonsaturatie van zulke deelvariëteiten glad is, en we geven een normaalvorm voor de geïnduceerde Poissonstructuur. Dit resultaat is een uitbreiding van enkele normaalvormstellingen in symplectische en Poissonmeetkunde. Als toepassing bewijzen we een uniciteitsresultaat aangaande coisotrope inbeddingen van Diracvariëteiten in Poissonvariëteiten.

Contents

Introduction

This thesis contains some results concerning submanifolds in Poisson geometry, with emphasis on normal forms and deformations. The upcoming chapters contain copies of the following three preprints, up to minor stylistic/linguistic modifications, some small corrections and extensions of the appendices.

1. *"Coisotropic submanifolds in b-symplectic geometry"*.
   Joint with Marco Zambon. Available on arXiv:1907.09251, and will appear in the Canadian Journal of Mathematics.

2. *"Deformations of Lagrangian submanifolds in log-symplectic manifolds"*.
   Joint with Marco Zambon. Available on arXiv:2009.01146.

3. *"The Poisson saturation of regular submanifolds"*.
   Available on arXiv:2011.12650.

The parts of this thesis that are not contained in the papers listed above are §2.6.3 in the Appendix of Chapter 2 and §3.8.2 in the Appendix of Chapter 3.

In this introduction, we give some background information about Poisson and Dirac geometry, and we outline the main results that are obtained in this thesis.

Poisson geometry

Poisson geometry has its origins in the Hamiltonian formulation of classical mechanics. Say we want to describe the movement of a particle in $\mathbb{R}^n$. Its state at time t is given by a point $(q(t), p(t))$ in the phase space $\mathbb{R}^{2n}$, where $q = (q_1, \ldots, q_n)$ are the position coordinates and $p = (p_1, \ldots, p_n)$ are the momentum coordinates. The way in which the state of the particle evolves with time depends on the Hamiltonian function $H = H(p, q, t)$, which is usually the total energy of the system. The time evolution of the system is determined by Hamilton's equations:

$$\begin{cases} \dot{q}_i = \frac{\partial H}{\partial p_i} \\ \dot{p}_i = -\frac{\partial H}{\partial q_i} \end{cases} . \tag{1}$$

Let us introduce a new product for functions $f, g \in C^\infty(\mathbb{R}^{2n})$, which was discovered by Poisson [P] in 1809, by the rule

$$\{f, g\} := \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right). \tag{2}$$

We refer to this product as the **standard Poisson bracket**. The equations (1) can now be conveniently rewritten as follows:

$$\dot{x}_i = \{H, x_i\},$$

where x_i denotes any of the coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$. Moreover, we notice that the operator $\{H, \cdot\}$ is a derivation of the product, so that it corresponds with a vector field, called the **Hamiltonian vector field**

$$X_H = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} \right).$$

Rewriting Hamilton's equations (1) in terms of this vector field gives

$$\dot{x}(t) = X_H(x(t)).$$

In other words, the evolution of the system is determined by the integral curves of the vector field X_H. This way, the earliest instances of Poisson geometry provide a geometric theory for classical mechanics.

It was Jacobi who identified the main properties of the standard Poisson bracket (2). The theory was formalized further by Lichnerowicz [L] in the late 1970s, but it was only after the foundational paper [W2] of Alan Weinstein that Poisson geometry took off as an independent field of study. We now introduce the basics of Poisson geometry; most of what follows can be found, for instance, in [DZ].

Poisson manifolds

Definition. A **Poisson structure** on a manifold M is a Lie bracket $\{\cdot, \cdot\}$ on the algebra of smooth functions $C^\infty(M)$ that is a derivation in each entry:

$$\{f, gh\} = \{f, g\}h + g\{f, h\} \qquad \forall f, g, h \in C^\infty(M).$$

Equivalently, a Poisson structure is given by a bivector field $\Pi \in \Gamma(\wedge^2 TM)$ satisfying $[\Pi, \Pi] = 0$, where $[\cdot, \cdot]$ is the Schouten bracket of multivector fields. The Schouten bracket $[\cdot, \cdot]$ is a natural extension of the usual Lie bracket of vector fields. The Poisson bracket $\{\cdot, \cdot\}$ and the Poisson bivector Π are related by the formula

$$\{f, g\} = \Pi(df, dg).$$

Geometrically, a Poisson manifold can be thought of as a manifold partitioned into symplectic submanifolds. More precisely, the Poisson bivector gives rise to a bundle map

$$\Pi^\sharp : T^*M \to TM : \alpha \mapsto \iota_\alpha \Pi, \tag{3}$$

whose image is an integrable (singular) distribution. So M inherits a partition into connected, immersed submanifolds, called leaves, such that the leaf $\mathcal{O}$ through a point $p \in M$ satisfies

$$T_p\mathcal{O} = \Pi^\sharp(T_p^*M).$$

Moreover, each leaf has an induced symplectic structure: the Poisson manifold (M, Π) carries a **symplectic foliation**. It is important to stress that this foliation is singular in general, i.e. the leaves may have varying dimension.

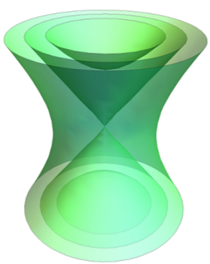

Figure 1: The symplectic foliation for $\Pi = x\partial_y \wedge \partial_z + y\partial_z \wedge \partial_x - z\partial_x \wedge \partial_y$.

Examples

- Symplectic structures $\omega \in \Omega^2(M)$ are exactly the non-degenerate Poisson structures $\Pi \in \Gamma(\wedge^2 TM)$, i.e. those Poisson structures Π for which the bundle map (3) is invertible. Indeed, there is a correspondence between non-degenerate two-forms $\omega \in \Omega^2(M)$ and non-degenerate bivector fields $\Pi \in \Gamma(\wedge^2 TM)$, given by

$$\omega^\flat = \pm(\Pi^\sharp)^{-1},$$

where the plus or minus sign depends merely on convention[1]. Under this correspondence, the closedness condition $d\omega = 0$ is equivalent with the Poisson condition $[\Pi, \Pi] = 0$.

For instance, the Poisson bracket (2) corresponds with the standard symplectic structure on $\mathbb{R}^{2n}$.

- Every manifold M is trivially Poisson for the zero Poisson structure.

- If $(\mathfrak{g}, [\cdot,\cdot])$ is a Lie algebra, then $\mathfrak{g}^*$ has a canonical Poisson bracket $\{\cdot,\cdot\}_{\mathfrak{g}}$ defined by
$$\{f, h\}_{\mathfrak{g}}(\xi) = \langle [d_\xi f, d_\xi h], \xi \rangle.$$
Here we identify $(\mathfrak{g}^*)^* \cong \mathfrak{g}$. The associated Poisson bivector $\Pi_{\mathfrak{g}}$ satisfies
$$\Pi_{\mathfrak{g}}(\xi) = \xi \circ [\cdot,\cdot] \in \wedge^2 \mathfrak{g}^* \cong \wedge^2 (T_\xi \mathfrak{g}^*).$$
The symplectic foliation of this Poisson structure is given by the orbits of the coadjoint action. Figure 1 above displays the case $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$.

Maps and vector fields

A map between Poisson manifolds $\varphi : (M, \Pi_M) \to (N, \Pi_N)$ is a **Poisson map** if the pullback
$$\varphi^* : \big(C^\infty(N), \{\cdot,\cdot\}_N\big) \to \big(C^\infty(M), \{\cdot,\cdot\}_M\big)$$
is a Lie algebra morphism. It is equivalent to require that the bivector fields Π_M and Π_N are φ-related, i.e.
$$(\wedge^2 d_p \varphi)(\Pi_M)_p = (\Pi_N)_{\varphi(p)} \qquad \forall p \in M.$$

A **symplectic realization** of (M, Π) is a surjective Poisson submersion from a symplectic manifold (Σ, ω^{-1}) onto (M, Π).

A vector field X on a Poisson manifold (M, Π) is a **Poisson vector field** if its flow ϕ_X^t consists of Poisson diffeomorphisms, or equivalently, if $\pounds_X \Pi = 0$. Every function $f \in C^\infty(M)$ gives rise to a Poisson vector field X_f via
$$X_f = \Pi^\sharp(df),$$
called the **Hamiltonian vector field** associated with f. The foliation of a Poisson manifold is generated by its Hamiltonian vector fields. Two points

[1]We make clear in each chapter which convention we use. In Chapters 1 and 2, we use the minus sign. In Chapter 3 however, we chose to use the plus sign for convenience, since that chapter relies on results proved in papers that adopt the latter convention.

$p, q \in M$ lie in the same leaf if q can be obtained by applying finitely many Hamiltonian flows to p:

$$q = \phi^1_{X_{f_k}} \circ \cdots \circ \phi^1_{X_{f_1}}(p).$$

Equivalently, q lies in the same leaf as p if there exists a time-dependent Hamiltonian whose flow takes p to q:

$$q = \phi^1_{X_{f_t}}(p).$$

There is a specific kind of Poisson vector field that is of special importance in this thesis. Upon choosing a volume form $\mu \in \Omega^{top}(M)$, there is a vector field V^{μ}_{mod} that is uniquely determined by

$$\mathcal{L}_{X_f}\mu = V^{\mu}_{mod}(f)\mu \qquad \forall f \in C^{\infty}(M).$$

This vector field is a Poisson vector field, called the **modular vector field** associated with μ [W3]. Choosing a different volume form changes the modular vector field by a Hamiltonian vector field, since

$$V^{g\mu}_{mod} = V^{\mu}_{mod} - X_{\ln|g|}. \tag{4}$$

Poisson cohomology

Any Poisson manifold (M, Π) carries a differential

$$d_{\Pi} : \Gamma(\wedge^k TM) \to \Gamma(\wedge^{k+1} TM) : \xi \mapsto [\Pi, \xi],$$

whose cohomology is called the **Poisson cohomology**. In low degrees, these cohomology groups $H^{\bullet}_{\Pi}(M)$ have geometric interpretations. For our purposes, it suffices to know that $H^1_{\Pi}(M)$ is the quotient of the space of Poisson vector fields by the space of Hamiltonian vector fields:

$$H^1_{\Pi}(M) = \frac{\{X \in \Gamma(TM) :\ \mathcal{L}_X \Pi = 0\}}{\{\Pi^{\sharp}(df) :\ f \in C^{\infty}(M)\}}.$$

The equation (4) shows that the Poisson cohomology class $[V^{\mu}_{mod}] \in H^1_{\Pi}(M)$ is intrinsically defined, i.e. it is independent of the chosen volume form. This is the **modular class** of (M, Π). The modular class vanishes exactly when there is a volume form on M that is invariant under all Hamiltonian vector fields. In that case (M, Π) is called unimodular. For instance, a symplectic manifold (M^{2n}, ω) is unimodular; an invariant volume form is given by $\wedge^n \omega$.

In case the Poisson structure Π is symplectic, then the Poisson cohomology of (M, Π) is canonically isomorphic with the de Rham cohomology of M. We should note however that in general, the computation of Poisson cohomology is by no means standard at this point due to lack of general methods.

Cotangent Lie algebroid

For a Poisson manifold (M,Π), it is in some respects possible to work with its cotangent bundle T^*M as if it were the tangent bundle TM. Both are related by the bundle map $\Pi^\sharp$ (see (3)). The reason why this works is that also $\Gamma(T^*M)$ carries a Lie bracket $[\cdot,\cdot]_\Pi$, defined by

$$[\alpha,\beta]_\Pi := \pounds_{\Pi^\sharp(\alpha)}\beta - \pounds_{\Pi^\sharp(\beta)}\alpha - d(\Pi(\alpha,\beta)).$$

The map $\Pi^\sharp$ and the bracket $[\alpha,\beta]_\Pi$ are related through the Leibniz rule

$$[\alpha,f\beta]_\Pi = f[\alpha,\beta]_\Pi + \pounds_{\Pi^\sharp(\alpha)}(f)\beta.$$

In the spirit of using T^*M as a tangent bundle, the symplectic leaves of (M,Π) can be described in terms of paths with “contravariant speed”. Namely, two points $p,q\in M$ lie in the same leaf when they are joined by a **cotangent path** $(a(t),\gamma(t))$, meaning that $\gamma:[0,1]\to M$ is a path joining p and q, and $a:[0,1]\to T^*M$ is a path lying over γ such that

$$\Pi^\sharp(a(t)) = \gamma'(t).$$

All of this fits in the more general notion of **Lie algebroid**. A Lie algebroid is a vector bundle $A\to M$, together with a Lie bracket $[\cdot,\cdot]$ on its space of sections $\Gamma(A)$ and a bundle map $\rho: A\to TM$, satisfying the compatibility condition

$$[X,fY] = f[X,Y] + \pounds_{\rho(X)}(f)Y \qquad \forall X,Y\in\Gamma(A),\ f\in C^\infty(M). \tag{5}$$

The map $\rho: A\to TM$ is called the **anchor map**. The identity (5) implies in particular that $\rho:\Gamma(A)\to\mathfrak{X}(M)$ is a Lie algebra morphism.

Standard examples of Lie algebroids are tangent bundles, Lie algebras, and involutive distributions. Also the cotangent bundle of a Poisson manifold (M,Π) inherits a Lie algebroid structure: what we described above is nothing else but the **cotangent Lie algebroid** $(T^*M,\Pi^\sharp,[\cdot,\cdot]_\Pi)$ of (M,Π).

Any Lie algebroid $A\to M$ comes with an integrable (singular) distribution $\rho(A)$, which gives rise to a singular foliation on the base M. Two points $p,q\in M$ lie in the same leaf exactly when they are joined by an **A-path** $(a(t),\gamma(t))$ [CrFe]. This means that $\gamma:[0,1]\to M$ is a path joining p and q, and $a:[0,1]\to A$ is a path lying over γ such that

$$\rho(a(t)) = \gamma'(t).$$

We also remark here that a Lie algebroid $(A,\rho,[\cdot,\cdot])$ has an associated complex of differential forms $\big(\Gamma(\wedge^\bullet A^*),d_A\big)$, where the differential is defined by the familiar Koszul formula

$$d_A\alpha(X_0,\ldots,X_k) = \sum_{i=0}^{k}(-1)^i \pounds_{\rho(X_i)}\big(\alpha(X_0,\ldots,\hat{X}_i,\ldots,X_k)\big) + \sum_{0\le i<j\le k}(-1)^{i+j}\alpha([X_i,X_j],X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_k).$$

The associated cohomology $H^\bullet(A)$ is the **Lie algebroid cohomology** of A. For the cotangent Lie algebroid $(T^*M,\Pi^\sharp,[\cdot,\cdot]_\Pi)$ of a Poisson manifold (M,Π), this is just the Poisson cohomology as introduced above.

One case is of special interest in Chapter 2. As we remarked before, if $\mathcal{F}$ is a (regular) foliation on M then $T\mathcal{F}$ is a Lie algebroid (the anchor is the inclusion map $T\mathcal{F} \hookrightarrow TM$ and the bracket is just the restriction of the Lie bracket of vector fields). The associated differential complex $\big(\Gamma(\wedge^\bullet T^*\mathcal{F}), d_\mathcal{F}\big)$ consists of **foliated differential forms**. Its cohomology is the **foliated cohomology**, which we will denote by $H^\bullet(\mathcal{F})$.

Submanifolds

Given a Poisson manifold (M,Π), there are various interesting ways in which a submanifold $X\subset(M,\Pi)$ can interact with the Poisson tensor Π. Different classes of submanifolds $X\subset(M,\Pi)$ are conveniently described in terms of the "Π-orthogonal" of X, which is defined as

$$TX^{\perp_\Pi} := \Pi^\sharp(TX^0).$$

Geometrically, the Π-orthogonal $T_xX^{\perp_\Pi}$ at a point $x\in X$ is the symplectic orthogonal of $T_xX\cap T_x\mathcal{O}$ in the symplectic vector space $T_x\mathcal{O}$, where $\mathcal{O}$ is the symplectic leaf through x. In particular, if Π is symplectic then this notion reduces to the usual symplectic orthogonal of X.

We now list the types of submanifolds that occur in this thesis. For a more complete overview of submanifolds in Poisson geometry, we refer to [CFM], [Z].

- A **Poisson submanifold** is a submanifold $X\subset(M,\Pi)$ for which Π is tangent to X, or equivalently, for which $TX^{\perp_\Pi}=0$. The restriction $\Pi|_X$ defines a Poisson structure on X.

- A **Lagrangian submanifold** is a submanifold $X\subset(M,\Pi)$ such that for all $x\in X$,
$$T_xX^{\perp_\Pi} = T_xX\cap T_x\mathcal{O},$$

where $\mathcal{O}$ is the symplectic leaf through x. This means that $T_xX \cap T_x\mathcal{O}$ is a Lagrangian subspace in the symplectic vector space $T_x\mathcal{O}$. Lagrangian submanifolds are the main objects of study in Chapter 2.

- A **coisotropic submanifold** is a submanifold $X \subset (M, \Pi)$ such that for all $x \in X$,
$$(TX^{\perp_\Pi})_x \subset T_xX \cap T_x\mathcal{O},^2$$
where $\mathcal{O}$ is the symplectic leaf through x. This means that $T_xX \cap T_x\mathcal{O}$ is a coisotropic subspace in the symplectic vector space $T_x\mathcal{O}$. We discuss coisotropic submanifolds in Chapter 1.
- We call a submanifold $X \subset (M, \Pi)$ **regular** if $TX^{\perp_\Pi}$ has constant rank. Regular submanifolds are the central concept of Chapter 3.
- A **transversal** is a submanifold $X \subset (M, \Pi)$ transverse to the leaves. Equivalently, $TX^{\perp_\Pi}$ has constant rank equal to $codim(X)$.
- A **Poisson transversal** is a submanifold $X \subset (M, \Pi)$ that intersects each leaf transversally and symplectically, that is
$$TM|_X = TX \oplus TX^{\perp_\Pi}.$$
The submanifold X inherits a Poisson structure from (M, Π). We briefly encounter Poisson transversals in Chapter 3.
- A **pre-Poisson submanifold** is a submanifold $X \subset (M, \Pi)$ for which $TX + TX^{\perp_\Pi}$ has constant rank. These are Poisson analogs of constant rank submanifolds in symplectic geometry. Pre-Poisson submanifolds will also be touched upon in Chapter 3.

As for any geometric structure, an important line of research is the construction of normal forms for Poisson structures. The aim is to find convenient descriptions for them near certain submanifolds. We recall here that in symplectic geometry, the Darboux theorem gives a universal local model for symplectic manifolds of a fixed dimension. In Poisson geometry however there is no such model, hence even the search for normal forms around points is interesting.

A crucial result in this respect is **Weinstein's splitting theorem** [W2], which states that near a point $p \in M$ where $\mathrm{rank}(\Pi_p) = 2k$, one can find coordinates

[2] The standard definition of coisotropic submanifold is a submanifold $X \subset (M, \Pi)$ for which $TX^{\perp_\Pi} \subset TX$. The second condition in our definition, that $TX^{\perp_\Pi}$ is also tangent to the leaves, is automatically satisfied. We wrote the definition this way to make the analogy with the definition of Lagrangian submanifold.

$(x_1, \ldots, x_k, y_1, \ldots, y_k, q_1, \ldots, q_l)$ centered at p such that

$$\Pi = \sum_{i=1}^{k} \partial_{x_i} \wedge \partial_{y_i} + \sum_{1 \leq i < j \leq l} \phi_{i,j}(q) \partial_{q_i} \wedge \partial_{q_j} \tag{6}$$

and $\phi_{i,j}(0) = 0$. This formula shows that locally around a point $p \in M$, the Poisson manifold (M, Π) is a product of a symplectic manifold with a Poisson manifold whose Poisson tensor vanishes at p. This has important consequences for the construction of normal forms around points. First, it shows that around regular points (i.e. points that have a neighborhood on which the Poisson structure is of constant rank) there is a local model. Second, since we can "peel off" coordinates that give a partial local form (the first summand in (6)), we only need to find normal forms for Poisson structures around points $p \in M$ where Π_p vanishes. This leads us to the linearization of Poisson structures, an interesting but hard problem that was partially solved by Conn [Con] and is still an active domain of research [MZ].

Log-symplectic structures

While symplectic structures are the nicest instances of Poisson structures, an arbitrary Poisson manifold is much more complicated than a symplectic one. When trying to obtain explicit results, it is therefore instructive to look at classes of Poisson structures that are "not too far from being symplectic". Various types of mildly degenerate Poisson structures have been introduced and studied in recent years. We address in detail one such class, consisting of so-called log-symplectic structures, which play an important role in this thesis. Their first appearance is in the work of Nest-Tsygan [NT] (in the context of manifolds with boundary) and Radko classified these structures on compact oriented surfaces [R]. Since their systematic study in arbitrary dimension by Guillemin-Miranda-Pires [GMP], log-symplectic structures have attracted a lot of attention from the Poisson community.

Definition. An even-dimensional Poisson manifold (M^{2n}, Π) is called **log-symplectic** if $\wedge^n \Pi$ is transverse to the zero section of the line bundle $\wedge^{2n} TM$.

This notion encompasses symplectic structures, since those are exactly the Poisson structures Π for which $\wedge^n \Pi$ is nowhere zero.

A log-symplectic structure is symplectic everywhere, except at points of its **singular locus** $Z := (\wedge^n \Pi)^{-1}(0)$. When non-empty, Z is a smooth hypersurface in M because of the transversality requirement $\wedge^n \Pi \pitchfork 0$.

The singular locus Z is a Poisson submanifold of (M, Π), and its induced Poisson structure $\Pi|_Z$ is regular of corank-one. This Poisson structure has a special feature: it has a Poisson vector field transverse to the leaves. Such a vector field is obtained (in the orientable case) by restricting a modular vector field of (M, Π) to Z. Equivalent to the existence of a transverse Poisson vector field is the existence of a cosymplectic structure (θ, η) inducing $\Pi|_Z$. This means that the underlying foliation of $\Pi|_Z$ is the kernel of the closed one-form $\theta \in \Omega^1(Z)$, and the closed two-form $\eta \in \Omega^2(Z)$ extends the symplectic form on each leaf.

Example. On $\mathbb{R}^{2n}$ with coordinates $(x_1, y_1, \dots, x_n, y_n)$, the following is a log-symplectic structure:

$$\Pi = \partial_{x_1} \wedge y_1 \partial_{y_1} + \sum_{i=2}^{n} \partial_{x_i} \wedge \partial_{y_i}. \tag{7}$$

The singular locus is $Z = \{y_1 = 0\}$, with Poisson structure $\Pi|_Z = \sum_{i=2}^{n} \partial_{x_i} \wedge \partial_{y_i}$. Its leaves are the level sets of x_1, and ∂_{x_1} is indeed a transverse Poisson vector field. Interestingly, any log-symplectic structure has a coordinate expression like (7) around any point of its singular locus.

An important trait of log-symplectic structures is that they can be studied using symplectic geometry. In fact, they can be seen as symplectic structures on a certain Lie algebroid that serves as a replacement of the tangent bundle. This point of view uses the language of **b-geometry**, which we now explain. Here b stands for boundary, as this formalism was first used by Melrose in the context of differential operators on manifolds with boundary [Me].

The objects of study in b-geometry are so-called **b-manifolds**, which are pairs (M, Z) consisting of a manifold M and a hypersurface $Z \subset M$. Associated with such a pair (M, Z), there is a vector bundle ${}^bTM$ called the **b-tangent bundle**, whose sections are the vector fields on M that are tangent to Z. Since the b-tangent bundle is a Lie algebroid, there is an associated complex $\big(\Gamma(\wedge^\bullet({}^bT^*M), {}^bd\big)$ of differential forms on ${}^bTM$, which we call **b-forms**.

Working in coordinates $(x_1, \dots, x_n)$ adapted to $Z = \{x_1 = 0\}$, a local frame for the b-tangent bundle ${}^bTM$ is given by

$$\{x_1 \partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_n}\}.$$

Looking at the coordinate expression (7), we see that Π can be viewed as a non-degenerate element $\Pi \in \Gamma(\wedge^2({}^bTM))$ satisfying $[\Pi, \Pi] = 0$. By taking inverses, this is the same thing as a non-degenerate closed b-two-form $\omega \in \Gamma(\wedge^2({}^bT^*M))$, i.e. a **b-symplectic form**. For instance, the b-symplectic form corresponding with the log-symplectic structure (7) is

$$\omega = dx_1 \wedge \frac{dy_1}{y_1} + \sum_{i=2}^{n} dx_i \wedge dy_i.$$

This is a symplectic form with a logarithmic singularity, which justifies the name "log-symplectic". In conclusion, a log-symplectic structure can be viewed as a symplectic form on the b-tangent bundle, which enables the use of symplectic techniques in the study of log-symplectic structures.

The names "log-symplectic" and "b-symplectic" are used interchangeably in the literature, but we stress that they refer to the same objects. To leave no room for confusion, we emphasize:

$$\textbf{log-symplectic} = \textbf{b-symplectic}.$$

We will adopt the terminology "log-symplectic" when using the Poisson point of view (i.e. when working with the bivector field $\Pi \in \Gamma(\wedge^2 TM)$). We adopt the terminology "b-symplectic" when we use the b-geometry point of view (i.e. when working with the singular two-form $\omega \in \Gamma(\wedge^2({}^bT^*M))$).

Dirac geometry

We now recall some background information concerning Dirac geometry, which is an important ingredient in Chapter 3. Dirac structures first appeared in the work of Ted Courant [Cou], who was a student of Alan Weinstein at the time. Dirac geometry is useful because it provides a unifying framework for several geometric structures: foliations, closed two-forms and Poisson structures. Another advantage is that passing to Dirac geometry provides us with operations that are generally not available in the geometries just described. For instance, there is no sensible notion of pullback in Poisson geometry; however, under appropriate conditions, a Poisson structure can be pulled back as a Dirac structure, and in favorable cases the result is again a Poisson structure. For more information about Dirac manifolds, we recommend [B].

Dirac structures

Given a manifold M, the vector bundle $TM \oplus T^*M$ carries the following operations:

- a symmetric pairing $\langle\langle \cdot, \cdot \rangle\rangle$ on each fiber, given by
$$\langle\langle V + \alpha, W + \beta \rangle\rangle := \alpha(W) + \beta(V).$$
- the Dorfman bracket $[\![\cdot, \cdot]\!]$ on its space of sections, defined by
$$[\![V + \alpha, W + \beta]\!] := [V, W] + \pounds_V \beta - \iota_W d\alpha.$$

Definition. A subbundle $L \subset TM \oplus T^*M$ is **Lagrangian** if it is maximally isotropic with respect to $\langle\langle \cdot, \cdot \rangle\rangle$, i.e. $\langle\langle L, L \rangle\rangle = 0$ and $\mathrm{rank}(L) = \dim M$.

A **Dirac structure** is a Lagrangian subbundle $L \subset TM \oplus T^*M$ that is involutive with respect to $[\![\cdot, \cdot]\!]$, i.e. $[\![\Gamma(L), \Gamma(L)]\!] \subset \Gamma(L)$.

The Dorfman bracket $[\![\cdot, \cdot]\!]$ satisfies the "Jacobi identity in Leibniz form"

$$[\![a_1, [\![a_2, a_3]\!]]\!] = [\![[\![a_1, a_2]\!], a_3]\!] + [\![a_2, [\![a_1, a_3]\!]]\!] \qquad \forall a_1, a_2, a_3 \in \Gamma(TM \oplus T^*M), \tag{8}$$

but it is not skew-symmetric. However, its restriction to a Dirac structure $L \subset TM \oplus T^*M$ is skew-symmetric, and therefore (8) reduces to the usual Jacobi identity on elements of $\Gamma(L)$. Hence, $(\Gamma(L), [\![\cdot, \cdot]\!])$ is a Lie algebra. Even more is true: denoting by $pr_T : L \to TM$ the projection to the tangent bundle, we have that $(\Gamma(L), pr_T, [\![\cdot, \cdot]\!])$ is a **Lie algebroid**.

In particular, the distribution $pr_T(L) \subset TM$ gives rise to a singular foliation on M. We call it the **presymplectic foliation** associated with L, since each leaf has an induced closed two-form. Two points $p, q \in M$ lie in the same leaf when they are joined by an L-path, i.e. when there exists a path $\gamma : [0, 1] \to M$ joining p and q, and a path $a : [0, 1] \to L$ covering γ such that

$$pr_T(a(t)) = \gamma'(t).$$

Examples

- <u>Foliations:</u> A distribution $D \subset TM$ defines a Lagrangian subbundle
$$L_D := D \oplus D^0$$
of $(TM \oplus T^*M, \langle\langle \cdot, \cdot \rangle\rangle)$. This subbundle is a Dirac structure exactly when D is involutive, i.e. when D is the tangent distribution of a foliation. Foliations are those Dirac structures L for which $L = pr_T(L) \oplus pr_{T^*}(L)$.

- <u>Closed two-forms:</u> A two-form $\omega \in \Omega^2(M)$ defines a Lagrangian subbundle
$$Gr(\omega) := \{v + \iota_v \omega : v \in TM\}$$
of $(TM \oplus T^*M, \langle\langle \cdot, \cdot \rangle\rangle)$. This subbundle is a Dirac structure exactly when $d\omega = 0$. Closed two-forms are the Dirac structures L that are transverse to T^*M, i.e. those for which $pr_T(L) = TM$.

- <u>Poisson structures:</u> A bivector $\Pi \in \mathfrak{X}^2(M)$ defines a Lagrangian subbundle
$$L_\Pi := \{\Pi^\sharp(\alpha) + \alpha : \alpha \in T^*M\}$$
of $(TM \oplus T^*M, \langle\langle \cdot, \cdot \rangle\rangle)$. This subbundle is a Dirac structure exactly when Π is Poisson. Poisson structures are the Dirac structures L that are transverse to TM, i.e. those for which $pr_{T^*}(L) = T^*M$.

Maps in Dirac geometry

A map between Dirac manifolds $\varphi : (M_1, L_1) \to (M_2, L_2)$ is called **forward Dirac** if for all $p \in M_1$:

$$(L_2)_{\varphi(p)} = \{d_p\varphi(v) + \alpha : \ v + (d_p\varphi)^*\alpha \in (L_1)_p\}.$$

Similarly, the map $\varphi : (M_1, L_1) \to (M_2, L_2)$ is called **backward Dirac** if for all points $p \in M_1$:

$$(L_1)_p = \{v + (d_p\varphi)^*\alpha : \ d_p\varphi(v) + \alpha \in (L_2)_{\varphi(p)}\}. \tag{9}$$

A Poisson map $\varphi : (M_1, \Pi_1) \to (M_2, \Pi_2)$ is forward Dirac. If $\omega \in \Omega^2(M_2)$ is a closed two-form, then $\varphi : (M_1, Gr(\varphi^*\omega)) \to (M_2, Gr(\omega))$ is backward Dirac.

There is a notion of pullback of Dirac structures that we will frequently use. If $\varphi : M_1 \to (M_2, L_2)$ is a smooth map into a Dirac manifold (M_2, L_2) such that the family of vector spaces

$$d_p\varphi(T_pM_1) + pr_T(L_2)_{\varphi(p)} \subset T_{\varphi(p)}M_2$$

has constant dimension for $p \in M_1$, then the right hand side of (9) defines a Dirac structure on M_1. We denote this **pullback Dirac structure** by

$$(\varphi^*L_2)_p := \{v + (d_p\varphi)^*\alpha \in T_pM \oplus T_p^*M : \ d_p\varphi(v) + \alpha \in (L_2)_{\varphi(p)}\}.$$

This construction is relevant also in Poisson geometry. We give two examples:

- If $i : X \hookrightarrow (M, \Pi)$ is a Poisson transversal, then the pullback Dirac structure i^*L_Π defines a Poisson structure on X.
- If $i : L \hookrightarrow (M, \Pi)$ is a regular Lagrangian, then the pullback Dirac structure i^*L_Π is the foliation integrating the distribution $TL^{\perp_\Pi}$ on L.

Operations in Dirac geometry

We mention two ways of constructing new Dirac structures out of old ones.

First, if (M, L) is a Dirac manifold and $\omega \in \Omega^2(M)$ is a closed two-form, then we can define a new Dirac structure

$$L^\omega := \{v + \alpha + \iota_v\omega : \ v + \alpha \in L\},$$

called the **gauge transformation** of L by ω. The presymplectic foliations of L and L^ω have the same leaves, but the presymplectic forms of L^ω are obtained by adding the restriction of ω to those of L.

Second, if (M, L) is a Dirac manifold and $\lambda \neq 0$ is a real number, then we can define the **rescaled Dirac structure**

$$\lambda L := \{v + \lambda\alpha : \ v + \alpha \in L\}.$$

Geometrically, this corresponds with taking the presymplectic foliation of L and rescaling the presymplectic forms by λ. As a particular case, the **opposite Dirac structure** of L is $-L := (-1)L$.

Overview of the results

We motivate the problems that are considered in this thesis, and we briefly explain our main results. In a nutshell, we prove three normal form theorems for Poisson structures around certain kinds of submanifolds, and we use one of them to study the deformation theory of the submanifolds in question. The following overview is based on the introductory texts of the individual chapters.

Chapter 1

The central objects of study in Chapter 1 are coisotropic submanifolds of log-symplectic manifolds. Our motivation is the prominent role played by coisotropic submanifolds in Poisson geometry; for instance, they arise naturally as graphs of Poisson maps, they support Lie subalgebroids of the cotangent Lie algebroid, and they admit a natural quotient which inherits a Poisson structure.

We study two classes of coisotropic submanifolds of log-symplectic manifolds, which we call **b-coisotropic** and **strong b-coisotropic**. These submanifolds inherit a specified hypersurface from the log-symplectic structure, so they can be studied using the b-geometry point of view. We show that they enjoy some properties that can be seen as b-geometric versions of well-known statements about coisotropic submanifolds in symplectic and Poisson geometry.

b-coisotropic submanifolds

We call a submanifold C of a log-symplectic manifold (M, Z, Π) **b-coisotropic** if it is coisotropic and transverse to the singular locus Z. In particular, such a submanifold is itself a b-manifold $(C, C \cap Z)$. Prop. 1.3.4 shows that the coisotropicity condition can be rephrased as $({}^bTC)^{\omega} \subset {}^bTC$, where ω is the b-symplectic form corresponding with Π, indicating that these submanifolds are natural coisotropic objects in b-symplectic geometry.

We show that b-coisotropic submanifolds satisfy some b-geometric enhancements of classical results concerning coisotropic submanifolds in Poisson geometry. First, noticing that the b-cotangent bundle ${}^bT^*M$ of a b-symplectic manifold (M,Z,ω) is a Lie algebroid, we show that the b-conormal bundle $({}^bTC)^0 \subset {}^bT^*M$ of a b-coisotropic submanifold C is a Lie subalgebroid (Prop. 1.3.6).
Second, while the product of log-symplectic manifolds is not log-symplectic in general, a suitable blow-up of the product is. We show that graphs of suitable Poisson maps between log-symplectic manifolds are b-coisotropic submanifolds, once lifted to this blow-up (Prop. 1.3.8).

Our main result concerning b-coisotropic submanifolds is an extension of Gotay's theorem [G] from symplectic geometry. As a consequence, we obtain a normal form around them (Prop. 1.3.15).

Theorem 1A. *Given a b-coisotropic submanifold $C \overset{i}{\hookrightarrow} (M,Z,\omega)$, choose a complement G so that ${}^bTC = {}^bTC^\omega \oplus G$ and denote by $j : ({}^bTC^\omega)^* \hookrightarrow {}^bT^*C$ the inclusion. A neighborhood of C in (M,Z,ω) is b-symplectomorphic with a neighborhood of C in $\big(({}^bTC^\omega)^*, \Omega_G\big)$. Here Ω_G is a b-symplectic form on a neighborhood of the zero section, given by*

$$\Omega_G := {}^bp^*({}^bi^*\omega) + {}^bj^*\omega_{can},$$

where $p : ({}^bTC^\omega)^* \to C$ is the bundle projection and ω_{can} denotes the canonical b-symplectic form on ${}^bT^*C$.

In particular, around a b-coisotropic submanifold C, the b-symplectic form ω is determined by its pullback to C, up to b-symplectomorphism.

Strong b-coisotropic submanifolds

We then focus attention on a specific kind of b-coisotropic submanifolds. We call a submanifold C of a log-symplectic manifold (M,Z,Π) **strong b-coisotropic** if it is coisotropic and transverse to the leaves of (M,Π).

The main reason for looking at this kind of coisotropic submanifolds is the fact that their characteristic distribution $TC^{\perp_\Pi} \subset TC$ is regular, which opens the door for reduction. Indeed, a well-known fact in Poisson geometry states that when the quotient of a coisotropic submanifold by its characteristic distribution is smooth, it inherits a Poisson structure. We observe that, applying this procedure to a strong b-coisotropic submanifold of a log-symplectic manifold, the Poisson structure on the quotient is again log-symplectic (Prop. 1.4.6).

Proposition 1B. *Let C be a strong b-coisotropic submanifold of a log-symplectic manifold. If the quotient of C by its characteristic distribution is smooth, then the reduced Poisson structure is again log-symplectic.*

Group actions $G \curvearrowright (M, Z, \Pi)$ admitting a moment map $J : M \to \mathfrak{g}^*$ provide a source of examples for this reduction procedure. Under appropriate assumptions, the zero level set $J^{-1}(0)$ is strong b-coisotropic, and applying Theorem 1B yields a log-symplectic structure on the quotient $J^{-1}(0)/G$ (Cor. 1.4.10).

At last, we reverse the procedure by realizing compact log-symplectic manifolds through moment map reduction for S^1-actions (Prop. 1.4.14). We work out the details in the case of the two-sphere (Cor. 1.4.16).

Chapter 2

In Chapter 2, we again deal with log-symplectic manifolds, but now we are interested in Lagrangian submanifolds, and more precisely their deformation theory. Our motivation comes from symplectic geometry, as we now explain.

Recall that, by Weinstein's Lagrangian neighborhood theorem [W1], a neighborhood of a Lagrangian submanifold L in a symplectic manifold (M, ω) is symplectomorphic with a neighborhood of L in (T^*L, ω_{can}). To study the deformations of the Lagrangian submanifold L, it is convenient to work in the local model (T^*L, ω_{can}). The following is well-known [MS]:

- The graph of a section $\alpha \in \Gamma(T^*L)$ is Lagrangian in (T^*L, ω_{can}) exactly when $d\alpha = 0$.
- Two Lagrangian sections $\alpha_0, \alpha_1 \in \Gamma(T^*L)$ are interpolated by a family of Lagrangian sections $(\alpha_t)_{t \in [0,1]}$ generated by a Hamiltonian isotopy exactly when $[\alpha_0] = [\alpha_1]$ in $H^1(L)$.

So the moduli space of Lagrangian deformations under Hamiltonian equivalence can be identified with (a neighborhood in) the first de Rham cohomology $H^1(L)$. In particular, it is smooth and even finite dimensional if L is compact. These results spark the following question, which we pursue in Chapter 2:

Is the deformation theory of Lagrangian submanifolds in log-symplectic geometry as well-behaved as in symplectic geometry?

For Lagrangians transverse to the singular locus Z, the answer is easily seen to be positive [K], since b-geometry applies in that case. Indeed, specializing Theorem 1A to Lagrangian submanifolds gives a b-geometric version of Weinstein's Lagrangian neighborhood theorem, which implies that Lagrangian deformations of L are classified by ${}^bH^1(L) \cong H^1(L) \oplus H^0(L \cap Z)$.

We focus on Lagrangians at the other extreme, i.e. those contained in the singular locus. For such a Lagrangian $L \subset Z$, there are two possibilities: either it is $(n-1)$-dimensional and its connected components lie inside symplectic leaves of Z, or it is n-dimensional and it is transverse to the leaves of Z. The former are less interesting for our purposes; evidently their deformations stay inside Z, so in effect we would be dealing with a deformation problem inside a nice corank-one Poisson manifold of cosymplectic type. That is why we only look at **Lagrangian submanifolds $\mathbf{L^n}$ contained in the singular locus of Z of an orientable log-symplectic manifold** $(\mathbf{M^{2n}}, \mathbf{\Pi})$. Such a Lagrangian inherits a foliation $\mathcal{F}_L$, which will play an important role in the sequel.

Since the submanifolds considered in this chapter are opposite to those in the previous chapter, also the techniques involved are different: *b*-geometry does not apply due to lack of transversality, so we use the Poisson geometric point of view instead. Despite being polar opposites, we note that a Lagrangian $L^n \subset Z$ can often be deformed into one transverse to Z. The result is then a *b*-coisotropic submanifold, which is strong *b*-coisotropic only when it does not intersect Z.

Normal form

To facilitate the study of Lagrangian deformations, we first construct a normal form for the log-symplectic structure around Lagrangian submanifolds in the singular locus. This is done in two steps:

1. The local model for a Poisson manifold around a Lagrangian L transverse to the leaves is $(T^*\mathcal{F}_L, \Pi_{can})$ (Prop. 2.2.9). Here Π_{can} is a canonical Poisson structure, obtained out of the symplectic structure on T^*L.
2. It is well-known [GMP],[O] that the local model for an orientable log-symplectic manifold (M, Z, Π) around the singular locus Z is
$$\big(Z \times \mathbb{R}, V_{mod}|_Z \wedge t\partial_t + \Pi|_Z\big),$$
where V_{mod} is a modular vector field on (M, Π) and t is the $\mathbb{R}$-coordinate.

The local model in 2. only depends on the Poisson cohomology class of $V_{mod}|_Z$, up to Poisson diffeomorphisms in a neighborhood of Z. Combining 1. and 2., we obtain a normal form around L which depends on a class in $H^1_{\Pi_{can}}(T^*\mathcal{F}_L)$. We compute this cohomology group in terms of data attached to L (Cor. 2.3.5):

$$H^1(\mathcal{F}_L) \times \frac{\mathfrak{X}(L)^{\mathcal{F}_L}}{\Gamma(T\mathcal{F}_L)} \xrightarrow{\sim} H^1_{\Pi_{can}}(T^*\mathcal{F}_L) : ([\gamma], [X]) \mapsto \Big[\Pi^\sharp_{can}(p^*\gamma) + \widetilde{X}\Big].$$

This way, the class $[V_{mod}|_Z] \in H^1_{\Pi|_Z}(Z)$ is encoded by $([\gamma], [X])$, where γ is a closed foliated one-form on L and X is a vector field on L that preserves $\mathcal{F}_L$ and is nowhere tangent to it. The conclusion is the following normal form theorem.

Theorem 2A. *The log-symplectic structure is isomorphic around L with*

$$\Big(U \subset T^*\mathcal{F}_L \times \mathbb{R}, \left(\Pi_{can}^\sharp(p^*\gamma) + \widetilde{X}\right) \wedge t\partial_t + \Pi_{can}\Big),$$

where U is a neighborhood of the zero section L. Here $p : T^*\mathcal{F}_L \to L$ is the projection and $\widetilde{X}$ denotes the cotangent lift of X to $T^*\mathcal{F}_L$.

Algebraic aspects of the deformation problem

We use this normal form to study the Lagrangian deformations of L. We can characterize them algebraically as follows (Thm. 2.4.3, Cor. 2.4.10).

Theorem 2B. *Lagrangian deformations $\mathcal{C}^1$-close to L are exactly the graphs of sections (α, f) of the vector bundle $T^*\mathcal{F}_L \times \mathbb{R} \to L$ satisfying the quadratic equation*

$$\begin{cases} d_{\mathcal{F}_L}\alpha = 0 \\ d_{\mathcal{F}_L} f + f(\gamma - \pounds_X \alpha) = 0, \end{cases} \tag{10}$$

where $d_{\mathcal{F}_L}$ denotes the foliated de Rham differential and γ, X are as above. Further, this equation is the Maurer-Cartan equation of a DGLA.

The DGLA in question is obtained by specializing an L_∞-algebra introduced in greater generality by Cattaneo and Felder [CaFe] to our situation. It is supported on the graded vector space $\Gamma\left(\wedge^\bullet\left(T^*\mathcal{F}_L \times \mathbb{R}\right)\right) = \Gamma\left(\wedge^\bullet T^*\mathcal{F}_L \oplus \wedge^{\bullet-1} T^*\mathcal{F}_L\right)$ and its structure maps $(d, [\![\cdot,\cdot]\!])$ are defined by

$$\begin{aligned} d : \Gamma\left(\wedge^k\left(T^*\mathcal{F}_L \times \mathbb{R}\right)\right) &\to \Gamma\left(\wedge^{k+1}\left(T^*\mathcal{F}_L \times \mathbb{R}\right)\right) : \\ &(\alpha,\beta) \mapsto \left(-d_{\mathcal{F}_L}\alpha, -d_{\mathcal{F}_L}\beta - \gamma\wedge\beta\right), \end{aligned} \tag{11}$$

$$\begin{aligned} [\![\cdot,\cdot]\!] : \Gamma\left(\wedge^k\left(T^*\mathcal{F}_L \times \mathbb{R}\right)\right) \otimes \Gamma\left(\wedge^l\left(T^*\mathcal{F}_L \times \mathbb{R}\right)\right) &\to \Gamma\left(\wedge^{k+l}\left(T^*\mathcal{F}_L \times \mathbb{R}\right)\right) : \\ (\alpha,\beta)\otimes(\delta,\epsilon) &\mapsto \left(0, \pounds_X\alpha \wedge \epsilon - (-1)^{kl}\pounds_X\delta\wedge\beta\right). \end{aligned}$$

Notice that in the second component of (11), an exotic differential appears:

$$d_{\mathcal{F}_L}\bullet + \gamma\wedge\bullet : \Gamma(\wedge^k T^*\mathcal{F}_L) \to \Gamma(\wedge^{k+1}T^*\mathcal{F}_L),$$

obtained by twisting the foliated de Rham differential with the closed foliated one-form γ. We denote its cohomology by $H^\bullet_\gamma(\mathcal{F}_L)$, and we refer to it as the **foliated Morse-Novikov cohomology**.

Geometric aspects of the deformation problem

We use Theorem 2B above to study some geometric aspects of the deformation problem. We restrict ourselves to Lagrangians L that are compact and connected, which has consequences for the foliation $\mathcal{F}_L$: either it is the foliation by fibers of a fibration $L \to S^1$, or all leaves are dense (see §2.6.3). By considering both cases separately, we are able to obtain explicit results.

First, we look for conditions under which small deformations of the Lagrangian necessarily stay inside the singular locus. We notice that if γ is foliated exact (i.e. if there is a modular vector field tangent to L), then one can always smoothly deform the Lagrangian to one outside the singular locus (Prop. 2.5.2). At the opposite end of the spectrum, we have (Cor. 2.5.5, Prop. 2.5.10):

Theorem 2C. *Assume L is compact and connected.*

i) Suppose $\mathcal{F}_L$ is the fiber foliation of a fiber bundle $p : L \to S^1$. If for every leaf B of $\mathcal{F}_L$ we have $[\gamma|_B] \neq 0 \in H^1(B)$, then $\mathcal{C}^1$-small deformations of L necessarily stay inside the singular locus.

ii) Suppose $\mathcal{F}_L$ has dense leaves, and that $H^1(\mathcal{F}_L)$ is finite dimensional. If $\gamma \in \Omega^1_{cl}(\mathcal{F}_L)$ is not exact, then $\mathcal{C}^\infty$-small deformations of L necessarily stay inside the singular locus.

To clarify the statement above, we remark that we identify Lagrangian deformations with their corresponding sections of $T^*\mathcal{F}_L \times \mathbb{R}$, and that the space of sections $\Gamma(T^*\mathcal{F}_L \times \mathbb{R})$ carries natural $\mathcal{C}^k$- and $\mathcal{C}^\infty$-topologies.
The results of Theorem 2C rely on a computation of the zeroth foliated Morse-Novikov cohomology (Thm. 2.4.16). To see why, note that for a Lagrangian deformation $(\alpha, f) \in \Gamma(T^*\mathcal{F}_L \times \mathbb{R})$ of L, the function f indicates whether $\text{Graph}(\alpha, f)$ escapes from the singular locus. The equations (10) show that $f \in H^0_{\gamma - \pounds_X \alpha}(\mathcal{F}_L)$, so we should require that this cohomology group is zero for small enough $\alpha \in \Omega^1_{cl}(\mathcal{F}_L)$. The case with dense leaves in Theorem 2C is subtle; we show that the finite dimensionality assumption on $H^1(\mathcal{F}_L)$ is necessary, exhibiting an example in which $\mathcal{F}_L$ is the Kronecker foliation on the torus for which the slope is a Liouville number $\lambda \in \mathbb{R} \setminus \mathbb{Q}$ (Lemma 2.5.8 et seq.).

Second, we study obstructedness of first order deformations. A first order deformation of L is a solution (α, f) of the linearization of the equations (10), i.e.

$$\begin{cases} d_{\mathcal{F}_L}\alpha = 0 \\ d_{\mathcal{F}_L} f + f\gamma = 0 \end{cases}.$$

We call such (α, f) formally/smoothly **unobstructed** if it is tangent to a formal/smooth path of Lagrangian deformations of L. We show that the deformation problem is formally (hence smoothly) obstructed in general, by applying the classical Kuranishi criterium: we exhibit an example for which the **Kuranishi map**

$$Kr : H^1(\mathcal{F}_L) \oplus H^0_\gamma(\mathcal{F}_L) \to H^1_\gamma(\mathcal{F}_L)$$

is not identically zero (Ex. 2.5.11). In general, the Kuranishi criterium is a necessary – but not sufficient – condition for formal unobstructedness. In our situation however, we obtain the following striking result (Prop. 2.5.18, Cor. 2.5.20, Rem. 2.5.14).

Theorem 2D. *Let L be compact and connected. The following are equivalent:*

- *A first order deformation (α, f) is smoothly unobstructed,*
- $Kr\big([(\alpha, f)]\big) = 0$*,*
- $\pounds_X\alpha$ *is foliated exact on* $L \setminus \mathcal{Z}_f$*,*
- α *extends to a closed one-form on* $L \setminus \mathcal{Z}_f$*.*

Here $\mathcal{Z}_f$ denotes the zero locus of f.

The second condition above is remarkable because $Kr\big([(\alpha, f)]\big)$ is the obstruction to constructing the quadratic term in a formal power series solution of (10) prolonging (α, f). Somehow, its vanishing is enough to ensure the existence of a *smooth* path of solutions to (10) that prolongs (α, f). The third condition above is more useful for practical purposes because, while the Kuranishi map takes values in the exotic cohomology group $H^1_\gamma(\mathcal{F}_L)$, this statement concerns ordinary foliated cohomology. The last condition above seems interesting because it is independent of the data (γ, X) coming from the modular vector field.

At last, we address moduli spaces of deformations. From a geometric point of view, it is natural to identify Lagrangian deformations of L if they are related by a Hamiltonian isotopy. We show that this equivalence relation coincides with the gauge equivalence of Maurer-Cartan elements of the DGLA mentioned after Theorem 2B. As a consequence, the resulting moduli space $\mathcal{M}^{Ham}$ has formal tangent space at $[L]$ given by

$$T_{[L]}\mathcal{M}^{Ham} = H^1(\mathcal{F}_L) \oplus H^0_\gamma(L). \tag{12}$$

Usually this is an infinite dimensional vector space, whereas the formal tangent space at classes $[L]$ for L lying in $M \setminus Z$ is finite dimensional for compact L. Hence, the moduli space $\mathcal{M}^{Ham}$ is usually not smooth at $[L]$. Moreover, the formal tangent space (12) is never zero, indicating that Hamiltonian equivalence is too restrictive for rigidity purposes. That is why we also consider the more flexible equivalence relation by Poisson isotopies. We compute the formal tangent space $T_{[L]}\mathcal{M}^{Poiss}$ in Prop. 2.5.30, and we show that rigidity does occur in this setting. In fact, if the Lagrangian L is compact with dense leaves, then infinitesimal rigidity (i.e. vanishing of $T_{[L]}\mathcal{M}^{Poiss}$) implies rigidity with respect to the $\mathcal{C}^{\infty}$-topology (Prop. 2.5.34).

Chapter 3

Unlike Chapters 1 and 2, Chapter 3 is general Poisson geometric. We study regular submanifolds of arbitrary Poisson manifolds, i.e. those submanifolds $X \subset (M, \Pi)$ for which $TX^{\perp_\Pi}$ has constant rank. Our motivation again comes from symplectic geometry. It is a well-known fact that, given a symplectic manifold (M, ω) and a submanifold $X \subset M$, the restriction $\omega|_X$ determines ω up to symplectomorphism on a neighborhood of X [W1]. Classical results by Gotay [G] and Marle [Ma] refine this statement for coisotropic submanifolds and, more generally, for constant rank submanifolds, respectively:

- A neighborhood of a coisotropic submanifold $i : X \hookrightarrow (M, \omega)$ is characterized up to symplectomorphism by the pullback $i^*\omega$.
- A neighborhood of a constant rank submanifold $i : X \hookrightarrow (M, \omega)$ is characterized up to symplectomorphism by the pullback $i^*\omega$ and the symplectic vector bundle $\big(TX^{\omega}/(TX^{\omega} \cap TX), \omega\big)$.

By contrast, given a Poisson manifold (M, Π) and a submanifold $X \subset M$, one cannot expect a neighborhood of X to be determined up to Poisson diffeomorphism by the restriction $\Pi|_X$. Since $\Pi|_X$ only contains information "in the leafwise direction along X", we are led to consider the **saturation** of X, i.e. the union of the leaves intersecting X. The main problem is that the saturation fails to be smooth for arbitrary submanifolds X. To remedy this, we note that the saturation is traced out by Hamiltonian flows in directions normal to X, hence it is natural to impose the following regularity condition on X.

Definition. We call an embedded submanifold $X \subset (M, \Pi)$ **regular** if the map $pr \circ \Pi^{\sharp} : T^*M|_X \to TM|_X/TX$ has constant rank.

It is equivalent to require that $TX^{\perp_\Pi}$ has constant rank, which is the definition used earlier in this introduction. Extreme examples are transversals and Poisson submanifolds of (M, Π), and we show that any regular submanifold is obtained by intersecting such submanifolds (Prop. 3.2.10). Note that if Π is symplectic, then any submanifold of (M, Π) is regular. Our main result about regular submanifolds is the fact that their local saturation is smooth (Thm. 3.2.6).

Theorem 3A. *If $X \subset (M, \Pi)$ is a regular submanifold, then there exists a neighborhood V of X such that the saturation of X inside $(V, \Pi|_V)$ is an embedded Poisson submanifold.*

We refer to this Poisson submanifold as the **local Poisson saturation** of X. We proceed by constructing a normal form for the local Poisson saturation. The local model lives on the vector bundle $(TX^{\perp_\Pi})^*$, and it depends on two choices:

1. A choice of complement W to $TX^{\perp_\Pi}$ inside $TM|_X$. Such a choice yields an inclusion $j : (TX^{\perp_\Pi})^* \hookrightarrow T^*M|_X$.

2. A choice of closed two-form η on a neighborhood of X in $(TX^{\perp_\Pi})^*$, with prescribed restriction $\eta|_X = -\sigma - \tau$ along the zero section $X \subset (TX^{\perp_\Pi})^*$. Here $\sigma \in \Gamma(\wedge^2 TX^{\perp_\Pi})$ and $\tau \in \Gamma(T^*X \otimes TX^{\perp_\Pi})$ are bilinear forms defined as follows, for $\xi_1, \xi_2 \in \big(T_xX^{\perp_\Pi}\big)^*$ and $v_1, v_2 \in T_xX$:

$$\sigma(\xi_1, \xi_2) = \Pi\big(j(\xi_1), j(\xi_2)\big),$$
$$\tau\big((v_1, \xi_1), (v_2, \xi_2)\big) = \langle v_1, j(\xi_2)\rangle - \langle v_2, j(\xi_1)\rangle.$$

To such a complement W and closed extension η, we associate a Dirac structure on the total space of $pr : (TX^{\perp_\Pi})^* \to X$, defined by pulling back Π and gauge transforming with η:

$$(pr^*(i^*L_\Pi))^\eta.$$

This Dirac structure is in fact Poisson on a neighborhood U of X (Prop. 3.3.1), we denote it by $(U, \Pi(W, \eta))$. This Poisson structure is the local model for the local Poisson saturation of X (Prop. 3.3.2 and Thm. 3.4.2).

Theorem 3B. *Let $X \subset (M, \Pi)$ be a regular submanifold. A neighborhood of X in its local Poisson saturation is Poisson diffeomorphic with the local model $\big(U, \Pi(W, \eta)\big)$.*

The proof of Theorem 3B relies on the theory of dual pairs in Dirac geometry, as developed in [FM2]. We make two observations concerning this theorem. First, it shows that the local Poisson saturation of a regular submanifold X is determined around X by the restriction $\Pi|_X$. In particular, if the Poisson structure Π is symplectic, then Theorem 3B recovers the aforementioned result in symplectic geometry stating that a neighborhood of any submanifold $X \subset (M, \omega)$ is determined by the restriction $\omega|_X$ [W1]. Second, since X is a transversal in its local Poisson saturation, the normal form around Dirac transversals [BLM],[FM2] shows that the local Poisson saturation is determined around X by the pullback Dirac structure, up to diffeomorphisms and exact gauge transformations. Our Theorem 3B is consistent with this fact.

For special classes of regular submanifolds $X \subset (M, \Pi)$, good choices for W, η make the normal form more explicit. This way, we obtain the following.

- We recover the normal form around Poisson transversals [FM1],[BLM].

- We obtain a Poisson version of Gotay's theorem, showing that the local Poisson saturation of a regular coisotropic submanifold $i : X \hookrightarrow (M, \Pi)$ is determined around X by the pullback Dirac structure i^*L_Π (Cor. 3.5.2).
- We obtain a Poisson version of Marle's constant rank theorem, showing that the local Poisson saturation of a regular pre-Poisson submanifold $i : X \hookrightarrow (M, \Pi)$ is determined around X by the pullback Dirac structure i^*L_Π and the restriction of Π to $(TX^{\perp_\Pi})^*/(TX^{\perp_\Pi} \cap TX)^*$ (Cor. 3.5.4).

The Poisson version of Gotay's theorem stated above relates Chapter 3 with Chapters 1 and 2. First, recall that Theorem 1A is a b-geometric version of Gotay's theorem, which gives a normal form around b-coisotropic submanifolds of log-symplectic manifolds. Such submanifolds are regular exactly when they are *strong* b-coisotropic, in which case also the second bullet point above gives a normal form around them. We show that these normal forms are consistent (see §3.8.2). Second, a Lagrangian submanifold L transverse to the leaves of any Poisson manifold (M, Π) is regular, and the normal form around them that we used in Chapter 2 (Prop. 2.2.9) is a straightforward consequence of the second bullet point above (see again §3.8.2).

We apply our Poisson version of Gotay's theorem to the coisotropic embedding problem of Dirac manifolds (M, L) into Poisson manifolds (N, Π). The exact conditions for the existence of such an embedding are known [CZ]; a uniqueness result concerning these embeddings was conjectured in [CZ] but only proved under additional regularity assumptions on (M, L). We show that this conjecture follows from our Poisson version of Gotay's theorem (Prop. 3.6.1), which settles the uniqueness in full generality.

At last, we define a notion of regular submanifold in Dirac geometry. We prove a Dirac version of Theorem 3A, showing that the saturation of a regular submanifold inside an open neighborhood is an embedded invariant submanifold (Thm. 3.7.5). We refer to this invariant submanifold as the **local Dirac saturation** of X. We give a normal form for it, which shows that its induced Dirac structure is determined around X by the pullback i^*L, up to diffeomorphisms and exact gauge transformations (Cor. 3.7.7), a result which also follows from the normal form around Dirac transversals [BLM], [FM2].

Bibliography

[B] H. Bursztyn, *A brief introduction to Dirac manifolds*, Geometric and topological methods for quantum field theory, Cambridge University Press, p. 4-38, 2013.

[BLM] H. Bursztyn, H. Lima and E. Meinrenken, *Splitting theorems for Poisson and related structures*, J. Reine Angew. Math. **2019**(754), p. 281-312, 2019.

[CaFe] A.S. Cattaneo and G. Felder, *Relative formality theorem and quantisation of coisotropic submanifolds*, Adv. Math. **208**(2), p. 521-548, 2007.

[CZ] A.S. Cattaneo and M. Zambon, *Coisotropic embeddings in Poisson manifolds*, Trans. Amer. Math. Soc. **361**(7), p. 3721-3746, 2009.

[Con] J.F. Conn, *Normal forms for smooth Poisson structures*, Annals of Math. **121**(3), p. 565-593, 1985.

[Cou] T.J. Courant, *Dirac manifolds*, Trans. Amer. Math. Soc. **319**(2), p. 631-661, 1990.

[CrFe] M. Crainic and R.L. Fernandes, *Integrability of Lie brackets*, Annals of Math. **157**(2), p. 575-620, 2003.

[CFM] M. Crainic, R.L. Fernandes and I. Mărcuţ, *Lectures on Poisson Geometry*, In preparation.

[DZ] J.-P. Dufour and N.T. Zung, *Poisson Structures and their Normal Forms*, Progress in Mathematics **242**, Birkhäuser Basel, 2005.

[FM1] P. Frejlich and I. Mărcuţ, *The normal form theorem around Poisson transversals*, Pacific J. Math. **287**(2), p. 371-391, 2017.

[FM2] P. Frejlich and I. Mărcuţ, *On dual pairs in Dirac geometry*, Math. Z. **289** (1-2), p. 171-200, 2018.

[G] M. J. Gotay, *On coisotropic imbeddings of presymplectic manifolds*, Proc. Amer. Math. Soc. **84**(1), p. 111-114, 1982.

[GMP] V. Guillemin, E. Miranda and A.R. Pires, *Symplectic and Poisson geometry on b-manifolds*, Adv. Math. **264**, p. 864-896, 2014.

[K] C. Kirchhoff-Lukat, *Aspects of Generalized Geometry: Branes with Boundary, Blow-ups, Brackets and Bundles*, Ph.D. Thesis, University of Cambridge, 2018.

[L] A. Lichnerowicz, *Les variétés de Poisson et leurs algèbres de Lie associées*, J. Differential Geom. **12**(2), p. 253-300, 1977.

[MZ] I. Mărcuţ and F. Zeiser, *The Poisson linearization problem for the Lie algebra* $\mathfrak{sl}_2(\mathbb{C}) \cong \mathfrak{so}(3,1)$, In preparation.

[Ma] C.-M. Marle, *Sous-variétés de rang constant d'une variété symplectique*, Astérisque **107-108**, p. 69-86, 1983.

[MS] D. McDuff and D. Salamon, *Introduction to Symplectic Topology*, Oxford Graduate Texts in Mathematics **27**, 3rd edition, 2017.

[Me] R.B. Melrose, *The Atiyah-Patodi-Singer index theorem*, Research Notes in Mathematics **4**, A.K. Peters, Wellesley, 1993.

[NT] R. Nest and B. Tsygan, *Formal deformations of symplectic manifolds with boundary*, J. Reine Angew. Math. **1996**(481), p. 27-54, 1996.

[O] B. Osorno Torres, *Codimension-one symplectic foliations: constructions and examples*, Ph.D. thesis, Utrecht University, 2015.

[P] S.-D. Poisson, *Sur la variation des constantes arbitraires dans les questions de mécanique*, J. Ecole Polytechnique **8**(15), p. 266-344, 1809.

[R] O. Radko, *A classification of topologically stable Poisson structures on a compact oriented surface*, J. Symplectic Geom. **1**(3), p. 523-542, 2001.

[W1] A. Weinstein, *Symplectic manifolds and their Lagrangian submanifolds*, Adv. Math. **6**(3), p. 329-346, 1971.

[W2] A. Weinstein, *The local structure of Poisson manifolds*, J. Differential Geom. **18**(3), p. 523-557, 1983.

[W3] A. Weinstein, *The modular automorphism group of a Poisson manifold*, J. Geom. Phys. **23**(3-4), p. 379-394, 1997.

[Z] M. Zambon, *Submanifolds in Poisson geometry: a survey*, Complex and Differential Geometry, Springer, p. 403-420, 2011.

Chapter 1

Coisotropic submanifolds in b-symplectic geometry

This chapter contains a joint paper with Marco Zambon, titled *"Coisotropic submanifolds in b-symplectic geometry"*. The paper is available on arXiv:1907.09251, and will appear in the Canadian Journal of Mathematics.

Abstract - We study coisotropic submanifolds of b-symplectic manifolds. We prove that b-coisotropic submanifolds (those transverse to the degeneracy locus) determine the b-symplectic structure in a neighborhood, and provide a normal form theorem. This extends Gotay's theorem in symplectic geometry. Further, we introduce strong b-coisotropic submanifolds and show that their coisotropic quotient, which locally is always smooth, inherits a reduced b-symplectic structure.

1.1 Introduction

In symplectic geometry, an important and interesting class of submanifolds are the coisotropic ones. They are the submanifolds C satisfying $TC^{\Omega} \subset TC$, where TC^{Ω} denotes the symplectic orthogonal of the tangent bundle TC. They arise for instance as zero level sets of moment maps, and in mechanics as those submanifolds that are given by first class constraints (see Dirac's theory of constraints). The notion of coisotropic submanifolds extends to the wider realm of Poisson geometry, and it plays an important role there too: for instance, a map is a Poisson morphism if and only if its graph is coisotropic, and coisotropic submanifolds admit canonical quotients which inherit a Poisson structure.

The Poisson structures which are non-degenerate at every point are exactly the symplectic ones. Relaxing slightly the non-degeneracy condition, one obtains Poisson structures (M,Π) for which the top power $\wedge^n\Pi$ is transverse to the zero section of the line bundle $\wedge^{2n}TM$ (here $\dim(M)=2n$): they are called *log-symplectic* structures. They are symplectic outside the vanishing set of $\wedge^n\Pi$, a hypersurface which inherits a codimension-one symplectic foliation. Log-symplectic structures are studied systematically by Guillemin-Miranda-Pires in [GMP], and turn out to be equivalent to *b-symplectic* structures. The latter are defined on manifolds M with a choice of codimension-one submanifold Z, as follows: they are non-degenerate sections ω of $\wedge^2({}^bTM)^*$ which are closed w.r.t. the de Rham differential, where ${}^bTM$ is the b-tangent bundle (a Lie algebroid over M which encodes Z). In other words, they are the analogue of symplectic forms if one replaces the tangent bundle with the b-tangent bundle. Because of this, various phenomena in symplectic geometry have counterparts for log-symplectic manifolds.

This chapter is devoted to coisotropic submanifolds of log-symplectic manifolds. We single out two classes, which we call *b-coisotropic* and *strong b-coisotropic*. We prove that certain properties of coisotropic submanifolds in symplectic geometry – properties which certainly do not carry over to arbitrary coisotropic submanifolds of log-symplectic manifolds – do carry over to the above classes. Moreover, we show that these classes of submanifolds enjoy some properties that are b-geometric enhancements of well-known facts about coisotropic submanifolds in Poisson geometry. We now elaborate on this.

Main results. Let (M,Z,ω) be a b-symplectic manifold, and denote by Π the corresponding Poisson tensor on M. We consider two classes of submanifolds which are coisotropic (in the sense of Poisson geometry) with respect to Π.

A submanifold of M is called *b-coisotropic* if it is coisotropic and a b-submanifold (i.e. transverse to Z). An equivalent characterization is the following: a b-submanifold C such that $({}^bTC)^{\omega} \subset {}^bTC$. The latter formulation makes apparent that this notion is very natural in b-symplectic geometry. Section 1.3 is devoted to the class of b-coisotropic submanifolds. We show that the b-conormal bundle of a b-coisotropic submanifold is a Lie subalgebroid. We also show that for Poisson maps between log-symplectic manifolds compatible with the corresponding hypersurfaces, the graphs are b-coisotropic submanifolds, once "lifted" to a suitable blow-up [GL]. Both of these statements are b-geometric analogs of well-known facts about coisotropic submanifolds in Poisson geometry. Next, in Theorem 1.3.13 we show that Gotay's theorem in symplectic geometry [G] extends to b-coisotropic submanifolds in b-symplectic geometry. The main consequence is a normal form theorem for the b-symplectic structure around such submanifolds:

Theorem 1A. *A neighborhood of a b-coisotropic submanifold $C \overset{i}{\hookrightarrow} (M, Z, \omega)$ is b-symplectomorphic to the following model:*

$$(\textit{a neighborhood of the zero section in } E^*,\ \Omega),$$

where $E := \ker({}^b i^* \omega)$ denotes the kernel of the pullback of ω to C, and Ω is a b-symplectic form which is constructed out of the pullback ${}^b i^* \omega$ and is canonical up to neighborhood equivalence (see equation (1.15) *for the precise formula).*

Such a normal form can be used to study effectively the deformation theory of C as a coisotropic submanifold. We point out that in the special case of Lagrangian submanifolds, the above result is a version of Weinstein's tubular neighborhood theorem, and was already obtained by Kirchhoff-Lukat [Ki, Theorem 5.18].

In Section 1.4 we consider the following subclass of the b-coisotropic submanifolds. A submanifold C is called *strong b-coisotropic* if it is coisotropic and transverse to all the symplectic leaves of (M, Π) it meets. We remark that Lagrangian submanifolds intersecting the degeneracy locus Z never satisfy this definition. The main feature of strong b-coisotropic submanifolds is that the characteristic distribution

$$D := \Pi^\sharp \left(TC^0\right),$$

is *regular*, with rank equal to *codim*(C). Recall the following fact in Poisson geometry: when the quotient of a coisotropic submanifold by its characteristic distribution is a smooth manifold, then it inherits a Poisson structure, called the reduced Poisson structure. We show (see Proposition 1.4.6 for the full statement):

Proposition 1B. *Let C be a strong b-coisotropic submanifold of a b-symplectic manifold. If the quotient C/D by the characteristic distribution is smooth, then the reduced Poisson structure is again b-symplectic.*

Instances of the above result arise when a connected Lie group acts on a b-symplectic manifold with equivariant moment map, in the sense of Poisson geometry, and C is the zero level set of the latter, see Corollary 1.4.10. At the end of the chapter we provide examples of b-symplectic quotients, and – by reversing the procedure – in Corollary 1.4.16 we realize any b-symplectic structure on the 2-dimensional sphere as such a quotient.

In order to state and prove these results, in Section 1.2 we collect some facts about b-geometry. A few of them are new, to the best of our knowledge, and are of independent interest. More specifically, in Lemma 1.2.10 we show that, while the anchor map of the b-tangent bundle does not admit a canonical splitting, distributions tangent to Z do have a canonical lift to the b-tangent bundle. In Proposition 1.2.19 we provide a version of the b-Moser theorem relative to a b-submanifold, which we could not find elsewhere in the literature.

1.2 Background on b-geometry

In this section, we address the formalism of b-geometry, which originated from work of Melrose [Me] in the context of manifolds with boundary. We review some of the main concepts, including b-symplectic structures, and we prove some preliminary results that will be used in the body of this chapter.

1.2.1 b-manifolds and b-maps

We first introduce the objects and morphisms of the b-category, following [GMP].

Definition 1.2.1. A *b-manifold* is a pair (M, Z) consisting of a manifold M and a codimension-one submanifold $Z \subset M$.

Given a b-manifold (M, Z), we denote by ${}^b\mathfrak{X}(M)$ the set of vector fields on M that are tangent to Z. Note that ${}^b\mathfrak{X}(M)$ is a locally free $C^\infty(M)$-module, with generators

$$x_1\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n}$$

in a coordinate chart $(x_1, \ldots, x_n)$ adapted to $Z = \{x_1 = 0\}$. Thanks to the Serre-Swan theorem, these b-vector fields give rise to a vector bundle ${}^bTM$.

Definition 1.2.2. Let (M, Z) be a b-manifold. The *b-tangent bundle* ${}^bTM$ is the vector bundle over M satisfying $\Gamma\left({}^bTM\right) = {}^b\mathfrak{X}(M)$.

The inclusion ${}^b\mathfrak{X}(M) \subset \mathfrak{X}(M)$ induces a vector bundle map $\rho : {}^bTM \to TM$, which is an isomorphism away from Z. Restricting to Z, we get a bundle epimorphism $\rho|_Z : {}^bTM|_Z \to TZ$, which gives rise to a trivial line bundle $\mathbb{L} := \operatorname{Ker}(\rho|_Z)$. Indeed, $\mathbb{L}$ is canonically trivialized by the normal b-vector field

$\xi \in \Gamma(\mathbb{L})$, which is locally given by $x\partial_x$ where x is any local defining function for Z. So at any point $p \in Z$, we have a short exact sequence

$$0 \to \mathbb{L}_p \hookrightarrow {}^bT_pM \xrightarrow{\rho} T_pZ \to 0, \tag{1.1}$$

but this sequence does not split canonically. Since ${}^b\mathfrak{X}(M)$ is a Lie subalgebra of $\mathfrak{X}(M)$, it inherits a natural Lie bracket $[\cdot,\cdot]$. The data $(\rho, [\cdot,\cdot])$ endow ${}^bTM$ with a Lie algebroid structure. The map ρ is called the anchor of ${}^bTM$.

Definition 1.2.3. Let (M, Z) be a b-manifold. The *b-cotangent bundle* ${}^bT^*M$ is the dual bundle of ${}^bTM$.

In coordinates $(x_1, \dots, x_n)$ adapted to $Z = \{x_1 = 0\}$, the b-cotangent bundle ${}^bT^*M$ has local frame

$$\frac{dx_1}{x_1}, dx_2, \dots, dx_n.$$

We will denote the set $\Gamma\left(\wedge^k\left({}^bT^*M\right)\right)$ of Lie algebroid k-forms by ${}^b\Omega^k(M)$, and we refer to them as b-k-forms. The space ${}^b\Omega^\bullet(M)$ is endowed with the Lie algebroid differential ${}^bd$, which is determined by the fact that the restriction $\left({}^b\Omega^k(M), {}^bd\right) \to \left(\Omega^k(M \setminus Z), d\right)$ is a chain map. Note that the anchor ρ induces an injective map $\rho^* : \Omega^k(M) \to {}^b\Omega^k(M)$, which allows us to view honest de Rham forms as b-forms.

Definition 1.2.4. Given two b-manifolds (M_1, Z_1) and (M_2, Z_2), a *b-map* $f : (M_1, Z_1) \to (M_2, Z_2)$ is a smooth map $f : M_1 \to M_2$ such that f is transverse to Z_2 and $f^{-1}(Z_2) = Z_1$.

Given a b-map $f : (M_1, Z_1) \to (M_2, Z_2)$, the pullback $f^* : \Omega^\bullet(M_2) \to \Omega^\bullet(M_1)$ extends to an algebra morphism ${}^bf^* : {}^b\Omega^\bullet(M_2) \to {}^b\Omega^\bullet(M_1)$, see [Kl, Proof of Proposition 3.5.2]. That is, we have a commutative diagram

$$\begin{array}{ccc}
{}^b\Omega^\bullet(M_2) & \xrightarrow{\ {}^bf^*\ } & {}^b\Omega^\bullet(M_1) \\
\uparrow{\scriptstyle \rho_2^*} & & \uparrow{\scriptstyle \rho_1^*} \\
\Omega^\bullet(M_2) & \xrightarrow{\ f^*\ } & \Omega^\bullet(M_1)
\end{array}.$$

This b-pullback has the expected properties: the assignment $f \mapsto {}^bf^*$ is functorial, and the b-pullback ${}^bf^*$ commutes with the b-differential ${}^bd$.

We can now define the Lie derivative of a b-form $\omega \in {}^b\Omega^k(M)$ in the direction of a b-vector field $X \in {}^b\mathfrak{X}(M)$ by the usual formula

$$\pounds_X\omega = \left.\frac{d}{dt}\right|_{t=0} {}^b\rho_t^*\omega,$$

where the b-pullback is well-defined since the flow $\{\rho_t\}$ of X consists of b-diffeomorphisms. Cartan's formula is still valid:

$$\pounds_X \omega = {}^b d \iota_X \omega + \iota_X {}^b d\omega.$$

Dual to the b-pullback ${}^b f^*$, a b-map $f : (M_1, Z_1) \to (M_2, Z_2)$ induces a b-derivative ${}^b f_* : {}^b TM_1 \to {}^b TM_2$, which is the unique bundle map ${}^b TM_1 \to {}^b TM_2$ that makes the following diagram commute [Kl, Proposition 3.5.2]:

$$\begin{array}{ccc} {}^b TM_1 & \xrightarrow{\;{}^b f_*\;} & {}^b TM_2 \\ \big\downarrow \rho_1 & & \big\downarrow \rho_2 \\ TM_1 & \xrightarrow{\;f_*\;} & TM_2 \end{array} \quad . \tag{1.2}$$

At each point $p \in M_1$, the derivative $(f_*)_p$ and the b-derivative $\left({}^b f_*\right)_p$ have the same rank, by the next result proved in [CK].

Lemma 1.2.5. *Let $f : (M_1, Z_1) \to (M_2, Z_2)$ be a b-map. The anchor ρ_1 of ${}^b TM_1$ induces an isomorphism $(\rho_1)_p : Ker\left({}^b f_*\right)_p \to Ker(f_*)_p$ for all $p \in M_1$.*

We finish this subsection by observing that, if a b-vector field can be pushed forward by the derivative f_* of a b-map f, then its lift to a section of the b-tangent bundle can be pushed forward by the b-derivative ${}^b f_*$.

Lemma 1.2.6. *Let $f : (M_1, Z_1) \to (M_2, Z_2)$ be a surjective b-map, and assume that $\overline{Y} \in \Gamma({}^b TM_1)$ is such that $Y := \rho_1(\overline{Y})$ pushes forward to some element $W \in \mathfrak{X}(M_2)$. Then ${}^b f_*(\overline{Y})$ is a well-defined section of ${}^b TM_2$, and it equals the unique element $\overline{W} \in \Gamma\left({}^b TM_2\right)$ satisfying $\rho_2(\overline{W}) = W$.*

Proof. Since f is a b-map, we have that $W \in \mathfrak{X}(M_2)$ is tangent to Z_2, so indeed $W = \rho_2(\overline{W})$ for unique $\overline{W} \in \Gamma({}^b TM_2)$. Now, first consider $p \in M_1 \setminus Z_1$. Commutativity of the diagram (1.2) implies that

$$\rho_2\left(\left({}^b f_*\right)_p\left(\overline{Y}_p\right)\right) = (f_*)_p\left(\rho_1\left(\overline{Y}_p\right)\right) = (f_*)_p\left(Y_p\right) = W_{f(p)}.$$

But we also have $\rho_2\left(\overline{W}_{f(p)}\right) = W_{f(p)}$, so that injectivity of ρ_2 at $f(p) \in M_2 \setminus Z_2$ implies $\left({}^b f_*\right)_p\left(\overline{Y}_p\right) = \overline{W}_{f(p)}$. Next, we choose $p \in Z_1$. Since f is a b-map, we can take a (one-dimensional) slice S through p transverse to Z_1, such that the restriction $f|_S : S \to f(S)$ is a diffeomorphism. Since $\left({}^b f_*\right)\big|_S$ is a vector bundle map covering the diffeomorphism $f|_S$, the expression $\left({}^b f_*\right)\big|_S\left(\overline{Y}\big|_S\right)$ is well-defined and smooth. Moreover, it is equal to $\overline{W}|_{f(S)}$ on the dense subset $f(S) \setminus (f(S) \cap Z_2) \subset f(S)$, as we just proved. By continuity, the equality $\left({}^b f_*\right)\big|_S\left(\overline{Y}\big|_S\right) = \overline{W}|_{f(S)}$ holds on all of $f(S)$, so that in particular $\left({}^b f_*\right)_p\left(\overline{Y}_p\right) = \overline{W}_{f(p)}$. This concludes the proof. $\square$

1.2.2 b-submanifolds

Given a b-manifold (M, Z), a submanifold $C \subset M$ transverse to Z inherits a b-manifold structure with distinguished hypersurface $C \cap Z$. Such submanifolds are therefore the natural subobjects in the b-category.

Definition 1.2.7. A *b-submanifold* C of a b-manifold (M, Z) is a submanifold $C \subset M$ that is transverse to Z.

Let $C \subset (M, Z)$ be a b-submanifold. The inclusion $i : (C, C \cap Z) \hookrightarrow (M, Z)$ of b-manifolds induces a canonical map ${}^b i_* : {}^bTC \to {}^bTM$ that is injective by Lemma 1.2.5. This allows us to view ${}^bTC$ as a Lie subalgebroid of ${}^bTM$. In particular, we have the following fact.

Lemma 1.2.8. *If $C \subset (M, Z)$ is a b-submanifold, then $\mathbb{L}_p \subset {}^bT_pC$ for all $p \in C \cap Z$. Here $\mathbb{L}$ denotes the line bundle introduced in* (1.1).

Proof. We denote the anchor maps by $\tilde{\rho} : {}^bTC \to TC$ and $\rho : {}^bTM \to TM$, and we put $\widetilde{\mathbb{L}} := \text{Ker}\,(\tilde{\rho}|_{C\cap Z})$ and $\mathbb{L} = \text{Ker}\,(\rho|_Z)$ as before. If $i : (C, C \cap Z) \hookrightarrow (M, Z)$ denotes the inclusion, then we get a commutative diagram with exact rows, for points $p \in C \cap Z$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{L}_p & \hookrightarrow & {}^bT_pM & \xrightarrow{\rho} & T_pZ & \longrightarrow & 0 \\ & & & & \uparrow {\scriptstyle \left({}^b i_*\right)_p} & & \uparrow {\scriptstyle \left((i|_{C\cap Z})_*\right)_p} & & \\ 0 & \longrightarrow & \widetilde{\mathbb{L}}_p & \hookrightarrow & {}^bT_pC & \xrightarrow{\tilde{\rho}} & T_p(C \cap Z) & \longrightarrow & 0 \end{array} \quad . \tag{1.3}$$

We obtain $\left({}^b i_*\right)_p \left(\widetilde{\mathbb{L}}_p\right) = \mathbb{L}_p$: the inclusion "$\subset$" holds by the above diagram, and the equality follows by dimension reasons since $\left({}^b i_*\right)_p$ is injective. In particular, $\mathbb{L}_p$ is contained in the image of $\left({}^b i_*\right)_p$, as we wanted to show. □

The notions of b-map and b-submanifold are compatible, as we now show.

Lemma 1.2.9. *Let $f : (M_1, Z_1) \to (M_2, Z_2)$ be a b-map, and assume that we have b-submanifolds $C_1 \subset (M_1, Z_1)$ and $C_2 \subset (M_2, Z_2)$ such that $f(C_1) \subset C_2$.*

a) Restricting f gives a b-map

$$f|_{C_1} : (C_1, C_1 \cap Z_1) \to (C_2, C_2 \cap Z_2).$$

b) Further, $\left({}^b f_*\right)\big|_{{}^bTC_1} = {}^b\left(f|_{C_1}\right)_*$.

Proof. a) We first note that

$$\left(f|_{C_1}\right)^{-1}(C_2 \cap Z_2) = C_1 \cap f^{-1}(C_2 \cap Z_2) = C_1 \cap f^{-1}(C_2) \cap f^{-1}(Z_2)$$
$$= C_1 \cap f^{-1}(C_2) \cap Z_1 = C_1 \cap Z_1,$$

since f is a b-map and $C_1 \subset f^{-1}(C_2)$. Next, choosing $p \in C_1 \cap Z_1$, we have to show that

$$(f_*)_p (T_p C_1) + T_{f(p)} (C_2 \cap Z_2) = T_{f(p)} C_2. \tag{1.4}$$

We clearly have the inclusion "$\subset$". For the reverse, we choose $v \in T_{f(p)}C_2$. By transversality $f \pitchfork Z_2$, we know that $(f_*)_p (T_p M_1) + T_{f(p)} Z_2 = T_{f(p)} M_2$. So we have $v = (f_*)_p (x) + y$ for some $x \in T_p M_1$ and $y \in T_{f(p)} Z_2$. Next, since $C_1 \pitchfork Z_1$, we have $T_p C_1 + T_p Z_1 = T_p M_1$ so that $x = x_1 + x_2$ for some $x_1 \in T_p C_1$ and $x_2 \in T_p Z_1$. So we have

$$v = (f_*)_p (x_1) + \left[(f_*)_p (x_2) + y\right]. \tag{1.5}$$

The term in square brackets clearly lies in $T_{f(p)} Z_2$, and being equal to $v - (f_*)_p (x_1)$ it also lies in $T_{f(p)} C_2$. So it lies in $T_{f(p)} (C_2 \cap Z_2)$, using the transversality $C_2 \pitchfork Z_2$. So the decomposition (1.5) is as required in (1.4).

b) Denoting by $i_1 : (C_1, C_1 \cap Z_1) \hookrightarrow (M_1, Z_1)$ and $i_2 : (C_2, C_2 \cap Z_2) \hookrightarrow (M_2, Z_2)$ the inclusions, we have $f \circ i_1 = i_2 \circ f|_{C_1}$. Hence by functoriality, we obtain that ${}^b f_* \circ {}^b (i_1)_* = {}^b (i_2)_* \circ {}^b \left(f|_{C_1}\right)_*$, which implies the claim. □

1.2.3 Distributions on b-manifolds

We saw that the short exact sequence (1.1) does not split canonically. However, its restriction to suitable distributions does split.

Lemma 1.2.10. *Let (M, Z) be a b-manifold with anchor map $\rho : {}^b TM \to TM$.*

a) Given a distribution D on M that is tangent to Z, there exists a canonical splitting $\sigma \colon D \to {}^b TM$ of the anchor ρ.

b) Let $\mathcal{D}$ denote the set of distributions on M tangent to Z, and let $\mathcal{S}$ consist of the subbundles of ${}^b TM$ intersecting trivially $\ker(\rho)$. Then there is a bijection

$$\mathcal{D} \to \mathcal{S} : D \mapsto \sigma(D),$$

where the splitting σ is as in a). The inverse map reads $D' \mapsto \rho(D')$.

Proof. a) One checks that the inclusion $\Gamma(D) \subset \Gamma\left({}^bTM\right)$ induces a well-defined vector bundle map

$$\sigma : D \to {}^bTM : v \mapsto X_p,$$

where $X \in \Gamma(D)$ is any extension of $v \in D_p$. This map σ satisfies $\rho \circ \sigma = \mathrm{Id}_D$, so in particular $\rho(\sigma(D)) = D$.

b) We only have to show that if D' is a subbundle of ${}^bTM$ intersecting trivially $\ker(\rho)$, then $\sigma(\rho(D')) = D'$. Denote $D := \rho(D')$, a distribution on M tangent to Z. The canonical splitting $\sigma\colon D \to {}^bTM$ is injective, and D and D' have the same rank, hence it suffices to show that $\sigma(D) \subset D'$. If X is a section of D, then $X = \rho(Y)$ for unique $Y \in \Gamma\left(D'\right)$. We get

$$\rho(\sigma(X)) = X = \rho(Y),$$

and since the anchor ρ is injective on sections, this implies that $\sigma(X) = Y$. □

Corollary 1.2.11. *Let $f : (M_1, Z_1) \to (M_2, Z_2)$ be a b-map of constant rank. Notice that Ker(f_*) is a distribution on M_1 that is tangent to Z_1. It satisfies*

$$\sigma\left(\mathit{Ker}(f_*)\right) = \mathit{Ker}\left({}^bf_*\right),$$

where $\sigma : \mathit{Ker}(f_*) \to {}^bTM_1$ denotes the canonical splitting of the anchor ρ_1.

Proof. Under the bijection of Lemma 1.2.10 b), $\mathrm{Ker}(f_*)$ corresponds to $\mathrm{Ker}\left({}^bf_*\right)$, as a consequence of Lemma 1.2.5. □

1.2.4 Vector bundles in the b-category

If (M, Z) is a b-manifold and $\pi : E \to M$ is a vector bundle, then $(E, E|_Z)$ is naturally a b-manifold and the projection $\pi : (E, E|_Z) \to (M, Z)$ is a b-map. Along the zero section $M \subset E$, the b-tangent bundle ${}^bTE$ splits canonically.

Lemma 1.2.12. *Let (M, Z) be a b-manifold and $\pi : E \to M$ a vector bundle. Then at points $p \in M$ we have a canonical decomposition*

$${}^bT_pE \cong {}^bT_pM \oplus E_p.$$

Proof. Denote by $VE := \mathrm{Ker}(\pi_*)$ the vertical bundle. By Corollary 1.2.11 there is a canonical lift $\sigma : VE \hookrightarrow {}^bTE$ such that $\sigma(VE) = \mathrm{Ker}({}^b\pi_*)$. So we get a short exact sequence of vector bundles over E

$$0 \longrightarrow VE \overset{\sigma}{\hookrightarrow} {}^bTE \xrightarrow{\widetilde{{}^b\pi_*}} \pi^*\left({}^bTM\right) \to 0. \tag{1.6}$$

Here

$$\pi^*\left({}^bTM\right) = \left\{(e, v) \in E \times {}^bTM : \pi(e) = pr(v)\right\}$$

is the pullback of the vector bundle $pr : {}^bTM \to M$ by π, and the surjective vector bundle map

$$\widetilde{{}^b\pi_*} : {}^bTE \to \pi^*\left({}^bTM\right),\ (e,v) \mapsto \left(e, \left({}^b\pi_*\right)_e(v)\right)$$

is induced by the b-map $\pi : (E, E|_Z) \to (M, Z)$.

Restricting (1.6) to the zero section $M \subset E$ gives a short exact sequence of vector bundles over M:

$$0 \longrightarrow E \hookrightarrow {}^bTE|_M \xrightarrow{{}^b\pi_*} {}^bTM \to 0.$$

This sequence splits canonically through the map ${}^bi_* : {}^bTM \to {}^bTE|_M$ induced by the inclusion $i : (M, Z) \hookrightarrow (E, E|_Z)$. □

The next result makes use of the decomposition introduced in Lemma 1.2.12.

Lemma 1.2.13. *a) Let $\pi : (E, E|_Z) \to (M, Z)$ be a vector bundle over the b-manifold (M, Z). Denote by ρ and $\widetilde{\rho}$ the anchor maps of ${}^bTM$ and ${}^bTE$ respectively. Under the decomposition of Lemma 1.2.12, we have that the map*

$$\widetilde{\rho}|_M : {}^bTE|_M \cong {}^bTM \oplus E \longrightarrow TE|_M \cong TM \oplus E$$

equals $\rho \oplus Id_E$.

b) Let φ be a morphism of vector bundles over b-manifolds covering a b-map f:

$$\begin{array}{ccc} (E_1, E_1|_{Z_1}) & \xrightarrow{\varphi} & (E_2, E_2|_{Z_2}) \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ (M_1, Z_1) & \xrightarrow{f} & (M_2, Z_2) \end{array} \quad . \tag{1.7}$$

Then φ is a b-map, and its b-derivative along the zero section

$${}^b\varphi_*|_M : {}^bTE_1|_M \cong {}^bTM_1 \oplus E_1 \to {}^bTE_2|_M \cong {}^bTM_2 \oplus E_2$$

is equal to ${}^bf_* \oplus \varphi$.

Proof. a) Since M is a b-submanifold of $(E, E|_Z)$, we have that ${}^bTM$ is a Lie subalgebroid of ${}^bTE$. In particular, $\widetilde{\rho}$ and ρ agree on ${}^bTM$. Next, we know that $\widetilde{\rho}$ takes $E \subset {}^bTE|_M$ isomorphically to $E \subset TE|_M$, thanks to Lemma 1.2.5 applied to π. To see that $\widetilde{\rho}|_E = \mathrm{Id}_E$, we choose $v \in E_p$ and extend it to $V \in \Gamma(VE)$. Denote by $\sigma : VE \hookrightarrow {}^bTE$ the canonical splitting of $\widetilde{\rho}$, as in the proof of Lemma 1.2.12. Then $\widetilde{\rho}(v) = [\widetilde{\rho}(\sigma(V))]_p = V_p = v$.

b) It is routine to check that φ is a b-map, so we only prove the second statement. Taking the b-derivative of both sides of the equality $\pi_2 \circ \varphi = f \circ \pi_1$ at a point $p \in M_1$, we know that $\left({}^b\pi_2\right)_* \left({}^b\varphi_*(E_1)_p\right) = {}^b f_* \left(\left({}^b\pi_1\right)_* (E_1)_p\right) = 0$, since $(E_1)_p = \text{Ker}\left[\left({}^b\pi_1\right)_*\right]_p$. Hence ${}^b\varphi_*(E_1)_p \subset \text{Ker}\left[\left({}^b\pi_2\right)_*\right]_{f(p)} = (E_2)_{f(p)}$ by the proof of Lemma 1.2.12. Using a) and the diagram (1.2), we have a commutative diagram

$$\begin{array}{ccc} {}^bT_pE_1 \cong {}^bT_pM_1 \oplus (E_1)_p & \xrightarrow{{}^b\varphi_*} & {}^bT_{f(p)}E_2 \cong {}^bT_{f(p)}M_2 \oplus (E_2)_{f(p)} \\ \Big\downarrow{\scriptstyle (\rho_1 \oplus \text{Id})} & & \Big\downarrow{\scriptstyle (\rho_2 \oplus \text{Id})} \\ T_pE_1 \cong T_pM_1 \oplus (E_1)_p & \xrightarrow{\varphi_*} & T_{f(p)}E_2 \cong T_{f(p)}M_2 \oplus (E_2)_{f(p)} \end{array} . \tag{1.8}$$

It implies that

$${}^b\varphi_*\big|_{(E_1)_p} = \varphi_*\big|_{(E_1)_p} = \varphi\big|_{(E_1)_p} .$$

Finally, ${}^b\varphi_*|_{{}^bTM_1} = {}^b f_*$ holds by Lemma 1.2.9 b). □

1.2.5 Log-symplectic and b-symplectic structures

The b-geometry formalism can be used to describe a certain class of Poisson structures, called log-symplectic structures. These can indeed be regarded as symplectic structures on the b-tangent bundle.

Definition 1.2.14. i) A *Poisson structure* on a manifold M is a bivector field $\Pi \in \Gamma\left(\wedge^2 TM\right)$ such that the bracket $\{f, g\} = \Pi(df, dg)$ is a Lie bracket on $C^\infty(M)$. Equivalently, the bivector field Π must satisfy $[\Pi, \Pi] = 0$, where $[\cdot, \cdot]$ is the Schouten-Nijenhuis bracket of multivector fields.

ii) A smooth map $f : (M_1, \Pi_1) \to (M_2, \Pi_2)$ is a *Poisson map* if the pullback $f^* : (C^\infty(M_2), \{\cdot, \cdot\}_2) \to (C^\infty(M_1), \{\cdot, \cdot\}_1)$ is a Lie algebra homomorphism.

The bivector Π induces a bundle map $\Pi^\sharp : T^*M \to TM$ by

$$\left\langle \Pi_p^\sharp(\alpha), \beta \right\rangle = \Pi_p(\alpha, \beta) \qquad \forall \alpha, \beta \in T_p^*M,$$

and the *rank* of Π at $p \in M$ is defined as the rank of the linear map $\Pi_p^\sharp$. Poisson structures of full rank correspond with symplectic structures via $\omega \leftrightarrow -\Pi^{-1}$.

For every $f \in C^\infty(M)$, the operator $\{f, \cdot\}$ is a derivation of $C^\infty(M)$. The corresponding vector field $X_f = \Pi^\sharp(df)$ is the *Hamiltonian vector field* of f. Any Poisson manifold (M, Π) comes with a (singular) distribution $\text{Im}\left(\Pi^\sharp\right)$, generated by the Hamiltonian vector fields. This distribution is integrable (in the sense of Stefan-Sussman) and each leaf $\mathcal{O}$ of the associated foliation has an induced symplectic structure $\omega_{\mathcal{O}} := -\left(\Pi|_{\mathcal{O}}\right)^{-1}$.

Definition 1.2.15. A Poisson structure Π on a manifold M^{2n} is called *log-symplectic* if $\wedge^n\Pi$ is transverse to the zero section of the line bundle $\wedge^{2n}TM$.

Note that a log-symplectic structure Π is of full rank everywhere, except at points lying in the set $Z := (\wedge^n\Pi)^{-1}(0)$, called the singular locus of Π. If Z is nonempty, then it is a smooth hypersurface by the transversality condition, and we call Π bona fide log-symplectic. In that case, Z is a Poisson submanifold of (M, Π) with an induced Poisson structure that is regular of corank-one. If Z is empty, then Π defines a symplectic structure on M.

Since log-symplectic structures come with a specified hypersurface, it seems plausible that they have a b-geometric interpretation. As it turns out, log-symplectic structures are exactly the symplectic structures of the b-category.

Definition 1.2.16. A *b-symplectic form* on a b-manifold (M^{2n}, Z) is a ${}^b d$-closed and non-degenerate b-two-form $\omega \in {}^b\Omega^2(M)$.

Here non-degeneracy means that the map $\omega^\flat : {}^bTM \to {}^bT^*M$ is an isomorphism, or equivalently that $\wedge^n\omega$ is a nowhere vanishing element of ${}^b\Omega^{2n}(M)$.

Example 1.2.17. [GMP, Example 9] In analogy with the symplectic case, the b-cotangent bundle ${}^bT^*M$ of a b-manifold (M, Z) is b-symplectic in a canonical way. Note that $\left({}^bT^*M, {}^bT^*M\big|_Z\right)$ is naturally a b-manifold, and that the bundle projection $\pi : \left({}^bT^*M, {}^bT^*M\big|_Z\right) \to (M, Z)$ is a b-map. The tautological b-one-form $\theta \in {}^b\Omega^1\left({}^bT^*M\right)$ is defined by

$$\theta_\xi(v) = \left\langle \xi, \left({}^b\pi_*\right)_\xi(v) \right\rangle,$$

where $\xi \in {}^bT^*_{\pi(\xi)}M$ and $v \in {}^bT_\xi\left({}^bT^*M\right)$. Its differential $-{}^bd\theta$ is a b-symplectic form on ${}^bT^*M$. To see this, choose coordinates $(x_1, \ldots, x_n)$ on M adapted to $Z = \{x_1 = 0\}$, and let $(y_1, \ldots, y_n)$ denote the fiber coordinates on ${}^bT^*M$ with respect to the local frame $\left\{\frac{dx_1}{x_1}, dx_2, \ldots, dx_n\right\}$. The tautological b-one form is then given by

$$\theta = y_1\frac{dx_1}{x_1} + \sum_{i=2}^n y_i dx_i,$$

with exterior derivative

$$-{}^bd\theta = \frac{dx_1}{x_1} \wedge dy_1 + \sum_{i=2}^n dx_i \wedge dy_i.$$

A log-symplectic structure on M with singular locus Z is nothing else but a b-symplectic structure on the b-manifold (M, Z), see [GMP, Proposition 20]. Indeed, if we are given a b-symplectic form ω on (M, Z), then its negative inverse

${}^b\Pi^\sharp := -\left(\omega^\flat\right)^{-1} : {}^bT^*M \to {}^bTM$ defines a b-bivector field ${}^b\Pi \in \Gamma\left(\wedge^2\left({}^bTM\right)\right)$, and applying the anchor map ρ yields a bivector field $\Pi := \rho\left({}^b\Pi\right) \in \Gamma\left(\wedge^2 TM\right)$ that is log-symplectic with singular locus Z. Conversely, a log-symplectic structure Π on M with singular locus Z lifts uniquely under ρ to a non-degenerate b-bivector field ${}^b\Pi$, whose negative inverse is a b-symplectic form on (M, Z). These processes are summarized in the following diagram:

$$\begin{array}{ccc} {}^bT^*M & \overset{-\omega^\flat}{\underset{{}^b\Pi^\sharp}{\rightleftarrows}} & {}^bTM \\ \uparrow{\scriptstyle \rho^*} & & \downarrow{\scriptstyle \rho} \\ T^*M & \xrightarrow{\Pi^\sharp} & TM \end{array} . \tag{1.9}$$

We will switch between the b-symplectic and the log-symplectic (i.e. Poisson) viewpoint, depending on which one is the most convenient.

1.2.6 A relative b-Moser theorem

We'll need a relative Moser theorem in the b-symplectic setting. First, we prove a b-geometric version of the relative Poincaré lemma [C, Proposition 6.8].

Lemma 1.2.18. *Let (M, Z) be a b-manifold such that $Z \subset M$ is closed, and let $C \subset (M, Z)$ be a b-submanifold. Denote by $i : (C, C \cap Z) \hookrightarrow (M, Z)$ the inclusion. If $\beta \in {}^b\Omega^k(M)$ is ${}^bd$-closed and ${}^bi^*\beta = 0$, then there exist a neighborhood U of C and $\eta \in {}^b\Omega^{k-1}(U)$ such that*

$$\begin{cases} {}^bd\eta = \beta|_U \\ \eta|_C = 0 \end{cases} .$$

Proof. We adapt the proof of [C, Proposition 6.8]. We first choose a suitable tubular neighborhood of C that is compatible with the hypersurface Z. Due to transversality $C \pitchfork Z$, we can pick a complement V to TC in $TM|_C$ such that $V_p \subset T_pZ$ for all $p \in C \cap Z$. Fix a Riemannian metric g for which $Z \subset (M, g)$ is totally geodesic (e.g. [Mi, Lemma 6.8]). The associated exponential map then establishes a b-diffeomorphism between a neighborhood of C in $(V, V|_{C\cap Z})$ and a neighborhood of C in (M, Z).

So we may work instead on the total space of $\pi : (V, V|_{C\cap Z}) \to (C, C \cap Z)$. Define a retraction of V onto C by $r : V \times [0,1] \to V : (p, v, t) \mapsto (p, tv)$, and notice that the r_t are b-maps. The associated time-dependent vector field X_t

is given by $X_t(p, v) = \frac{1}{t}v$, which is a b-vector field that vanishes along C. It follows that we get a well-defined b-de Rham homotopy operator

$$I : {}^b\Omega^k(V) \to {}^b\Omega^{k-1}(V) : \alpha \mapsto \int_0^1 {}^b r_t^*(\iota_{X_t}\alpha)dt,$$

which satisfies

$${}^b r_1^*\alpha - {}^b r_0^*\alpha = {}^b dI(\alpha) + I({}^b d\alpha). \tag{1.10}$$

Since $r_1 = \mathrm{Id}$ and $r_0 = i \circ \pi$, the formula (1.10) implies that $\beta = {}^b dI(\beta)$. Setting $\eta := I(\beta)$ finishes the proof of the lemma. □

Proposition 1.2.19 (Relative b-Moser theorem)**.** *Let (M, Z) be a b-manifold with $Z \subset M$ closed, and let $C \subset (M, Z)$ be a b-submanifold. If ω_0 and ω_1 are b-symplectic forms on (M, Z) with $\omega_0|_C = \omega_1|_C$, then there is a b-diffeomorphism φ between neighborhoods of C such that $\varphi|_C = Id$ and ${}^b\varphi^*\omega_1 = \omega_0$.*

Proof. Consider the convex combination $\omega_t := \omega_0 + t(\omega_1 - \omega_0)$ for $t \in [0, 1]$. There exists a neighborhood U of C such that ω_t is non-degenerate on U for all $t \in [0, 1]$. Shrinking U if necessary, Lemma 1.2.18 yields $\eta \in {}^b\Omega^1(U)$ such that $\omega_1 - \omega_0 = {}^b d\eta$ and $\eta|_C = 0$. As in the usual Moser trick, it now suffices to solve the equation

$$\iota_{X_t}\omega_t + \eta = 0$$

for $X_t \in {}^b\mathfrak{X}(U)$, which is possible by non-degeneracy of ω_t. The b-vector fields X_t thus obtained vanish along C since $\eta|_C = 0$. Further shrinking U if necessary, we can integrate the X_t to an isotopy $\{\phi_t\}_{t\in[0,1]}$ defined on U. Note that the ϕ_t are b-diffeomorphisms that restrict to the identity on C. By the usual Moser argument, we have ${}^b\phi_1{}^*\omega_1 = \omega_0$, so setting $\varphi := \phi_1$ finishes the proof. □

Remark 1.2.20. We learnt from Ralph Klaasse that the work in progress [KL] contains a version of Proposition 1.2.19 that holds in the more general setting of symplectic Lie algebroids.

1.3 b-coisotropic submanifolds and the b-Gotay theorem

This section is devoted to coisotropic submanifolds of b-symplectic manifolds that are transverse to the degeneracy hypersurface. The main result is Theorem 1.3.13, a b-symplectic version of Gotay's theorem, which implies a normal form statement around such submanifolds. This can be used, for instance, to study the deformation theory of b-coisotropic submanifolds.

1.3.1 b-coisotropic submanifolds

In this subsection we introduce b-coisotropic submanifolds and we discuss some of their main features. First recall the definition of a coisotropic submanifold in Poisson geometry.

Definition 1.3.1. Let (M, Π) be a Poisson manifold with associated Poisson bracket $\{\cdot,\cdot\}$. A submanifold $C \subset M$ is *coisotropic* if the following equivalent conditions hold:

a) $\Pi^\sharp\left(TC^0\right) \subset TC$, where $TC^0 \subset T^*M|_C$ denotes the annihilator of TC.

b) $\{\mathcal{I}_C, \mathcal{I}_C\} \subset \mathcal{I}_C$, where $\mathcal{I}_C := \{f \in C^\infty(M) : f|_C = 0\}$ denotes the vanishing ideal of C.

c) $T_pC \cap T_p\mathcal{O}$ is a coisotropic subspace of the symplectic vector space $\left(T_p\mathcal{O}, -(\Pi|_\mathcal{O})_p^{-1}\right)$ for all $p \in C$, where $\mathcal{O}$ is the symplectic leaf through p.

The singular distribution $\Pi^\sharp\left(TC^0\right)$ on C appearing above is called the *characteristic distribution.* In case $\Pi = -\omega^{-1}$ is symplectic, the coisotropicity condition becomes $TC^\omega \subset TC$.

Definition 1.3.2. Let (M, Z, ω) be a b-symplectic manifold, with corresponding Poisson bivector field Π. A submanifold C of M is called *b-coisotropic* if it is coisotropic with respect to Π and a b-submanifold (i.e. transverse to Z).

Remark 1.3.3. A b-coisotropic submanifold $C^n \subset (M^{2n}, Z, \Pi)$ of middle dimension is necessarily Lagrangian, i.e. $T_pC \cap T_p\mathcal{O}$ is a Lagrangian subspace of the symplectic vector space $\left(T_p\mathcal{O}, -(\Pi|_\mathcal{O})_p^{-1}\right)$ for all $p \in C$, where $\mathcal{O}$ denotes the symplectic leaf through p. Indeed, at points away from Z there is nothing to prove. At points $p \in C \cap Z$, we have

$$\dim\left(T_pC \cap T_p\mathcal{O}\right) \leq \dim\left(T_pC \cap T_pZ\right) = n - 1,$$

where the last equality follows from transversality $C \pitchfork Z$. On the other hand, $T_pC \cap T_p\mathcal{O}$ is at least $(n-1)$-dimensional, being a coisotropic subspace of the $(2n-2)$-dimensional symplectic vector space $T_p\mathcal{O}$. So $\dim(T_pC \cap T_p\mathcal{O}) = n - 1$, which proves the claim.

Definition 1.3.2 can be rephrased in terms of the b-symplectic form ω: a b-coisotropic submanifold is precisely a b-submanifold C such that $({}^bTC)^\omega \subset {}^bTC$.

Proposition 1.3.4. *Let C be a b-submanifold of a b-symplectic manifold (M, Z, ω). Then C is coisotropic if and only if $({}^bTC)^\omega \subset {}^bTC$.*

Notice that the latter condition states that ${}^bTC$ is a coisotropic subbundle of the symplectic vector bundle $({}^bTM|_C, \omega|_C)$.

Proof. If C is coisotropic, then at points of $C\cap(M\setminus Z)$ we have that $TC^{\omega}\subset TC$, i.e. $({}^bTC)^{\omega}\subset {}^bTC$. By continuity, this inclusion of subbundles holds at all points of C. Conversely, if this inclusion holds on C, it follows that $C\cap(M\setminus Z)$ is coisotropic in $M\setminus Z$, and using characterization b) in Definition 1.3.1 we see that C is coisotropic in M. □

We give an alternative description of the characteristic distribution of a b-coisotropic submanifold.

Lemma 1.3.5. *Let C be any b-submanifold of a b-symplectic manifold (M,Z,ω), and let $\rho:{}^bTM\to TM$ denote the anchor of ${}^bTM$, so that $\Pi=\rho\left(-\omega^{-1}\right)$ is the Poisson bivector corresponding with ω. Then*

$$\rho\left(\left({}^bTC\right)^{\omega}\right)=\Pi^{\sharp}\left(TC^0\right). \tag{1.11}$$

Proof. At points $p\in C\setminus(C\cap Z)$, the equality (1.11) holds by symplectic linear algebra. So let $p\in C\cap Z$. Denote by ${}^b\Pi:=-\omega^{-1}\in\Gamma\left(\wedge^2\left({}^bTM\right)\right)$ the lift of Π as a b-bivector field. Note that

$$\left({}^bT_pC\right)^{\omega_p}=\left(\omega_p^{\flat}\right)^{-1}\left(\left({}^bT_pC\right)^0\right)={}^b\Pi^{\sharp}\left(\left({}^bT_pC\right)^0\right), \tag{1.12}$$

where the annihilator is taken in ${}^bT_p^*M$. We now assert:

<u>Claim:</u> $\left({}^bT_pC\right)^0=\rho_p^*\left(T_pC^0\right).$

To prove the claim, we first note that the dimensions of both sides agree since

$$\operatorname{Ker}(\rho_p^*)\cap T_pC^0=\operatorname{Im}(\rho_p)^0\cap T_pC^0=T_pZ^0\cap T_pC^0=(T_pZ+T_pC)^0=\{0\},$$

where the last equality holds by transversality $C\pitchfork Z$. Now it is enough to show that the inclusion "$\supset$" holds, which is clearly the case since $\rho_p({}^bT_pC)\subset T_pC$.

We thus obtain

$$\rho_p\left(\left({}^bT_pC\right)^{\omega_p}\right)=\left(\rho_p\circ{}^b\Pi_p^{\sharp}\circ\rho_p^*\right)\left(T_pC^0\right)=\Pi_p^{\sharp}\left(T_pC^0\right),$$

where in the first equality we used (1.12) and the claim just proved, and in the second we used the diagram (1.9). □

A general fact in Poisson geometry is that the conormal bundle of any coisotropic submanifold is a Lie subalgebroid of the cotangent Lie algebroid. We now show that the b-geometry version of this fact holds for b-coisotropic submanifolds.

Proposition 1.3.6. *Let (M, Z, ω) be a b-symplectic manifold with corresponding Poisson bivector field Π. Recall that ${}^bT^*M$ is a Lie algebroid (endowed with the Lie bracket induced by ${}^b\Pi$), fitting in the diagram of Lie algebroids* (1.9)*. Let C be a b-coisotropic submanifold.*

a) $({}^bTC)^\circ$ is a Lie subalgebroid of ${}^bT^*M$.

b) $({}^bTC)^\circ$ fits in a diagram of Lie subalgebroids of the diagram (1.9)*:*

$$\begin{array}{ccc} ({}^bTC)^\circ & \underset{{}^b\Pi^\sharp}{\overset{-\omega^\flat}{\underset{\simeq}{\rightleftarrows}}} & ({}^bTC)^\omega \\ {\scriptstyle \rho^*}\uparrow\wr & & \downarrow{\scriptstyle \rho} \\ TC^\circ & \xrightarrow{\Pi^\sharp} & TC \end{array} \quad . \tag{1.13}$$

Proof. The diagram (1.13) is a diagram of vector subbundles of diagram (1.9), by the claim in the proof of Lemma 1.3.5 and by equation (1.12). For a), since the morphism ${}^b\Pi^\sharp$ in diagram (1.9) is an isomorphism of Lie algebroids, it suffices to show that $({}^bTC)^\omega$ is a Lie subalgebroid of ${}^bTM$. Since $({}^bTC)^\omega$ is the kernel of the closed b-2-form ${}^bi^*\omega$, a standard Cartan calculus computation shows that this is indeed the case. It is well-known that TC° and TC are also Lie subalgebroids, proving b). ☐

1.3.2 Examples of b-coisotropic submanifolds

We now exhibit some examples of b-coisotropic submanifolds. The main result of this subsection is Proposition 1.3.8, which shows that graphs of suitable Poisson maps between log-symplectic manifolds give rise to b-coisotropic submanifolds, once lifted to a certain blow-up.

Examples 1.3.7. a) Given a log-symplectic manifold (M, Z, Π), any hypersurface of M transverse to Z is b-coisotropic.

b) Let (M, Ω) be a symplectic manifold, whose non-degenerate Poisson structure we denote $\Pi_M := -\Omega^{-1}$, and let (N, Π_N) be a log-symplectic manifold with singular locus Z. Then $(M \times N, \Pi_M - \Pi_N)$ is log-symplectic with singular locus $M \times Z$. Given a Poisson map $\phi : (M, \Pi_M) \to (N, \Pi_N)$ transverse to Z, we have that $\mathrm{Graph}(\phi) \subset (M \times N, \Pi_M - \Pi_N)$ is b-coisotropic. As a concrete example, consider for instance

$$\phi : \left(\mathbb{R}^4, \sum_{i=1}^{2} \partial_{x_i} \wedge \partial_{y_i}\right) \to \left(\mathbb{R}^2, x\partial_x \wedge \partial_y\right) : (x_1, y_1, x_2, y_2) \mapsto (y_1, x_2 - x_1y_1).$$

We will now prove Proposition 1.3.8. We start recalling some facts from [GL, §2.1]. Given a manifold M and a closed submanifold L of codimension ≥ 2, one can construct a new manifold by replacing L with the projectivization of its normal bundle. The resulting manifold $Bl_L(M)$, the *real projective blow-up* of M along L, comes with a map

$$p\colon Bl_L(M) \to M$$

which restricts to a diffeomorphism $Bl_L(M) \setminus p^{-1}(L) \to M \setminus L$. Further, let $S \subset M$ be a submanifold which intersects cleanly L, i.e. $S \cap L$ is a submanifold with $T(S \cap L) = TS \cap TL$. Then S can be "lifted" to a submanifold of $Bl_L(M)$, namely the closure of the inverse image of $S \setminus L$ under p:

$$\overline{\overline{S}} := \overline{p^{-1}(S \setminus L)}.$$

Now let (M_i, Z_i, Π_i) be log-symplectic manifolds, for $i = 1, 2$. The product $M_1 \times M_2$ is not log-symplectic in general,[1] but [P], [GL, §2.2]

$$X := Bl_{Z_1 \times Z_2}(M_1 \times M_2) \setminus (\overline{\overline{M_1 \times Z_2}} \cup \overline{\overline{Z_1 \times M_2}}) \tag{1.14}$$

is log-symplectic with singular locus the exceptional divisor $p^{-1}(Z_1 \times Z_2)$, and the blow-down map $p\colon X \to M_1 \times \hat{M}_2$ is Poisson, where $\hat{M}_2$ denotes $(M_2, -\Pi_2)$.

Proposition 1.3.8. *Assume that* $f\colon (M_1, Z_1, \Pi_1) \to (M_2, Z_2, \Pi_2)$ *is a Poisson map with* $f(Z_1) \subset Z_2$*. Then* $\overline{\overline{graph(f)}}$ *is a b-coisotropic submanifold of the log-symplectic manifold* X *defined in* (1.14).

Proof. The intersection $graph(f) \cap (Z_1 \times Z_2)$ is clean, since it coincides with $graph(f|_{Z_1})$ by the assumption $f(Z_1) \subset Z_2$. So $graph(f)$ can be "lifted" to X.

The resulting submanifold $\overline{\overline{graph(f)}}$ is coisotropic: $graph(f)$ is coisotropic in $M_1 \times \hat{M}_2$ because f is a Poisson map, so $p^{-1}(graph(f) \setminus Z_1 \times Z_2)$ is coisotropic in X (since p is a Poisson diffeomorphism away from the exceptional divisor), and the same holds for its closure.

To finish the proof, we have to show that $\overline{\overline{graph(f)}}$ is transverse to the exceptional divisor $E := p^{-1}(Z_1 \times Z_2)$. Let $(x_i^{(1)})$ be local coordinates on

[1] However it fits in a slight generalization of the notion of log-symplectic structure used in this thesis: indeed $(M_1 \times Z_2) \cup (Z_1 \times M_2)$ is a normal crossing divisor, and vector fields tangent to it give rise to a Lie algebroid to which the Poisson structure on $M_1 \times \hat{M}_2$ lifts in a non-degenerate way (we thank Aldo Witte for pointing this out to us). One can check that if $f\colon M_1 \to M_2$ is a Poisson map transverse to Z_2, then $graph(f)$ intersects transversely both $M_1 \times Z_2$ and $Z_1 \times M_2$. This statement generalizes Example 1.3.7 b) and can be viewed as an analogue of Proposition 1.3.8.

M_1 such that $Z_1 = \{x_1^{(1)} = 0\}$, and similarly let $(x_j^{(2)})$ be local coordinates on M_2 such that $Z_2 = \{x_1^{(2)} = 0\}$. Then

$$(x_i^{(1)}, f^*(x_j^{(2)}) - x_j^{(2)})$$

are local coordinates on $M_1 \times M_2$ that are adapted $graph(f)$, but also to

$$(Z_1 \times Z_2) = \{x_1^{(1)} = 0, f^*(x_1^{(2)}) - x_1^{(2)} = 0\},$$

due to the hypothesis $f(Z_1) \subset Z_2$. Hence we can apply Lemma 1.3.9 below, which yields the desired transversality. □

The proof of Proposition 1.3.8 uses the following statement, for which we could not find a reference in the literature.

Lemma 1.3.9. *Let m, n be non-negative integers. Consider $\mathbb{R}^{n+m}$ with standard coordinates $x_1, \dots, x_n, y_1, \dots, y_m$, and the subspaces*

$$\begin{aligned} Z :=& \{0\} \times \mathbb{R}^m, \\ S :=& (\mathbb{R}^k \times \{0\}) \times (\mathbb{R}^l \times \{0\}) \end{aligned}$$

where $k \leq n$ and $l \leq m$. Then, in the blow-up $Bl_Z(\mathbb{R}^{n+m})$, the submanifold $\overline{\overline{S}}$ interesects transversely the exceptional divisor E.

Proof. We have

$$Bl_Z(\mathbb{R}^{n+m}) = \overline{\left\{ \Big((x,y), [x] \Big) : x \in \mathbb{R}^n \setminus \{0\}, y \in \mathbb{R}^m \right\}} \subset \mathbb{R}^{n+m} \times \mathbb{R}P^{n-1},$$

where $[\cdot]$ denotes the class in projective space. Notice that by taking the closure we are adding exactly the exceptional divisor

$$E = \{0\} \times \mathbb{R}^m \times \mathbb{R}P^{n-1}.$$

We have

$$\overline{\overline{S}} = \overline{\left\{ \Big((x_1, 0, y_1, 0), [(x_1, 0)] \Big) : x_1 \in \mathbb{R}^k \setminus \{0\}, y_1 \in \mathbb{R}^l \right\}}.$$

By taking the closure we are adding exactly

$$\{0\} \times (\mathbb{R}^l \times \{0\}) \times \mathbb{R}P^{k-1} = \overline{\overline{S}} \cap E.$$

For every point $p \in \overline{\overline{S}} \cap E$ there is a curve of the form

$$\gamma \colon t \mapsto \Big((tx_1, 0, y_1, 0), [(x_1, 0)] \Big)$$

lying in $\overline{\overline{S}}$ with $\gamma(0) = p$, and clearly $\frac{d}{dt}|_0 \gamma(t) \notin T_p E$. Since $\frac{d}{dt}|_0 \gamma(t) \in T_p \overline{\overline{S}}$ and E has codimension 1, we obtain that $T_p E + T_p \overline{\overline{S}} = T_p Bl_Z(\mathbb{R}^{n+m})$. □

Remark 1.3.10. One can show that for any pair of submanifolds L and S intersecting cleanly, around any point of the intersection there exist local coordinates of the ambient manifold M that are simultaneously adapted to both submanifolds. Lemma 1.3.9 then implies that, with the same notation as before, $\overline{\overline{S}}$ intersects transversely the hypersurface $p^{-1}(L)$ of $Bl_L(M)$.

1.3.3 b-coisotropic embeddings and the b-Gotay theorem

If $C \overset{i}{\hookrightarrow} (M, Z, \omega)$ is b-coisotropic, then $(C, C \cap Z, {}^b i^*\omega)$ is b-presymplectic by Proposition 1.3.4, i.e. the b-two-form ${}^b i^*\omega \in {}^b\Omega^2(C)$ is closed of constant rank. Conversely, in this subsection we prove that any b-presymplectic manifold embeds b-coisotropically into a b-symplectic manifold, which is unique up to neighborhood equivalence. In other words, we show a version of Gotay's theorem for b-coisotropic submanifolds. For Lagrangian submanifolds, this becomes a version of Weinstein's tubular neighborhood theorem, which was already obtained in [Ki, Theorem 5.18].

As a consequence, *a b-coisotropic submanifold $C \subset (M, Z, \omega)$ determines ω (up to b-symplectomorphism) in a neighborhood of C.* Notice that arbitrary coisotropic submanifolds of the log-symplectic manifold (M, Z, Π) do not satisfy this property: for instance Z is a coisotropic (even Poisson) submanifold, and by [GMP] the additional data consisting of a certain element of $H^1_\Pi(Z)$ is necessary in order to determine the b-symplectic structure in a neighborhood of Z.

Definition 1.3.11. A *b-presymplectic form* on a b-manifold is a b-two-form which is closed and of constant rank.

Definition 1.3.12. Let (M_1, Z_1, ω) be a b-manifold with a b-presymplectic form $\omega \in {}^b\Omega^2(M_1)$. A *$b$-coisotropic embedding* of (M_1, Z_1, ω) into a b-symplectic manifold (M_2, Z_2, Ω) is a b-map $\phi : (M_1, Z_1) \to (M_2, Z_2)$ such that ϕ is an embedding and

i) ${}^b\phi^*\Omega = \omega$.

ii) $\phi(M_1)$ is b-coisotropic in (M_2, Z_2, Ω).

We will prove the following Gotay theorem in the b-symplectic setting.

Theorem 1.3.13 (The b-Gotay theorem)**.** *Let (C, Z_C, ω_C) be a b-manifold with a b-presymplectic form $\omega_C \in {}^b\Omega^2(C)$. We then have the following:*

a) C embeds b-coisotropically into a b-symplectic manifold,

b) the embedding is unique up to b-symplectomorphism in a tubular neighborhood of C, fixing C pointwise.

We divide the proof of Theorem 1.3.13 into several steps. We roughly follow the reasoning from the symplectic case, as presented in [G]. We start by constructing a b-symplectic thickening of the b-presymplectic manifold C, from which item $a)$ of Theorem 1.3.13 will follow.

Proposition 1.3.14. *Denote by E the vector bundle Ker$(\omega_C) \subset {}^bTC$. There is a b-symplectic structure Ω_G on a neighborhood of the zero section $C \subset E^*$.*

Proof. Fix a complement G to E in ${}^bTC$, and let $j : E^* \hookrightarrow {}^bT^*C$ be the induced inclusion. Clearly, $j(E^*) = G^0$. The projection $\pi : (E^*, E^*|_{Z_C}) \to (C, Z_C)$ and the inclusion $j : (E^*, E^*|_{Z_C}) \to \left({}^bT^*C, {}^bT^*C|_{Z_C}\right)$ are both b-maps, so we can define a b-two-form Ω_G on $(E^*, E^*|_{Z_C})$ by

$$\Omega_G := {}^b\pi^*\omega_C + {}^bj^*\omega_{can}. \tag{1.15}$$

Here ω_{can} denotes the canonical b-symplectic form on ${}^bT^*C$ as in Example 1.2.17, and the subscript G is used to stress that the definition depends on the choice of complement G. We want to show that Ω_G is b-symplectic on a neighborhood of $C \subset (E^*, E^*|_{Z_C})$. As Ω_G is clearly b-closed, it suffices to prove that Ω_G is non-degenerate at points $p \in C$.

<u>Claim:</u> Under the decomposition ${}^bT_p\left({}^bT^*C\right) \cong {}^bT_pC \oplus {}^bT^*_pC$, of Lemma 1.2.12, the canonical b-symplectic form is the usual pairing

$$(\omega_{can})_p\,(v+\alpha, w+\beta) = \langle v, \beta\rangle - \langle w, \alpha\rangle. \tag{1.16}$$

This claim can be checked in coordinates $\omega_{can} = \frac{dx_1}{x_1} \wedge dy_1 + \sum_{i=2}^n dx_i \wedge dy_i$, noting that y_i is a linear coordinate on fibers ${}^bT^*_pC$, i.e. $y_i \in \left({}^bT^*_pC\right)^* \cong {}^bT_pC$.

Consider now the decomposition

$${}^bT_pE^* \cong {}^bT_pC \oplus E^*_p = E_p \oplus G_p \oplus E^*_p \tag{1.17}$$

given by Lemma 1.2.12. Using Lemma 1.2.13 $b)$ we have $\left({}^bj_*\right)_p = \mathrm{Id}_{{}^bT_pC} \oplus j|_{E^*_p}$. Hence under the decomposition (1.17) we have

$$\begin{aligned}\left({}^bj^*\omega_{can}\right)_p(v+w+\alpha, x+y+\beta) &= (\omega_{can})_p\left(v+w+j(\alpha), x+y+j(\beta)\right)\\ &= \langle v+w, j(\beta)\rangle - \langle x+y, j(\alpha)\rangle\\ &= \langle v, j(\beta)\rangle - \langle x, j(\alpha)\rangle,\end{aligned}$$

using the above claim and recalling that $j(E^*_p) = G^0_p$. In matrix notation,

$$\left({}^bj^*\omega_{can}\right)_p = \begin{array}{c} \\ E_p \\ G_p \\ E^*_p \end{array} \begin{array}{c} \begin{array}{ccc} E_p & G_p & E^*_p \end{array} \\ \begin{pmatrix} 0 & 0 & A \\ 0 & 0 & 0 \\ -A^T & 0 & 0 \end{pmatrix} \end{array}, \tag{1.18}$$

for some matrix A of full rank. Similarly we have $\left({}^b\pi_*\right)_p = \mathrm{Id}_{{}^bT_pC} \oplus 0$, applying Lemma 1.2.13 *b*) to π (regarded as a vector bundle map). Hence, under (1.17) we get

$$\left({}^b\pi^*\omega_C\right)_p (v + w + \alpha, x + y + \beta) = (\omega_C)_p (v + w, x + y),$$

so that we get a matrix representation of the form

$$\left({}^b\pi^*\omega_C\right)_p = \begin{array}{c} \\ E_p \\ G_p \\ E_p^* \end{array} \begin{array}{c} \begin{array}{ccc} E_p & G_p & E_p^* \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array}, \tag{1.19}$$

where we also use that $E = \mathrm{Ker}(\omega_C)$. Note that the matrix B in (1.19) is of full rank since the restriction of $(\omega_C)_p$ to G_p is non-degenerate. Combining (1.18) and (1.19), we have that

$$(\Omega_G)_p = \left({}^b\pi^*\omega_C\right)_p + \left({}^bj^*\omega_{can}\right)_p = \begin{array}{c} \\ E_p \\ G_p \\ E_p^* \end{array} \begin{array}{c} \begin{array}{ccc} E_p & G_p & E_p^* \end{array} \\ \begin{pmatrix} 0 & 0 & A \\ 0 & B & 0 \\ -A^T & 0 & 0 \end{pmatrix} \end{array}, \tag{1.20}$$

which is of full rank. Therefore, Ω_G is non-degenerate at points $p \in C \subset E^*$. □

Proof of Theorem 1.3.13 a). The inclusion $(C, Z_C, \omega_C) \overset{i}{\hookrightarrow} (E^*, E^*|_{Z_C}, \Omega_G)$ is a b-coisotropic embedding, i.e.

i) ${}^bi^*\Omega_G = \omega_C$,

ii) ${}^bTC^{\Omega_G} \subset {}^bTC$.

We have ${}^bi^*\Omega_G = {}^b(\pi \circ i)^*\omega_C + {}^b(j \circ i)^*\omega_{can} = \omega_C + {}^b(j \circ i)^*\omega_{can}$. Note that $j \circ i$ is the inclusion of C into ${}^bT^*C$, so that ${}^b(j \circ i)^*\omega_{can} = 0$ since C is b-Lagrangian in $\left({}^bT^*C, \omega_{can}\right)$. This proves that item i) holds.

To check ii), we let $p \in C$ and choose $v + w + \alpha \in E_p \oplus G_p \oplus E_p^* \cong {}^bT_pE^*$ lying in ${}^bT_pC^{\Omega_G}$. Let $x \in E_p \subset {}^bT_pC$ be arbitrary. Thanks to (1.20), we then have

$$0 = (\Omega_G)_p (x, v + w + \alpha) = (\Omega_G)_p (x, \alpha),$$

which forces that $\alpha = 0$ due to non-degeneracy of $(\Omega_G)_p$ on $E_p \times E_p^*$. Hence $v + w + \alpha = v + w$ lies in $E_p \oplus G_p = {}^bT_pC$, as desired. □

The uniqueness statement *b*) of Theorem 1.3.13 is an immediate consequence of the following proposition, to which we devote the rest of this subsection.

Proposition 1.3.15. *Let (M, Z, ω) be a b-symplectic manifold and C a b-coisotropic submanifold, with induced b-presymplectic form $\omega_C \in {}^b\Omega^2(C)$. Denote $E := Ker(\omega_C)$ and fix a splitting ${}^bTC = E \oplus G$. Then there is a b-symplectomorphism τ between a neighborhood of $C \subset (M, Z, \omega)$ and a neighborhood of $C \subset (E^*, E^*|_{C\cap Z}, \Omega_G)$, with $\tau|_C = Id_C$.*

Proof. Since the restriction $\omega|_{G\times G}$ is non-degenerate, we have a decomposition ${}^bTM|_C = G \oplus G^\omega$ as symplectic vector bundles. Note that E is a Lagrangian subbundle of (G^ω, ω), since

$$E^\omega \cap G^\omega = (E \oplus G)^\omega = {}^bTC^\omega = {}^bTC^\omega \cap {}^bTC = E. \tag{1.21}$$

We fix a Lagrangian complement V to E in (G^ω, ω), i.e. $G^\omega = E \oplus V$.

The idea of the proof is to construct a b-diffeomorphism between neighborhoods of C in M and E^* – obtained by composing b-diffeomorphisms to a neighborhood in V – whose b-derivative at points of C pulls back Ω_G to ω, and then apply a Moser argument. We first prove a b-geometry version of the tubular neighborhood theorem, in which V plays the role of the normal bundle to C.

Claim 1: There exists a b-diffeomorphism ϕ between a neighborhood of C in $(V, V|_{C\cap Z})$ and a neighborhood of C in (M, Z), with ${}^b\phi_*\big|_C = \mathrm{Id}_{{}^bTM|_C}$.

We will construct this map in two steps:

$$V \underset{(1)}{\longrightarrow} \rho(V) \underset{(2)}{\longrightarrow} M.$$

First, let $\rho : {}^bTM \to TM$ denote the anchor map of ${}^bTM$ and notice that its restriction to V is injective. To see this, recall the decomposition

$${}^bTM|_C = G \oplus G^\omega = G \oplus E \oplus V = {}^bTC \oplus V \tag{1.22}$$

and note that $\mathrm{Ker}(\rho|_C) \subset {}^bTC$ by Lemma 1.2.8, so $\mathrm{Ker}(\rho)$ intersects V trivially. As such, we get a b-diffeomorphism $\rho : (V, V|_{C\cap Z}) \to (\rho(V), \rho(V)|_{C\cap Z})$.

Second, the distribution $\rho(V)$ is complementary to TC, i.e.

$$TM|_C = TC \oplus \rho(V).$$

Indeed, by what we just showed, we have at any point $p \in C$ that

$$\dim(T_pM) = \dim(T_pC) + \dim(V_p) = \dim(T_pC) + \dim(\rho(V_p)).$$

Moreover, if $v \in V_p$ is such that $\rho(v) \in T_pC$, then $v \in {}^bT_pC \cap V_p = \{0\}$. Now fix a Riemannian metric g on M such that $Z \subset (M, g)$ is totally geodesic (e.g. [Mi, Lemma 6.8]). The corresponding exponential map $\exp_g$ takes a neighborhood of

$C \subset \rho(V)$ diffeomorphically onto a neighborhood of $C \subset M$. Moreover the fibers of $\rho(V)$ over $C \cap Z$ are mapped into Z, since $\rho(V_p) \in T_pZ$ for $p \in C \cap Z$ and Z is totally geodesic. Therefore, the map $\exp_g : (\rho(V), \rho(V)|_{C\cap Z}) \to (M, Z)$ is a b-diffeomorphism between neighborhoods of C.

We now show that $\phi := \exp_g \circ \rho \colon V \to M$ has the claimed property. That is, we show that $\left[{}^b(\exp_g \circ \rho)_*\right]\big|_C$ is the identity map on ${}^bTV|_C \cong {}^bTC \oplus V = {}^bTM|_C$, by checking that it acts as the identity on sections. We will need the commutative diagram

$$\begin{array}{ccc} {}^bTC \oplus V & \xrightarrow{\;{}^b(\exp_g \circ \rho)_*\;} & {}^bTM|_C \\ \Big\downarrow{\scriptstyle \rho \oplus \mathrm{Id}_V} & & \Big\downarrow{\scriptstyle \rho} \\ TC \oplus V & \xrightarrow{\;(\exp_g \circ \rho)_*\;} & TM|_C \end{array} , \tag{1.23}$$

which implicitly uses a) of Lemma 1.2.13. We will also use that for all $q \in C$ the ordinary derivative reads

$$\left[(\exp_g \circ \rho)_*\right]_q : T_qV \cong T_qC \oplus V_q \to T_qM = T_qC \oplus \rho(V_q) : w + v \mapsto w + \rho(v). \tag{1.24}$$

For a section $X + Y \in \Gamma\left({}^bTC \oplus V\right)$ we now compute

$$\begin{aligned} \rho\left[{}^b(\exp_g \circ \rho)_*(X + Y)\right] &= (\exp_g \circ \rho)_*(\rho(X) + Y) \\ &= \rho(X) + \rho(Y) \\ &= \rho(X + Y) \end{aligned}$$

using (1.23) in the first equality and (1.24) in the second. Since the anchor ρ is injective on sections, this implies that ${}^b(\exp_g \circ \rho)_*(X + Y) = X + Y$, as desired. This finishes the proof of Claim 1.

Next, the map

$$\psi : V \to E^*,\ v \mapsto -\iota_v\omega|_E$$

is an isomorphism of vector bundles covering Id_C, whence a b-diffeomorphism between the total spaces (For the injectivity, note that $\iota_v\omega|_E = 0$ implies that $v \in E^\omega \cap G^\omega = E$ as in (1.21), so that $v \in V \cap E = \{0\}$). The composition $\psi \circ \phi^{-1} : (M, Z) \to (E^*, E^*|_{C\cap Z})$ is a b-diffeomorphism between neighborhoods of C, with $\left(\psi \circ \phi^{-1}\right)\big|_C = \mathrm{Id}_C$.

Claim 2: This b-diffeomorphism satisfies $\left[{}^b(\psi \circ \phi^{-1})^*\Omega_G\right]\big|_C = \omega|_C$.

As before, let $\pi : E^* \to C$ denote the bundle projection, and let $j : E^* \hookrightarrow {}^bT^*C$ be the inclusion induced by the splitting ${}^bTC = E \oplus G$. Since $\psi \colon V \to E^*$ is a vector bundle morphism covering Id_C, by Lemma 1.2.13 b) we have that

$$^b\psi_*|_C \colon {}^bTV|_C \cong {}^bTC \oplus V \to {}^bTE^*|_C \cong {}^bTC \oplus E^*$$

equals $Id_{^bTC} \oplus \psi$. Furthermore $^b\phi_*\big|_C = \mathrm{Id}_{^bTM|_C}$ by Claim 1. Therefore, for $p \in C$ and $x_i + v_i \in {}^bT_pC \oplus V_p = {}^bT_pM$, we have

$$\left[^b(\psi \circ \phi^{-1})^*\Omega_G\right]_p (x_1 + v_1, x_2 + v_2) = (\Omega_G)_p \left(x_1 + \psi(v_1), x_2 + \psi(v_2)\right). \quad (1.25)$$

Recalling equation (1.15) and applying Lemma 1.2.13 *b*) as in the proof of Proposition 1.3.14, the right hand side of (1.25) can be rewritten as follows:

$$\begin{aligned} &\omega_p(x_1, x_2) + (\omega_{can})_p (x_1 + j(\psi(v_1)), x_2 + j(\psi(v_2))) \\ &\quad = \omega_p(x_1, x_2) + \langle x_1, j(\psi(v_2))\rangle - \langle x_2, j(\psi(v_1))\rangle \\ &\quad = \omega_p(x_1, x_2) + \langle e_1, \psi(v_2)\rangle - \langle e_2, \psi(v_1)\rangle \\ &\quad = \omega_p(x_1, x_2) + \omega_p(e_1, v_2) + \omega_p(v_1, e_2) \\ &\quad = \omega_p(x_1 + v_1, x_2 + v_2), \end{aligned}$$

first using equation (1.16), then writing $x_i = e_i + g_i \in E_p \oplus G_p = {}^bT_pC$, and using in the last equality that V is a Lagrangian subbundle of (G^ω, ω). This finishes the proof of Claim 2.

Applying Proposition 1.2.19 (relative *b*-Moser) yields a *b*-diffeomorphism f, defined on a neighborhood of $C \subset (M, Z)$, such that $^bf^*\left(^b(\psi \circ \phi^{-1})^*\Omega_G\right) = \omega$ and $f|_C = \mathrm{Id}_C$. So setting $\tau := \psi \circ \phi^{-1} \circ f$ finishes the proof. □

1.4 Strong b-coisotropic submanifolds and b-symplectic reduction

We consider a subclass of *b*-coisotropic submanifolds in *b*-symplectic manifolds, namely, the coisotropic submanifolds that are transverse to the symplectic leaves they meet. The main observation is that their characteristic distribution has constant rank, and the quotient (whenever smooth) by this distribution inherits a *b*-symplectic form (Proposition 1.4.6).

1.4.1 Strong b-coisotropic submanifolds

In Subsection 1.3.1 we have seen that a *b*-coisotropic submanifold $C \subset (M, Z, \omega)$ comes with a characteristic distribution

$$D := \rho\left(^bTC^\omega\right) = \Pi^\sharp\left(TC^0\right).$$

In general, D fails to be regular. To force that D has constant rank, we have to impose a condition on C that is stronger than b-coisotropicity.

Definition 1.4.1. A submanifold C of a log-symplectic manifold (M, Z, Π) is called *strong b-coisotropic* if it is coisotropic (with respect to Π) and transverse to all the symplectic leaves of (M, Π) it meets.

To justify this definition, we note that

$$\begin{aligned} \Pi_p^\sharp\big|_{T_pC^0} \text{ is injective} &\Leftrightarrow \operatorname{Ker}\left(\Pi_p^\sharp\right) \cap T_pC^0 = \{0\} \\ &\Leftrightarrow T_p\mathcal{O}^0 \cap T_pC^0 = \{0\} \\ &\Leftrightarrow \left(T_p\mathcal{O} + T_pC\right)^0 = \{0\} \\ &\Leftrightarrow T_p\mathcal{O} + T_pC = T_pM, \end{aligned} \tag{1.26}$$

where $\mathcal{O}$ denotes the symplectic leaf through p. The last equation is exactly the transversality condition of Definition 1.4.1. Consequently, we have:

Proposition 1.4.2. *Let $C \subset (M, Z, \Pi)$ be a coisotropic submanifold. Then C is strong b-coisotropic iff the characteristic distribution of C is regular, with rank equal to codim(C).*

Lemma 1.3.5 immediately implies:

Corollary 1.4.3. *Let $C \subset (M, Z, \omega)$ be a strong b-coisotropic submanifold. Then its characteristic distribution is tangent to Z, and corresponds to ${}^bTC^\omega$ under the bijection of Lemma 1.2.10 b).*

Remark 1.4.4. If C is a strong b-coisotropic submanifold of (M^{2n}, Z, Π) intersecting Z, then necessarily $dim(C) \geq n + 1$. Indeed, if $\mathcal{O}$ denotes the symplectic leaf through $p \in C \cap Z$, then we have

$$\begin{aligned} \dim(C) &= \dim\left(T_p\mathcal{O} + T_pC\right) + \dim\left(T_p\mathcal{O} \cap T_pC\right) - \dim(\mathcal{O}) \\ &= \dim\left(T_p\mathcal{O} \cap T_pC\right) + 2 \\ &\geq n + 1, \end{aligned}$$

where the last inequality holds since $T_p\mathcal{O} \cap T_pC$ is a coisotropic subspace of the $(2n-2)$-dimensional symplectic vector space $T_p\mathcal{O}$. Alternatively, one can observe that a middle-dimensional b-coisotropic submanifold $C^n \subset (M^{2n}, Z, \omega)$ is b-Lagrangian (i.e. ${}^bTC^\omega = {}^bTC$). Its characteristic distribution satisfies

$$\dim\left(D_p\right) = \begin{cases} \dim(C) - 1 & \text{if } p \in C \cap Z \\ \dim(C) & \text{else} \end{cases},$$

so C can't be strong b-coisotropic when it intersects Z, due to Proposition 1.4.2.

1.4.2 Coisotropic reduction in b-symplectic geometry

We now adapt coisotropic reduction to the b-symplectic category. It is well-known that, given a coisotropic submanifold C of a Poisson manifold M, its quotient $\underline{C}$ by the characteristic distribution is again Poisson, when it is smooth. More precisely, the vanishing ideal $\mathcal{I}_C$ is a Poisson subalgebra of $(C^\infty(M), \{\cdot,\cdot\})$, and denoting by $\mathcal{N}(\mathcal{I}_C) := \{f \in C^\infty(M) : \{f, \mathcal{I}_C\} \subset \mathcal{I}_C\}$ its Poisson normalizer, we have that $\mathcal{N}(\mathcal{I}_C)/\mathcal{I}_C$ is a Poisson algebra. As an algebra it is canonically isomorphic to the algebra of smooth functions on the quotient $\underline{C}$, so it endows the latter with a Poisson structure, called the *reduced Poisson structure.*

Remark 1.4.5. If the Poisson structure on M is non-degenerate, i.e. corresponds to a symplectic form $\omega \in \Omega^2(M)$, the reduced Poisson structure on $\underline{C}$ is also non-degenerate. Indeed [SW], it corresponds to the symplectic form ω_{red} on $\underline{C}$ obtained by symplectic coisotropic reduction, i.e. the unique one such that $q^*\omega_{red} = i^*\omega$, where $q\colon C \to \underline{C}$ is the projection and $i\colon C \to M$ is the inclusion.

Proposition 1.4.6 (Coisotropic reduction)**.** *Let C be a strong b-coisotropic submanifold of a b-symplectic manifold (M, Z, ω, Π). Then $D := \Pi^\sharp\left(TC^0\right)$ is a (constant rank) involutive distribution on C. Assume that $\underline{C} := C/D$ has a smooth manifold structure, such that the projection $q : C \to \underline{C}$ is a submersion. Then $\underline{C}$ inherits a b-symplectic structure $\underline{\Omega}$, determined by*

$$ {}^b q^* \underline{\Omega} = {}^b i^* \omega, \tag{1.27} $$

where $i : C \hookrightarrow M$ is the inclusion. Its corresponding log-symplectic structure is exactly the reduced Poisson structure on $\underline{C}$ obtained from Π.

Proof. We know that D has constant rank, by Proposition 1.4.2. As for involutivity, first note that D is generated by Hamiltonians $X_h|_C$ of functions $h \in \mathcal{I}_C$. On such generators, we have

$$ [X_{h_1}|_C\,, X_{h_2}|_C] = [X_{h_1}, X_{h_2}]|_C = X_{\{h_1,h_2\}}\big|_C\,, $$

where $\{h_1, h_2\} \in \mathcal{I}_C$ due to coisotropicity of C. Hence D is involutive.

The quotient $\underline{C \cap Z} := (C \cap Z)/D$ is a smooth submanifold of $\underline{C}$, since for every slice S in C transverse to D, the intersection $S \cap Z$ is a smooth slice in $C \cap Z$ transverse to D. The leaf space $(\underline{C}, \underline{C \cap Z})$ is a b-manifold, and the projection $q : (C, C \cap Z) \to (\underline{C}, \underline{C \cap Z})$ is a b-map. For $p \in C$, we have an exact sequence

$$ 0 \to D_p \hookrightarrow T_pC \overset{(q_*)_p}{\longrightarrow} T_{q(p)}\underline{C} \to 0, $$

which corresponds with an exact sequence on the level of b-tangent spaces

$$ 0 \to \left({}^bT_pC\right)^{\omega_p} \hookrightarrow {}^bT_pC \overset{\left({}^bq_*\right)_p}{\longrightarrow} {}^bT_{q(p)}\underline{C} \to 0. \tag{1.28} $$

To see this, consider the canonical splitting $\sigma : D \to {}^bTC$ of the anchor $\rho : {}^bTC \to TC$, as constructed in Lemma 1.2.10 a), and notice that

$$\operatorname{Ker}\left(\left({}^bq_*\right)_p\right) = \sigma\left(\operatorname{Ker}\left(q_*\right)_p\right) = \sigma(D_p) = \left({}^bT_pC\right)^{\omega_p},$$

using Corollary 1.2.11 in the first and Corollary 1.4.3 in the third equality.

Since q is a surjective submersion, it admits sections, hence for every sufficiently small open subset $U \subset \underline{C}$ there is a submanifold $S \subset C$ transverse to D such that $q|_S \colon S \to U$ is a diffeomorphism. At points $p \in S$ we have

$${}^bT_pC = \left({}^bT_pC\right)^{\omega_p} \oplus {}^bT_pS$$

due to the sequence (1.28). This implies that ${}^bi_S^*\omega_C$ is a b-symplectic form on S, where $i_S : S \hookrightarrow C$ is the inclusion and ω_C is the restriction of ω to C. Denote by $\tau : U \to S$ the inverse of $q|_S : S \to U$. Then $\underline{\Omega} := {}^b\tau^*\left({}^bi_S^*\omega_C\right)$ is b-symplectic on U. Away from $\underline{C \cap Z}$, this b-2-form agrees with the symplectic form obtained by symplectic coisotropic reduction from $\omega|_{M\setminus Z}$. Denote by $-\underline{\Omega}^{-1}$ the non-degenerate b-bivector on U corresponding to $\underline{\Omega}$. Away from $\underline{C \cap Z}$, the log-symplectic structure $\underline{\rho}(-\underline{\Omega}^{-1})$ agrees with the reduced Poisson structure, by Remark 1.4.5. By continuity, the same is true on the whole of U. As U was arbitrary, the reduced Poisson structure on $\underline{C}$ is log-symplectic, and the above reasoning shows that the corresponding b-symplectic form satisfies equation (1.27). This finishes the proof. □

Examples 1.4.7. a) Let $i\colon B \hookrightarrow (M, Z)$ be a b-submanifold. A quick check in coordinates shows[2] that ${}^bT^*M|_B$ is strong b-coisotropic in ${}^bT^*M$. Its quotient $\underline{{}^bT^*M|_B}$ is canonically b-symplectomorphic to ${}^bT^*B$. To see this, consider the surjective submersion

$$\varphi : {}^bT^*M|_B \to {}^bT^*B : \alpha_p \mapsto \left({}^bi_*\right)_p^* \alpha_p$$

and notice that the fibers of φ coincide with the leaves of the characteristic distribution on ${}^bT^*M|_B$. We get a b-diffeomorphism $\overline{\varphi} : \underline{{}^bT^*M|_B} \to {}^bT^*B$. To see that this is in fact a b-symplectomorphism, we note that the tautological b-one-forms on ${}^bT^*M$ and ${}^bT^*B$ are related by

$${}^b\varphi^*\theta_B = {}^bj^*\theta_M, \tag{1.29}$$

where $j :{}^b T^*M|_B \hookrightarrow {}^bT^*M$ is the inclusion. Recall that the b-symplectic form $\underline{\Omega}$ on $\underline{{}^bT^*M|_B}$ is determined by the relation ${}^bq^*\underline{\Omega} = {}^bj^*\omega_M$, where $q :{}^b T^*M|_B \to \underline{{}^bT^*M|_B}$ is the projection (cf. (1.27)). Hence to conclude that $\overline{\varphi}$ is b-symplectic, we have to show that ${}^bq^*\left({}^b\overline{\varphi}^*\omega_B\right) = {}^bj^*\omega_M$. But this is immediate from (1.29) since $\overline{\varphi} \circ q = \varphi$.

[2]The converse is also true. If ${}^bT^*M|_B$ is strong b-coisotropic in ${}^bT^*M$, then ${}^bT^*M|_B$ is transverse to ${}^bT^*M|_Z$, which implies that B is transverse to Z, i.e. that B is a b-submanifold.

b) Given a b-manifold (M, Z), let K be an involutive distribution on M tangent to Z. Thanks to Lemma 1.2.10 a) we can view K as a subbundle $\sigma(K)$ of ${}^bTM$. Its annihilator $\sigma(K)^0$ is strong b-coisotropic in ${}^bT^*M$, and the quotient $\underline{\sigma(K)^0}$ is ${}^bT^*(M/K)$, whenever M/K is smooth. We give a proof of this fact in the particular case of a group action, see Corollary 1.4.13.

1.4.3 Moment map reduction in b-symplectic geometry

Recall that, given an action of a Lie group G on a Poisson manifold (M, Π), a *moment map* is a smooth map $J : M \to \mathfrak{g}^*$ satisfying the condition

$$\Pi^\sharp\left(dJ^x\right) = v_x \qquad \forall x \in \mathfrak{g}. \tag{1.30}$$

Here $J^x : M \to \mathbb{R} : p \mapsto \langle J(p), x\rangle$ is the x-component of J and the vector field v_x is the infinitesimal generator of the action corresponding with $x \in \mathfrak{g}$, i.e.

$$v_x(p) = \left.\frac{d}{dt}\right|_{t=0} \exp(-tx) \cdot p.$$

We will consider *equivariant* moment maps $J : M \to \mathfrak{g}^*$, i.e. those that intertwine the action $G \curvearrowright M$ and the coadjoint action $G \curvearrowright \mathfrak{g}^*$. Such a moment map $J : M \to \mathfrak{g}^*$ is automatically a Poisson map [V, Proposition 7.30], when $\mathfrak{g}^*$ is endowed with its canonical Lie-Poisson structure [CW, Section 3]. In view of Proposition 1.4.6, we recall a general fact about equivariant moment maps.

Lemma 1.4.8. *Let G be a Lie group acting on a Poisson manifold (M, Π) with equivariant moment map $J : M \to \mathfrak{g}^*$. Assume that the action is locally free on $J^{-1}(0)$. We then have the following:*

a) $J^{-1}(0)$ is a coisotropic submanifold of (M, Π).

b) $J^{-1}(0)$ is transverse to all symplectic leaves of (M, Π) it meets.

c) the characteristic distribution $\Pi^\sharp\left(T(J^{-1}(0))^0\right)$ on $J^{-1}(0)$ coincides with the tangent distribution to the orbits of $G \curvearrowright J^{-1}(0)$.

Remark 1.4.9. (i) When (M, Π) is a log-symplectic manifold, Lemma 1.4.8 implies that the level set $J^{-1}(0)$ is a strong b-coisotropic submanifold.

(ii) When G a torus, there is a more flexible notion of moment map [GMPS, Definition 22] for log-symplectic manifolds. The smooth level sets of such moment maps are not strong b-coisotropic submanifolds in general. Indeed they can even fail to be transverse to the degeneracy locus Z (see [GMPS, Example 23] for an instance where Z itself is such a level set).

For the sake of completeness, we give a proof of Lemma 1.4.8. Items a) and c) also follow from well-known facts in symplectic geometry, by restricting the G-action to each symplectic leaf (whenever G is connected) and using item b).

Proof. a) We show that 0 is a regular value of J. Choosing $p \in J^{-1}(0)$, it is enough to prove that the restriction $d_pJ : \mathrm{Im}(\Pi_p^\sharp) \subset T_pM \to \mathfrak{g}^*$ is surjective. To this end, assume that $\xi \in \mathfrak{g}$ annihilates $d_pJ(\mathrm{Im}(\Pi_p^\sharp))$. We then get for all $\alpha \in T_p^*M$ that

$$\langle \alpha, (v_\xi)_p \rangle = \left\langle \alpha, \Pi_p^\sharp \left(d_pJ^\xi\right)\right\rangle = -\left\langle d_pJ^\xi, \Pi_p^\sharp(\alpha)\right\rangle = -\left\langle d_pJ\left(\Pi_p^\sharp(\alpha)\right), \xi\right\rangle = 0,$$

so $(v_\xi)_p = 0$. Since the action $G \curvearrowright J^{-1}(0)$ is locally free, this implies $\xi = 0$. It follows that $d_pJ\big(\mathrm{Im}(\Pi_p^\sharp)\big) = \mathfrak{g}^*$, so 0 is a regular value of J. In particular, $J^{-1}(0)$ is a submanifold of M. The coisotropicity of $J^{-1}(0)$ follows since it is the preimage of a symplectic leaf $\{0\} \subset \mathfrak{g}^*$ under a Poisson map.

b) Let $\mathcal{O}$ denote the symplectic leaf through $p \in J^{-1}(0)$. By the computation (1.26), it suffices to prove that $\Pi_p^\sharp\big|_{[T_pJ^{-1}(0)]^0}$ is injective. Since 0 is a regular value, this annihilator is given by $\left[T_pJ^{-1}(0)\right]^0 = \{d_pJ^x : x \in \mathfrak{g}\}$. We now have a composition of maps

$$\begin{aligned} \mathfrak{g} &\longrightarrow \left[T_pJ^{-1}(0)\right]^0 \longrightarrow \Pi_p^\sharp\left(\left[T_pJ^{-1}(0)\right]^0\right) \\ x &\;\mapsto\quad d_pJ^x \quad\mapsto\quad \Pi_p^\sharp\left(d_pJ^x\right) = (v_x)_p\,, \end{aligned}$$

that is injective because the action $G \curvearrowright J^{-1}(0)$ is locally free. Consequently, also $\Pi_p^\sharp\big|_{[T_pJ^{-1}(0)]^0}$ is injective.

c) We have

$$\Pi_p^\sharp\left(\left[T_pJ^{-1}(0)\right]^0\right) = \left\{\Pi_p^\sharp\left(d_pJ^x\right) : x \in \mathfrak{g}\right\} = \{(v_x)_p : x \in \mathfrak{g}\}\,,$$

which is exactly the tangent space of the G-orbit through p. □

Combining Proposition 1.4.6 with Lemma 1.4.8, we obtain a moment map reduction statement in the b-symplectic category. The case $G = S^1$ was already addressed in [GLPR, Proposition 7.8].

Corollary 1.4.10 (Moment map reduction). *Consider an action of a connected Lie group G on a b-symplectic manifold (M, Z, Π) with equivariant moment map $J\colon M \to \mathfrak{g}^*$. Assume the action is free and proper on $J^{-1}(0)$. Then $J^{-1}(0)$ is a strong b-coisotropic submanifold, and its reduction $J^{-1}(0)/G$ is b-symplectic.*

Remark 1.4.11. The fact that $J^{-1}(0)/G$ is b-symplectic follows already from [MPR, Theorem 3.11], taking $A = {}^bTM$ there. (The hypothesis made there, that $(J_*)_x \circ \rho_x \colon {}^bT_xM \to \mathfrak{g}^*$ has constant rank for all $x \in J^{-1}(0)$, is satisfied since $J^{-1}(0)$ is transverse to Z). In that reference the authors develop a reduction theory for level sets of arbitrary regular values $\mu \in \mathfrak{g}^*$ satisfying the constant rank hypothesis, their statement is thus more general than the reduction statement in our Corollary 1.4.10.

Exact b-symplectic forms

As a particular case of the previous construction, we consider the b-symplectic analog of a well-known fact in symplectic geometry. Recall that, if a Lie group G acts on an exact symplectic manifold $(M, -d\theta)$ and θ is invariant under the action, then $J : M \to \mathfrak{g}^*$ defined by

$$J^x = -\iota_{v_x}\theta \tag{1.31}$$

is an equivariant moment map for the action (in the sense of (1.30)). For a proof, see [AM, Theorem 4.2.10]. A similar result holds in b-symplectic geometry.

Lemma 1.4.12 (Exact b-symplectic forms)**.** *Suppose (M, Z) is a b-manifold with exact b-symplectic form $\omega = -{}^bd\theta$. If $\phi : G \times M \to M$ is a Lie group action preserving Z and $\theta \in {}^b\Omega^1(M)$, then an equivariant moment map $J : M \to \mathfrak{g}^*$ is given by $J^x = -\iota_{V_x}\theta$. Here $V_x \in \Gamma\left({}^bTM\right)$ is the lift of the infinitesimal generator $v_x \in \Gamma(TM)$ under the anchor ρ.*

Proof. Clearly $J : M \to \mathfrak{g}^*$ is a smooth map. Restricting the action to the symplectic manifold $\left(M \setminus Z, \omega|_{M\setminus Z}\right)$, we know that $G \curvearrowright \left(M \setminus Z, -d\theta|_{M\setminus Z}\right)$ admits a moment map given by $J|_{M\setminus Z}$. Hence the equality $\Pi^\sharp\left(dJ^x\right) = v_x$ holds on the dense subset $M \setminus Z$, and as both sides are smooth on M, it holds on all of M. Similarly, since $J|_{M\setminus Z}$ is equivariant, also J itself is equivariant. □

An example of Corollary 1.4.10 and Lemma 1.4.12 is b-cotangent bundle reduction. Let us recall the picture in symplectic geometry: given an action $G \curvearrowright M$, its cotangent lift $G \curvearrowright (T^*M, -d\theta_{can})$ preserves the tautological one-form θ_{can} and therefore it comes with an equivariant moment map $J : T^*M \to \mathfrak{g}^*$ given by (1.31)

$$\langle J(\alpha_q), x\rangle = -\langle \alpha_q, v_x(q)\rangle.$$

Here v_x is the infinitesimal generator of $G \curvearrowright M$ corresponding with $x \in \mathfrak{g}$. If the action $G \curvearrowright M$ is free and proper, then symplectic reduction gives $J^{-1}(0)/G \cong T^*(M/G)$. Indeed, in some detail, there is a well-defined map

$$\varphi : J^{-1}(0) \to T^*(M/G),\ \alpha_q \mapsto \tilde{\alpha}_{pr(q)},$$

where $pr : M \to M/G$ denotes the projection and

$$\tilde{\alpha}_{pr(q)} : T_{pr(q)}(M/G) \cong \frac{T_q M}{T_q(G \cdot q)} \to \mathbb{R},\ [v] \mapsto \alpha_q(v).$$

Since the fibers of φ coincide with the orbits of $G \curvearrowright J^{-1}(0)$, there is an induced bijection $\overline{\varphi} : J^{-1}(0)/G \to T^*(M/G)$, which is in fact a symplectomorphism (see [MMOPR, Theorem 2.2.2]). We now prove a b-geometric analog of this.

Corollary 1.4.13 (Group actions on b-cotangent bundles)**.** *Given a b-manifold (M, Z) and a connected Lie group G, assume that $\phi : G \times M \to M$ is a free and proper action that preserves Z. Denote by $\Phi : G \times {}^bT^*M \to {}^bT^*M$ the b-cotangent lift of this action, that is*

$$\langle \Phi_g(\alpha_q), v\rangle = \left\langle \alpha_q, \left[{}^b\left(\phi_{g^{-1}}\right)_*\right]_{\phi_g(q)} v \right\rangle$$

for $\alpha_q \in {}^bT^*_qM$ and $v \in {}^bT_{\phi_g(q)}M$. Then Φ is also free and proper, and it preserves the hypersurface ${}^bT^*M|_Z$. The action Φ has a canonical equivariant moment map J, and $J^{-1}(0)/G$ is canonically b-symplectomorphic to ${}^bT^*(M/G)$.

Proof. Denote the infinitesimal generators of ϕ by $v_x = \rho_M(V_x) \in \mathfrak{X}(M)$ and those of Φ by $\overline{v}_x = \rho_{({}^bT^*M)}\left(\overline{V}_x\right) \in \mathfrak{X}\left({}^bT^*M\right)$, where $x \in \mathfrak{g}$. One checks that they are related via

$$\pi_*(\overline{v}_x) = v_x, \tag{1.32}$$

where $\pi : {}^bT^*M \to M$ denotes the projection. Since the action Φ preserves the tautological b-one form $\theta \in {}^b\Omega^1\left({}^bT^*M\right)$, Lemma 1.4.12 gives an equivariant moment map $J : {}^bT^*M \to \mathfrak{g}^*$ defined by $J^x = -\iota_{\overline{V}_x}\theta$. Explicitly, one has

$$-\langle J(\xi_p), x\rangle = \left(\iota_{\overline{V}_x}\theta\right)(\xi_p) = \theta_{\xi_p}\left(\overline{V}_x\right)_{\xi_p} = \left\langle \xi_p, \left({}^b\pi_*\right)_{\xi_p}\left(\overline{V}_x\right)_{\xi_p}\right\rangle = \left\langle \xi_p, (V_x)_p\right\rangle, \tag{1.33}$$

where the last equality uses (1.32) and Lemma 1.2.6. Denoting by K the tangent distribution to the orbits of $G \curvearrowright M$ and by $\sigma : K \hookrightarrow {}^bTM$ the splitting of the anchor $\rho_M : {}^bTM \to TM$ obtained via Lemma 1.2.10 a), the equality (1.33) shows that

$$J^{-1}(0) = \sigma(K)^0. \tag{1.34}$$

We now perform reduction on $J^{-1}(0)$ as in Corollary 1.4.10. Because the projection map $pr : (M, Z) \to (M/G, Z/G)$ is a b-submersion with kernel $\mathrm{Ker}(pr_*) = K$, Corollary 1.2.11 implies that $\mathrm{Ker}({}^bpr_*) = \sigma(K)$, and therefore

$${}^bT_{pr(q)}(M/G) \cong \frac{{}^bT_qM}{\sigma(K_q)}. \tag{1.35}$$

It is now clear from (1.34) and (1.35) that b-covectors in $J^{-1}(0)$ descend to M/G, i.e. we get a well-defined map

$$\varphi : J^{-1}(0) \to {}^bT^*(M/G),\ \alpha_q \mapsto \tilde{\alpha}_{pr(q)},$$

where

$$\tilde{\alpha}_{pr(q)} : {}^bT_{pr(q)}(M/G) \cong \frac{{}^bT_qM}{\sigma(K_q)} \to \mathbb{R},\ [v] \mapsto \alpha_q(v).$$

It is easy to check that φ is a surjective submersion with connected fibers. From symplectic geometry we know that the fibers of φ and the orbits of the G-action $G \curvearrowright J^{-1}(0)$ coincide on the open dense subset $J^{-1}(0) \setminus \left(J^{-1}(0) \cap {}^bT^*M\big|_Z\right)$ of $J^{-1}(0)$. By continuity, the corresponding tangent distributions must agree on all of $J^{-1}(0)$, and so the same holds for the foliations integrating them. Therefore, the map φ descends to a smooth bijective b-map

$$\overline{\varphi} : J^{-1}(0)/G \to {}^bT^*(M/G).$$

Being a bijective submersion, $\overline{\varphi}$ is a diffeomorphism. The restriction of $\overline{\varphi}$ to the complement of $\left(J^{-1}(0) \cap {}^bT^*M\big|_Z\right)/G$, endowed with the symplectic structure obtained by symplectic (i.e. coisotropic) reduction, is a symplectomorphism onto its image. Hence, by Proposition 1.4.6, $\overline{\varphi}$ is a b-symplectomorphism . □

Circle bundles

We find examples for Proposition 1.4.6 and Corollary 1.4.10 by "reverse engineering".

Proposition 1.4.14. *Let (N, ω) be a b-symplectic manifold, which we assume to be compact. Let $q\colon C \to N$ be a principal S^1-bundle, with connection $\theta \in \Omega^1(C)$. Denote by $\sigma \in \Omega^2(N)$ the closed 2-form satisfying $d\theta = q^*\sigma$.*

(i) The following is a is b-symplectic manifold:

$$\left(C \times I,\ \ \tilde{\omega} := dt \wedge p^*\theta + (t-1)p^*q^*\sigma + {}^bp^*\,{}^bq^*\omega\right).$$

Here I is an interval around 1 with coordinate t, and $p\colon C \times I \to C$ denotes the projection map.

(ii) $C \times \{1\}$ is a strong b-coisotropic submanifold, and the reduced b-symplectic manifold (as in Proposition 1.4.6) is isomorphic to (N, ω).

We make a few observations about $\tilde{\omega}$. The summand of $\tilde{\omega}$ containing σ is necessary to ensure that $\tilde{\omega}$ is ${}^bd$-closed. In the special case that C is the trivial S^1-bundle $N \times S^1$, choosing $\theta = d\rho$ for ρ the angle "coordinate" on S^1 (so $\sigma = 0$), the above lemma delivers the product of the b-symplectic manifold (N, ω) and of the symplectic manifold $(I \times S^1, dt \wedge \theta)$.

In the special case that ω equals the closed 2-form σ, we have $\tilde{\omega} = d(tp^*\theta)$, which can be interpreted as the prequantization of σ when the latter is symplectic.

Remark 1.4.15. By the above proposition, we actually recover (N, ω) by moment map reduction, as in Corollary 1.4.10. Indeed, S^1 acts on $C \times I$ (trivially on the second factor) preserving the b-symplectic form $\tilde{\omega}$ (since θ is S^1-invariant). An equivariant moment map is $J(x,t) = t - 1$, hence $C \times \{1\} = J^{-1}(0)$.

Proof. (i) To check that $\tilde{\omega}$ is ${}^b d$-closed, notice that its first two summands can be written as $d(tp^*\theta) - p^*q^*\sigma$, which is closed since σ is closed.
For every real number t sufficiently close to 1, $(t-1)\sigma + \omega$ is a b-symplectic form on N, so its n-th power (where $\dim(N) = 2n$) is a nowhere-vanishing element of ${}^b\Omega^{2n}(N)$. This implies that $\tilde{\omega}^{n+1}$ is a nowhere-vanishing element of ${}^b\Omega^{2(n+1)}(C \times I)$, shrinking I if necessary. Hence $\tilde{\omega}$ is b-symplectic.

(ii) Denote by $Z \subset N$ the singular hypersurface of ω. Then the singular hypersurface of $\tilde{\omega}$ is $p^{-1}(q^{-1}(Z)) \subset C \times I$, which is transverse to $C \times \{1\}$. Therefore the latter is a b-submanifold, and is coisotropic since it has codimension one. If $i \colon C \times \{1\} \to C \times I$ denotes the inclusion, then we have ${}^b i^*\tilde{\omega} = {}^b q^*\omega$. One consequence is that ${}^bT(C \times \{1\})^{\tilde{\omega}} = \ker({}^b i^*\tilde{\omega}) = \ker({}^b q_*)$. Applying the anchor ρ, we obtain that the characteristic distribution $\rho\left({}^bT(C \times \{1\})^{\tilde{\omega}}\right)$ of $C \times \{1\}$ is given by $\ker(q_*)$. Since the latter has constant rank one, Proposition 1.4.2 yields that $C \times \{1\}$ is a strong b-coisotropic submanifold. A second consequence is that the reduced b-symplectic manifold is isomorphic to (N, ω). □

A concrete instance of the construction of Proposition 1.4.14 is the following.

Corollary 1.4.16. *Let h be any smooth function on $\mathbb{C}P^1$ that vanishes transversely along a hypersurface. On $\mathbb{C}^2$ consider the differential forms $\Omega := i(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)$ (twice the standard symplectic form) and $\alpha := \bar{z}_1 dz_1 + \bar{z}_2 dz_2$, and denote by r the radius.*

(i) In a neighborhood of the unit sphere S^3, the following is a b-symplectic form:

$$\tilde{\omega} = \frac{1}{r^2}\left(-1 + \frac{1}{P^*h}\right)\left(-\frac{i}{r^2}(\alpha \wedge \bar{\alpha}) + \Omega\right) + \Omega, \tag{1.36}$$

where $P \colon \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}P^1$ is the projection.

(ii) The unit sphere S^3 is a strong b-coisotropic submanifold, and the reduced b-symplectic manifold is $(\mathbb{C}P^1, \frac{1}{h}\sigma)$ where σ is twice the Fubini-Study symplectic form.

Remark 1.4.17. The diagonal action of S^1 on the above neighborhood of the unit sphere S^3 in $\mathbb{C}^2$ preserves $\tilde{\omega}$ and has moment map given by $v \mapsto ||v||^2 - 1$. This follows from Remark 1.4.15 and the proof below.

Proof. On $\mathbb{R}^4 = \mathbb{C}^2$, consider the 1-form $\tilde{\theta} = \sum_{j=1}^{2} x_j dy_j - y_j dx_j$. Note that $d\tilde{\theta} = 2\sum_{j=1}^{2} dx_j \wedge dy_j = \Omega$. Consider the unit sphere S^3, and denote by $q\colon S^3 \to \mathbb{C}P^1$ the principal bundle given by the diagonal action of $U(1)$ (the Hopf fibration). Then $\theta := i^*\tilde{\theta}$ is a connection 1-form on S^3, where i is the inclusion, and $d\theta = q^*\sigma$, where σ is the symplectic form on $\mathbb{C}P^1$ obtained from Ω by coisotropic reduction. Consider the b-symplectic form $\omega := \frac{1}{h}\sigma$ on $\mathbb{C}P^1$. Applying Proposition 1.4.14 to $S^3 \times I \xrightarrow{p} S^3 \xrightarrow{q} \mathbb{C}P^1$ yields a b-symplectic form $\tilde{\omega}$ on $S^3 \times I$, defined by

$$\tilde{\omega} = dt \wedge p^*\theta + \left(t - 1 + \frac{1}{p^*q^*h}\right) p^*q^*\sigma. \tag{1.37}$$

We now make $\tilde{\omega}$ more explicit. Denote by $p'\colon \mathbb{C}^2 \setminus \{0\} \to S^3$ the projection $v \mapsto v/||v||$, let r denote the radius function $v \mapsto ||v||$. Then $p'^*(\theta) = \tilde{\theta}/r^2$, since the Euler vector field E satisfies $\iota_E\tilde{\theta} = 0$ and $\mathcal{L}_E(\tilde{\theta}/r^2) = 0$. Hence, using $q^*\sigma = d\theta$ and $d\tilde{\theta} = \Omega$ we obtain

$$p'^*q^*\sigma = d(\tilde{\theta}/r^2) = \frac{1}{r^2}\left(-2\frac{dr}{r} \wedge \tilde{\theta} + \Omega\right).$$

Using $\tilde{\theta} = Im(\alpha)$ and $r^2 = z_1\bar{z}_1 + z_2\bar{z}_2$ we get $-2\frac{dr}{r} \wedge \tilde{\theta} = -\frac{i}{r^2}(\alpha \wedge \bar{\alpha})$. If we now use the identification $(a, t) \mapsto \sqrt{t}a$ between $S^3 \times I$ and a neighborhood of S^3 in $\mathbb{C}^2$ (so $t = r^2$), then the expression (1.37) becomes (1.36). □

Remark 1.4.18. We show directly from its definition (1.36) that $\tilde{\omega}$ satisfies the transversality requirement for b-symplectic forms. As $\left(-\frac{i}{r^2}(\alpha \wedge \bar{\alpha}) + \Omega\right)^{\wedge 2}$ vanishes, one obtains $\tilde{\omega}^{\wedge 2} = -2(1 - \frac{1}{r^2} + \frac{1}{r^2}\frac{1}{P^*h})dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2$. The dual 4-vector field is thus transverse to the zero section, in a neighborhood of the unit sphere S^3.

Example 1.4.19. We display an example of a function h on $\mathbb{C}P^1$ which vanishes on the circle $\mathbb{R}P^1 \subset \mathbb{C}P^1$. The function $g := Im(\bar{z}_1 z_2) = x_1y_2 - y_1x_2$ on S^3 is $U(1)$-invariant, hence it descends to a function h on $\mathbb{C}P^1$, which is readily seen to vanish exactly on $\mathbb{R}P^1$. It vanishes linearly there: using homogeneous the coordinate $w := z_2/z_1$ on the open subset $\{[z_1 : z_2] : z_1 \neq 0\}$ of $\mathbb{C}P^1$, we have[3] $h = \frac{Im(w)}{1+|w|^2}$, which vanishes with non-zero derivative on $\{Im(w) = 0\}$. Since g is quadratic, we have $p'^*g = g/r^2$, hence the coefficient $\frac{1}{r^2}\left(-1 + \frac{1}{P^*h}\right)$ in equation (1.36) reads

$$\left(-\frac{1}{r^2} + \frac{1}{Im(\bar{z}_1 z_2)}\right).$$

[3]To see this, first notice that on S^3 we have $\bar{z}_1 z_2 = (\bar{z}_1 z_2)/(\bar{z}_1 z_1 + \bar{z}_2 z_2)$, and then divide numerator and denominator by $\bar{z}_1 z_1$.

Bibliography

[AM] R. Abraham and J. Marsden, *Foundations of Mechanics*, Addison-Wesley, 2nd edition, 1987.

[C] A. Cannas da Silva, *Lectures on Symplectic Geometry*, Lecture Notes in Mathematics **1764**, Springer-Verlag Berlin, 2001.

[CW] A. Cannas da Silva and A. Weinstein, *Geometric Models for Noncommutative Algebras*, Berkeley Mathematics Lecture Notes series **10**, American Mathematical Society, 1999.

[CK] G. Cavalcanti and R. Klaasse, *Fibrations and log-symplectic structures*, J. Symplectic Geom. **17**(3), p. 603-638, 2019.

[G] M. J. Gotay, *On coisotropic imbeddings of presymplectic manifolds*, Proc. Amer. Math. Soc. **84**(1), p. 111-114, 1982.

[GL] M. Gualtieri and S. Li, *Symplectic groupoids of log symplectic manifolds*, Int. Math. Res. Not. **2014**(11), p. 3022-3074, 2014.

[GLPR] M. Gualtieri, S. Li, A. Pelayo and T. Ratiu, *The tropical momentum map: a classification of toric log symplectic manifolds*, Math. Ann. **367**(3-4), p. 1217-1258, 2016.

[GMP] V. Guillemin, E. Miranda and A.R. Pires, *Symplectic and Poisson geometry on b-manifolds*, Adv. Math. **264**, p. 864-896, 2014.

[GMPS] V. Guillemin, E. Miranda, A.R. Pires and G. Scott, *Toric actions on b-symplectic manifolds*, Int. Math. Res. Not. **2015**(14), p. 5818-5848, 2015.

[Ki] C. Kirchhoff-Lukat, *Aspects of Generalized Geometry: Branes with Boundary, Blow-ups, Brackets and Bundles*, Ph.D. Thesis, University of Cambridge, 2018.

[Kl] R. Klaasse, *Geometric Structures and Lie Algebroids*, Ph.D. Thesis, Utrecht University, 2017.

[KL] R. Klaasse and M. Lanius, *Poisson cohomology of almost-regular Poisson structures*, In preparation.

[MPR] J.C. Marrero, E. Padron and M. Rodriguez-Olmos, *Reduction of a symplectic-like Lie algebroid with momentum map and its application to fiberwise linear Poisson structures*, J. Phys. A **45**(16), p. 165-201, 2012.

[MMOPR] J. Marsden, G. Misiolek, J.-P. Ortega, M. Perlmutter and T. Ratiu, *Hamiltonian Reduction by Stages*, Springer, 2007.

[Me] R.B. Melrose, *The Atiyah-Patodi-Singer index theorem*, Research Notes in Mathematics **4**, A.K. Peters, Wellesley, 1993.

[Mi] J. Milnor, *Lectures on the h-cobordism theorem*, Princeton University Press, 1965.

[P] A. Polishchuk, *Algebraic geometry of Poisson brackets*, J. Math. Sci. **84**(5), p. 1413-1444, 1997.

[SW] J. Sniatycki and A. Weinstein, *Reduction and quantization for singular momentum mappings*, Lett. Math. Phys. **7**(2), p. 155-161, 1983.

[V] I. Vaisman, *Lectures on the Geometry of Poisson Manifolds*, Progress in Mathematics **118**, Birkhäuser Basel, 1994.

Chapter 2

Deformations of Lagrangian submanifolds in log-symplectic manifolds

This chapter is based on joint work with Marco Zambon. It contains the preprint *"Deformations of Lagrangian submanifolds in log-symplectic manifolds"*, which is available on arXiv:2009.01146.

Abstract - This chapter is devoted to deformations of Lagrangian submanifolds contained in the singular locus of a log-symplectic manifold. We prove a normal form result for the log-symplectic structure around such a Lagrangian, which we use to extract algebraic and geometric information about the Lagrangian deformations. We show that the deformation problem is governed by a DGLA, we discuss whether the Lagrangian admits deformations not contained in the singular locus, and we give precise criteria for unobstructedness of first order deformations. We also address equivalences of deformations, showing that the gauge equivalence relation of the DGLA corresponds with the geometric notion of equivalence by Hamiltonian isotopies. We discuss the corresponding moduli space, and we prove a rigidity statement for the more flexible equivalence relation by Poisson isotopies.

2.1 Introduction

Symplectic manifolds are a key concept in modern geometry and physics. A fundamental role in symplectic geometry is played by the distinguished class of Lagrangian submanifolds, as emphasized in Weinstein's symplectic creed [W2]: *Everything is a Lagrangian submanifold.*

The deformation theory of Lagrangian submanifolds is well-behaved: as a consequence of Weinstein's Lagrangian neighborhood theorem [W1], deformations of a Lagrangian submanifold L correspond with small closed one-forms on L, and the moduli space under equivalence by Hamiltonian isotopies can be identified with the first de Rham cohomology group $H^1(L)$.

Poisson manifolds are intimately related with symplectic geometry. The non-degenerate Poisson manifolds are exactly the symplectic ones. If one relaxes the non-degeneracy condition, replacing it with a transverse vanishing condition, one obtains a larger class of Poisson manifolds, called log-symplectic manifolds: they are symplectic outside of their singular locus, which is a codimension-one submanifold. Their first appearance occurs in the work of Nest-Tsygan [NT]. The study of their geometry was initiated by Radko [R], who classified two-dimensional log-symplectic manifolds (nowadays called *Radko surfaces*). Since the systematic study of their geometry in arbitrary dimension by Guillemin-Miranda-Pires [GMP2], log-symplectic manifolds have attracted lots of attention. One reason for this is that, despite the presence of singularities, they behave like symplectic manifolds in many respects. For instance, Mărcuţ-Osorno Torres [MO] showed that, on a compact manifold M, the space of log-symplectic structures $\mathcal{C}^1$-close to a given one (modulo $\mathcal{C}^1$-small diffeomorphisms) is smooth and finite dimensional, parametrized by the second b-cohomology of M.

This work originated from the following question: *in log-symplectic geometry, is the deformation theory of Lagrangian submanifolds as nicely behaved as in symplectic geometry?*

For Lagrangian submanifolds L transverse to the singular locus of the log-symplectic manifold, the answer is easily seen to be positive, as shown by Kirchhoff-Lukat [K]: a neighborhood of L is equivalent to the b-cotangent bundle of L, and the Lagrangian deformations of L (modulo Hamiltonian isotopy) are parametrized by the first b-cohomology group of L. In particular, the moduli space of Lagrangian deformations is smooth and finite dimensional for compact Lagrangians L.

This chapter focuses on the opposite extreme: we assume that the Lagrangian submanifold L^n is *contained in the singular locus* Z of an orientable log-symplectic manifold M^{2n}. Note that the b-calculus developed by Melrose

[Me], which is one of the main tools in log-symplectic geometry, does not apply in our setting, due to the complete lack of transversality to Z.

The main geometric questions we address are:

1) Can $L \subset Z$ be deformed smoothly to Lagrangian submanifolds not contained in Z?
2) Can a first order deformation of L be extended to a smooth path of Lagrangian deformations?
3) Is the moduli space of Lagrangian deformations – under the equivalence by Hamiltonian isotopies – smooth at L?

For "many" Lagrangian submanifolds L, the answer to 1) is positive, ensuring that the deformation problem we consider does not boil down to the case of symplectic geometry. The answer to 3) is typically negative, in contrast to the symplectic case. The answer to 2) is striking, and displays a behaviour that comes close to the symplectic case: first order deformations are generally obstructed, but if an obvious quadratic obstruction vanishes, then they can be extended to a smooth path of deformations.

Summary of results. As in many deformation problems in geometry, the first step consists in providing a normal form for the log-symplectic structure in a neighborhood of the Lagrangian L. Notice that as L is contained in the singular locus, it carries a codimension-one foliation $\mathcal{F}_L$. Our normal form around L (Cor. 2.2.18) is constructed in two steps: we combine a normal form statement around Lagrangian submanifolds transverse to the symplectic leaves of an arbitrary Poisson manifold (Prop. 2.2.9) with the normal form around the singular locus Z of a log-symplectic manifold (M,Π) due to Guillemin-Miranda-Pires [GMP2], [O]. Since the latter involves the modular class of (M,Π), we also need to express the first Poisson cohomology of a neighborhood of L in the singular locus Z in terms of L alone (Cor. 2.3.5). The modular class is then encoded by two objects attached to L:

a) A class in $H^1(\mathcal{F}_L)$, the first foliated de Rham cohomology.
   We fix a representative $\gamma \in \Omega^1_{cl}(\mathcal{F}_L)$.
b) An element of $\mathfrak{X}(L)^{\mathcal{F}_L}/\Gamma(T\mathcal{F}_L) \cong H^0(\mathcal{F}_L)$.
   We fix a representative $X \in \mathfrak{X}(L)^{\mathcal{F}_L}$, a vector field on L that preserves the foliation and is nowhere tangent to it.

Theorem 2A. *The log-symplectic structure in a tubular neighborhood of L is isomorphic to*

$$\big(U \subset T^*\mathcal{F}_L \times \mathbb{R},\ \ (V_{vert} + V_{lift}) \wedge t\partial_t + \Pi_{can}\big).$$

Here U is a neighborhood of the zero section L, Π_{can} is the canonical Poisson structure on the cotangent bundle $T^*\mathcal{F}_L$ of the foliation $\mathcal{F}_L$, and t denotes the *coordinate on $\mathbb{R}$. Further, V_{vert} is the vertical fiberwise constant vector field on $T^*\mathcal{F}_L$ which corresponds to $\gamma \in \Gamma(T^*\mathcal{F}_L)$ under the natural identification, and V_{lift} is the cotangent lift of X.*

The above normal form theorem gives an explicit model in which the Lagrangian deformations of L can be investigated. We can characterize algebraically the Lagrangian deformations of L, as follows (Thm. 2.4.3, Cor. 2.4.10):

Theorem 2B. *Lagrangian deformations $\mathcal{C}^1$-close to L are exactly the graphs of sections (α, f) of the vector bundle $T^*\mathcal{F}_L \times \mathbb{R} \to L$ satisfying the quadratic equation*

$$\begin{cases} d_{\mathcal{F}_L}\alpha = 0 \\ d_{\mathcal{F}_L}f + f(\gamma - \pounds_X\alpha) = 0, \end{cases}$$

where $d_{\mathcal{F}_L}$ denotes the foliated de Rham differential and γ, X are as above. Further, this equation is the Maurer-Cartan equation of a DGLA.

The differential graded Lie algebra mentioned above is the one introduced in greater generality by Cattaneo-Felder [CF], and to ensure that it captures the Lagrangian deformations we need to check that the Poisson structure of Thm. 2A is fiberwise entire.

In turn, Thm. 2B has several geometric consequences. Before explaining them, we discuss briefly two of the tools we use. First, when L is compact and connected, the following dichotomy about the foliation $\mathcal{F}_L$ is well-known [C, Theorem 9.3.13]: either it is the foliation associated to a fibration $L \to S^1$, or all leaves are dense. This allows us to prove several statements in the compact case by considering the two cases separately. Second, the linear part of the above Maurer-Cartan equation reads

$$d_{\mathcal{F}_L}\alpha = 0, \qquad d^{\gamma}_{\mathcal{F}_L}f = 0 \tag{2.1}$$

where $d^{\gamma}_{\mathcal{F}_L}f = d_{\mathcal{F}_L}f + f\gamma$ denotes the foliated de Rham differential *twisted* by γ. The cohomology associated to $d^{\gamma}_{\mathcal{F}_L}$ is the foliated Morse-Novikov cohomology $H^{\bullet}_{\gamma}(\mathcal{F}_L)$. We will compute it in degree 0 for compact L (Prop. 2.4.16). The ordinary (untwisted) foliated cohomology will be denoted by $H^{\bullet}(\mathcal{F}_L)$.

If the modular vector field can be chosen to be tangent to L– this happens exactly when $[\gamma] = 0 \in H^1(\mathcal{F}_L)$– then it is easy to see that L can be deformed smoothly to Lagrangian submanifolds outside of the singular locus Z. At the opposite end of the spectrum we have (Cor. 2.5.5, Prop. 2.5.10):

Theorem 2C. *Assume L is compact and connected.*

i) Suppose $\mathcal{F}_L$ is the fiber foliation of a fiber bundle $p : L \to S^1$. If for every leaf B of $\mathcal{F}_L$ we have $[\gamma|_B] \neq 0 \in H^1(B)$, then $\mathcal{C}^1$-small deformations of L necessarily stay inside the singular locus.

ii) Suppose $\mathcal{F}_L$ has dense leaves, and that $H^1(\mathcal{F}_L)$ is finite dimensional. If $\gamma \in \Omega^1_{cl}(\mathcal{F}_L)$ is not exact, then $\mathcal{C}^\infty$-small deformations of L necessarily stay inside the singular locus.

The finite dimensionality assumption in *ii)* above is necessary: we show this exhibiting an example, in which L is the 2-torus and $\mathcal{F}_L$ a Kronecker foliation for which the slope $\lambda \in \mathbb{R} \setminus \mathbb{Q}$ is a Liouville number. The proof of these statements relies on some functional analysis and Fourier analysis.

A first order deformation is a solution of eq. (2.1), the linear part of the Maurer-Cartan equation. The deformation problem is obstructed in general: there are first order deformations which do not extend to a (formal or smooth) path of Lagrangian deformations. This is detected by the classical Kuranishi criterium: given a first order deformation (α, f), where $\alpha \in \Omega^1(\mathcal{F}_L)$ and $f \in C^\infty(L)$, the class $Kr\big([(\alpha, f)]\big)$ might not vanish. This class lives in the first foliated Morse-Novikov cohomology group $H^1_\gamma(\mathcal{F}_L)$. For a general deformation problem, the Kuranishi criterium is a necessary – but not sufficient – condition to extend a first order deformation to a formal curve of deformations. In the case at hand however, we have the following striking result (Prop. 2.5.18, Cor. 2.5.20):

Theorem 2D. *Let L be compact and connected. The following are equivalent:*

- *A first order deformation (α, f) of L is* smoothly *unobstructed,*
- $Kr\big([(\alpha, f)]\big) = 0$,
- *α extends to a closed one-form on $L \setminus \mathcal{Z}_f$, the complement of the zero locus of f.*

Notice that the last condition is independent of the data (X, γ) encoding the modular vector field.

Finally, we address moduli spaces. From a geometric point of view, it is natural to identify two $\mathcal{C}^1$-small Lagrangian deformations of L if they are related by a Hamiltonian isotopy of the ambient log-symplectic manifold (M, Π). We show that this is exactly the equivalence relation that the DGLA of Thm. 2B induces on Maurer-Cartan elements (Prop. 2.5.26). Thus by eq. (2.1), the resulting moduli space $\mathcal{M}^{Ham}$ has formal tangent space at $[L]$ given by

$$T_{[L]}\mathcal{M}^{Ham} = H^1(\mathcal{F}_L) \oplus H^0_\gamma(\mathcal{F}_L).$$

For most choices of L, this is an infinite dimensional vector space, while the formal tangent space to $\mathcal{M}^{Ham}$ at Lagrangians contained in $M \setminus Z$ is finite dimensional (at least if L is compact). Hence, for most choices of L, the moduli space is not smooth at $[L]$. We also exhibit some choices of L at which the moduli space is smooth.

The same phenomenon occurs for the moduli space $\mathcal{M}^{Poiss}$ obtained replacing Hamiltonian isotopies by Poisson isotopies (Prop. 2.5.30). When L is compact with dense leaves, we show that L being infinitesimally rigid under Poisson isotopies (i.e. $T_{[L]}\mathcal{M}^{Poiss} = 0$) implies that L is rigid in the following sense: any sufficiently $\mathcal{C}^\infty$-small deformation of L is related to L by a Poisson diffeomorphism isotopic to the identity (Prop. 2.5.34).

Organization of the chapter. In §2.2 and §2.3 we provide the geometric background and prove the normal form given in Theorem 2A. In §2.4 and §2.5 we address the deformations of Lagrangian submanifolds in log-symplectic manifolds, exhibiting the underlying algebraic structure and drawing several geometric consequences. We refer to the introductory text of the individual sections for more details.

2.2 Lagrangian submanifolds in Poisson geometry

In this section, we first recall some concepts in Poisson geometry and we introduce the notion of Lagrangian submanifold. Then we prove a normal form for Poisson structures around Lagrangian submanifolds intersecting the symplectic leaves transversely (Prop. 2.2.9), which can be seen as an extension of Weinstein's Lagrangian neighborhood theorem from symplectic geometry. Our main motivation is the study of Lagrangian submanifolds contained in the singular locus of a log-symplectic manifold. In §2.2.3-§2.2.4 we use the aforementioned result to find local and semilocal normal forms around them (Prop. 2.2.17 and Cor. 2.2.18).

2.2.1 Poisson structures

Definition 2.2.1. A *Poisson structure* on a manifold M is a bivector field $\Pi \in \Gamma(\wedge^2 TM)$ satisfying $[\Pi, \Pi] = 0$, where $[\cdot, \cdot]$ is the Schouten-Nijenhuis bracket of multivector fields.

The Schouten-Nijenhuis bracket on $\Gamma(\wedge^\bullet TM)$ is a natural extension of the Lie bracket of vector fields, which turns $\Gamma(\wedge^\bullet TM)[1]$ into a graded Lie algebra [DZ, Section 1.8].

The bivector field Π induces a bundle map $\Pi^\sharp : T^*M \to TM$, given by contraction of Π with covectors. The rank of Π at a point $p \in M$ is defined to be the rank of the linear map $\Pi^\sharp_p : T^*_pM \to T_pM$. A Poisson structure is called regular if its rank is the same at all points. Poisson structures $\Pi \in \Gamma(\wedge^2 TM)$ of full rank correspond with symplectic structures $\omega \in \Gamma(\wedge^2 T^*M)$ via $\omega \leftrightarrow -\Pi^{-1}$. In general, a Poisson manifold (M, Π) comes with an integrable singular distribution $\mathrm{Im}(\Pi^\sharp)$. Each leaf $\mathcal{O}$ of the associated foliation has an induced symplectic structure, given by $\omega_{\mathcal{O}} = -\left(\Pi|_{\mathcal{O}}\right)^{-1}$.

A map $\Phi : (M, \Pi_M) \to (N, \Pi_N)$ between Poisson manifolds is a Poisson map if Π_M and Π_N are Φ-related, i.e. $\left(\wedge^2 d_p\Phi\right)\left(\Pi_M\right)_p = \left(\Pi_N\right)_{\Phi(p)}$ for all $p \in M$. A vector field X on a Poisson manifold (M, Π) is called Poisson if its flow consists of Poisson diffeomorphisms, or equivalently, if $\pounds_X \Pi = 0$. Each function $f \in C^\infty(M)$ determines a Poisson vector field $X_f := \Pi^\sharp(df)$, called the Hamiltonian vector field of f. The characteristic distribution $\mathrm{Im}(\Pi^\sharp)$ of a Poisson manifold (M, Π) is generated by its Hamiltonian vector fields.

Thanks to the graded Jacobi identity of the Schouten-Nijenhuis bracket $[\cdot, \cdot]$, the operator $[\Pi, \cdot] : \Gamma\left(\wedge^\bullet TM\right) \to \Gamma\left(\wedge^{\bullet+1} TM\right)$ squares to zero. The cohomology of the resulting cochain complex $\left(\Gamma\left(\wedge^\bullet TM\right), [\Pi, \cdot]\right)$ is the Poisson cohomology of (M, Π), which we denote by $H^\bullet_\Pi(M)$. The cohomology groups in low degrees have geometric interpretations, see for instance [DZ, Section 2.1]. We will only encounter the first cohomology group $H^1_\Pi(M)$, which is the quotient of the space of Poisson vector fields by the space of Hamiltonian vector fields.

The modular class of (M, Π) is a distinguished element in $H^1_\Pi(M)$ which will play a key role in this chapter. It is defined as follows: upon choosing a volume form $\mu \in \Omega^{top}(M)$, there is a unique vector field $V^\mu_{mod} \in \mathfrak{X}(M)$ such that for all $f \in C^\infty(M)$, one has

$$\pounds_{X_f} \mu = V^\mu_{mod}(f) \mu.$$

The vector field V^μ_{mod} is called the modular vector field associated with μ. One can check that V^μ_{mod} is a Poisson vector field, and that choosing a different volume form $\mu' = g\mu$ changes the modular vector field V^μ_{mod} by a Hamiltonian vector field:

$$V^{\mu'}_{mod} = V^\mu_{mod} - X_{\ln|g|}. \tag{2.2}$$

So the Poisson cohomology class $[V^\mu_{mod}] \in H^1_\Pi(M)$ is intrinsically defined; it is called the modular class of (M, Π). A Poisson manifold is called unimodular if its modular class vanishes. If M is not orientable, one can still define the modular class using densities instead of volume forms. In this thesis, we will only work with modular vector fields on orientable manifolds. For more on the modular class, see [W3].

We also recall some useful notions from contravariant geometry [CM1]. The general idea behind contravariant geometry on Poisson manifolds (M, Π) is to replace the tangent bundle TM by the cotangent bundle T^*M, using the bundle map $\Pi^\sharp : T^*M \to TM$.

Definition 2.2.2. For a Poisson manifold (M, Π), a Poisson spray $\chi \in \mathfrak{X}(T^*M)$ is a vector field on T^*M that satisfies the following properties:

i) $p_*\chi(\xi) = \Pi^\sharp(\xi)$ for all $\xi \in T^*M$,

ii) $m_t^*\chi = t\chi$ for all $t > 0$,

where $p : T^*M \to M$ is the projection map and $m_t : T^*M \to T^*M$ denotes fiberwise multiplication by t.

Property *ii)* above implies that χ vanishes on M, so that there exists a neighborhood $U \subset T^*M$ of M where the flow ϕ_χ of χ is defined up to time 1. The contravariant exponential map of χ is defined as

$$\exp_\chi : U \subset T^*M \to M : \xi \mapsto p \circ \phi_\chi^1(\xi).$$

The properties of the Poisson spray imply that $\exp_\chi$ fixes M and that its derivative at points $x \in M$ is given by

$$d_x \exp_\chi : T_xM \oplus T_x^*M :\to T_xM : (v, \xi) \mapsto v + \Pi_x^\sharp(\xi).$$

By property *i)* above, $\exp_\chi$ maps the fiber $U \cap T_x^*M$ into the symplectic leaf through x. Poisson sprays exist for any Poisson manifold (M, Π). They proved to be useful in the construction of symplectic realizations [CM1] and normal forms [FM], for instance.

2.2.2 Lagrangian submanifolds of Poisson manifolds

Lagrangian submanifolds

We now introduce Lagrangian submanifolds, which are the main objects of study in this chapter. We will use the following definition [V], [GU].

Definition 2.2.3. A submanifold L of a Poisson manifold (M, Π) will be called *Lagrangian* if the following equivalent conditions hold at all points $p \in L$:

i) $T_pL \cap T_p\mathcal{O} \subset \left(T_p\mathcal{O}, (\omega_\mathcal{O})_p\right)$ is a Lagrangian subspace, in the sense of symplectic linear algebra.

ii) $\Pi_p^\sharp\left(T_pL^0\right) = T_pL \cap T_p\mathcal{O}$, where $T_pL^0 \subset T_p^*M$ is the annihilator of T_pL.

Here $(\mathcal{O}, \omega_\mathcal{O})$ denotes the symplectic leaf through the point p.

In case (M, Π) is symplectic, this definition reduces to the usual notion of Lagrangian submanifold in symplectic geometry. Another special case of interest is when the manifold L has clean intersection with the leaves of (M, Π); then L is Lagrangian in M exactly when its intersection with each leaf is Lagrangian inside the leaf, in the sense of symplectic geometry.

Coisotropic submanifolds of a Poisson manifold (M, Π) are defined similarly, replacing “Lagrangian" by “coisotropic" in i) and replacing equality by the inclusion $\subset$ in ii). While coisotropic submanifolds have received lots of attention, Lagrangian submanifolds only appear rarely in the context of Poisson geometry. In this regard, there seems to be no standard definition for Lagrangian submanifolds $L \subset (M, \Pi)$. Another definition that appears in the literature uses the condition $\Pi^\sharp(TL^0) = TL$ (e.g. [D]). Notice that the latter definition is more restrictive than our Definition 2.2.3, since it imposes that connected components of L are contained in symplectic leaves and are Lagrangian therein.

Examples 2.2.4. a) The symplectic foliation associated with the Lie-Poisson structure on $\mathfrak{so}_3^* \cong \mathbb{R}^3$ consists of concentric spheres of radius $r \geq 0$. So a plane in $\mathfrak{so}_3^*$ is Lagrangian exactly when it passes through the origin.

b) Let (M, Π) be a regular Poisson manifold of rank $2k$, and $\Phi : (M, \Pi) \to (N, 0)$ a proper surjective Poisson submersion of maximal rank, which amounts to $\dim N = \dim M - k$. Assuming that the fibers of Φ are connected, they are Lagrangian tori inside the symplectic leaves of (M, Π) [DDFP, Thm. 2.6].

c) Let G be a Lie group acting on a Poisson manifold (M, Π) with equivariant moment map $J : M \to \mathfrak{g}^*$. Assume the action is locally free on $J^{-1}(0)$. Then $J^{-1}(0) \subset (M, \Pi)$ is coisotropic and transverse to the symplectic leaves (see Lemma 1.4.8). If the leaves it meets have dimension equal to $2 \dim \mathfrak{g}$, then $J^{-1}(0)$ is Lagrangian.

d) It is well-known that the graph of a Poisson map $\Phi : (M_1, \Pi_1) \to (M_2, \Pi_2)$ is coisotropic in the product $(M_1 \times M_2, \Pi_1 - \Pi_2)$. If additionally Φ restricts to an immersion on each leaf of (M_1, Π_1), then its graph is in fact Lagrangian.

Normal forms

We will establish a normal form around Lagrangian submanifolds $L \subset (M, \Pi)$ that are transverse to the symplectic leaves, extending Weinstein's Lagrangian neighborhood theorem [W1] from symplectic geometry. This is done in Prop. 2.2.9 below. The following lemma reduces the problem to Lagrangian submanifolds of regular Poisson manifolds.

Lemma 2.2.5. *Let (M, Π) be a Poisson manifold, and $L \subset (M, \Pi)$ a Lagrangian submanifold transverse to the symplectic leaves. Then there exists a neighborhood U of L such that $\Pi|_U$ is regular.*

Proof. The conditions that L be Lagrangian and transverse to the leaves of (M, Π) determine the dimension of the leaves that L meets. Indeed, if $p \in L$ and $\mathcal{O}$ is the leaf through p, then

$$\begin{aligned}\dim(T_pL) + \dim(T_p\mathcal{O}) &= \dim(T_pL + T_p\mathcal{O}) + \dim(T_pL \cap T_p\mathcal{O}) \\ &= \dim(T_pM) + \frac{1}{2}\dim(T_p\mathcal{O}),\end{aligned}$$

so that $\dim(\mathcal{O}) = 2(\dim(M) - \dim(L))$. It now suffices to show that there is an open neighborhood U of L that is contained in the saturation of L (i.e. the union of the leaves that intersect L).

To construct such a neighborhood, fix a Poisson spray $\chi \in \mathfrak{X}(T^*M)$. Let $E := \Pi^\sharp(TL^0)$, which is a vector bundle of rank $\dim(M) - \dim(L)$ because of the transversality requirement. Choosing a complement to E in $TM|_L$, we get a direct sum decomposition

$$T^*M|_L = E^* \oplus E^0. \tag{2.3}$$

We claim that the contravariant exponential map

$$\exp_\chi : E^* \to M$$

maps a neighborhood $V \subset E^*$ of L diffeomorphically onto a neighborhood $U \subset M$ of L. By property *i)* in Definition 2.2.2, this neighborhood U is then automatically contained in the saturation of L. To prove the claim, it suffices to show injectivity of the derivative of $\exp_\chi$ along the zero section

$$d_x \exp_\chi : T_xL \oplus E_x^* \to T_xM : (v, \xi) \mapsto v + \Pi_x^\sharp(\xi). \tag{2.4}$$

To do so, note that if $\Pi_x^\sharp(\xi) = -v \in T_xL$, then $\xi \in \left(\Pi_x^\sharp\right)^{-1}(T_xL) = E_x^0$. But also $\xi \in E_x^*$, so that $\xi = 0$ because of the direct sum (2.3). This implies that also $v = 0$, which proves injectivity of the map (2.4). This finishes the proof. □

So in the following, we may assume that L is Lagrangian in a regular Poisson manifold (M, Π). In the next lemma, we put the foliation of (M, Π) in normal form around L, and we construct the local model for the Poisson structure Π.

Lemma 2.2.6. *Let (M, Π) be a regular Poisson manifold with symplectic foliation $(\mathcal{F}, \omega)$. Let $L \subset (M, \Pi)$ be a Lagrangian submanifold transverse to the leaves of $\mathcal{F}$, and denote by $\mathcal{F}_L$ the induced foliation on L. We then have:*

a) There is a foliated diffeomorphism ϕ between a neighborhood of L in $(M, \mathcal{F})$ and a neighborhood of L in $(T^*\mathcal{F}_L, p^*\mathcal{F}_L)$, with $\phi|_L = Id$. Here $T^*\mathcal{F}_L$ denotes the union of the cotangent bundles of the leaves of $\mathcal{F}_L$, and $p^*\mathcal{F}_L$ is the pullback foliation of $\mathcal{F}_L$ by the bundle projection $p : T^*\mathcal{F}_L \to L$.

b) There is a canonical Poisson structure Π_{can} on the total space $T^*\mathcal{F}_L$ which gives rise to the foliation $p^*\mathcal{F}_L$.

Proof. a) By definition, $T\mathcal{F}_L$ is a Lagrangian subbundle of the symplectic vector bundle $(T\mathcal{F}|_L, \omega|_L)$. Let V be a Lagrangian complement of $T\mathcal{F}_L$, i.e. $T\mathcal{F}|_L = T\mathcal{F}_L \oplus V$. The leafwise symplectic form ω gives an isomorphism of vector bundles

$$-\omega^\flat : V \to T^*\mathcal{F}_L. \tag{2.5}$$

Next, by choosing a fiber metric g on the vector bundle $T\mathcal{F}$, we obtain a foliated exponential map $\exp_{\mathcal{F}} : U \subset T\mathcal{F} \to M$ [CC, Example 3.3.9]. For each leaf $\mathcal{O}$ of $\mathcal{F}$, we have that $\exp_{\mathcal{F}} : U \cap T\mathcal{O} \to \mathcal{O}$ is the usual exponential map of $(\mathcal{O}, g|_{T\mathcal{O}})$. Since $V \subset T\mathcal{F}|_L$ is a complement to TL in $TM|_L$, the map $\exp_{\mathcal{F}}$ gives a local diffeomorphism between neighborhoods of L

$$\exp_{\mathcal{F}} : V \to M. \tag{2.6}$$

Composing (2.5) and (2.6) gives a diffeomorphism between neighborhoods of L that matches the leaves of $\mathcal{F}$ with those of $p^*\mathcal{F}_L$. Clearly, this map restricts to the identity on L.

b) We claim that the canonical Poisson structure Π_{T^*L} on T^*L pushes forward under the restriction map $r : T^*L \to T^*\mathcal{F}_L$, and that $\Pi_{can} := r_* (\Pi_{T^*L})$ satisfies the requirement. This is readily checked in coordinates. Take a foliated chart $(x_1, \ldots, x_k, x_{k+1} \ldots, x_n)$ on L such that plaques of $\mathcal{F}_L$ are level sets of $(x_{k+1}, \ldots, x_n)$, and let $(y_1, \ldots, y_n)$ be the associated fiber coordinates on T^*L. Then the restriction map $r : T^*L \to T^*\mathcal{F}_L$ is just the projection onto the first $n+k$ coordinates, which implies that $\Pi_{T^*L} = \sum_{i=1}^n \partial_{x_i} \wedge \partial_{y_i}$ pushes forward to a Poisson structure

$$r_* (\Pi_{T^*L}) = \sum_{i=1}^k \partial_{x_i} \wedge \partial_{y_i}.$$

Clearly, the Poisson manifold $(T^*\mathcal{F}_L, \Pi_{can})$ decomposes into symplectic leaves as follows:

$$(T^*\mathcal{F}_L, \Pi_{can}) = \coprod_{\mathcal{O} \in \mathcal{F}_L} (T^*\mathcal{O}, \omega_{T^*\mathcal{O}}), \tag{2.7}$$

where $\omega_{T^*\mathcal{O}}$ denotes the canonical symplectic form on $T^*\mathcal{O}$. This finishes the proof of the lemma. □

We now proceed by showing that (M, Π) and $(T^*\mathcal{F}_L, \Pi_{can})$ are Poisson diffeomorphic near L. If $\phi : (M, \mathcal{F}) \to (T^*\mathcal{F}_L, p^*\mathcal{F}_L)$ denotes the diffeomorphism constructed in Lemma 2.2.6 (defined on a neighborhood of L), then we have that

$$(\phi_*\Pi)|_L = \Pi_{can}|_L. \tag{2.8}$$

This can be checked by direct computation, but instead we refer to the proof of Weinstein's Lagrangian neighborhood theorem in [W1], as we are just applying Weinstein's construction leaf by leaf. In some detail, we consider the restriction $\phi : (\mathcal{S}, \omega_{\mathcal{S}}) \to \left(T^*(L \cap \mathcal{S}), \omega_{T^*(L\cap\mathcal{S})}\right)$ for each leaf $\mathcal{S} \in \mathcal{F}$, and the usual argument of the Lagrangian neighborhood theorem shows that $\phi^*\omega_{T^*(L\cap\mathcal{S})}$ and $\omega_{\mathcal{S}}$ agree along $L \cap \mathcal{S}$. This immediately implies the equality (2.8).

Having established (2.8), we need an appropriate version of Moser's theorem in order to construct a Poisson diffeomorphism between neighborhoods of L in (M, Π) and $(T^*\mathcal{F}_L, \Pi_{can})$. This in turn requires a foliated version of the relative Poincaré lemma. Both statements already appeared in the literature (see [DI, Prop. 3.3] and [CM2, Lemma 5]); we state them here for convenience.

Lemma 2.2.7. *Let $(N, \mathcal{F})$ be a foliated manifold, and let $p : M \to N$ be a vector bundle over N. Denote by $\mathcal{F}' := p^*(\mathcal{F})$ the pullback foliation of $\mathcal{F}$. Suppose that $\alpha \in \Gamma\left(\wedge^k T^*\mathcal{F}'\right)$ is a closed foliated k-form whose pullback $i^*\alpha$ to $(N, \mathcal{F})$ vanishes. Then there exists a foliated $(k-1)$-form $\beta \in \Gamma\left(\wedge^{k-1} T^*\mathcal{F}'\right)$ such that $d_{\mathcal{F}'}\beta = \alpha$ and $\beta|_N = 0$.*

Lemma 2.2.8. *Let $(M, \mathcal{F}, \omega)$ be a symplectic foliation. Consider a foliated 1-form $\alpha \in \Omega^1(\mathcal{F})$ satisfying $\alpha|_N = (d_{\mathcal{F}}\alpha)|_N = 0$ for a submanifold $N \subset M$. Then $\omega + d_{\mathcal{F}}\alpha$ is non-degenerate in a neighborhood U of N, and the resulting symplectic foliation $(U, \mathcal{F}|_U, \omega|_U + (d_{\mathcal{F}}\alpha)|_U)$ is isomorphic around N to $(M, \mathcal{F}, \omega)$ by a foliated diffeomorphism that is the identity on N.*

Altogether, we obtain the following normal form around Lagrangian submanifolds transverse to the symplectic leaves of a Poisson manifold.

Proposition 2.2.9 (Local model around Lagrangians transverse to leaves)**.** *Given a Poisson manifold (M, Π), let $L \subset (M, \Pi)$ be a Lagrangian submanifold transverse to the symplectic leaves. Denote by $\mathcal{F}_L$ the induced foliation on L. Then a neighborhood of L in (M, Π) is Poisson diffeomorphic with a neighborhood of L in $(T^*\mathcal{F}_L, \Pi_{can})$, by a diffeomorphism that restricts to the identity on L.*

Proof. By Lemma 2.2.5, we can assume that (M, Π) is regular, with underlying foliation $\mathcal{F}$. By Lemma 2.2.6 and (2.8), there exists a foliated diffeomorphism between neighborhoods of L, $\phi : U \subset (M, \mathcal{F}) \to V \subset (T^*\mathcal{F}_L, p^*\mathcal{F}_L)$, satisfying

$$(\phi_*\Pi)|_L = \Pi_{can}|_L \quad \text{and} \quad \phi|_L = \mathrm{Id}.$$

Denote by $\omega, \tilde{\omega} \in \Omega^2(p^*\mathcal{F}_L|_V)$ the leafwise symplectic forms on $V \subset T^*\mathcal{F}_L$ corresponding with the Poisson structures Π_{can} and $\phi_*\Pi$, respectively. Since $\tilde{\omega} - \omega$ is closed and the restriction $(\tilde{\omega} - \omega)|_L$ vanishes, we can apply Lemma 2.2.7: shrinking V if necessary, we get that $\tilde{\omega} - \omega = d_{p^*\mathcal{F}_L}\beta$ for some $\beta \in \Omega^1(p^*\mathcal{F}_L|_V)$ satisfying $\beta|_L = 0$. Lemma 2.2.8 gives an isomorphism of symplectic foliations $\psi : (V, p^*\mathcal{F}_L|_V, \tilde{\omega}|_V) \to \left(\psi(V), p^*\mathcal{F}_L|_{\psi(V)}, \omega|_{\psi(V)}\right)$ such that $\psi|_L = \mathrm{Id}$, again shrinking V if necessary. The map $\psi \circ \phi : (U, \Pi|_U) \to \left(\psi(V), \Pi_{can}|_{\psi(V)}\right)$ now satisfies the criteria. □

Remark 2.2.10. One can also obtain Proposition 2.2.9 by applying some more general results that appeared in [CZ]. There one shows the following:

- [CZ, Theorem 8.1] Let (M, D) be a smooth Dirac manifold. If $D \cap TM$ has constant rank, then (M, D) can be embedded coisotropically into a Poisson manifold (P, Π). Explicitly, denote $E := D \cap TM$ and define P to be the total space of the vector bundle $\pi : E^* \to M$. Choosing a complement to E inside TM gives an embedding $i : E^* \hookrightarrow T^*M$. Then the Dirac structure $(\pi^* D)^{i^* \omega_{T^*M}}$, obtained pulling back D along π and applying the gauge transformation by $i^* \omega_{T^*M}$, defines a Poisson structure Π on a neighborhood of M in E^*. It has the desired properties: $M \subset (P, \Pi)$ is coisotropic and the Dirac structure D_Π pulls back to D on M.
- [CZ, Proposition 9.4] Given a Dirac manifold (M, D) for which $D \cap TM$ has constant rank k, let (P_1, Π_1) and (P_2, Π_2) be Poisson manifolds of dimension $\dim(M) + k$ in which (M, D) embeds coisotropically. Also assume that the presymplectic leaves of (M, D) have constant dimension. Then (P_1, Π_1) and (P_2, Π_2) are Poisson diffeomorphic around M.

In our situation, we have a Lagrangian submanifold $i : L \hookrightarrow (M, \Pi)$ transverse to the symplectic leaves of (M, Π), so the pullback $i^* D_\Pi$ is a smooth Dirac structure on L. Moreover, $i^* D_\Pi \cap TL$ has constant rank since it is given by $\Pi^\sharp(TL^0) = T\mathcal{F}_L$. The procedure in described in the first bullet point above then yields exactly the local model $(T^* \mathcal{F}_L, \Pi_{can})$.

Now $(L, i^* D_\Pi)$ is embedded coisotropically in (M, Π) and in $(T^* \mathcal{F}_L, \Pi_{can})$, both of which have dimension equal to $\dim(L) + rk(T\mathcal{F}_L)$. The presymplectic leaves of $(L, i^* D_\Pi)$ have constant dimension, since they are just the leaves of $\mathcal{F}_L$. Applying the second bullet point above then shows that (M, Π) and $(T^* \mathcal{F}_L, \Pi_{can})$ are Poisson diffeomorphic around L.

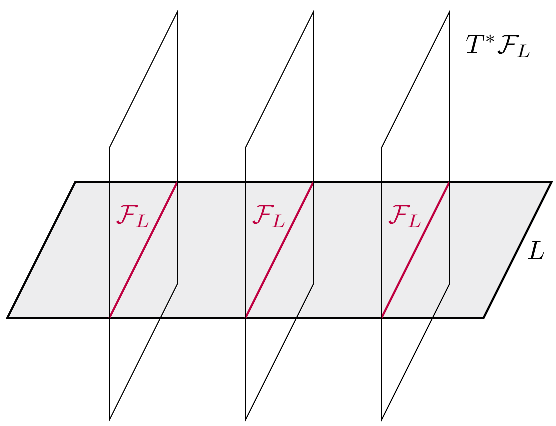

Figure 2.1: The foliation $\mathcal{F}_L$ and vector bundle $T^* \mathcal{F}_L$.

Proposition 2.2.9 implies that $\mathcal{C}^1$-small deformations of a Lagrangian $L \subset (M, \Pi)$ transverse to the leaves correspond with Lagrangian sections of $(T^*\mathcal{F}_L, \Pi_{can})$. Thanks to the decomposition (2.7), these can be studied using well-known results from symplectic geometry about Lagrangian sections in cotangent bundles. We obtain that deformations of $L \subset (M, \Pi)$ are classified by the first foliated cohomology group $H^1(\mathcal{F}_L)$.

Corollary 2.2.11. *Given a Poisson manifold (M, Π), let $L \subset (M, \Pi)$ be a Lagrangian submanifold transverse to the symplectic leaves. Denote by $\mathcal{F}_L$ the induced foliation on L.*

- *The graph of $\alpha \in \Gamma(T^*\mathcal{F}_L)$ is Lagrangian in $(T^*\mathcal{F}_L, \Pi_{can})$ iff. $d_{\mathcal{F}_L}\alpha = 0$.*
- *Two Lagrangian sections $\alpha_0, \alpha_1 \in \Gamma(T^*\mathcal{F}_L)$ are interpolated by a smooth family (α_s) of Lagrangian sections that is generated by a Hamiltonian isotopy exactly when $[\alpha_0] = [\alpha_1]$ in $H^1(\mathcal{F}_L)$.*

2.2.3 Log-symplectic structures

The rest of this chapter is devoted to a specific class of Poisson structures, called log-symplectic structures. These are generically symplectic, except at some singularities where the bivector drops rank in a controlled way.

Definition 2.2.12. A Poisson structure Π on a manifold M^{2n} is called *log-symplectic* if $\wedge^n\Pi$ is transverse to the zero section of the line bundle $\wedge^{2n}TM$.

A log-symplectic structure Π is symplectic everywhere, except at points lying in the set $Z := (\wedge^n\Pi)^{-1}(0)$, called the singular locus of (M, Π). If Z is nonempty, then it is a smooth hypersurface by the transversality condition. In that case, Z is a Poisson submanifold of (M, Π) with an induced Poisson structure that is regular of corank-one.

The geometry of the singular locus $(Z, \Pi|_Z)$ has some nice features. The foliation of $\Pi|_Z$ is unimodular, i.e. defined by a closed one-form $\theta \in \Omega^1(Z)$, and the leafwise symplectic form extends to a closed two-form $\omega \in \Omega^2(Z)$. The pair (θ, ω) defines a cosymplectic structure on Z. The existence of such a pair is equivalent with the existence of a Poisson vector field on Z that is transverse to the leaves of $\Pi|_Z$ [GMP1]. One can obtain such a vector field by restricting a modular vector field on (M, Π) to Z [GMP2].

Example 2.2.13. The standard example of a log-symplectic manifold is $\mathbb{R}^{2n}$ with coordinates $(x_1, y_1, \ldots, x_n, y_n)$ and Poisson structure

$$\Pi = \partial_{x_1} \wedge y_1\partial_{y_1} + \sum_{i=2}^{n} \partial_{x_i} \wedge \partial_{y_i}.$$

By Weinstein's splitting theorem, any log-symplectic structure looks like this near a point in its singular locus. In this example, the vector field ∂_{x_1} is the modular vector field corresponding with the volume form $\sum_{i=1}^{n} dx_i \wedge dy_i$. It is indeed tangent to the singular locus $Z = \{y_1 = 0\}$ and transverse to the symplectic leaves of $\Pi|_Z$, which are the level sets of the x_1-coordinate.

The importance of modular vector fields is apparent in the following normal form result, which describes the log-symplectic structure in a neighborhood of its singular locus [GMP2], [O, Prop. 4.1.2].

Proposition 2.2.14 (Local form around singular locus)**.** *Let* Π *be a log-symplectic structure on an orientable manifold* M*, with singular locus* $(Z, \Pi|_Z)$*. Let* $V_{mod} \in \mathfrak{X}(M)$ *be a modular vector field on* M*. Then there is a tubular neighborhood* $U \subset Z \times \mathbb{R}$ *of* Z*, in which* Z *corresponds to* $t = 0$*, such that*

$$\Pi|_U = V_{mod}|_Z \wedge t\partial_t + \Pi|_Z.$$

Recall from Chapter 1 that log-symplectic structures can alternatively be viewed as symplectic forms on a suitable Lie algebroid. To any b-manifold (M, Z) consisting of a manifold M and a hypersurface $Z \subset M$, one can associate a Lie algebroid ${}^bTM$ whose sections are the vector fields on M that are tangent to Z. Lie algebroid 2-forms $\omega \in \Gamma\left(\wedge^2\left({}^bT^*M\right)\right)$ that are closed and non-degenerate are called b-symplectic forms. Having a log-symplectic structure Π on M with singular locus Z is equivalent to having a b-symplectic form on (M, Z) [GMP2]. This point of view allows one to study log-symplectic structures using symplectic techniques. We refer to §1.2.5 for more details.

2.2.4 Lagrangian submanifolds of log-symplectic manifolds

We now focus on Lagrangian submanifolds L of log-symplectic manifolds (M, Z, Π). Lagrangians transverse to the degeneracy locus Z can be treated using the b-geometry point of view, which reduces their study to symplectic geometry. Indeed, the submanifold L is naturally a b-manifold $(L, L \cap Z)$ and the condition that L be Lagrangian (in the sense of Def. 2.2.3) is equivalent with the requirements

$$\begin{cases} {}^bi^*\omega = 0 \\ \dim(L) = \frac{1}{2}\dim(M) \end{cases},$$

where ω is the b-symplectic form defined by Π and $i : (L, L \cap Z) \hookrightarrow (M, Z)$ is the inclusion. In [K], one shows that a neighborhood of L in (M, ω) is b-symplectomorphic with a neighborhood of L in its b-cotangent bundle ${}^bT^*L$, endowed with the canonical b-symplectic form. As a consequence, the moduli space of Lagrangian deformations of L under Hamiltonian equivalence can be identified with the first b-cohomology group ${}^bH^1(L)$. All of this is in complete analogy with what happens in symplectic geometry.

We will consider Lagrangians at the other extreme, i.e. those that are contained in the singular locus of a log-symplectic manifold (M^{2n}, Z, Π). If L is such a Lagrangian and $\mathcal{O}^{2n-2}$ is the leaf through $p \in L$, then we have

$$\dim(T_pL) = \dim(T_pL + T_p\mathcal{O}) - n + 1,$$

where $2n - 2 \leq \dim(T_pL + T_p\mathcal{O}) \leq 2n - 1$. So either $\dim(L) = n - 1$ and connected components of L lie inside symplectic leaves, or $\dim(L) = n$ and L is transverse to the leaves in Z. In the rest of this chapter, we will deal with Lagrangians of the second kind:

middle dimensional Lagrangian submanifolds contained in the singular locus.

Remark 2.2.15. More generally, instead of middle dimensional Lagrangian submanifolds, one could consider middle dimensional coisotropic submanifolds $C \subset (M, Z, \Pi)$. Although these two notions coincide for submanifolds transverse to the degeneracy locus Z, they are not equivalent in general – in particular, they are not equivalent in the setup we consider.
An example of middle dimensional coisotropic C contained in Z which is not Lagrangian, is the following. Take $M = \mathbb{R}^4$ and $\Pi = \partial_{x_1} \wedge y_1\partial_{y_1} + \partial_{x_2} \wedge \partial_{y_2}$, take C given by the constraints $x_1 - y_2^3 = 0$ and $y_1 = 0$. It is coisotropic because the Poisson bracket of these constraints is y_1, thus again a constraint. It is not Lagrangian because $T_pC = T_p\mathcal{O}$ at points p of C where y_2 vanishes, where $\mathcal{O}$ denotes the (2-dimensional) symplectic leaf through p.

Example 2.2.16. In the local model $\left(\mathbb{R}^{2n}, x_1, y_1, \ldots, x_n, y_n\right)$ with its standard log-symplectic structure $\Pi = \partial_{x_1} \wedge y_1\partial_{y_1} + \sum_{i=2}^n \partial_{x_i} \wedge \partial_{y_i}$, the submanifold $L = \{y_1 = \cdots = y_n = 0\}$ is Lagrangian of middle dimension, contained in the singular locus $\{y_1 = 0\}$.

Example 2.2.16 is in fact the local model for any Lagrangian $L^n \subset Z \subset (M^{2n}, \Pi)$.

Proposition 2.2.17 (Local form around a point). *Let (M^{2n}, Z, Π) be a log-symplectic manifold and let $L^n \subset Z$ be a Lagrangian submanifold. Around any point $p \in L$, there exist coordinates $(x_1, y_1, \ldots, x_n, y_n)$ such that*

$$\begin{cases} Z = \{y_1 = 0\} \\ \Pi = \partial_{x_1} \wedge y_1\partial_{y_1} + \sum_{i=2}^n \partial_{x_i} \wedge \partial_{y_i} \\ L = \{y_1 = \cdots = y_n = 0\} \end{cases}.$$

Proof. Applying Prop. 2.2.9 and Prop. 2.2.14 locally around p shows that there exists a coordinate system $(U; x_1, t, x_2, y_2 \dots, x_n, y_n)$ such that

$$\Pi|_U = V_{mod}|_{U\cap Z} \wedge t\partial_t + \sum_{i=2}^{n} \partial_{x_i} \wedge \partial_{y_i}. \tag{2.9}$$

Here V_{mod} is a locally defined modular vector field, $L = \{t = y_2 = \cdots = y_n = 0\}$ and $Z = \{t = 0\}$. If we write $V_{mod}|_{U\cap Z}$ in coordinates as

$$V_{mod}|_{U\cap Z} = z(x,y)\partial_{x_1} + \sum_{i=2}^{n} g_i(x,y)\partial_{x_i} + \sum_{i=2}^{n} h_i(x,y)\partial_{y_i},$$

then requiring that $V_{mod}|_{U\cap Z}$ is Poisson yields that $z(x,y)$ only depends on x_1. Now $V_{mod}|_{U\cap Z} - z(x_1)\partial_{x_1}$ is a Poisson vector field tangent to the leaves, so it is locally Hamiltonian. This implies that, changing to a different modular vector field, we may assume

$$V_{mod}|_{U\cap Z} = z(x_1)\partial_{x_1}.$$

Note here that $z(x_1)$ is nowhere zero since $V_{mod}|_{U\cap Z}$ is transverse to the leaves. This allows us to define a new coordinate ξ by

$$\xi := \int \frac{1}{z(x_1)} dx_1.$$

In the new coordinate system $(\xi, t, x_2, y_2, \dots, x_n, y_n)$, the expression (2.9) becomes

$$\Pi = \partial_\xi \wedge t\partial_t + \sum_{i=2}^{n} \partial_{x_i} \wedge \partial_{y_i},$$

so these coordinates satisfy the criteria. This finishes the proof. ☐

Given a Lagrangian L^n contained in the singular locus Z of a log-symplectic manifold (M^{2n}, Π), Prop. 2.2.9 describes a neighborhood of L in $(Z, \Pi|_Z)$ and Prop. 2.2.14 describes a neighborhood of Z in (M^{2n}, Π). Combining the two propositions, we get the following normal form around $L^n \subset (M^{2n}, \Pi)$.

Corollary 2.2.18 (Local form around a Lagrangian in the singular locus)**.** *Let (M^{2n}, Z, Π) be an orientable log-symplectic manifold, and make a choice of modular vector field V_{mod} on M. Let $L^n \subset Z$ be a Lagrangian submanifold, and denote by $\mathcal{F}_L$ the induced*[1] *foliation on L. Then a neighborhood of L in (M, Π)*

[1]The leaves of the codimension-one foliation $\mathcal{F}_L$ are the connected components of the intersections of L with the symplectic leaves of Z.

can be identified with a neighborhood of L in the vector bundle $T^*\mathcal{F}_L \times \mathbb{R} \to L$, endowed with the log-symplectic structure

$$\widetilde{\Pi} := V \wedge t\partial_t + \Pi_{can}. \tag{2.10}$$

Here t is the coordinate on $\mathbb{R}$, and V is the image of $V_{mod}|_Z$ under the Poisson diffeomorphism $(Z, \Pi|_Z) \to (T^*\mathcal{F}_L, \Pi_{can})$ between neighborhoods of L constructed in Prop. 2.2.9.

Remark 2.2.19. The vector field V in (2.10) is only defined on a neighborhood W of L in $T^*\mathcal{F}_L$. Note that there is some freedom in the formula (2.10), in the sense that there we can replace V by any Poisson vector field representing the Poisson cohomology class $[V]$.
To see this, take any representative $V - X_f$ of $[V]$, for some function f defined on W. Note that V is a modular vector field of $\widetilde{\Pi}$, with respect to the volume form Ω on $W \times \mathbb{R}$ that is uniquely determined by requiring that $\langle \Omega, \wedge^n \widetilde{\Pi} \rangle = t$. If $\tilde{f}$ is an extension of f to $W \times \mathbb{R}$, then also $V - X_{\tilde{f}}$ is a modular vector field of $\widetilde{\Pi}$, with respect to the volume form $e^{\tilde{f}}\Omega$ on $W \times \mathbb{R}$. Proposition 2.2.14 now implies that replacing V by $V - X_f$ in (2.10) gives a log-symplectic structure that is Poisson diffeomorphic to $\widetilde{\Pi}$ in a neighborhood of $W \subset W \times \mathbb{R}$.

An arbitrary representative of the modular class has little to do with the Lagrangian L; we will remedy this in the next section. One could hope to find a representative of $[V]$ that is tangent to L. This amounts to finding a modular vector field, defined on a neighborhood of L in (M, Π), that is tangent to L. This can always be done locally near a point, as a consequence of Prop. 2.2.17 (namely, the vector field ∂_{x_1} in the statement of the proposition is modular and tangent to L). Globally however, this may fail, as we now show.

Example 2.2.20 (No modular vector field is tangent)*.* Consider the manifold $\mathbb{R} \times S^1 \times \mathbb{T}^2$ with coordinates $(t, \tau, \theta_1, \theta_2)$ and log-symplectic structure

$$\Pi = (\partial_\tau + \partial_{\theta_2}) \wedge t\partial_t + \partial_{\theta_1} \wedge \partial_{\theta_2}.$$

The submanifold $L := \{t = \theta_2 = 0\} \cong S^1 \times S^1$ is Lagrangian inside the singular locus $Z = S^1 \times \mathbb{T}^2$. Note that $\partial_\tau + \partial_{\theta_2}$ is a modular vector field for Π (associated with the volume form $d\theta_1 \wedge d\theta_2 \wedge dt \wedge d\tau$). If there existed a modular vector field tangent to L (defined near L), then its restriction to Z would look like

$$\partial_\tau + \partial_{\theta_2} + (\partial_{\theta_1} \wedge \partial_{\theta_2})^\sharp (df) = \partial_\tau + \left(1 + \frac{\partial f}{\partial \theta_1}\right)\partial_{\theta_2} - \frac{\partial f}{\partial \theta_2}\partial_{\theta_1}$$

for some $f \in C^\infty(Z)$, where

$$\left.\left(1 + \frac{\partial f}{\partial \theta_1}\right)\right|_{\theta_2 = 0} = 0. \tag{2.11}$$

But then, fixing any value of τ and denoting by $i : \{\tau\} \times S^1 \times \{0\} \hookrightarrow Z$ the inclusion, we get

$$0 = \int_{\{\tau\}\times S^1\times\{0\}} i^* df = \int_{\{\tau\}\times S^1\times\{0\}} \frac{\partial f}{\partial \theta_1} d\theta_1 = -\int_{\{\tau\}\times S^1\times\{0\}} d\theta_1 = -2\pi,$$

using Stokes' theorem, and (2.11) in the third equality. So there is no modular vector field tangent to L.

2.3 Poisson vector fields on the cotangent bundle of a foliation

Let L be a manifold and $\mathcal{F}_L$ a foliation on L. Denote by Π_{can} the canonical Poisson structure on $T^*\mathcal{F}_L$ (as in b) of Lemma 2.2.6). This section treats Poisson vector fields on $(T^*\mathcal{F}_L, \Pi_{can})$. We show that every class in the first Poisson cohomology group of $(T^*\mathcal{F}_L, \Pi_{can})$ admits a convenient representative (Thm. 2.3.2), and use this to compute explicitly the first Poisson cohomology group (Cor. 2.3.5). At the beginning of §2.4, we apply these results to the modular vector field of a log-symplectic manifold, and we find a convenient representative of the class $[V]$ in (2.10).

2.3.1 Convenient representatives

We denote by

$$\mathfrak{X}(L)^{\mathcal{F}_L} := \left\{W \in \mathfrak{X}(L) : [W, \Gamma(T\mathcal{F}_L)] \subset \Gamma(T\mathcal{F}_L)\right\}$$

the Lie subalgebra of vector fields on L whose flow preserves the foliation $\mathcal{F}_L$.

Lemma 2.3.1. *Let $W \in \mathfrak{X}(L)^{\mathcal{F}_L}$ and let $r : (T^*L, \Pi_{T^*L}) \to (T^*\mathcal{F}_L, \Pi_{can})$ denote the restriction. We then have the following:*

(i) The cotangent lift of W pushes forward via $r\colon T^*L \to T^*\mathcal{F}_L$ to a Poisson vector field on $T^*\mathcal{F}_L$, which we denote by $\widetilde{W}$.

(ii) When W lies in $\Gamma(T\mathcal{F}_L)$, the vector field $\widetilde{W}$ is Hamiltonian.

Proof. We denote by $p_{T^*\mathcal{F}_L} : T^*\mathcal{F}_L \to L$ and $p_{T^*L} : T^*L \to L$ the projections.

(i) Let $W_{T^*L} \in \mathfrak{X}(T^*L)$ denote the cotangent lift of W. To show that it pushes forward via r, we need to show that its action on functions preserves $r^*(C^\infty(T^*\mathcal{F}_L))$. It suffices to consider fiberwise constant and fiberwise linear functions on $T^*\mathcal{F}_L$. The fiberwise constant ones are of the form $p^*_{T^*\mathcal{F}_L}g$ for $g \in C^\infty(L)$. Since $p_{T^*L} = p_{T^*\mathcal{F}_L} \circ r$, we have

$$W_{T^*L}(r^*(p^*_{T^*\mathcal{F}_L}g)) = W_{T^*L}\,(p^*_{T^*L}g) = p^*_{T^*L}(W(g)) = r^*(p^*_{T^*\mathcal{F}_L}(W(g))).$$

Next, fiberwise linear functions on $T^*\mathcal{F}_L$ look like

$$h_X : T^*\mathcal{F}_L \to \mathbb{R} : (p,\alpha) \mapsto \langle \alpha, X(p)\rangle$$

for $X \in \Gamma(T\mathcal{F}_L)$. Clearly, one has a commutative diagram

$$\begin{array}{ccc} C^\infty_{lin}(T^*\mathcal{F}_L) & \xhookrightarrow{r^*} & C^\infty_{lin}(T^*L) \\ {\scriptstyle h_\bullet}\uparrow & & \uparrow{\scriptstyle h_\bullet} \\ \Gamma(T\mathcal{F}_L) & \xhookrightarrow{i} & \Gamma(TL) \end{array} \quad . \tag{2.12}$$

Recall that for the standard symplectic structure on T^*L, the Poisson bracket satisfies $\{h_X, h_Y\} = -h_{[X,Y]}$, for $X, Y \in \Gamma(TL)$. Moreover, the cotangent lift W_{T^*L} is minus the Hamiltonian vector field of h_W (see e.g. [CCS, §2]). So for $X \in \Gamma(T\mathcal{F}_L)$ we get

$$W_{T^*L}(r^*h_X) = W_{T^*L}\left(h_{i(X)}\right) = -X_{h_W}\left(h_{i(X)}\right) = -\{h_W, h_{i(X)}\} = h_{[W,i(X)]}.$$

The vector field $[W, i(X)]$ lies in $\Gamma(T\mathcal{F}_L)$ by assumption, so that $W_{T^*L}(r^*h_X)$ lies in $r^*(C^\infty_{lin}(T^*\mathcal{F}_L))$. This shows that W_{T^*L} pushes forward under r.

The vector field $\widetilde{W}$ is Poisson since the cotangent lift W_{T^*L} is a symplectic vector field and r is a Poisson map.

(ii) If W lies in $\Gamma(T\mathcal{F}_L)$, then we have

$$\widetilde{W} = r_*\big(i(W)\big)_{T^*L} = r_*\left(-X_{h_{i(W)}}\right) = -r_*\left(X_{r^*h_W}\right) = -X_{h_W},$$

where in the second equality we used the above comment about Hamiltonian vector fields, and in the third we used the commutativity of the diagram (2.12). □

The rest of this section is devoted to the following theorem, which provides convenient representatives for first Poisson cohomology classes, and its consequences.

Theorem 2.3.2. *Let $(L, \mathcal{F}_L)$ be a foliated manifold. Consider the standard Poisson structure Π_{can} on the total space of the vector bundle $p\colon T^*\mathcal{F}_L \to L$. Fix a class in $H^1_{\Pi_{can}}(T^*\mathcal{F}_L)$. Then there exists a representative $Y \in \mathfrak{X}(T^*\mathcal{F}_L)$ such that* *(i) Y is p-projectable and $p_*Y \in \mathfrak{X}(L)^{\mathcal{F}_L}$,*

(ii) the vector field[2] $Y - \widetilde{p_*Y}$ is vertical and constant on each fiber of p, and $Y - \widetilde{p_*Y}$ is closed when viewed as[3] a foliated 1-form on $(L, \mathcal{F}_L)$.

Notice that given a class in $H^1_{\Pi_{can}}(T^*\mathcal{F}_L)$, a representative Y as in Thm. 2.3.2 is by no means unique: adding to Y a Hamiltonian vector field of the form $\widetilde{W} + X_{p^*g}$ for $W \in \Gamma(T\mathcal{F}_L)$ and $g \in C^\infty(L)$ gives a representative of the same class that still satisfies the requirements of Thm. 2.3.2 (see Cor. 2.3.5 below).

Example 2.3.3. Consider the plane $L = \mathbb{R}^2$ with coordinates x, y, and the foliation $\mathcal{F}_L$ given by the lines $\{x = const\}$. Then $T^*\mathcal{F}_L$ is $\mathbb{R}^3$ with coordinates x, y, z, with vector bundle projection $p = (x, y)\colon \mathbb{R}^3 \to \mathbb{R}^2$ and Poisson structure $\Pi_{can} = \partial_y \wedge \partial_z$. An arbitrary Poisson vector field has the form

$$U = f(x)\partial_x + g\partial_y + k\partial_z,$$

where $g, k \in C^\infty(\mathbb{R}^3)$ satisfy $\partial_y g = -\partial_z k$. This vector field is not p-projectable in general, because g might depend on z. However, if $h(x, y, z) := \int_0^z g(x, y, t)dt$, we obtain a function on $\mathbb{R}^3$ such that

$$Y := U + X_h = f(x)\partial_x + (k + \partial_y h)\partial_z$$

is p-projectable. Notice that $p_*Y = f(x)\partial_x$ lies in $\mathfrak{X}(L)^{\mathcal{F}_L}$. Moreover, since the partial derivative $\partial_z(k+\partial_y h)$ vanishes, the vertical vector field $V := (k+\partial_y h)\partial_z$ is indeed constant on each fiber of p. Regarding V as a foliated 1-form on $(L, \mathcal{F}_L)$ yields $(k + \partial_y h)dy$, which is closed due to dimension reasons.

To prove Thm. 2.3.2, we need a few general statements about cotangent bundles.

Lemma 2.3.4. *Let N be a manifold. Consider its cotangent bundle T^*N with the standard symplectic form ω and bundle projection p_{T^*N}.*

(i) Let $Y \in \mathfrak{X}(T^*N)$ be a symplectic vector field[4]. Then there is $h \in C^\infty(T^*N)$ such that $Y + X_h$ is a vertical vector field.

(ii) Let $V \in \mathfrak{X}(T^*N)$ be a vertical symplectic vector field. Then V must be constant on each fiber. It is closed when viewed as an element of $\Gamma(T^*N) = \Omega^1(N)$.

Proof. (i) Consider the foliation $\mathcal{F}_{fiber}$ of T^*N by fibers of the projection p_{T^*N}. Denote by i_{fiber} the inclusion of its tangent distribution into the tangent bundle of T^*N. Since the 1-form $\iota_Y\omega \in \Omega^1(T^*N)$ is closed, its

[2] The lift $\widetilde{p_*Y}$ was defined in Lemma 2.3.1.
[3] I.e., when viewed as a section of $p : T^*\mathcal{F}_L \to L$.
[4] Part (i) holds more generally when the pullback of $\iota_Y\omega$ to each fiber of p_{T^*N} is closed.

pullback $i^*_{fiber}(\iota_Y\omega)$ is closed as a foliated 1-form. It is foliated exact, as the leaves of $\mathcal{F}_{fiber}$ are just fibers of a vector bundle (choosing the primitives on each fiber to vanish on the zero section, they assemble to a smooth function on T^*N, c.f. Lemma 2.2.7). So $i^*_{fiber}(\iota_Y\omega)$ equals $d_{\mathcal{F}_{fiber}}h$ for some $h \in C^\infty(T^*N)$, which implies that $\iota_Y\omega - dh \in \Omega^1(T^*N)$ pulls back to zero under i_{fiber}. As the fibers are Lagrangian, this means that $\omega^{-1}(\iota_Y\omega - dh) = Y + X_h$ is a vertical vector field on T^*N.

(ii) The 1-form $\iota_V\omega$ is closed because V is a symplectic vector field. For any vertical vector field W we have that $\iota_W\iota_V\omega = 0$ and $\pounds_W(\iota_V\omega) = 0$, so $\iota_V\omega = -p^*_{T^*N}\alpha$ for a unique, closed $\alpha \in \Omega^1(N)$. Writing in local coordinates $\alpha = \sum_i f_i(q)dq_i$, in the corresponding canonical coordinates on T^*N we have

$$V = -(\omega^{-1})(p^*_{T^*N}\alpha) = \sum_i f_i(q)\partial_{p_i},$$

showing that V is constant along the fibers. This formula also shows that V, when regarded as an element of $\Gamma(T^*N) = \Omega^1(N)$, is precisely the closed 1-form α. This finishes the proof. □

Proof of Thm. 2.3.2. Let $U \in \mathfrak{X}(T^*\mathcal{F}_L)$ be any representative of the given class in $H^1_{\Pi_{can}}(T^*\mathcal{F}_L)$. Let $U_0 \in \mathfrak{X}(L)$ be given by $(U_0)(x) := (d_x p)(U(x))$ at each point $x \in L$. So $(U_0)(x)$ is just the T_xL-component of $U(x)$ w.r.t. the canonical splitting $T_x(T^*\mathcal{F}_L) = T_xL \oplus T^*_x\mathcal{F}_L$.

We first show that $U_0 \in \mathfrak{X}(L)^{\mathcal{F}_L}$. Since this is a local statement, it suffices to consider open subsets of L whose quotient by the restriction of $\mathcal{F}_L$ is a smooth manifold and show that the restriction of U_0 projects to a vector field on the leaf space. By abuse of notation, we denote such an open subset by L. Since the leaves of the symplectic foliation $\mathcal{F}_{sympl}$ of $T^*\mathcal{F}_L$ are the preimages under p of the leaves of $\mathcal{F}_L$, there is a canonical diffeomorphism of leaf spaces

$$T^*\mathcal{F}_L/\mathcal{F}_{sympl} \cong L/\mathcal{F}_L,$$

induced by the vector bundle projection $p\colon T^*\mathcal{F}_L \to L$ (or equivalently, by the inclusion of the zero section). Since U is a Poisson vector field, it projects under $T^*\mathcal{F}_L \to T^*\mathcal{F}_L/\mathcal{F}_{sympl}$ to some vector field U_{quot}. Restricting to points of the zero section L, we see that U_0 is projectable under $L \to L/\mathcal{F}_L$ (to the same vector field U_{quot}).

By Lemma 2.3.1, U_0 lifts to a Poisson vector field $\widetilde{U_0}$ on $T^*\mathcal{F}_L$. The Poisson vector field $U - \widetilde{U_0}$ is tangent to the leaves of $T^*\mathcal{F}_L$. Indeed, since the statement is local, we can again work on suitable open subsets of L and use that both U and $\widetilde{U_0}$ are projectable to the same vector field under $T^*\mathcal{F}_L \to T^*\mathcal{F}_L/\mathcal{F}_{sympl}$.

We now apply Lemma 2.3.4 $i)$ smoothly to the leaves of $\mathcal{F}_{sympl}$ and the vector field $U - \widetilde{U_0}$. More precisely, by the proof of Lemma 2.3.4 $i)$, if ω denotes the leafwise symplectic form on $T^*\mathcal{F}_L$, then we find a function $h \in C^\infty(T^*\mathcal{F}_L)$ such that the pullback of $\iota_{U-\widetilde{U_0}}\omega - d_{\mathcal{F}_{sympl}}h$ to the fibers of p is zero. It follows that

$$-\Pi^\sharp_{can}\left(\iota_{U-\widetilde{U_0}}\omega - d_{\mathcal{F}_{sympl}}h\right) = U - \widetilde{U_0} + X_h$$

is vertical, i.e. tangent to the p-fibers. This has two consequences. First, we can apply Lemma 2.3.4 ii) to conclude that this vector field is constant on each fiber, and is closed when viewed as a foliated 1-form on $(L, \mathcal{F}_L)$. Second, $U + X_h$ is p-projectable and it projects to the same vector field as $\widetilde{U_0}$, namely $U_0 \in \mathfrak{X}(L)$. Hence $Y := U + X_h$ is a representative of the class $H^1_{\Pi_{can}}(T^*\mathcal{F}_L)$ with the required properties. This proves the theorem. □

2.3.2 The first Poisson cohomology

Using Thm. 2.3.2, we can compute the first Poisson cohomology of $(T^*\mathcal{F}_L, \Pi_{can})$. In the following, $H^\bullet(\mathcal{F}_L)$ denotes the cohomology of the foliated differential forms along the leaves of $\mathcal{F}_L$.

Corollary 2.3.5. *Let $(L, \mathcal{F}_L)$ be a foliated manifold and denote by Π_{can} the standard Poisson structure on the total space of the vector bundle $p\colon T^*\mathcal{F}_L \to L$. There is a linear isomorphism*

$$\begin{aligned}\Phi\colon H^1_{\Pi_{can}}(T^*\mathcal{F}_L) &\to \mathfrak{X}(L)^{\mathcal{F}_L}/\Gamma(T\mathcal{F}_L) \times H^1(\mathcal{F}_L)\\ [Y] &\mapsto \quad \big([p_*Y], [Y - \widetilde{p_*Y}]\big),\end{aligned} \tag{2.13}$$

where the representative Y satisfies the properties in Thm. 2.3.2.

Notice that $\mathfrak{X}(L)^{\mathcal{F}_L}/\Gamma(T\mathcal{F}_L)$ agrees with the space of vector fields on $L/\mathcal{F}_L$, whenever the latter quotient is smooth.

Proof. We first show that the map Φ is well-defined. For this, due to Thm. 2.3.2, we only need to show that the above assignment is independent of the choice of representative. Equivalently, since the expression in (2.13) depends linearly on Y, we have to show that if Y is a Hamiltonian vector field on $T^*\mathcal{F}_L$ satisfying the properties in Thm. 2.3.2, then p_*Y lies in $\Gamma(T\mathcal{F}_L)$ and $Y - \widetilde{p_*Y}$ is exact when viewed as a foliated 1-form on $(L, \mathcal{F}_L)$.

Being Hamiltonian, Y is tangent to the symplectic foliation of $T^*\mathcal{F}_L$, so p_*Y is tangent to the foliation $\mathcal{F}_L$. Hence $\widetilde{p_*Y}$ is a Hamiltonian vector field, by Lemma 2.3.1. Being the difference of two Hamiltonian vector fields, the vertical and fiberwise constant vector field $V := Y - \widetilde{p_*Y}$ is Hamiltonian. Denote by $F \in C^\infty(T^*\mathcal{F}_L)$ a Hamiltonian function for V, so that for each leaf $\mathcal{O}$ of $\mathcal{F}_L$ we have $\iota_V \omega_{T^*\mathcal{O}} = -d(F|_{T^*\mathcal{O}})$. Regarding the vertical constant vector field V as a foliated 1-form yields $\alpha \in \Omega^1(\mathcal{F}_L)$, determined by

$$\iota_V \omega_{T^*\mathcal{O}} = -p^*_{T^*\mathcal{O}}(\alpha|_{\mathcal{O}}), \tag{2.14}$$

see the proof of Lemma 2.3.4 *(ii)*. In particular, F is constant along the fibers of $p\colon T^*\mathcal{F}_L \to L$, i.e. $F = p^*(F|_L)$. Thus $\alpha = d_{\mathcal{F}_L}F$, so it is foliated exact.

We show that Φ is surjective. Let $W \in \mathfrak{X}(L)^{\mathcal{F}_L}$. Then its lift $\widetilde{W}$ is a Poisson vector field on $T^*\mathcal{F}_L$, by Lemma 2.3.1 *i)*. Let $\alpha \in \Omega^1(\mathcal{F}_L)$ be a closed foliated 1-form. Denote by V the corresponding vertical fiberwise constant vector field on $T^*\mathcal{F}_L$. Then V is a Poisson vector field, because it is tangent to the symplectic leaves of $T^*\mathcal{F}_L$ and its restriction to each symplectic leaf is a symplectic vector field, by eq. (2.14). Hence $\widetilde{W} + V$ is a Poisson vector field on $T^*\mathcal{F}_L$. By construction it satisfies the properties of Thm. 2.3.2, and its Poisson cohomology class maps under Φ to $([W], [\alpha])$.

We show that Φ is injective. Let Y be a Poisson vector field on $T^*\mathcal{F}_L$ satisfying the properties in Thm. 2.3.2, so that p_*Y lies in $\Gamma(T\mathcal{F}_L)$ and $V := Y - \widetilde{p_*Y}$ is exact when viewed as a foliated 1-form on $(L, \mathcal{F}_L)$. By Lemma 2.3.1 *ii)*, $\widetilde{p_*Y}$ is a Hamiltonian vector field. Let $\alpha = d_{\mathcal{F}_L} f \in \Omega^1(\mathcal{F}_L)$ be the exact foliated 1-form corresponding to V, where $f \in C^\infty(L)$. Then eq. (2.14) implies that $V = \Pi^\sharp_{can}(p^*(df))$, showing that V is a Hamiltonian vector field. Hence $Y = \widetilde{p_*Y} + V$ is Hamiltonian, so $[Y] = 0$. □

We discuss the isomorphism (2.13) in two particular cases.

Example 2.3.6. i) Suppose $\mathcal{F}_L$ is the foliation of L by points. Then $T^*\mathcal{F}_L$ is just L with the zero Poisson structure, and the map Φ reduces to the identity map on $\mathfrak{X}(L)$.

ii) On the other extreme, suppose $\mathcal{F}_L$ is the one-leaf foliation of L. Then $T^*\mathcal{F}_L$ is the cotangent bundle T^*L with its standard symplectic form, and

$$\Phi\colon H^1_{\Pi_{can}}(T^*L) \to H^1(L).$$

Since Φ is an isomorphism, every class in $H^1_{\Pi_{can}}(T^*L)$ admits a representative V which is a vertical fiberwise constant vector field (c.f. Lemma 2.3.4). The image of this class under Φ is $[\alpha] \in H^1(L)$, where α is just V regarded as a 1-form. The inverse map Φ^{-1} reads $[\alpha] \mapsto -[\omega^{-1}(p^*\alpha)]$, by eq. (2.14), i.e. it is the composition of the natural isomorphism $p^*\colon H^1(L) \to H^1(T^*L)$ and the isomorphism $H^1(T^*L) \cong H^1_{\Pi_{can}}(T^*L)$ from de Rham to Poisson cohomology carried by every symplectic manifold.

Remark 2.3.7. In case the foliation $\mathcal{F}_L$ on L is of codimension-one, we can compare our Corollary 2.3.5 with some results that appeared in [O].

i) In [O, Prop. 1.4.7], one proves the following: if (M,Π) is a corank-one Poisson manifold and $(\mathcal{F},\omega)$ denotes its symplectic foliation, then there is a long exact sequence

$$\cdots \to H^{k-2}(\mathcal{F},\nu) \xrightarrow{\partial} H^k(\mathcal{F}) \xrightarrow{\Pi} H^k_{\Pi}(M) \to H^{k-1}(\mathcal{F},\nu) \xrightarrow{\partial} H^{k+1}(\mathcal{F}) \to \cdots \tag{2.15}$$

Here $\nu := TM/T\mathcal{F}$ denotes the normal bundle of the foliation, and $H^\bullet(\mathcal{F},\nu)$ is the cohomology of the complex $\big(\Gamma(\wedge^\bullet T^*\mathcal{F}\otimes\nu), d_\nabla\big)$, where the differential d_∇ is induced by the Bott connection

$$\nabla : \Gamma(T\mathcal{F}) \times \Gamma(\nu) \to \Gamma(\nu) : \nabla_X \overline{N} = \overline{[X,N]}.$$

The connecting map ∂ is, up to sign, given by the cup product with the leafwise variation $var_\omega \in H^2(\mathcal{F},\nu^*)$ of ω [O, Def. 1.2.14], which vanishes when ω extends to a globally defined closed 2-form on M.

Specializing to our situation, assume $(L,\mathcal{F}_L)$ is a codimension-one foliation. Then $(T^*\mathcal{F}_L,\Pi_{can})$ is a corank-one Poisson manifold with symplectic foliation $(\mathcal{F}_{sympl},\omega)$. The leafwise symplectic form $\omega \in \Gamma(\wedge^2 T^*\mathcal{F}_{sympl})$ extends to a closed 2-form on $T^*\mathcal{F}_L$. Indeed, a closed extension of ω is given by $q^*\omega_{T^*L}$, where $q : T^*\mathcal{F}_L \to T^*L$ is any splitting of the restriction map $r : T^*L \to T^*\mathcal{F}_L$ and ω_{T^*L} is the canonical symplectic form on T^*L. So the connecting map ∂ in (2.15) is zero, which implies in particular that

$$H^1_{\Pi_{can}}(T^*\mathcal{F}_L) \cong H^1(\mathcal{F}_{sympl}) \oplus H^0(\mathcal{F}_{sympl},\nu). \tag{2.16}$$

This is equivalent with our isomorphism in Corollary 2.3.5. Firstly, we have that $H^1(\mathcal{F}_{sympl}) \cong H^1(\mathcal{F}_L)$ by homotopy invariance. Secondly, as $H^0(\mathcal{F}_{sympl},\nu) = \mathfrak{X}(T^*\mathcal{F}_L)^{\mathcal{F}_{sympl}}/\Gamma(T\mathcal{F}_{sympl})$, we have an isomorphism

$$\mathfrak{X}(L)^{\mathcal{F}_L}/\Gamma(T\mathcal{F}_L) \to H^0(\mathcal{F}_{sympl},\nu) : [X] \mapsto \overline{\widetilde{X}},$$

where $\widetilde{X}$ is the lift of X as defined in Lemma 2.3.1. To see that this map is well-defined, just note that $\widetilde{X} \in \mathfrak{X}(T^*\mathcal{F}_L)^{\mathcal{F}_{sympl}}$, being a Poisson vector field. Injectivity is clear, for if $\widetilde{X}$ is tangent to $\mathcal{F}_{sympl}$, then its projection $p_*\widetilde{X} = X$ is tangent to $\mathcal{F}_L$. As for surjectivity, if $\overline{U} \in H^0(\mathcal{F}_{sympl},\nu)$ then $\overline{U} = \overline{\widetilde{U_0}}$ as in the proof of Theorem 2.3.2, where $U_0 \in \mathfrak{X}(L)^{\mathcal{F}_L}$. So the isomorphism (2.16) is equivalent with the one from Corollary 2.3.5:

$$H^1_{\Pi_{can}}(T^*\mathcal{F}_L) \cong H^1(\mathcal{F}_L) \oplus \mathfrak{X}(L)^{\mathcal{F}_L}/\Gamma(T\mathcal{F}_L). \tag{2.17}$$

ii) In case $(L, \mathcal{F}_L)$ is a unimodular codimension-one foliation, then we can further simplify the isomorphism (2.17). Indeed, if $\theta \in \Omega^1(L)$ is a closed defining one-form for $\mathcal{F}_L$, then we get an isomorphism

$$\mathfrak{X}(L)^{\mathcal{F}_L}/\Gamma(T\mathcal{F}_L) \to H^0(\mathcal{F}_L) : [V] \mapsto \theta(V).$$

An alternative argument, building on i) above, is the following. Since also $\mathcal{F}_{sympl}$ is unimodular, the representation of $T\mathcal{F}_{sympl}$ on ν given by the Bott connection is isomorphic with the trivial representation of $T\mathcal{F}_{sympl}$ on the trivial $\mathbb{R}$-bundle $T^*\mathcal{F}_L \times \mathbb{R}$ (see [O, Lemma 1.5.15]). So in (2.16), we get $H^0(\mathcal{F}_{sympl}, \nu) \cong H^0(\mathcal{F}_{sympl}) \cong H^0(\mathcal{F}_L)$.

We will now upgrade Corollary 2.3.5 to an isomorphism of Lie algebras. Note that the Lie bracket on $\mathfrak{X}(L)$ restricts to $\mathfrak{X}(L)^{\mathcal{F}_L}$ thanks to the Jacobi identity. Since $\Gamma(T\mathcal{F}_L)$ is a Lie algebra ideal of $\left(\mathfrak{X}(L)^{\mathcal{F}_L}, [\cdot,\cdot]\right)$, the Lie bracket passes to the quotient $\mathfrak{X}(L)^{\mathcal{F}_L}/\Gamma(T\mathcal{F}_L)$. We get a representation of this Lie algebra on the vector space $H^1(\mathcal{F}_L)$, namely

$$\rho : \frac{\mathfrak{X}(L)^{\mathcal{F}_L}}{\Gamma(T\mathcal{F}_L)} \to \mathrm{End}\left(H^1(\mathcal{F}_L)\right) : [X] \mapsto \pounds_X \bullet . \tag{2.18}$$

Here the Lie derivative

$$\pounds_X \alpha := \frac{d}{dt}\bigg|_{t=0} \phi_t^* \alpha \tag{2.19}$$

of $\alpha \in \Omega^1(\mathcal{F}_L)$ along X makes sense since the flow ϕ_t of X preserves the foliation $\mathcal{F}_L$. Clearly, the map (2.18) is well-defined: for any $X \in \mathfrak{X}(L)^{\mathcal{F}_L}$, the Lie derivative $\pounds_X$ acts on $H^1(\mathcal{F}_L)$ since it commutes with the foliated differential $d_{\mathcal{F}_L}$. Moreover, if $X \in \Gamma(T\mathcal{F}_L)$, then $\pounds_X$ acts trivially in cohomology thanks to Cartan's magic formula. The fact that ρ is a Lie algebra morphism is simply the identity $\pounds_{[X,Y]} = \pounds_X \circ \pounds_Y - \pounds_Y \circ \pounds_X$ for $X, Y \in \mathfrak{X}(L)^{\mathcal{F}_L}$.

Proposition 2.3.8. *Let L be a manifold and $\mathcal{F}_L$ a foliation on L. Let Π_{can} denote the standard Poisson structure on the total space of $p : T^*\mathcal{F}_L \to L$. The map Φ constructed in Corollary 2.3.5 becomes an isomorphism of Lie algebras*

$$\Phi : \left(H^1_{\Pi_{can}}(T^*\mathcal{F}_L), [\cdot,\cdot]\right) \to \left(\mathfrak{X}(L)^{\mathcal{F}_L}/\Gamma(T\mathcal{F}_L) \ltimes_\rho H^1(\mathcal{F}_L), [\cdot,\cdot]_\rho\right),$$

where $[\cdot,\cdot]$ is the usual the Lie bracket of vector fields and $[\cdot,\cdot]_\rho$ is the semidirect product bracket induced by the Lie algebra representation ρ defined in (2.18).

To prove Prop. 2.3.8, it is convenient to rewrite the action (2.18) in terms of vertical fiberwise constant vector fields on $T^*\mathcal{F}_L$ instead of foliated one-forms on $(L, \mathcal{F}_L)$. To do so, we use the correspondence

$$\left(\Omega^\bullet(\mathcal{F}_L), d_{\mathcal{F}_L}\right) \to \left(\mathfrak{X}^\bullet_{vert.const.}(T^*\mathcal{F}_L), [\Pi_{can}, \cdot]\right) : \alpha \mapsto \left(\wedge^\bullet \Pi^\sharp_{can}\right)(p^*\alpha), \tag{2.20}$$

which is an isomorphism of cochain complexes up to a global sign, i.e. it matches $d_{\mathcal{F}_L}$ with $-[\Pi_{can}, \cdot]$ (see for instance [DZ, Lemma 2.1.3]).

Lemma 2.3.9. *For every $X \in \mathfrak{X}(L)^{\mathcal{F}_L}$, the correspondence* (2.20) *matches $\pounds_X$ and $\big[\widetilde{X}, \cdot\big]$, where $\widetilde{X}$ is the lift as described in Lemma 2.3.1.*

Proof. For every foliated differential form $\alpha \in \Omega^k(\mathcal{F}_L)$ we have to show that

$$(\wedge^k \Pi^\sharp_{can})\left(p^*\left(\pounds_X \alpha\right)\right) = \big[\widetilde{X}, (\wedge^k \Pi^\sharp_{can})(p^*\alpha)\big]. \tag{2.21}$$

The left hand side of this equality, using (2.19), reads

$$\left.\frac{d}{dt}\right|_{t=0} (\wedge^k \Pi^\sharp_{can})\left((\phi_t \circ p)^*\alpha\right).$$

Since $p_*\widetilde{X} = X$, we have $\phi_t \circ p = p \circ \psi_t$, where ψ_t denotes the flow of $\widetilde{X}$. So

$$\begin{aligned}(\wedge^k \Pi^\sharp_{can})\left(p^*\left(\pounds_X \alpha\right)\right) &= \left.\frac{d}{dt}\right|_{t=0} (\wedge^k \Pi^\sharp_{can})\left(\psi_t^*(p^*\alpha)\right) \\ &= \left.\frac{d}{dt}\right|_{t=0} (\psi_{-t})_*\left((\wedge^k \Pi^\sharp_{can})(p^*\alpha)\right) \\ &= \left[\widetilde{X}, (\wedge^k \Pi^\sharp_{can})(p^*\alpha)\right],\end{aligned}$$

using in the second equality that ψ_t is a Poisson diffeomorphism of $(T^*\mathcal{F}_L, \Pi_{can})$. So the equality (2.21) holds, and this proves the lemma. ☐

Proof of Prop. 2.3.8. To avoid confusion with too many brackets, we denote equivalence classes by underlining the representatives. Fix $\underline{Y}, \underline{Z} \in H^1_{\Pi_{can}}(T^*\mathcal{F}_L)$ and assume that the representatives Y, Z satisfy the properties in Theorem 2.3.2. Then also their Lie bracket $[Y, Z]$ satisfies these properties: it is p-projectable, $p_*[Y, Z] \in \mathfrak{X}(L)^{\mathcal{F}_L}$ and

$$\begin{aligned}[Y, Z] - \widetilde{p_*[Y, Z]} &= [Y, Z] - [\widetilde{p_*Y}, \widetilde{p_*Z}] \\ &= \left[\widetilde{p_*Y} + \left(Y - \widetilde{p_*Y}\right), \widetilde{p_*Z} + \left(Z - \widetilde{p_*Z}\right)\right] - [\widetilde{p_*Y}, \widetilde{p_*Z}] \\ &= \left[\widetilde{p_*Y}, Z - \widetilde{p_*Z}\right] + \left[Y - \widetilde{p_*Y}, \widetilde{p_*Z}\right],\end{aligned} \tag{2.22}$$

using in the last equation that $Y - \widetilde{p_*Y}$ and $Z - \widetilde{p_*Z}$ are vertical and fiberwise constant. Lemma 2.3.9 shows in particular that both terms in (2.22) are vertical fiberwise constant Poisson vector fields, hence the same holds for $[Y, Z] - \widetilde{p_*[Y, Z]}$.

So $[Y, Z]$ meets the criteria of Theorem 2.3.2. We can therefore proceed as follows:

$$\begin{aligned}
\Phi\left([\underline{Y}, \underline{Z}]\right) &= \Phi\left(\underline{[Y, Z]}\right) \\
&= \left(\underline{p_*[Y, Z]}, \underline{[Y, Z] - \widetilde{p_*[Y, Z]}}\right) \\
&= \left(\underline{[p_*Y, p_*Z]}, \underline{\left[\widetilde{p_*Y}, Z - \widetilde{p_*Z}\right]} + \underline{\left[Y - \widetilde{p_*Y}, \widetilde{p_*Z}\right]}\right) \\
&= \left[\left(\underline{p_*Y}, \underline{Y - \widetilde{p_*Y}}\right), \left(\underline{p_*Z}, \underline{Z - \widetilde{p_*Z}}\right)\right]_\rho \\
&= \left[\Phi(\underline{Y}), \Phi(\underline{Z})\right]_\rho,
\end{aligned}$$

using the equation (2.22) in the third and Lemma 2.3.9 in the fourth equality. □

2.4 Deformations of Lagrangian submanifolds in log-symplectic manifolds: algebraic aspects

In this section, we address the algebra behind the deformation problem of a Lagrangian submanifold L^n contained in the singular locus of a log-symplectic manifold (M^{2n}, Z, Π). In §2.4.1-§2.4.2 we show that the deformation problem is governed by a DGLA, and we discuss the corresponding Maurer-Cartan equation (Thm. 2.4.3 and Cor. 2.4.10). We also compute the cohomology of the DGLA in degree one, by calculating the zeroth foliated Morse-Novikov cohomology in §2.4.3 (Thm. 2.4.16). This result will be used in the next section to extract geometric information about the deformations.

To set up the stage, we revisit Corollary 2.2.18, which states that a neighborhood of a Lagrangian submanifold L^n contained in the singular locus of an orientable log-symplectic manifold (M^{2n}, Z, Π) can be identified with a neighborhood of the zero section in $T^*\mathcal{F}_L \times \mathbb{R}$, endowed with the log-symplectic structure

$$\widetilde{\Pi} := V \wedge t\partial_t + \Pi_{can}. \tag{2.23}$$

Here V is defined on a neighborhood of L in $T^*\mathcal{F}_L$, and only its Poisson cohomology class $[V]$ is fixed, see Remark 2.2.19. We can use Theorem 2.3.2 to choose a convenient representative V that satisfies

$$V = V_{vert} + V_{lift},$$

where $V_{lift} := \widetilde{p_*V}$ is the cotangent lift of $p_*V \in \mathfrak{X}(L)^{\mathcal{F}_L}$ and $V_{vert} := V - \widetilde{p_*V}$ is vertical, fiberwise constant and closed as a section of $p : T^*\mathcal{F}_L \to L$. Indeed, although Theorem 2.3.2 is phrased for Poisson vector fields defined on all of $T^*\mathcal{F}_L$, it is clear that the proof still works if those vector fields are only defined on a neighborhood of L in $T^*\mathcal{F}_L$ whose intersection with each fiber is convex. We summarize the setup for the rest of this chapter:

Given a Lagrangian submanifold L^n contained in the singular locus Z of an orientable log-symplectic manifold (M^{2n}, Z, Π), denote by $\mathcal{F}_L$ the induced foliation on L. Fix an embedding $\psi : (Z, \Pi|_Z) \to (T^*\mathcal{F}_L, \Pi_{can})$ of a tubular neighborhood of L, as in Prop. 2.2.9. Denote by $[V]$ the image of $[V_{mod}|_Z]$ under this map, and assume that V is a representative that satisfies the assumptions of Thm. 2.3.2. The local model around L is then

$$\big(U \subset T^*\mathcal{F}_L \times \mathbb{R},\ \widetilde{\Pi} := (V_{vert} + V_{lift}) \wedge t\partial_t + \Pi_{can}\big),$$

where U is a neighborhood of the zero section L. We denote by

$$\gamma \in \Omega^1_{cl}(\mathcal{F}_L)$$

the closed leafwise one-form that is defined by considering V_{vert} as a section of $p : T^*\mathcal{F}_L \to L$, and we also write

$$X := p_* V \in \mathfrak{X}(L)^{\mathcal{F}_L}.$$

Remark 2.4.1. The Poisson cohomology class $[V]$ is completely determined by $[V_{mod}|_Z]$, i.e. it does not depend on the choice of tubular neighborhood embedding $\psi : (Z, \Pi|_Z) \to (T^*\mathcal{F}_L, \Pi_{can})$ of L. Hence, the same holds for the classes $([\gamma], [X]) \in H^1(\mathcal{F}_L) \times \mathfrak{X}(L)^{\mathcal{F}_L}/\Gamma(T\mathcal{F}_L)$. This is a consequence of the fact that any two tubular neighborhoods are isotopic; a proof can be made using the concrete isotopy constructed in the proof of [Hi, Theorem 5.3].

Slightly abusing notation, we will often denote the local model by $\big(T^*\mathcal{F}_L \times \mathbb{R}, \widetilde{\Pi}\big)$ although it is only defined on the neighborhood U. Throughout the rest of the chapter, U denotes this fixed neighborhood. We only make reference to it when strictly necessary.

2.4.1 The Maurer-Cartan equation

Studying $\mathcal{C}^1$-small deformations of L now amounts to studying Lagrangian sections in $\big(T^*\mathcal{F}_L \times \mathbb{R}, \widetilde{\Pi}\big)$, the vector bundle over L given by the Whitney sum of $T^*\mathcal{F}_L$ and the trivial $\mathbb{R}$-bundle. By the following little lemma, it is equivalent to look at coisotropic sections.

Lemma 2.4.2. *The graph of a section* $(\alpha, f) \in \Gamma(T^*\mathcal{F}_L \times \mathbb{R})$ *is coisotropic in* $\big(T^*\mathcal{F}_L \times \mathbb{R}, \widetilde{\Pi}\big)$ *iff. it is Lagrangian.*

Proof. We only have to check the forward implication at points $(\alpha(q), 0)$ inside the singular locus $T^*\mathcal{F}_L \times \{0\}$. The symplectic leaf of $\big(T^*\mathcal{F}_L \times \mathbb{R}, \widetilde{\Pi}\big)$ through $(\alpha(q), 0)$ is given by $p^{-1}(\mathcal{O}) \times \{0\}$, where $\mathcal{O}$ is the leaf of $\mathcal{F}_L$ through q. By assumption, the subspace

$$\begin{aligned} &T_{(\alpha(q),0)}\mathrm{Graph}(\alpha, f) \cap T_{(\alpha(q),0)}\big(p^{-1}(\mathcal{O}) \times \{0\}\big) \\ &\quad = \{(d_q\alpha)(v) : v \in T_q\mathcal{O} \text{ and } (d_q f)(v) = 0\} \end{aligned} \tag{2.24}$$

is coisotropic in $T_{(\alpha(q),0)}\big(p^{-1}(\mathcal{O}) \times \{0\}\big)$, so it is at least $(n-1)$-dimensional. But clearly the right hand side of (2.24) is at most $(n-1)$-dimensional, which shows that the subspace (2.24) is Lagrangian in $T_{(\alpha(q),0)}\big(p^{-1}(\mathcal{O}) \times \{0\}\big)$. $\square$

We now derive the equations that cut out coisotropic sections in $\big(T^*\mathcal{F}_L \times \mathbb{R}, \widetilde{\Pi}\big)$.

Theorem 2.4.3. *The graph of a section $(\alpha, f) \in \Gamma(T^*\mathcal{F}_L \times \mathbb{R})$ is coisotropic in $\big(T^*\mathcal{F}_L \times \mathbb{R}, \widetilde{\Pi}\big)$ exactly when*

$$\boxed{\begin{cases} d_{\mathcal{F}_L}\alpha = 0 \\ d_{\mathcal{F}_L} f + f(\gamma - \pounds_X\alpha) = 0 \end{cases} .} \tag{2.25}$$

Recall that any vector bundle $E \to L$ carries a family of natural maps $\wedge^\bullet P_E : \mathfrak{X}^\bullet(E) \to \Gamma(\wedge^\bullet E)$, given by restriction to L composed with the vertical projection $\Gamma\left(\wedge^\bullet TE|_L\right) \to \Gamma(\wedge^\bullet E)$. In particular, for $E := T^*\mathcal{F}_L \times \mathbb{R}$ we have the following map in degree two:

$$\wedge^2 P_E : \mathfrak{X}^2(E) \to \Gamma(\wedge^2 T^*\mathcal{F}_L) \oplus \Gamma(T^*\mathcal{F}_L).$$

Clearly, L is coisotropic with respect to $\widetilde{\Pi}$ if and only if $\wedge^2 P_E(\widetilde{\Pi}) = 0$. Below, we denote the bundle projections by $pr_E : E \to L$ and $pr_{T^*\mathcal{F}_L} : T^*\mathcal{F}_L \to L$.

Proof of Thm. 2.4.3. A section $(\alpha, f) \in \Gamma(E)$ gives rise to a diffeomorphism

$$\phi^{(-\alpha,-f)} : E \to E : (p, \xi, t) \mapsto \big(p, \xi - \alpha(p), t - f(p)\big)$$

which maps the graph of (α, f) to the zero section $L \subset E$. So it suffices to single out the sections (α, f) such that L is coisotropic with respect to $\phi^{(-\alpha,-f)}_*\widetilde{\Pi}$. This amounts to asking that

$$0 = \wedge^2 P_E\left(\phi^{(-\alpha,-f)}_*\big((V_{vert} + V_{lift}) \wedge t\partial_t + \Pi_{can}\big)\right). \tag{2.26}$$

We now simplify the expression (2.26) in two steps, identifying throughout vertical fiberwise constant vector fields on $T^*\mathcal{F}_L$ with foliated one-forms on $(L, \mathcal{F}_L)$ via the bijection (2.20).

Claim 1: $\wedge^2 P_E \left(\phi_*^{(-\alpha,-f)}\big((V_{vert}+V_{lift})\wedge t\partial_t\big)\right) = (0, f\gamma - f\pounds_X\alpha)\,.$ (*)

First, we note that

$$P_E\left(\phi_*^{(-\alpha,-f)}V_{vert}\right) = P_{T^*\mathcal{F}_L}(V_{vert}) = V_{vert}$$

and

$$P_E\left(\phi_*^{(-\alpha,-f)}t\partial_t\right) = P_E\big((t+pr_E^*f)\partial_t\big) = (pr_E^*f)\partial_t,$$

which yields the first term on the right in (*). Secondly, we have

$$\begin{aligned}\wedge^2 P_E\left(\phi_*^{(-\alpha,-f)}(V_{lift}\wedge t\partial_t)\right) &= P_E\left(\phi_*^{(-\alpha,-f)}V_{lift}\right)\wedge(pr_E^*f)\partial_t\\ &= P_{T^*\mathcal{F}_L}\left(\phi_*^{-\alpha}V_{lift}\right)\wedge(pr_E^*f)\partial_t,\end{aligned}$$

so Claim 1 follows if we show that

$$P_{T^*\mathcal{F}_L}\left(\phi_*^{-\alpha}V_{lift}\right) = -\pounds_X\alpha. \tag{2.27}$$

To do so, we compute

$$\begin{aligned}P_{T^*\mathcal{F}_L}\left(\phi_*^{-\alpha}V_{lift}\right) &= \left(\phi_*^{-\alpha}V_{lift}\right)\big|_L - (pr_{T^*\mathcal{F}_L})_*\left(\phi_*^{-\alpha}V_{lift}\right)\\ &= \left(\phi_*^{-\alpha}V_{lift}\right)\big|_L - (pr_{T^*\mathcal{F}_L})_*\left(V_{lift}\right)\\ &= \left(\phi_*^{-\alpha}V_{lift}\right)\big|_L - V_{lift}|_L\\ &= \int_0^1 \frac{d}{dt}\left(\phi_*^{-t\alpha}V_{lift}\right)\bigg|_L dt,\end{aligned} \tag{2.28}$$

using that $pr_{T^*\mathcal{F}_L}\circ\phi^{-\alpha} = pr_{T^*\mathcal{F}_L}$. Now, note that

$$\begin{aligned}\frac{d}{dt}\left(\phi_*^{-t\alpha}V_{lift}\right) &= \frac{d}{ds}\bigg|_{s=0}\phi_*^{-t\alpha}\left(\phi_*^{-s\alpha}V_{lift}\right)\\ &= \phi_*^{-t\alpha}\left(\frac{d}{ds}\bigg|_{s=0}\phi_*^{-s\alpha}V_{lift}\right)\\ &= \phi_*^{-t\alpha}\left([\alpha, V_{lift}]\right)\\ &= [\alpha, V_{lift}]\\ &= -[V_{lift},\alpha],\end{aligned} \tag{2.29}$$

In the fourth equality, we used that $[\alpha, V_{lift}]$ is vertical and fiberwise constant, which follows from Lemma 2.3.9. Therefore, the equality (2.28) becomes

$$P_{T^*\mathcal{F}_L}\left(\phi_*^{-\alpha}V_{lift}\right) = -\int_0^1 [V_{lift},\alpha]|_L\,dt = -\left[V_{lift},\alpha\right]|_L\,,$$

which is exactly (2.27) under the identification (2.20). This proves Claim 1.

Claim 2: $\wedge^2 P_E\left(\phi_*^{(-\alpha,-f)}\Pi_{can}\right) = (d_{\mathcal{F}_L}\alpha, d_{\mathcal{F}_L}f).$ (**)

Since $L \subset (T^*\mathcal{F}_L, \Pi_{can})$ is Lagrangian, we get $\wedge^2 P_E(\Pi_{can}) = \wedge^2 P_{T^*\mathcal{F}_L}\Pi_{can} = 0$, so the left hand side of (**) is equal to

$$\wedge^2 P_E\left(\phi_*^{(-\alpha,-f)}\Pi_{can} - \Pi_{can}\right). \tag{2.30}$$

We can decompose

$$\phi_*^{(-\alpha,-f)}\Pi_{can} - \Pi_{can} = A_t \wedge \partial_t + B_t \tag{2.31}$$

for some $A_t \in \mathfrak{X}(T^*\mathcal{F}_L)$ and $B_t \in \mathfrak{X}^2(T^*\mathcal{F}_L)$ depending smoothly on t. We find A_t by contracting with dt:

$$\begin{aligned} A_t &= -\left(\phi_*^{(-\alpha,-f)}\Pi_{can} - \Pi_{can}\right)^\sharp(dt) \\ &= -\left[\phi_*^{(-\alpha,-f)} \circ \Pi_{can}^\sharp \circ \left(\phi^{(-\alpha,-f)}\right)^*\right](dt) \\ &= -\phi_*^{(-\alpha,-f)}\left(\Pi_{can}^\sharp\left(d(t - pr_E^* f)\right)\right) \\ &= \phi_*^{(-\alpha,-f)}\left(X_{pr_{T^*\mathcal{F}_L}^* f}\right) \\ &= X_{pr_{T^*\mathcal{F}_L}^* f}, \end{aligned} \tag{2.32}$$

using that $X_{pr_{T^*\mathcal{F}_L}^* f}$ is vertical and fiberwise constant. Next, since

$$\begin{aligned} \pounds_{\partial_t}\left(\phi_*^{(-\alpha,-f)}\Pi_{can}\right) &= \left.\frac{d}{dt}\right|_{t=0} \phi_*^{(0,-t)}\left(\phi_*^{(-\alpha,-f)}\Pi_{can}\right) \\ &= \phi_*^{(-\alpha,-f)}\left(\left.\frac{d}{dt}\right|_{t=0}\phi_*^{(0,-t)}\Pi_{can}\right) \\ &= \phi_*^{(-\alpha,-f)}(\pounds_{\partial_t}\Pi_{can}) \\ &= 0, \end{aligned}$$

it follows that

$$\pounds_{\partial_t}B_t = \pounds_{\partial_t}\left(\phi_*^{(-\alpha,-f)}\Pi_{can} - \Pi_{can} - X_{pr_{T^*\mathcal{F}_L}^* f} \wedge \partial_t\right) = 0,$$

i.e. $B_t = B$ is independent of t. So B is equal to its pushforward under the projection $T^*\mathcal{F}_L \times \mathbb{R} \to T^*\mathcal{F}_L$, which yields

$$B = \phi_*^{-\alpha}\Pi_{can} - \Pi_{can} = \int_0^1 \frac{d}{dt}\left(\phi_*^{-t\alpha}\Pi_{can}\right)dt = \int_0^1 \pounds_\alpha \Pi_{can} dt = \pounds_\alpha \Pi_{can}. \tag{2.33}$$

Here the third equality follows from a computation similar to the one that led to (2.29). Inserting (2.32) and (2.33) into (2.31) gives

$$\phi_*^{(-\alpha,-f)}\Pi_{can} - \Pi_{can} = X_{pr^*_{T^*\mathcal{F}_L}f} \wedge \partial_t + \pounds_\alpha \Pi_{can}.$$

Applying the identification (2.20) now yields the conclusion of Claim 2:

$$\begin{aligned}\wedge^2 P_E\left(\phi_*^{(-\alpha,-f)}\Pi_{can} - \Pi_{can}\right) &= \wedge^2 P_E\left(X_{pr^*_{T^*\mathcal{F}_L}f} \wedge \partial_t + \pounds_\alpha \Pi_{can}\right)\\ &= (d_{\mathcal{F}_L}\alpha, d_{\mathcal{F}_L}f).\end{aligned}$$

Combining Claim 1 and Claim 2, we see that the requirement (2.26) is equivalent with the equations (2.25) in the statement of the theorem. □

Corollary 2.4.4. *Any Lagrangian section of $\left(T^*\mathcal{F}_L \times \mathbb{R}, \widetilde{\Pi}\right)$ can be connected to L by a smooth path of Lagrangian sections. In particular, the set of Lagrangian sections of $\left(T^*\mathcal{F}_L \times \mathbb{R}, \widetilde{\Pi}\right)$ is path connected for the compact-open topology.*

Proof. Let $(\alpha, f) \in \Gamma(T^*\mathcal{F}_L \times \mathbb{R})$ be a Lagrangian section. Fix a smooth function $\Psi \in C^\infty(\mathbb{R})$ satisfying $\Psi(s) = 0$ for $s \leq 0$, $0 < \Psi(s) < 1$ for $0 < s < 1$ and $\Psi(s) = 1$ for $s \geq 1$. Define $\Phi \in C^\infty(\mathbb{R})$ by putting $\Phi(s) := \Psi(s-1)$, and notice that $\Phi \cdot \Psi = \Phi$. Consequently, the smooth path $s \mapsto \big(\Psi(s)\alpha, \Phi(s)f\big)$ consists of Lagrangian sections, since $\big(\Psi(s)\alpha, \Phi(s)f\big)$ is a solution to the equations (2.25) for each value of $s \in \mathbb{R}$. Clearly, this path is continuous for the compact-open topology, it passes through the zero section at $s = 0$, and it reaches (α, f) at time $s = 2$. This proves the statement. □

Remark 2.4.5. We comment on the Maurer-Cartan equation (2.25).

i) Twisting the foliated de Rham differential with a closed element $\eta \in \Omega^1(\mathcal{F}_L)$ gives a differential

$$d^\eta_{\mathcal{F}_L} : \Omega^k(\mathcal{F}_L) \to \Omega^{k+1}(\mathcal{F}_L) : \alpha \mapsto d_{\mathcal{F}_L}\alpha + \eta \wedge \alpha. \tag{2.34}$$

The associated cohomology groups, which we denote by $H^k_\eta(\mathcal{F}_L)$, will be discussed in more detail later. If $\mathcal{F}_L$ is the one-leaf foliation on L, then we recover what is called the Morse-Novikov cohomology, which appears in the context of locally conformal symplectic structures [HR, Section 1].

ii) The Maurer-Cartan equation (2.25) shows that the problem of deforming L into a nearby Lagrangian Graph(α, f) can essentially be done in two steps. Indeed, one can solve the first (linear) equation in (2.25) for α, and then solve the second equation – which for fixed α becomes linear – for f. Geometrically, this amounts to the following. First, one deforms L inside the singular locus along the leafwise closed one-form α, and then one moves the obtained Lagrangian $L' = \text{Graph}(\alpha) \subset T^*\mathcal{F}_L$ in the direction normal to $T^*\mathcal{F}_L$ along the function $f \in H^0_{\gamma - \pounds_X \alpha}(\mathcal{F}_L)$.

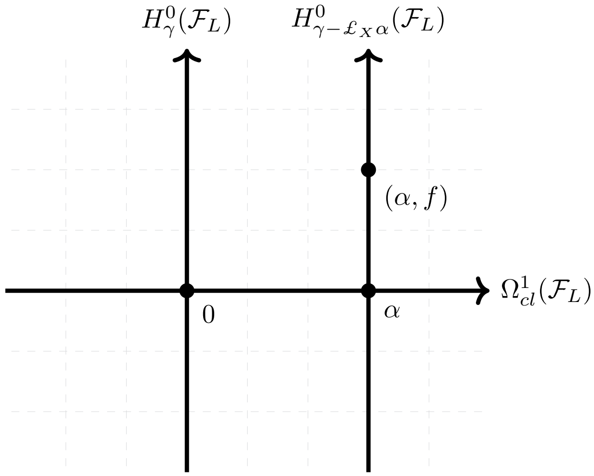

Figure 2.2: Deforming L into $\text{Graph}(\alpha, f)$.

So heuristically, it seems like deforming L into $\text{Graph}(\alpha) \subset T^*\mathcal{F}_L$ for closed $\alpha \in \Omega^1(\mathcal{F}_L)$ transforms γ into $\gamma - \pounds_X\alpha$. We will now make this precise.

Proposition 2.4.6. *Let $\psi : U_0 \subset (Z, \Pi|_Z) \to U_1 \subset (T^*\mathcal{F}_L, \Pi_{T^*\mathcal{F}_L})$ be a fixed tubular neighborhood embedding of L into $T^*\mathcal{F}_L$, where $\Pi_{T^*\mathcal{F}_L}$ denotes the canonical Poisson structure Π_{can}. Assume that $V = V_{vert} + V_{lift}$ is a representative of the Poisson cohomology class $\psi_*[V_{mod}|_Z]$ satisfying the requirements of Thm. 2.3.2, with associated data $(X, \gamma) \in \mathfrak{X}(L)^{\mathcal{F}_L} \times \Omega^1_{cl}(\mathcal{F}_L)$. Consider a Lagrangian $L' = Graph(\alpha) \subset U_1 \subset T^*\mathcal{F}_L$ for some closed $\alpha \in \Omega^1(\mathcal{F}_L)$. Then the following hold:*

i) There is a canonical diffeomorphism of affine bundles

$$(\Phi, \phi) : (T^*\mathcal{F}_L, \Pi_{T^*\mathcal{F}_L}) \to \left(T^*\mathcal{F}_{L'}, \Pi_{T^*\mathcal{F}_{L'}}\right),$$

which is a Poisson diffeomorphism between the total spaces and which fixes points of L', so that $\Phi \circ \psi$ is a tubular neighborhood embedding of $\psi^{-1}(L')$ into $\left(T^*\mathcal{F}_{L'}, \Pi_{T^*\mathcal{F}_{L'}}\right)$.

ii) The representative Φ_*V also satisfies the requirements of Thm. 2.3.2, and its associated data are

$$(X', \gamma') = \left(\phi_*X, \left(\phi^{-1}\right)^* (\gamma - \pounds_X\alpha)\right) \in \mathfrak{X}(L')^{\mathcal{F}_{L'}} \times \Omega^1_{cl}(\mathcal{F}_{L'}).$$

Proof. i) Since α is closed, the translation map

$$\phi^{-\alpha} : (T^*\mathcal{F}_L, \Pi_{T^*\mathcal{F}_L}) \overset{\sim}{\to} (T^*\mathcal{F}_L, \Pi_{T^*\mathcal{F}_L}) : (p,\xi) \mapsto (p, \xi - \alpha(p))$$

is a Poisson diffeomorphism; this follows from the computation (2.33) and the isomorphism (2.20). Its restriction to L', which coincides with the restriction of the vector bundle projection p_L to L', is a foliated diffeomorphism $\tau : (L', \mathcal{F}_{L'}) \overset{\sim}{\to} (L, \mathcal{F}_L)$, and the cotangent lift $T^*\tau$ of τ descends to a Poisson diffeomorphism

$$T^*_{\mathcal{F}}\tau : (T^*\mathcal{F}_L, \Pi_{T^*\mathcal{F}_L}) \overset{\sim}{\to} \left(T^*\mathcal{F}_{L'}, \Pi_{T^*\mathcal{F}_{L'}}\right).$$

In summary, we have a commutative diagram

$$\begin{array}{ccc} (T^*L', \Pi_{T^*L'}) & \xleftarrow[T^*\tau]{} & (T^*L, \Pi_{T^*L}) \\ \big\downarrow{\scriptstyle r_{L'}} & & \big\downarrow{\scriptstyle r_L} \\ \left(T^*\mathcal{F}_{L'}, \Pi_{T^*\mathcal{F}_{L'}}\right) & \xleftarrow[T^*_{\mathcal{F}}\tau]{} & (T^*\mathcal{F}_L, \Pi_{T^*\mathcal{F}_L}) \\ \big\downarrow{\scriptstyle p_{L'}} & & \big\downarrow{\scriptstyle p_L} \\ (L', \mathcal{F}_{L'}) & \xrightarrow{\tau} & (L, \mathcal{F}_L) \end{array} . \tag{2.35}$$

The affine bundle map $(\Phi, \phi) := \left(T^*_{\mathcal{F}}\tau \circ \phi^{-\alpha}, \tau^{-1}\right)$ meets the requirements.

ii) Clearly $\Phi_* V$ is $p_{L'}$-projectable, since V is p_L-projectable and Φ covers the diffeomorphism ϕ. Using that $p_L \circ \phi^{-\alpha} = p_L$, we have

$$(p_{L'})_* (\Phi_* V) = \left(p_{L'} \circ T^*_{\mathcal{F}}\tau \circ \phi^{-\alpha}\right)_* V = \left(\tau^{-1} \circ p_L \circ \phi^{-\alpha}\right)_* V = \left(\tau^{-1}\right)_* X.$$

Because the map $\tau^{-1} : (L, \mathcal{F}_L) \to (L', \mathcal{F}_{L'})$ is a foliated diffeomorphism and $X \in \mathfrak{X}(L)^{\mathcal{F}_L}$, we also have $\left(\tau^{-1}\right)_* X \in \mathfrak{X}(L')^{\mathcal{F}_{L'}}$. Moreover, it is clear that $\widetilde{(\tau^{-1})_* X} = (T^*_{\mathcal{F}}\tau)_* V_{lift}$ by functoriality. It remains to show that

$$\Phi_* V - \widetilde{(p_{L'})_*(\Phi_* V)} = \left(T^*_{\mathcal{F}}\tau \circ \phi^{-\alpha}\right)_* V - (T^*_{\mathcal{F}}\tau)_* V_{lift}$$

is vertical, fiberwise constant, and that it corresponds with the closed foliated one-form $\tau^* (\gamma - \pounds_X \alpha) \in \Omega^1(\mathcal{F}_{L'})$. We rewrite it as

$$\begin{aligned} &(T^*_{\mathcal{F}}\tau)_* \left[\left(\phi^{-\alpha}\right)_* (V - V_{lift})\right] + (T^*_{\mathcal{F}}\tau)_* \left[\left(\phi^{-\alpha}\right)_* V_{lift} - V_{lift}\right] \\ &\quad = (T^*_{\mathcal{F}}\tau)_* (V - V_{lift}) + (T^*_{\mathcal{F}}\tau)_* \left[\left(\phi^{-\alpha}\right)_* V_{lift} - V_{lift}\right], \end{aligned} \tag{2.36}$$

using that $V - V_{lift}$ is vertical and fiberwise constant. The computations done in (2.28) and (2.29) show that $\left(\phi^{-\alpha}\right)_* V_{lift} - V_{lift} = -[V_{lift}, \alpha]$, so it is vertical fiberwise constant and it corresponds with the closed one-form $-\pounds_X\alpha \in \Omega^1(\mathcal{F}_L)$ under (2.20). Since $V - V_{lift}$ corresponds with $\gamma \in \Omega^1(\mathcal{F}_L)$, we get that the vertical fiberwise constant vector field (2.36) indeed corresponds with the closed one-form $\tau^*(\gamma - \pounds_X\alpha)$. $\square$

2.4.2 The DGLA behind the deformation problem

We now show that the equations (2.25) obtained in Theorem 2.4.3 represent the Maurer-Cartan equation of a differential graded Lie algebra (DGLA) that governs the deformations of the Lagrangian $L \subset \left(T^*\mathcal{F}_L \times \mathbb{R}, \widetilde{\Pi}\right)$. To this end, recall the following.

Suppose $E \to C$ is a vector bundle and let Π be a Poisson structure on E such that C is coisotropic. Cattaneo and Felder showed in [CF] that the graded vector space $\Gamma\left(\wedge^\bullet E\right)[1]$ supports a canonical $L_\infty[1]$-algebra structure whose multibrackets $\lambda_k : \Gamma\left(\wedge^\bullet E\right)[1]^{\otimes^k} \to \Gamma\left(\wedge^\bullet E\right)[1]$ are defined by

$$\lambda_k(\xi_1 \otimes \cdots \otimes \xi_k) := \wedge P\left([\ldots[[\Pi, \xi_1], \xi_2]\ldots, \xi_k]\right). \tag{2.37}$$

Here the ξ_i are interpreted as vertical fiberwise constant multivector fields on E and the map $\wedge^\bullet P : \mathfrak{X}^\bullet(E) \to \Gamma(\wedge^\bullet E)$ is the restriction to C composed with the vertical projection $\Gamma\left(\wedge^\bullet TE|_L\right) \to \Gamma(\wedge^\bullet E)$. These structure maps λ_k only depend on the ∞-jet of Π along the submanifold C, so the $L_\infty[1]$-algebra usually does not carry enough information to codify Π in a neighborhood of C. Consequently, this $L_\infty[1]$-algebra fails to encode coisotropic deformations of C in general (see [S, Ex. 3.2]).

However, if the Poisson structure Π is analytic in the fiber directions, then the $L_\infty[1]$-algebra of Cattaneo-Felder does govern the smooth coisotropic deformation problem of C. In [SZ1], such bivector fields are called fiberwise entire, and there one proves the following [SZ1, Thm. 1.12].

Theorem 2.4.7. *Let $E \to C$ be a vector bundle and Π a fiberwise entire Poisson structure which is defined on a tubular neighborhood U of C in E. Suppose that C is coisotropic with respect to Π, and consider the $L_\infty[1]$-algebra associated with $C \subset (U, \Pi)$. For any section $\alpha \in \Gamma(E)$ such that Graph$(-\alpha)$ is contained in U, the Maurer-Cartan series $MC(\alpha)$ converges. Furthermore, for any such $\alpha \in \Gamma(E)$, the following are equivalent:*

- *i) The graph of $-\alpha$ is a coisotropic submanifold of (U, Π).*
- *ii) The Maurer-Cartan series $MC(\alpha)$ converges to zero.*

In the rest of this section, we show that the L_∞-algebra of Cattaneo-Felder associated with $\left(T^*\mathcal{F}_L \times \mathbb{R}, \widetilde{\Pi}\right)$ reduces to a DGLA, and that this DGLA governs the deformation problem of the Lagrangian L.

Lemma 2.4.8. *The Poisson structure* $\widetilde{\Pi} = (V_{vert} + V_{lift}) \wedge t\partial_t + \Pi_{can}$, *defined on a neighborhood* U *of* L *in* $T^*\mathcal{F}_L \times \mathbb{R}$, *is fiberwise entire.*

Proof. This is straightforward computation. Choose coordinates $(x_1, \dots, x_n)$ on L adapted to the foliation $\mathcal{F}_L$, such that plaques of $\mathcal{F}_L$ are level sets of x_1. Let $(y_1, \dots, y_n)$ be the corresponding fiber coordinates on T^*L. Then write

$$\Pi_{can} = \sum_{i=2}^{n} \partial_{x_i} \wedge \partial_{y_i}, \qquad V_{vert} = \sum_{j=2}^{n} f_j(x)\partial_{y_j}, \qquad p_*V = \sum_{j=1}^{n} h_j(x)\partial_{x_j},$$

where $p : T^*\mathcal{F}_L \to L$ is the projection and $h_1(x)$ only depends on x_1 since $p_*V \in \mathfrak{X}(L)^{\mathcal{F}_L}$. We then obtain

$$V_{lift} = \sum_{j=1}^{n} h_j(x)\partial_{x_j} - \sum_{i=2}^{n}\sum_{j=2}^{n} y_j \frac{\partial h_j}{\partial x_i}(x)\partial_{y_i}.$$

So the Poisson structure $\widetilde{\Pi}$ reads

$$\widetilde{\Pi} = \left(\sum_{j=2}^{n} f_j(x)\partial_{y_j} + \sum_{j=1}^{n} h_j(x)\partial_{x_j} - \sum_{i=2}^{n}\sum_{j=2}^{n} y_j \frac{\partial h_j}{\partial x_i}(x)\partial_{y_i} \right) \wedge t\partial_t + \sum_{i=2}^{n} \partial_{x_i} \wedge \partial_{y_i},$$

which is clearly a fiberwise entire bivector field. □

Lemma 2.4.9. *The* $L_\infty[1]$*-algebra* $\left(\Gamma(\wedge^\bullet(T^*\mathcal{F}_L \times \mathbb{R}))[1], \{\lambda_k\}\right)$ *of Cattaneo-Felder associated with* $\left(T^*\mathcal{F}_L \times \mathbb{R}, \widetilde{\Pi}\right)$ *corresponds to a DGLA-structure on the graded vector space* $\Gamma(\wedge^\bullet(T^*\mathcal{F}_L \times \mathbb{R}))$.

Proof. We will show that the multibrackets λ_k defined in (2.37) vanish for $k \geq 3$. Since the λ_k are multiderivations, it is enough to evaluate them on elements of $C^\infty(L)$ and $\Gamma(T^*\mathcal{F}_L \times \mathbb{R})$. As the λ_k have degree one, a degree counting argument shows that they can be non-zero only when evaluated on tuples of the form

$$(\sigma_1, \dots, \sigma_k), \quad (h, \sigma_1, \dots, \sigma_{k-1}) \text{ and } (h, h', \sigma_1, \dots, \sigma_{k-2}),$$

where $h, h' \in C^\infty(L)$ and $\sigma_1, \dots, \sigma_k \in \Gamma\left(T^*\mathcal{F}_L \times \mathbb{R}\right)$. Now choose sections $(\alpha, f), (\beta, g) \in \Gamma\left(T^*\mathcal{F}_L \times \mathbb{R}\right)$ and let $h \in C^\infty(L)$. Let $pr : T^*\mathcal{F}_L \times \mathbb{R} \to L$ denote the projection. If we show that the multivector fields

$$\begin{gathered} \left[\left[\widetilde{\Pi}, \alpha + pr^*(f)\partial_t\right], \beta + pr^*(g)\partial_t\right], \\ \left[\left[\widetilde{\Pi}, \alpha + pr^*(f)\partial_t\right], pr^*(h)\right] \end{gathered} \tag{2.38}$$

are vertical and fiberwise constant, then the above observation implies that $\lambda_k = 0$ whenever $k \geq 3$, since clearly $\left[\left[\widetilde{\Pi}, pr^*(h)\right], pr^*(h')\right] = 0$. One checks that

$$\begin{aligned}\left[\widetilde{\Pi}, \alpha + pr^*(f)\partial_t\right] &= \left[\,(V_{vert} + V_{lift}) \wedge t\partial_t + \Pi_{can}\ ,\ \alpha + pr^*(f)\partial_t\right] \\ &= -pr^*(f)V_{vert} \wedge \partial_t + [V_{lift}, \alpha] \wedge t\partial_t - pr^*(f)V_{lift} \wedge \partial_t \\ &\quad + [\Pi_{can}, \alpha] + [\Pi_{can}, pr^* f] \wedge \partial_t,\end{aligned}$$

where $-pr^*(f)V_{vert} \wedge \partial_t, [\Pi_{can}, \alpha]$ and $[\Pi_{can}, pr^* f] \wedge \partial_t$ are vertical and fiberwise constant. So only the second and third summand are relevant to compute the expressions (2.38), and we get

$$\begin{aligned}&\left[\left[\widetilde{\Pi}, \alpha + pr^*(f)\partial_t\right], \beta + pr^*(g)\partial_t\right] \\ &\quad = \left[[V_{lift}, \alpha] \wedge t\partial_t - pr^*(f)V_{lift} \wedge \partial_t\ ,\ \beta + pr^*(g)\partial_t\right] \\ &\quad = pr^*(f)[\beta, V_{lift}] \wedge \partial_t + pr^*(g)[\alpha, V_{lift}] \wedge \partial_t\end{aligned}$$

and

$$\begin{aligned}\left[\left[\widetilde{\Pi}, \alpha + pr^*(f)\partial_t\right], pr^*(h)\right] &= \left[[V_{lift}, \alpha] \wedge t\partial_t - pr^*(f)V_{lift} \wedge \partial_t\ ,\ pr^*(h)\right] \\ &= pr^*\big(fX(h)\big)\partial_t,\end{aligned}$$

where $X = pr_* V_{lift}$ as before. Using Lemma 2.3.9, we see that the multivector fields (2.38) are vertical and fiberwise constant, which proves the lemma. □

We now established the existence of a DGLA-structure supported on the graded vector space $\Gamma(\wedge^\bullet(T^*\mathcal{F}_L \times \mathbb{R}))$, which governs the deformations of L as a coisotropic submanifold. Thanks to Lemma 2.4.2, this DGLA in fact governs the Lagrangian deformation problem of L. We now provide more explicit descriptions for the structure maps of the DGLA.

Corollary 2.4.10. *The deformation problem of a Lagrangian submanifold L^n contained in the singular locus of an orientable log-symplectic manifold (M^{2n}, Z, Π) is governed by a DGLA supported on the graded vector space $\Gamma\left(\wedge^\bullet\left(T^*\mathcal{F}_L \times \mathbb{R}\right)\right) = \Gamma\left(\wedge^\bullet T^*\mathcal{F}_L \oplus \wedge^{\bullet-1} T^*\mathcal{F}_L\right)$, whose structure maps $(d, [\![\cdot,\cdot]\!])$ are defined by*

$$d : \Gamma\left(\wedge^k\left(T^*\mathcal{F}_L \times \mathbb{R}\right)\right) \to \Gamma\left(\wedge^{k+1}\left(T^*\mathcal{F}_L \times \mathbb{R}\right)\right) :$$

$$(\alpha, \beta) \mapsto \left(-d_{\mathcal{F}_L}\alpha, -d_{\mathcal{F}_L}\beta - \gamma \wedge \beta\right),$$

$$[\![\cdot,\cdot]\!] : \Gamma\left(\wedge^k\left(T^*\mathcal{F}_L \times \mathbb{R}\right)\right) \otimes \Gamma\left(\wedge^l\left(T^*\mathcal{F}_L \times \mathbb{R}\right)\right) \to \Gamma\left(\wedge^{k+l}\left(T^*\mathcal{F}_L \times \mathbb{R}\right)\right) :$$

$$(\alpha, \beta) \otimes (\delta, \epsilon) \mapsto \left(0, \pounds_X\alpha \wedge \epsilon - (-1)^{kl}\pounds_X\delta \wedge \beta\right).$$

Proof. We first write down explicitly the structure maps λ_1, λ_2 of the $L_\infty[1]$-algebra $\big(\Gamma(\wedge^\bullet(T^*\mathcal{F}_L \times \mathbb{R}))[1], \lambda_1, \lambda_2\big)$, as defined in (2.37). We then apply the décalage isomorphisms to obtain the DGLA $\big(\Gamma(\wedge^\bullet(T^*\mathcal{F}_L \times \mathbb{R})), d, [\![\cdot,\cdot]\!]\big)$ associated with it. In the computations below, we again identify elements of $\Gamma\big(\wedge^\bullet T^*\mathcal{F}_L\big)$ with vertical fiberwise constant multivector fields on $T^*\mathcal{F}_L$ via the isomorphism (2.20).

Choosing $(\alpha, \beta) \in \Gamma\left(\wedge^k (T^*\mathcal{F}_L \times \mathbb{R})\right)$ and $(\delta, \epsilon) \in \Gamma\left(\wedge^l (T^*\mathcal{F}_L \times \mathbb{R})\right)$, we have

$$\begin{aligned}\left[\widetilde{\Pi}, \alpha + \beta \wedge \partial_t\right] &= \left[(V_{vert} + V_{lift}) \wedge t\partial_t + \Pi_{can}, \alpha + \beta \wedge \partial_t\right] \\ &= (-1)^{k-1}[V_{lift}, \alpha] \wedge t\partial_t - V \wedge \beta \wedge \partial_t \\ &\qquad + [\Pi_{can}, \alpha] + [\Pi_{can}, \beta] \wedge \partial_t, \end{aligned} \tag{2.39}$$

which implies that

$$\lambda_1\big((\alpha, \beta)\big) = \left(-d_{\mathcal{F}_L}\alpha, -d_{\mathcal{F}_L}\beta - \gamma \wedge \beta\right). \tag{2.40}$$

Next, using the computation (2.39), we have

$$\begin{aligned}&\left[\left[\widetilde{\Pi}, \alpha + \beta \wedge \partial_t\right], \delta + \epsilon \wedge \partial_t\right] \\ &= \left[(-1)^{k-1}[V_{lift}, \alpha] \wedge t\partial_t - V \wedge \beta \wedge \partial_t + [\Pi_{can}, \alpha] + [\Pi_{can}, \beta] \wedge \partial_t, \delta + \epsilon \wedge \partial_t\right] \\ &= \left[(-1)^{k-1}[V_{lift}, \alpha] \wedge t\partial_t - V_{lift} \wedge \beta \wedge \partial_t, \delta + \epsilon \wedge \partial_t\right] \\ &= (-1)^k \left[V_{lift}, \alpha\right] \wedge \epsilon \wedge \partial_t - (-1)^{k(l-1)} \left[V_{lift}, \delta\right] \wedge \beta \wedge \partial_t,\end{aligned}$$

which implies that

$$\lambda_2\big((\alpha, \beta) \otimes (\delta, \epsilon)\big) = \Big(0, (-1)^k \pounds_X \alpha \wedge \epsilon - (-1)^{k(l-1)} \pounds_X \delta \wedge \beta\Big). \tag{2.41}$$

The décalage isomorphisms act as

$$\begin{aligned}(\alpha, \beta) &\mapsto (\alpha, \beta) \\ (\alpha, \beta) \otimes (\delta, \epsilon) &\mapsto (-1)^k (\alpha, \beta) \otimes (\delta, \epsilon),\end{aligned}$$

and applying them to (2.40) and (2.41) yields the result of the corollary. □

In more detail, the fact that this DGLA governs the deformations of L means the following. For convenience, we assume the neighborhood U of L in $T^*\mathcal{F}_L \times \mathbb{R}$ where $\widetilde{\Pi}$ is defined to be invariant under fiberwise multiplication by -1.

Then for any section $(\alpha, f) \in \Gamma(T^*\mathcal{F}_L \times \mathbb{R})$ whose graph lies inside U, we have

$$\begin{aligned}\text{Graph}(\alpha, f) \text{ is Lagrangian} &\Leftrightarrow \text{Graph}(\alpha, f) \text{ is coisotropic}\\ &\Leftrightarrow (-\alpha, -f) \text{ is a Maurer-Cartan element of}\\ &\quad\ \text{the } L_\infty[1]-\text{algebra } \big(\Gamma(T^*\mathcal{F}_L \times \mathbb{R})[1], \lambda_1, \lambda_2\big)\\ &\Leftrightarrow (\alpha, f) \text{ is a Maurer-Cartan element of}\\ &\quad\ \text{the DGLA } \big(\Gamma(T^*\mathcal{F}_L \times \mathbb{R}), d, [\![\cdot,\cdot]\!]\big)\\ &\Leftrightarrow \begin{cases} d_{\mathcal{F}_L}\alpha = 0\\ d_{\mathcal{F}_L}f + f(\gamma - \pounds_X\alpha) = 0\end{cases},\end{aligned}$$

where the first equivalence is Lemma 2.4.2 and the second one is Thm. 2.4.7. So we recover the equations (2.25) derived in Thm. 2.4.3 by direct computation.

Remark 2.4.11 (Formality). We do not know whether the DGLA in Corollary 2.4.10 is formal, i.e. L_∞-quasi-isomorphic to its cohomology $H^\bullet(\mathcal{F}_L) \oplus H_\gamma^{\bullet-1}(\mathcal{F}_L)$ with the induced graded Lie algebra structure. On one side, such a result would not be so surprising when L is compact, because of the following. Any graded Lie algebra $(H, [\cdot,\cdot])$ has the property that the Kuranishi map completely characterizes unobstructedess: a first order deformation A is unobstructed if and only[5] if $Kr(A) = 0$. When L is compact, we know that the DGLA in Corollary 2.4.10 satisfies this property, as a consequence of Prop. 2.5.18. Further, we expect this property to be invariant under L_∞-quasi-isomorphisms satisfying mild assumptions. We do not address the formality question any further here. A possible approach is to apply Manetti's formality criteria in Thm. 3.3 or Thm. 3.4 of [Ma].

Remark 2.4.12. We comment on the DGLA $\big(\Gamma(T^*\mathcal{F}_L \times \mathbb{R}), d, [\![\cdot,\cdot]\!]\big)$ introduced in Corollary 2.4.10 above.

i) One can write down this DGLA in more generality. Let $\big(A, \rho, [\cdot,\cdot]\big)$ be a Lie algebroid over a manifold M, and let ∇ be a flat A-connection on a line bundle $E \to M$. Let $D \in \text{Der}(A)$ be a derivation of A. Then there is an induced DGLA-structure $(d, [\![\cdot,\cdot]\!])$ on the graded vector space $\Gamma(\wedge^\bullet(A^* \oplus E)) = \Gamma(\wedge^\bullet A^*) \oplus \Gamma\big(\wedge^{\bullet-1}A^* \otimes E\big)$ defined by

$$\begin{aligned} d(\alpha, \varphi) &= (d_A\alpha, d_\nabla\varphi)\\ [\![(\alpha,\varphi),(\beta,\psi)]\!] &= \big(0, \pounds_D\alpha \wedge \psi - (-1)^{kl}\pounds_D\beta \wedge \varphi\big), \end{aligned} \tag{2.42}$$

for $(\alpha, \varphi) \in \Gamma\big(\wedge^k(A^* \oplus E)\big)$ and $(\beta, \psi) \in \Gamma\big(\wedge^l(A^* \oplus E)\big)$. Here the Lie derivative $\pounds_D$ is obtained extending the derivation on A^* dual to D.

[5]This is immediate, since if $Kr(A) = [A, A]$ vanishes then $t \mapsto tA$ is a curve of Maurer-Cartan elements.

ii) We discuss the structure of the DGLA $(\Gamma(\wedge^\bullet(A^* \oplus E)), d, [\![\cdot,\cdot]\!])$. The underlying cochain complex is a direct sum of complexes

$$(\Gamma(\wedge^\bullet A^*), d_A) \oplus (\Gamma(\wedge^\bullet A^* \otimes E)[-1], d_\nabla).$$

It can also be described as the cochain complex of differential forms on the Lie algebroid $A \oplus E^*$, the semidirect product of A by the representation on E^* given by the dual connection ∇^*. The underlying graded Lie algebra structure is the semidirect product of the abelian graded Lie algebras $\Gamma(\wedge^\bullet A^*)$ and $\Gamma(\wedge^\bullet A^* \otimes E)[-1]$ with respect to the action

$$\Gamma(\wedge^\bullet A^*) \to \mathrm{Der}\big(\Gamma(\wedge^\bullet A^* \otimes E)[-1]\big) : \alpha \mapsto \pounds_D \alpha \wedge \bullet.$$

iii) We can recover the DGLA $\big(\Gamma(T^*\mathcal{F}_L \times \mathbb{R}), d, [\![\cdot,\cdot]\!]\big)$ described in Corollary 2.4.10 by making the following choices in the general construction of i) above:

- Take the Lie algebroid $A := (T\mathcal{F}_L, -\iota, -[\cdot,\cdot])$, where $\iota : T\mathcal{F}_L \hookrightarrow TL$ is the inclusion and $[\cdot,\cdot]$ is the Lie bracket of vector fields. The Lie algebroid differential d_A on $\Gamma(\wedge^\bullet A^*)$ is then $-d_{\mathcal{F}_L}$.
- Let $D := [X, \cdot]$ be the derivation determined by $X \in \mathfrak{X}(L)^{\mathcal{F}_L}$.
- Let $E := L \times \mathbb{R} \to L$ be the trivial line bundle.
- Let the representation ∇ of A on E be defined by

$$\nabla_Y \bullet = \pounds_{-Y} \bullet - \gamma(Y) \bullet$$

for $Y \in \Gamma(A)$. Since γ is closed, this is indeed a representation, and the induced differential d_∇ on $\Gamma(\wedge^\bullet A^*)$ is given by

$$d_\nabla \bullet = -d_{\mathcal{F}_L} \bullet - \gamma \wedge \bullet.$$

2.4.3 On foliated Morse-Novikov cohomology

We now discuss the cohomology of the DGLA $\big(\Gamma(\wedge^\bullet(T^*\mathcal{F}_L \times \mathbb{R})), d, [\![\cdot,\cdot]\!]\big)$ in degree one, which in the notation of Remark 2.4.5 is given by $H^1(\mathcal{F}_L) \oplus H^0_\gamma(\mathcal{F}_L)$. We explicitly compute the second summand of this cohomology group for Lagrangians that are compact and connected. We first collect some foliated analogs of well-known facts about Morse-Novikov cohomology [HR, Section 1].

Lemma 2.4.13. *Let L be a manifold, $\mathcal{F}_L$ a foliation on L and $\eta \in \Omega^1(\mathcal{F}_L)$ a closed foliated one-form. As before, denote by $H^\bullet_\eta(\mathcal{F}_L)$ the cohomology groups of the differential $d^\eta_{\mathcal{F}_L}$ defined in* (2.34)*. We then have the following:*

i) If $[\eta] = [\eta'] \in H^1(\mathcal{F}_L)$, then $H^k_\eta(\mathcal{F}_L) \cong H^k_{\eta'}(\mathcal{F}_L)$. In particular, if $[\eta] = 0$ in $H^1(\mathcal{F}_L)$ then $H^k_\eta(\mathcal{F}_L) \cong H^k(\mathcal{F}_L)$.

ii) Assume $[\eta] \neq 0$ *in* $H^1(\mathcal{F}_L)$ *and let* $f \in H^0_\eta(\mathcal{F}_L)$*. Then there is a leaf* $\mathcal{O}$ *of* $\mathcal{F}_L$ *on which* f *vanishes identically.*

Proof. i) If $\eta' = \eta + d_{\mathcal{F}_L} g$ for $g \in C^\infty(L)$, then the following map is an isomorphism of cochain complexes:

$$\left(\Omega^\bullet(\mathcal{F}_L), d^{\eta'}_{\mathcal{F}_L}\right) \to \left(\Omega^\bullet(\mathcal{F}_L), d^{\eta}_{\mathcal{F}_L}\right) : \beta \mapsto e^g\beta. \tag{2.43}$$

ii) By assumption we have that

$$d_{\mathcal{F}_L} f + f\eta = 0. \tag{2.44}$$

If f would be nowhere zero, then we could write $\eta = -d_{\mathcal{F}_L} log|f|$, contradicting that η is not exact. So f must have a zero, say in the leaf $\mathcal{O} \in \mathcal{F}_L$. Consider the vanishing set $\mathcal{Z}_f := \{x \in \mathcal{O} : f(x) = 0\}$, which is nonempty and closed in $\mathcal{O}$. If we show that $\mathcal{Z}_f$ is also open in $\mathcal{O}$, then we reach the conclusion $f|_\mathcal{O} \equiv 0$, since $\mathcal{O}$ is connected.

To this end, let $x \in \mathcal{Z}_f$. Since $\eta|_\mathcal{O} \in \Omega^1(\mathcal{O})$ is closed, there exist a neighborhood U of x in $\mathcal{O}$ and $g \in C^\infty(U)$ such that $\eta|_U = dg$. Using the isomorphism (2.43) for the one-leaf foliation on U, we obtain that $d(e^g f|_U) = 0$. So $e^g f|_U$ is constant on U, and since $f(x) = 0$ we must have $e^g f|_U \equiv 0$. Consequently $f|_U \equiv 0$, which shows that $U \subset \mathcal{Z}_f$. So $\mathcal{Z}_f$ is open, and this finishes the proof. ☐

Remark 2.4.14. If we replace the hypothesis $[\eta] \neq 0$ in *ii)* of Lemma 2.4.13 by the stronger requirement that $\eta|_\mathcal{O} \in \Omega^1(\mathcal{O})$ be not exact for all leaves $\mathcal{O} \in \mathcal{F}_L$, then, restricting the equality (2.44) to each leaf $\mathcal{O}$, we obtain that $H^0_\eta(\mathcal{F}_L) = 0$.

We now specialize to compact, connected manifolds L endowed with a codimension-one foliation $\mathcal{F}_L$ defined by a nowhere vanishing closed one-form. Under these assumptions it is well-known [C, Thm. 9.3.13] that:

- either $(L, \mathcal{F}_L)$ is the fiber foliation of a fiber bundle $p : L \to S^1$,
- or all leaves of $\mathcal{F}_L$ are dense. $(\star)$

For completeness, we have included a proof of this fact in the Appendix. Recall that in the fibration case, the k-th cohomology groups of the fibers of $p : L \to S^1$ constitute a vector bundle $\mathcal{H}^k$ over S^1:

$$\mathcal{H}^k_q = H^k\left(p^{-1}(q)\right),$$

and one has

$$H^k(\mathcal{F}_L) \xrightarrow{\sim} \Gamma\left(\mathcal{H}^k\right) : [\alpha] \mapsto \left(\sigma_\alpha : q \mapsto \left[\alpha|_{p^{-1}(q)}\right]\right). \tag{2.45}$$

Using the identification

$$\mathfrak{X}(S^1) \xrightarrow{\sim} \frac{\mathfrak{X}(L)^{\mathcal{F}_L}}{\Gamma(T\mathcal{F}_L)} : Y \mapsto \overline{Y},$$

one can define a natural flat connection ∇ on the vector bundle $\mathcal{H}^k$ by the formula

$$\nabla_Y \sigma_\alpha := \sigma_{\pounds_{\overline{Y}}\alpha}, \tag{2.46}$$

for $\alpha \in \Omega^1_{cl}(\mathcal{F}_L)$ and $Y \in \mathfrak{X}(S^1)$. Note that ∇ is well-defined, because of Cartan's formula. If F denotes the typical fiber of $p : L \to S^1$ and $\{[\beta_1], \dots, [\beta_m]\}$ is a basis of $H^k(F)$, then in a local trivialization $U \times F$, the constant functions $[\beta_1], \dots, [\beta_m] \in C^\infty(U, H^k(F)) \cong \Gamma(\mathcal{H}^k|_U)$ form a local frame of flat sections.

The following lemma will be useful to compute the foliated Morse-Novikov cohomology in case the foliation is given by a fibration.

Lemma 2.4.15. *Let L be a compact manifold endowed with a foliation $\mathcal{F}_L$ that is the fiber foliation of a fiber bundle*[6] *$p : L \to S^1$. Let $\eta \in \Omega^1(\mathcal{F}_L)$ be a closed foliated one-form, denote by $\sigma_\eta \in \Gamma(\mathcal{H}^1)$ the section corresponding with $[\eta] \in H^1(\mathcal{F}_L)$ under* (2.45)*, and let $\mathcal{Z}_\eta := \sigma_\eta^{-1}(0)$. Then there exists a smooth function $g \in C^\infty(L)$ such that*

$$\eta|_{p^{-1}(q)} = d\left(g|_{p^{-1}(q)}\right) \qquad \textit{for all } q \in \mathcal{Z}_\eta.$$

Proof. By [MM, Lemma 2.28], we can fix an embedded loop $\tau : S^1 \to L$ transverse to the leaves of $\mathcal{F}_L$ which hits each leaf of $\mathcal{F}_L$ exactly once. Define a function h on $p^{-1}(\mathcal{Z}_\eta)$ by setting $h|_{p^{-1}(q)}$ to be the unique primitive of $\eta|_{p^{-1}(q)}$ that vanishes at the point $p^{-1}(q) \cap \tau(S^1)$. We claim that h extends to a smooth function $g \in C^\infty(L)$. To prove this, it suffices to show that around each point $x \in p^{-1}(\mathcal{Z}_\eta)$ there exist a neighborhood $U \subset L$ and a smooth function on U that agrees with h on $U \cap p^{-1}(\mathcal{Z}_\eta)$.

Let $x \in p^{-1}(q)$ for $q \in \mathcal{Z}_\eta$ and denote $y := p^{-1}(q) \cap \tau(S^1)$. Working in a local trivialization $V \times p^{-1}(q)$, choose a path $\gamma : (-\epsilon, 1+\epsilon) \to p^{-1}(q)$ such that $\gamma(0) = x$ and $\gamma(1) = y$, take a tubular neighborhood N of this path in $p^{-1}(q)$ and define $U := V \times N$. Since N is contractible, we have for each value of $v \in V$ that $\eta_v \in \Omega^1(N)$ is exact. Since one can choose primitives varying smoothly in v (see [GLSW]), it follows that $\eta|_U$ is foliated exact. Choose any primitive $k \in C^\infty(U)$ of $\eta|_U$. Shrinking V if necessary, we can assume that each fiber $\{v\} \times N$ intersects the loop $\tau(S^1)$. Define a map $\phi : U \to U \cap \tau(S^1)$ by setting $\phi(z)$ to be the intersection point of $\tau(S^1)$ with the fiber through z. Then setting $\widetilde{h} := k - \phi^*\left(k|_{U \cap \tau(S^1)}\right)$, we obtain a primitive of $\eta|_U$ that vanishes

[6] Note that under these assumptions, L is automatically connected.

along $U \cap \tau(S^1)$. Uniqueness of such primitives implies that $\widetilde{h}$ agrees with h wherever both of them are defined. This shows that h can be extended to a smooth function $g \in C^\infty(L)$. □

We can now compute the zeroth foliated Morse-Novikov cohomology group.

Theorem 2.4.16. *Let $(L, \mathcal{F}_L)$ be a compact, connected manifold endowed with a codimension-one foliation defined by a closed one-form. Let $\eta \in \Omega^1(\mathcal{F}_L)$ be a closed foliated one-form.*

i) Assume $\mathcal{F}_L$ is the fiber foliation of a fiber bundle $p : L \to S^1$. Then we have

$$H^0_\eta(\mathcal{F}_L) \cong \{f \in C^\infty(S^1) : f \cdot \sigma_\eta = 0\},$$

where $\sigma_\eta \in \Gamma(\mathcal{H}^1)$ denotes the section corresponding with $[\eta] \in H^1(\mathcal{F}_L)$ under the correspondence (2.45).

ii) Assume all leaves of $\mathcal{F}_L$ are dense. Then

$$H^0_\eta(\mathcal{F}_L) = \begin{cases} \mathbb{R} & \text{if } \eta \text{ is foliated exact} \\ 0 & \text{otherwise} \end{cases}.$$

Proof. i) Fix a smooth function $g \in C^\infty(L)$ as constructed in Lemma 2.4.15 and define

$$\Psi : H^0_\eta(\mathcal{F}_L) \to \{f \in C^\infty(S^1) : f \cdot \sigma_\eta = 0\} : h \mapsto e^g h.$$

We first check that Ψ is well-defined. Choosing $h \in H^0_\eta(\mathcal{F}_L)$, we must show that $e^g h$ is constant along the leaves of $\mathcal{F}_L$, and that the induced function on the leaf space S^1 lies in the annihilator ideal of $\sigma_\eta \in \Gamma(\mathcal{H}^1)$. Note that for any $q \in S^1$, we have

$$d\left(h|_{p^{-1}(q)}\right) + h|_{p^{-1}(q)}\, \eta|_{p^{-1}(q)} = 0.$$

In case $\sigma_\eta(q) = 0$, then $\eta|_{p^{-1}(q)} = d(g|_{p^{-1}(q)})$ and the isomorphism (2.43) implies that

$$(e^g h)|_{p^{-1}(q)} \in H^0(p^{-1}(q)) = \mathbb{R}.$$

Next, assume that $\sigma_\eta(q) \neq 0$, i.e. $\eta|_{p^{-1}(q)}$ is not exact. Then $h|_{p^{-1}(q)} \equiv 0$ by applying *ii)* of Lemma 2.4.13 to the one-leaf foliation on $p^{-1}(q)$, and therefore $(e^g h)|_{p^{-1}(q)} \equiv 0$.

Clearly, Ψ is linear and injective. For surjectivity, we let $f \in C^\infty(S^1)$ be such that $f \cdot \sigma_\eta = 0$ and we have to check that $e^{-g} p^* f \in H^0_\eta(\mathcal{F}_L)$, i.e.

$$d_{\mathcal{F}_L}\left(\frac{p^* f}{e^g}\right) + \frac{p^* f}{e^g}\eta = 0. \tag{2.47}$$

On fibers $p^{-1}(q)$ with $\sigma_\eta(q) \neq 0$, the equality (2.47) is satisfied since p^*f vanishes there. On fibers $p^{-1}(q)$ with $\sigma_\eta(q) = 0$, we know that $\eta|_{p^{-1}(q)} = d(g|_{p^{-1}(q)})$, so that the left hand side of (2.47) becomes

$$-f(q)(e^{-g}|_{p^{-1}(q)})d(g|_{p^{-1}(q)}) + f(q)(e^{-g}|_{p^{-1}(q)})d(g|_{p^{-1}(q)}) = 0.$$

ii) This is an immediate consequence of Lemma 2.4.13. □

Example 2.4.17. Take $L = (S^1 \times S^1, \theta_1, \theta_2)$ and let $\mathcal{F}_L$ be the foliation by fibers of the projection $(S^1 \times S^1, \theta_1, \theta_2) \to (S^1, \theta_1)$. In order to compute the cohomology group $H^0_\eta(\mathcal{F}_L)$ for closed $\eta \in \Omega^1(\mathcal{F}_L)$, we can choose a convenient representative of $[\eta] \in H^1(\mathcal{F}_L)$, by *i*) of Lemma 2.4.13.

In this respect, notice that every class $[g(\theta_1,\theta_2)d\theta_2] \in H^1(\mathcal{F}_L)$ has a unique representative of the form $h(\theta_1)d\theta_2$. Namely, setting $h(\theta_1) := \frac{1}{2\pi}\int_{S^1} g(\theta_1,\theta_2)d\theta_2$, we have

$$\int_{S^1} [g(\theta_1,\theta_2) - h(\theta_1)]\, d\theta_2 = 0,$$

which implies that there exists $k(\theta_1,\theta_2) \in C^\infty(S^1 \times S^1)$ such that

$$g(\theta_1,\theta_2) - h(\theta_1) = \frac{\partial k}{\partial \theta_2}(\theta_1,\theta_2).$$

This implies that

$$g(\theta_1,\theta_2)d\theta_2 - h(\theta_1)d\theta_2 = \frac{\partial k}{\partial \theta_2}(\theta_1,\theta_2)d\theta_2 = d_{\mathcal{F}_L}k.$$

Uniqueness of such representatives follows by integrating around circles $\{\theta_1\}\times S^1$.

Now, fix $\eta = h(\theta_1)d\theta_2$ in $\Omega^1(\mathcal{F}_L)$ and assume that $f \in H^0_\eta(\mathcal{F}_L)$. Then

$$0 = \frac{\partial f}{\partial \theta_2}d\theta_2 + f \cdot h(\theta_1)d\theta_2. \tag{2.48}$$

For fixed θ_1, the restriction of f to $\{\theta_1\} \times S^1$ reaches a maximum M and a minimum m. The equality (2.48) implies that

$$\begin{cases} M \cdot h(\theta_1) = 0 \\ m \cdot h(\theta_1) = 0 \end{cases}.$$

So either $h(\theta_1) = 0$ or $f|_{\{\theta_1\}\times S^1} \equiv 0$. Hence, we get that $f \cdot h(\theta_1) = 0$, and (2.48) then implies that also $\partial f/\partial\theta_2 = 0$. In conclusion, we get

$$\begin{aligned} H^0_{h(\theta_1)d\theta_2}(\mathcal{F}_L) &= \{f(\theta_1) : f(\theta_1)h(\theta_1)d\theta_2 = 0\} \\ &= \{f(\theta_1) : f(\theta_1)\cdot\sigma_{h(\theta_1)d\theta_2} = 0\}, \end{aligned}$$

using in the last equality that $\sigma_{h(\theta_1)d\theta_2}(\theta_1) = 0 \Leftrightarrow h(\theta_1)d\theta_2 = 0$. So we obtain the result that was predicted by *i*) of Theorem 2.4.16.

Remark 2.4.18. The example we have in mind throughout this subsection is of course that of a compact connected Lagrangian L^n contained in the singular locus Z of a log-symplectic manifold (M^{2n}, Z, Π). The induced foliation $\mathcal{F}_L$ on L is defined by a closed one-form, which is obtained by pulling back a closed defining one-form for the foliation on Z. So $(L, \mathcal{F}_L)$ is either the fiber foliation of a fiber bundle $L \to S^1$, or all leaves of $\mathcal{F}_L$ are dense.
Moreover, the foliation type of $\mathcal{F}_L$ is stable under small deformations of the Lagrangian L inside Z. To see this, we can work in the local model $p : T^*\mathcal{F}_L \to L$, where the total space $T^*\mathcal{F}_L$ is endowed with the pullback foliation $p^{-1}(\mathcal{F}_L)$. Any Lagrangian deformation L' of L is of the form $L' = \text{Graph}(\alpha)$ for some $\alpha \in \Omega^1_{cl}(\mathcal{F}_L)$, and the induced foliation $\mathcal{F}_{L'}$ is obtained by intersecting L' with the leaves of $p^{-1}(\mathcal{F}_L)$. Therefore, the map $p : (L', \mathcal{F}_{L'}) \to (L, \mathcal{F}_L)$ is a foliated diffeomorphism (with inverse $\alpha : (L, \mathcal{F}_L) \to (L', \mathcal{F}_{L'})$), which shows that $(L, \mathcal{F}_L)$ and $(L', \mathcal{F}_{L'})$ are of the same type.

2.5 Deformations of Lagrangian submanifolds in log-symplectic manifolds: geometric aspects

We present some geometric consequences of the algebraic results obtained in the previous section. We address three different geometric questions, each in a separate subsection, as we now outline. Throughout, we assume the set-up given at the beginning of §2.4.

§2.5.1 Deformations constrained to the singular locus We investigate when all sufficiently small deformations of the Lagrangian L are constrained to the singular locus. Prop. 2.5.2 gives a condition under which this does not happen. On the opposite extreme, in Cor. 2.5.5 and Prop. 2.5.10 we obtain positive results assuming that L is compact, by considering separately the case that L is the total space of a fibration and the case that L has a dense leaf. The latter case is subtle, and we show that the conclusion of Prop. 2.5.10 fails to hold if we remove a certain finite dimensionality assumption.

§2.5.2 Obstructedness of deformations We ask when infinitesimal deformations of the Lagrangian L can be extended to a smooth curve of Lagrangian deformations. A sufficient criterium is given in Prop. 2.5.13. (All smoothly unobstructed deformations arise this way under an additional assumption, see Lemma 2.5.17). Our main results, under the assumption that L is compact, are the computable "if and only if" criteria of Prop. 2.5.18 and Cor. 2.5.20.

§2.5.3 Equivalences of deformations and rigidity On the set of Lagrangian deformations of L there are two natural notions of equivalence: an algebraic one and a geometric one, given by Hamiltonian isotopies. In Prop. 2.5.26 we show that they coincide. We also show that there are no Lagrangian submanifolds which are infinitesimally rigid under Hamiltonian isotopies, so that the moduli space (which typically is not smooth) does not have any isolated points. This leads us to consider the more flexible equivalence relation given by Poisson isotopies. The formal tangent space of its moduli space is computed in Prop. 2.5.30. There do exist Lagrangians which are rigid under Poisson isotopies, as follows using Prop. 2.5.34.

Remark 2.5.1 (The local deformation problem)*.* We summarize here how our results specialize to the local deformation problem, i.e. to a Lagrangian L as in the local model of Prop. 2.2.17:

- L can be deformed smoothly to a Lagrangian submanifold outside of the singular locus (Remark 2.5.3).
- all first order deformations of L are smoothly unobstructed (Cor. 2.5.15).
- The space of local Lagrangian deformations modulo Hamiltonian isotopies is not smooth at $[L]$. Indeed, the formal tangent space at $[L]$ is isomorphic to $C^\infty(\mathbb{R})$ (see eq. (2.75)), while at Lagrangians contained in $M \setminus Z$ it is the zero vector space. The same is true if one replaces Hamiltonian isotopies by Poisson isotopies.

2.5.1 Deformations constrained to the singular locus

We now investigate whether it is always possible to find deformations of the Lagrangian L that escape from the singular locus. Working in the model $\big(U \subset T^*\mathcal{F}_L \times \mathbb{R}, V \wedge t\partial_t + \Pi_{can}\big)$, a sufficient condition is the existence of a representative of the fixed first Poisson cohomology class $[V]$ that is tangent to L. Below, we denote by $W := U \cap \{t = 0\} \subset T^*\mathcal{F}_L$ the neighborhood of L in $T^*\mathcal{F}_L$ where V is defined.

Proposition 2.5.2. *The Poisson cohomology class $[V] \in H^1_{\Pi_{can}}(W)$ has a representative tangent to L if and only if $[\gamma] = 0 \in H^1(\mathcal{F}_L)$. If these equivalent conditions hold, then there is a smooth path of Lagrangian deformations L_s starting at $L_0 = L$ which is not contained in the singular locus for $s > 0$.*

Proof. We start by showing that the conditions are equivalent. First assume that $V - X_g$ for $g \in C^\infty(W)$ is a representative of $[V]$ that is tangent to L. As before, let P denote the map that restricts vector fields on W to L and then takes their vertical component. By assumption, we then have

$$\begin{aligned} 0 = P(V - X_g) &= P(V_{vert} + V_{lift} - X_g) \\ &= \gamma - P(X_g - X_{p^*(i^*g)} + X_{p^*(i^*g)}) \\ &= \gamma - P(X_{p^*(i^*g)}) \\ &= \gamma - d_{\mathcal{F}_L}(i^*g). \end{aligned}$$

Here $p : W \to L$ and $i : L \hookrightarrow W$ denote the projection and inclusion, respectively, the fourth equality holds since L is coisotropic, and the last equality holds by the correspondence (2.20). This shows that $\gamma = d_{\mathcal{F}_L}(i^*g)$, and therefore $[\gamma] = 0 \in H^1(\mathcal{F}_L)$. Conversely, if $\gamma = d_{\mathcal{F}_L} g$ for some $g \in C^\infty(L)$, then $V - X_{p^*g}$ is a representative of $[V]$ that is tangent to L.

If the equivalent conditions hold, then by Remark 2.2.19 we can assume that $\gamma = 0$. The Maurer-Cartan equation (2.25) then shows that any path of the form $s \mapsto (0, sf)$ for a nonzero leafwise constant function $f \in C^\infty(L)$ consists of Lagrangian deformations of L that are no longer contained in the singular locus for $s > 0$. Alternatively, if $\gamma = d_{\mathcal{F}_L} g$ for some $g \in C^\infty(L)$, then for any nonzero leafwise constant function f on L, the path $s \mapsto (0, sfe^{-g})$ meets the criteria. This proves the proposition. □

Remark 2.5.3. For the local deformation problem we have $\gamma = 0$, see Prop. 2.2.17. Hence locally, every middle-dimensional Lagrangian submanifold contained in the singular locus can be deformed smoothly to one outside of the singular locus, by Prop. 2.5.2.

We will single out some Lagrangians whose deformations are constrained to the singular locus. We restrict ourselves to Lagrangians L that are compact and connected. Recalling the dichotomy ($\star$) from §2.4.3, these assumptions imply that either $(L, \mathcal{F}_L)$ is the fiber foliation of a fiber bundle $L \to S^1$ or the leaves of $\mathcal{F}_L$ are dense.

The fibration case

We need the following lemma about the map which, under the identification (2.45), assigns to a closed foliated one-form its cohomology class.

Lemma 2.5.4. *Let $(L, \mathcal{F}_L)$ be a compact manifold, where $\mathcal{F}_L$ is the fiber foliation of a fiber bundle $p : L \to S^1$. Then the following map is continuous for the $\mathcal{C}^0$-topology:*

$$\left(\Omega^1_{cl}(\mathcal{F}_L), \mathcal{C}^0\right) \to \left(\Gamma(\mathcal{H}^1), \mathcal{C}^0\right) : \alpha \mapsto \sigma_\alpha. \tag{2.49}$$

Proof. Let F denote the typical fiber of $p : L \to S^1$. Choose a basis $\{[\gamma_1], \ldots, [\gamma_n]\}$ of the first homology group $H_1(F;\mathbb{Z})$, where the representatives $\gamma_i : [0,1] \to F$ are smooth 1-cycles. Via the de Rham isomorphism

$$H^1_{dR}(F) \to \mathrm{Hom}\left(H_1(F;\mathbb{Z}), \mathbb{R}\right) : [\omega] \mapsto \left(\sum_{i=1}^n c_i[\gamma_i] \mapsto \sum_{i=1}^n c_i \int_{\gamma_i} \omega\right),$$

we can pick the dual basis $\{[\alpha_1], \ldots, [\alpha_n]\}$ of $H^1_{dR}(F)$, satisfying

$$\int_{\gamma_i} \alpha_j = \delta_{ij}.$$

This provides an isomorphism $H^1_{dR}(F) \cong \mathbb{R}^n$. Choose local trivialisations of $p : L \to S^1$ over open subsets $U_1, \ldots, U_r$ covering S^1, and let $V_1, \ldots, V_r$ be open subsets whose compact closures satisfy $\overline{V_i} \subset U_i$, such that $V_1, \ldots, V_r$ still cover S^1. Then locally the map (2.49) reads

$$\Omega^1_{cl}(\mathcal{F})\big|_{p^{-1}(U_i)} \to \Gamma\left(U_i \times H^1_{dR}(F)\right) \cong C^\infty(U_i, \mathbb{R})^n : \alpha_\theta \mapsto \left(\int_{\gamma_1} \alpha_\theta, \ldots, \int_{\gamma_n} \alpha_\theta\right).$$

Therefore the $\mathcal{C}^0$-norm of σ_α is

$$\sum_{1 \leq i \leq r} \sum_{1 \leq j \leq n} \sup_{\theta \in \overline{V_i}} \left| \int_{\gamma_j} \alpha_\theta \right|,$$

which can be made arbitrarily small by shrinking α in $\mathcal{C}^0$. Since the map (2.49) is linear, this proves the lemma. ☐

The following corollary states that, under hypotheses that are antipodal to those of Prop. 2.5.2, small deformations of L stay inside the singular locus Z.

Corollary 2.5.5. *Let L^n be a compact connected Lagrangian submanifold contained in the singular locus Z of an orientable log-symplectic manifold (M^{2n}, Z, Π). Assume that the induced foliation $\mathcal{F}_L$ on L is the fiber foliation of a fiber bundle $p : L \to S^1$, and that the section $\sigma_\gamma \in \Gamma\left(\mathcal{H}^1\right)$ is nowhere zero. Then $\mathcal{C}^1$-small deformations of L stay inside the singular locus Z.*

Proof. Clearly, we have a continuous map

$$\left(\Omega^1_{cl}(\mathcal{F}_L), \mathcal{C}^1\right) \to \left(\Omega^1_{cl}(\mathcal{F}_L), \mathcal{C}^0\right) : \alpha \mapsto \gamma - \pounds_X \alpha,$$

so composing with the map (2.49) gives a continuous map

$$\left(\Omega^1_{cl}(\mathcal{F}_L), \mathcal{C}^1\right) \to \left(\Gamma(\mathcal{H}^1), \mathcal{C}^0\right) : \alpha \mapsto \sigma_{\gamma - \pounds_X \alpha}.$$

Therefore, since $\sigma_\gamma \in \Gamma(\mathcal{H}^1)$ is nowhere zero, the same holds for $\sigma_{\gamma-\pounds_X\alpha} \in \Gamma(\mathcal{H}^1)$ provided that $\alpha \in \Omega^1_{cl}(\mathcal{F}_L)$ is sufficiently $\mathcal{C}^1$-small. By Theorem 2.4.16, this means that the cohomology group $H^0_{\gamma-\pounds_X\alpha}(\mathcal{F}_L)$ is zero for $\mathcal{C}^1$-small α. Looking at the Maurer-Cartan equation (2.25), this implies that $\mathcal{C}^1$-small deformations of L necessarily lie inside the singular locus. □

Remark 2.5.6. The assumption in Corollary 2.5.5 cannot be weakened. Clearly, if the interior of $\sigma_\gamma^{-1}(0)$ is nonempty, then by Theorem 2.4.16 i) there exists $f \in H^0_\gamma(\mathcal{F}_L)$ which is not identically zero, and $s \mapsto (0, sf)$ is a path of Lagrangian sections not inside the singular locus for $s > 0$.
Even if we ask that the support of σ_γ be all of S^1, the conclusion of Corollary 2.5.5 does not hold. Indeed, one can construct counterexamples where the vanishing set of $\sigma_{\gamma-\pounds_X\alpha}$ has nonempty interior, for arbitrarily $\mathcal{C}^1$-small $\alpha \in \Omega^1_{cl}(\mathcal{F}_L)$.

The following is an example of a Lagrangian L for which small deformations stay inside the singular locus. However there exist "long" paths of Lagrangian deformations that start at L and end at a Lagrangian that is no longer contained in the singular locus.

Example 2.5.7. Consider the manifold $(\mathbb{T}^2 \times \mathbb{R}^2, \theta_1, \theta_2, \xi_1, \xi_2)$ with log-symplectic structure

$$\Pi := (\partial_{\theta_1} - \partial_{\xi_2}) \wedge \xi_1 \partial_{\xi_1} + \partial_{\theta_2} \wedge \partial_{\xi_2}$$

and Lagrangian $L := \mathbb{T}^2 \times \{(0,0)\}$. Note that the leaves of $\mathcal{F}_L$ are the fibers of the fibration $(\mathbb{T}^2, \theta_1, \theta_2) \to (S^1, \theta_1)$. Considering (ξ_1, ξ_2) as the fiber coordinates on T^*L induced by the frame $\{d\theta_1, d\theta_2\}$, we have that $T^*\mathcal{F}_L = (\mathbb{T}^2 \times \mathbb{R}, \theta_1, \theta_2, \xi_2)$ with canonical Poisson structure $\partial_{\theta_2} \wedge \partial_{\xi_2}$. Therefore, using the notation established in the beginning of this section, we have

$$\begin{cases} X = \partial_{\theta_1} \\ \gamma = -d\theta_2 \end{cases}.$$

So the section $\sigma_\gamma \in \Gamma(\mathcal{H}^1)$ is nowhere zero, and Corollary 2.5.5 shows that $\mathcal{C}^1$-small deformations of the Lagrangian L stay inside the singular locus.

It is however possible to construct (large) deformations of L that don't lie in the singular locus $T^*\mathcal{F}_L \times \{0\} \subset T^*\mathcal{F}_L \times \mathbb{R}$, first deforming L inside the singular locus by large enough $\alpha \in \Omega^1_{cl}(\mathcal{F}_L)$ such that $H^0_{\gamma-\pounds_X\alpha}(\mathcal{F}_L)$ is no longer zero. To do so explicitly, note that $\big(g(\theta_1,\theta_2)d\theta_2, f(\theta_1,\theta_2)\big) \in \Omega^1(\mathcal{F}_L) \times C^\infty(L)$, gives rise to a Lagrangian section of $T^*\mathcal{F}_L \times \mathbb{R}$ exactly when

$$\frac{\partial f}{\partial \theta_2} - f = f \frac{\partial g}{\partial \theta_1}. \tag{2.50}$$

We construct a solution (g, f) to (2.50) with f not identically zero. For instance, let $f(\theta_1)$ be any bump function and let $H(\theta_1)$ be another bump function with

$H|_{\mathrm{supp}(f)} \equiv -1$ and $-1 \leq H(\theta_1) \leq 0$. Define $C := \int_{S^1} H(\theta_1)d\theta_1$, so $C > -2\pi$. Let $K := -C/(C+2\pi)$ and put $G(\theta_1) := K\big(1 + H(\theta_1)\big) + H(\theta_1)$. Notice that $G|_{\mathrm{supp}(f)} \equiv -1$, and since

$$\int_{S^1} G(\theta_1)d\theta_1 = K(2\pi + C) + C = 0,$$

there exists a periodic primitive $g(\theta_1)$ with $\partial g/\partial\theta_1 = G$. We check that (g, f) is a solution to the Maurer-Cartan equation (2.50): for $p \notin \mathrm{supp}(f)$ both sides of (2.50) evaluate to zero, whereas for $p \in \mathrm{supp}(f)$ both sides of (2.50) are equal to $-f(p)$. It is clear that $f \not\equiv 0$.

So first deforming L along $\alpha := g(\theta_1)d\theta_2$ and then moving outside of $T^*\mathcal{F}_L \times \{0\}$ along f gives a Lagrangian deformation that is no longer contained in the singular locus. As a sanity check, looking at $i)$ of Theorem 2.4.16, we notice that $H^0_{\gamma - \pounds_X\alpha}(\mathcal{F}_L)$ is indeed nonzero and that $f \in H^0_{\gamma-\pounds_X\alpha}(\mathcal{F}_L)$, since the section $\sigma_{\gamma-\pounds_X\alpha}$ vanishes on the support of f.

Moreover, the proof of Corollary 2.4.4 shows that this procedure can be done smoothly, in the sense that one can construct a smooth "long" path of Lagrangians that connects L with Lagrangians that are no longer contained in the singular locus. Concretely, let (g, f) be the solution to (2.50) just constructed, and let $\Psi : \mathbb{R} \to \mathbb{R}$ be any smooth function satisfying $\Psi(s) = 0$ for $s \leq 0$, $0 < \Psi(s) < 1$ for $0 < s < 1$ and $\Psi(s) = 1$ for $s \geq 1$. Take $\Phi : \mathbb{R} \to \mathbb{R}$ to be defined by $\Phi(s) = \Psi(s-1)$. Then the path $s \mapsto (\Psi(s)gd\theta_2, \Phi(s)f)$ consists of Lagrangian sections, it starts at L for $s = 0$, passes through $(\alpha, 0)$ at time $s = 1$, and it reaches $\mathrm{Graph}(\alpha, f)$ at time $s = 2$.

The Lagrangian $\mathrm{Graph}(\alpha, f)$ constructed above does not lie entirely outside of the singular locus. Interestingly, it is not possible to find such deformations of L. For if we assume by contradiction that (g, f) is a solution to (2.50) with f nowhere zero, then

$$\int_0^{2\pi} \frac{1}{f}\left(\frac{\partial f}{\partial\theta_2} - f\right) d\theta_1 = \int_0^{2\pi} \frac{\partial g}{\partial\theta_1} d\theta_1 = 0,$$

so that

$$\int_0^{2\pi} \frac{1}{f}\frac{\partial f}{\partial\theta_2} d\theta_1 = 2\pi. \tag{2.51}$$

But then we would get

$$0 = \int_{\mathbb{T}^2} d\big(\ln|f| d\theta_1\big) = \int_{\mathbb{T}^2} \frac{1}{f}\frac{\partial f}{\partial\theta_2} d\theta_2 \wedge d\theta_1 = -\int_0^{2\pi}\left(\int_0^{2\pi} \frac{1}{f}\frac{\partial f}{\partial\theta_2} d\theta_1\right) d\theta_2 = -4\pi^2,$$

using Stokes' theorem in the first and (2.51) in the last equality. This contradiction shows that f must have a zero, i.e. the Lagrangian $\mathrm{Graph}(\alpha, f)$ intersects the singular locus.

Alternatively, if there were $\alpha = g(\theta_1, \theta_2)d\theta_2$ and a function $f \in H^0_{\gamma - \pounds_X\alpha}(\mathcal{F}_L)$ that is nowhere zero, then Theorem 2.4.16 implies that $\sigma_{\gamma-\pounds_X\alpha} \equiv 0$. Therefore, $\gamma - \pounds_X\alpha$ is foliated exact, which implies that there exists a function $k \in C^\infty(\mathbb{T}^2)$ such that

$$-1 - \frac{\partial g}{\partial \theta_1} = \frac{\partial k}{\partial \theta_2}.$$

Integrating this equality against the standard volume form $d\theta_1 \wedge d\theta_2$ on the torus $\mathbb{T}^2$ gives a contradiction, since the left hand side integrates to $-4\pi^2$ and the right hand side integrates to zero.

The case with dense leaves

Corollary 2.5.5 has no counterpart for Lagrangians whose induced foliation $\mathcal{F}_L$ has dense leaves, at least not without additional assumptions. Indeed, looking at Theorem 2.4.16 and the Maurer-Cartan equation (2.25), we would need a positive answer to the following question:

If $\gamma \in \Omega^1_{cl}(\mathcal{F}_L)$ is not exact, is $\gamma - \pounds_X\alpha$ still not exact for small $\alpha \in \Omega^1_{cl}(\mathcal{F}_L)$?

Drawing inspiration from [B, Section 4], we construct an explicit counterexample which answers this question in the negative. Let $L = \left(\mathbb{T}^2, \theta_1, \theta_2\right)$ be the torus with Kronecker foliation $T\mathcal{F}_L = \mathrm{Ker}(d\theta_1 - \lambda d\theta_2)$, for $\lambda \in \mathbb{R} \setminus \mathbb{Q}$ irrational. A global frame for $T^*\mathcal{F}_L$ is given by $d\theta_2$, so that every foliated one-form looks like $f(\theta_1, \theta_2)d\theta_2$, which is automatically closed by dimension reasons. It is exact when there exists $g(\theta_1, \theta_2) \in C^\infty\left(\mathbb{T}^2\right)$ such that

$$f = \lambda\frac{\partial g}{\partial \theta_1} + \frac{\partial g}{\partial \theta_2}. \tag{2.52}$$

Expanding f and g in double Fourier series,

$$f(\theta_1, \theta_2) = \sum_{m,n\in\mathbb{Z}} f_{m,n}e^{2\pi i(n\theta_1 + m\theta_2)} \quad \text{and} \quad g(\theta_1, \theta_2) = \sum_{m,n\in\mathbb{Z}} g_{m,n}e^{2\pi i(n\theta_1 + m\theta_2)},$$

the equality (2.52) is equivalent with

$$f_{m,n} = 2\pi i(m + \lambda n)g_{m,n} \qquad \forall m, n \in \mathbb{Z}, \tag{2.53}$$

which implies in particular that $f_{0,0} = 0$. Note that the $g_{m,n}$ for $(m, n) \neq (0, 0)$ are uniquely determined by the relation (2.53) since λ is irrational.

Assume moreover that the slope $\lambda \in \mathbb{R} \setminus \mathbb{Q}$ is a Liouville number (see Definition 2.6.1 in the Appendix). In this case the foliated cohomology group $H^1(\mathcal{F}_L)$ is infinite dimensional [Hae], [MS, Chapter III], as one can construct smooth functions $f(\theta_1, \theta_2)$ in such a way that the $g_{m,n}$ defined in (2.53) are not the Fourier coefficients of a smooth function. We give an example of such a function $f(\theta_1, \theta_2)$ in part i) of the proof below.

Lemma 2.5.8. *Consider the torus $L = \left(\mathbb{T}^2, \theta_1, \theta_2\right)$ endowed with the Kronecker foliation $T\mathcal{F}_L = Ker(d\theta_1 - \lambda d\theta_2)$, for $\lambda \in \mathbb{R} \setminus \mathbb{Q}$ a Liouville number. There exist a non-exact foliated one-form $\gamma \in \Omega^1(\mathcal{F}_L)$ and $X \in \mathfrak{X}(L)^{\mathcal{F}_L}$ such that every $\mathcal{C}^\infty$-open neighborhood of 0 in $\Omega^1(\mathcal{F}_L)$ contains a one-form α for which $\gamma - \pounds_X \alpha$ is foliated exact.*

Proof. The proof is divided into two steps. In the first step, we construct $\gamma \in \Omega^1(\mathcal{F}_L)$. In the second step, we fix $X \in \mathfrak{X}(L)^{\mathcal{F}_L}$ and we construct a sequence of foliated one-forms α_k such that $\gamma - \pounds_X \alpha_k$ is exact for each value of k, and $\alpha_k \to 0$ in the Fréchet $\mathcal{C}^\infty$-topology.

i) We first have to find a foliated one-form $\gamma = f(\theta_1, \theta_2) d\theta_2$ that is not exact. Moreover, since we want to approach γ by means of exact one-forms, we need that the coefficient $f_{0,0} = \frac{1}{4\pi^2} \int_{\mathbb{T}^2} f d\theta_1 \wedge d\theta_2$ is zero. This can be done as follows. By Lemma 2.6.3 in the Appendix, for each integer $p \geq 1$, there exists a pair of integers (m_p, n_p) such that

$$|m_p + \lambda n_p| \leq \frac{1}{(|m_p| + |n_p|)^p}. \tag{2.54}$$

We can moreover assume that $(m_p, n_p) \neq (m_q, n_q)$ for $p \neq q$, and that $n_p \geq p$ (see Remark 2.6.4). We now define $f(\theta_1, \theta_2)$ by means of its Fourier coefficients $f_{m,n}$, setting

$$f_{m,n} = \begin{cases} (m_p + \lambda n_p) n_p & \text{if } (m,n) = (m_p, n_p) \\ 0 & \text{else} \end{cases}.$$

To see that these coefficients define a smooth function, we make the following estimate for $k \in \mathbb{N}$:

$$\begin{aligned} |f_{m_p,n_p}| \cdot \|(m_p, n_p)\|^k &= |m_p + \lambda n_p| \cdot n_p \cdot \|(m_p, n_p)\|^k \\ &\leq |m_p + \lambda n_p| \cdot (|m_p| + |n_p|) \cdot (|m_p| + |n_p|)^k \\ &= |m_p + \lambda n_p| \cdot (|m_p| + |n_p|)^{k+1} \\ &\leq \frac{1}{(|m_p| + |n_p|)^p} \cdot (|m_p| + |n_p|)^{k+1} \\ &\leq \left(\frac{1}{p}\right)^{p-k-1}, \end{aligned}$$

where the last inequality holds for $p \geq k+1$. This shows that $\sup_{(m,n)\in\mathbb{Z}^2} |f_{m,n}| \|(m,n)\|^k$ is finite for each value of $k \in \mathbb{N}$, and therefore $f(\theta_1,\theta_2)$ is indeed smooth. To see that $\gamma = f(\theta_1,\theta_2)d\theta_2$ is not exact, note that the Fourier coefficients of a primitive $g(\theta_1,\theta_2)$ are given by (2.53):

$$g_{m,n} = \frac{f_{m,n}}{2\pi i(m+\lambda n)} \qquad \text{for } (m,n) \neq (0,0). \tag{2.55}$$

Therefore

$$|g_{m_p,n_p}| = \frac{1}{2\pi} n_p \geq \frac{1}{2\pi} p,$$

which does not tend to zero for $p \to \infty$. So the $g_{m,n}$ defined in (2.55) are not the Fourier coefficients of a smooth function.

ii) We let $X := \partial_{\theta_1}$. Notice that $X \in \mathfrak{X}(L)^{\mathcal{F}_L}$ and that X is transverse to the leaves of $\mathcal{F}_L$. We now construct a sequence $\alpha_k \in \Omega^1_{cl}(\mathcal{F}_L)$ such that $\gamma - \pounds_X \alpha_k$ is exact and $\alpha_k \to 0$ in the $\mathcal{C}^\infty$-topology. For each integer $k \geq 1$ we define $\alpha_k = h_k(\theta_1,\theta_2)d\theta_2$, where $h_k(\theta_1,\theta_2)$ is given by its Fourier coefficients

$$h^k_{m,n} = \begin{cases} \left(\frac{m_p+\lambda n_p}{2\pi i}\right)\cdot \phi_k\left(\frac{1}{p}\right) & \text{if } (m,n) = (m_p,n_p) \\ 0 & \text{else} \end{cases}.$$

Here ϕ_k is a bump function on $\mathbb{R}$ that is identically equal to 1 on the interval $[0, \frac{1}{k}]$. As before, we see that h_k is a smooth function by the estimate

$$|h^k_{m_p,n_p}| \cdot \|(m_p,n_p)\|^l \leq \frac{1}{2\pi}|m_p+\lambda n_p| \cdot (|m_p|+|n_p|)^l \leq \frac{1}{2\pi}\left(\frac{1}{p}\right)^{p-l},$$

where the last inequality holds for $p \geq l$. Note that $\gamma - \pounds_X \alpha_k$ is indeed exact: it equals $(f - \partial_{\theta_1} h_k)d\theta_2$, and the Fourier coefficients of $f - \partial_{\theta_1} h_k$ are given by

$$f_{m,n} - 2\pi i \cdot n \cdot h^k_{m,n} = \begin{cases} (m_p+\lambda n_p)n_p\left(1-\phi_k\left(\frac{1}{p}\right)\right) & \text{if } (m,n) = (m_p,n_p) \\ 0 & \text{else} \end{cases}$$

only finitely many of which are nonzero. Finally, by letting k increase, we can make α_k as $\mathcal{C}^\infty$-small as desired. Indeed, for each integer l we have

$$\begin{aligned} \|h_k\|_l &\leq \sum_{0\leq j\leq l} \sum_{p\geq k} |m_p+\lambda n_p| \cdot \|(m_p,n_p)\|^j \cdot (2\pi)^{j-1} \\ &\leq \sum_{0\leq j\leq l} \sum_{p\geq k} \left(\frac{1}{p}\right)^{p-j} \cdot (2\pi)^{j-1}, \end{aligned} \tag{2.56}$$

where the last inequality holds for $k \geq l$. The expression (2.56) tends to zero for $k \to \infty$, since the inner sum is the tail of a convergent series for each value of $j \in \{0, \ldots, l\}$. □

The above construction gives a concrete counterexample to the version of Corollary 2.5.5 for Lagrangians $(L, \mathcal{F}_L)$ with dense leaves. We only have to realize $L = (\mathbb{T}^2, \theta_1, \theta_2)$ with $T\mathcal{F}_L = \mathrm{Ker}(d\theta_1 - \lambda d\theta_2)$ as a Lagrangian submanifold contained in the singular locus of some log-symplectic manifold. The normal form (2.23) tells us how to construct this log-symplectic manifold. If (ξ_1, ξ_2) are the fiber coordinates on T^*L, and ξ is the fiber coordinate on $T^*\mathcal{F}_L$ corresponding with the frame $\{d\theta_2\}$, then the restriction map reads

$$r : T^*L \to T^*\mathcal{F}_L : (\theta_1, \theta_2, \xi_1, \xi_2) \mapsto (\theta_1, \theta_2, \lambda\xi_1 + \xi_2),$$

and therefore the canonical Poisson structure on $T^*\mathcal{F}_L$ is

$$\Pi_{can} = r_* \left(\partial_{\theta_1} \wedge \partial_{\xi_1} + \partial_{\theta_2} \wedge \partial_{\xi_2}\right) = \left(\lambda\partial_{\theta_1} + \partial_{\theta_2}\right) \wedge \partial_\xi.$$

Let V denote the vertical Poisson vector field on $T^*\mathcal{F}_L$ defined by the one-form $\gamma \in \Gamma(T^*\mathcal{F}_L)$ constructed in Lemma 2.5.8, and let $X := \partial_{\theta_1}$. Then $V + X$ is a Poisson vector field on $(T^*\mathcal{F}_L, \Pi_{can})$ transverse to the symplectic leaves, so the following is a log-symplectic structure:

$$\left(T^*\mathcal{F}_L \times \mathbb{R}, (V + X) \wedge t\partial_t + (\lambda\partial_{\theta_1} + \partial_{\theta_2}) \wedge \partial_\xi\right),$$

and L is Lagrangian inside $T^*\mathcal{F}_L$ with induced foliation $\mathcal{F}_L$. The above argument shows that, for each integer $k \geq 0$, there exists arbitrarily $\mathcal{C}^k$-small $\alpha \in \Omega^1_{cl}(\mathcal{F}_L)$ for which $\gamma - \pounds_X\alpha$ is exact. By Theorem 2.4.16, there exists $f \in H^0_{\gamma - \pounds_X\alpha}(\mathcal{F}_L)$ not identically zero, where f can be made arbitrarily $\mathcal{C}^k$-small by rescaling with a nonzero constant. The Maurer-Cartan equation (2.25) now implies that $\mathrm{Graph}(\alpha, f)$ is an arbitrarily $\mathcal{C}^k$-small Lagrangian deformation of L that is not completely contained in the singular locus $T^*\mathcal{F}_L$.

Remark 2.5.9. In the above counterexample, it is crucial that the slope λ is a Liouville number. If $\mathcal{F}_L$ is the Kronecker foliation with generic (i.e. not Liouville) irrational slope λ, then $H^1(\mathcal{F}_L) = \mathbb{R}[d\theta_2]$. In this case, exactness is detected by integration, for if we denote

$$I : C^\infty(\mathbb{T}^2) \to \mathbb{R} : h(\theta_1, \theta_2) \mapsto \int_{\mathbb{T}^2} h(\theta_1, \theta_2) d\theta_1 \wedge d\theta_2,$$

then $hd\theta_2 \in \Omega^1(\mathcal{F}_L)$ being exact is equivalent with $h \in I^{-1}(0)$. Since integration is $\mathcal{C}^0$-continuous, it follows that the space of exact one-forms $\mathrm{Im}(d_{\mathcal{F}_L}) \subset \left(\Omega^1(\mathcal{F}_L), \mathcal{C}^0\right)$ is closed. Therefore, if we take $\gamma \in \Omega^1(\mathcal{F}_L)$ not exact, so that $H^0_\gamma(\mathcal{F}_L) = 0$, then also $\gamma - \pounds_X\alpha$ is not exact for $\mathcal{C}^1$-small α, so that still $H^0_{\gamma-\pounds_X\alpha}(\mathcal{F}_L) = 0$. This shows that, if in the above counterexample we take a generic slope $\lambda \in \mathbb{R} \setminus \mathbb{Q}$, then $\mathcal{C}^1$-small deformations of L do stay inside the singular locus.

The problem in the above counterexample is that the space of exact one-forms in $\Omega^1_{cl}(\mathcal{F}_L)$ is not closed with respect to the Fréchet $\mathcal{C}^\infty$-topology generated by $\mathcal{C}^k$-norms $\{\|\cdot\|_k\}_{k\geq 0}$. Under the additional assumption that $H^1(\mathcal{F}_L)$ is finite dimensional, this problem does not occur, and we obtain the following analog to Corollary 2.5.5.

Proposition 2.5.10. *Let L be a compact, connected Lagrangian whose foliation $\mathcal{F}_L$ has dense leaves. Assume that $H^1(\mathcal{F}_L)$ is finite dimensional and that $\gamma \in \Omega^1_{cl}(\mathcal{F}_L)$ is not exact. There is a neighborhood $\mathcal{V}$ of 0 in $\big(\Gamma(T^*\mathcal{F}_L \times \mathbb{R}), \mathcal{C}^\infty\big)$ such that if Graph(α, f) is Lagrangian for $(\alpha, f) \in \mathcal{V}$, then $f \equiv 0$.*

Proof. Consider the Fréchet space $\big(\Omega^1(\mathcal{F}_L), \mathcal{C}^\infty\big)$ and notice that the space of closed foliated one-forms $\Omega^1_{cl}(\mathcal{F}_L) \subset \big(\Omega^1(\mathcal{F}_L), \mathcal{C}^\infty\big)$ is closed. To see this, note that $d_{\mathcal{F}_L}$ is continuous with respect to the $\mathcal{C}^\infty$-topology and that $\{0\} \subset \big(\Omega^2(\mathcal{F}_L), \mathcal{C}^\infty\big)$ is closed since Fréchet spaces are Hausdorff. Consequently, $\big(\Omega^1_{cl}(\mathcal{F}_L), \mathcal{C}^\infty\big)$ is itself a Fréchet space. Moreover, the space of exact foliated one-forms $\mathrm{Im}(d_{\mathcal{F}_L}) \subset \big(\Omega^1_{cl}(\mathcal{F}_L), \mathcal{C}^\infty\big)$ is a closed subspace. Indeed, by assumption, the range of $d_{\mathcal{F}_L} : \big(C^\infty(L), \mathcal{C}^\infty\big) \to \big(\Omega^1_{cl}(\mathcal{F}_L), \mathcal{C}^\infty\big)$ has finite codimension, so it must be closed because of the open mapping theorem (see [B, Remark 3.2]).

Since γ is not foliated exact, there exists a $\mathcal{C}^\infty$-open neighborhood of γ consisting of non-exact one-forms. By continuity of the map

$$\big(\Omega^1_{cl}(\mathcal{F}_L), \mathcal{C}^\infty\big) \to \big(\Omega^1_{cl}(\mathcal{F}_L), \mathcal{C}^\infty\big) : \alpha \mapsto \gamma - \pounds_X\alpha,$$

we find a $\mathcal{C}^\infty$-open neighborhood $\mathcal{U}$ of 0 in $\Omega^1_{cl}(\mathcal{F}_L)$ such that $\gamma - \pounds_X\alpha$ is not exact for all $\alpha \in \mathcal{U}$. Take a $\mathcal{C}^\infty$-open subset $\mathcal{U}' \subset \Omega^1(\mathcal{F}_L)$ such that $\mathcal{U} = \mathcal{U}' \cap \Omega^1_{cl}(\mathcal{F}_L)$. We now define the $\mathcal{C}^\infty$-neighborhood $\mathcal{V}$ of 0 in $(\Gamma(T^*\mathcal{F}_L \times \mathbb{R}), \mathcal{C}^\infty)$ by

$$\mathcal{V} := \{(\alpha, f) \in \Gamma(T^*\mathcal{F}_L \times \mathbb{R}) : \alpha \in \mathcal{U}'\}.$$

To see that $\mathcal{V}$ satisfies the criteria, let $(\alpha, f) \in \mathcal{V}$ be such that Graph(α, f) is Lagrangian in $(T^*\mathcal{F}_L \times \mathbb{R}, \widetilde{\Pi})$. Then $\alpha \in \mathcal{U}$ and $f \in H^0_{\gamma-\pounds_X\alpha}(\mathcal{F}_L) = \{0\}$, by *ii)* of Theorem 2.4.16. This proves the proposition. $\square$

Since the $\mathcal{C}^\infty$-topology is generated by the increasing family of $\mathcal{C}^k$-norms, every $\mathcal{C}^\infty$-open neighborhood contains a $\mathcal{C}^k$-open neighborhood for some $k \in \mathbb{N}$. So shrinking the neighborhood $\mathcal{V}$ obtained in the above proposition, one can assume that it is a $\mathcal{C}^k$-neighborhood of the zero section, for some (unspecified) $k \in \mathbb{N}$.

2.5.2 Obstructedness of deformations

Recall that a deformation problem governed by a DGLA $(W, d, [\![\cdot,\cdot]\!])$ is called formally/smoothly unobstructed if every closed element $\alpha \in W_1$ – i.e. every first order deformation – can be extended to a formal/smooth curve of Maurer-Cartan elements. A way to detect obstructedness is by means of the Kuranishi map

$$Kr : H^1(W) \to H^2(W) : [\alpha] \mapsto \left[[\![\alpha, \alpha]\!]\right],$$

for if $Kr([\alpha])$ does not vanish, then α is formally (and therefore also smoothly) obstructed [OP, Theorem 11.4].

For the deformation problem of a Lagrangian L^n contained in the singular locus of a log-symplectic manifold (M^{2n}, Z, Π), a first order deformation is a pair $(\alpha_1, f_1) \in \Gamma\left(T^*\mathcal{F}_L \times \mathbb{R}\right)$ such that

$$\begin{cases} d_{\mathcal{F}_L}\alpha_1 = 0 \\ d_{\mathcal{F}_L}f_1 + f_1\gamma = 0 \end{cases}. \tag{2.57}$$

Clearly, first order deformations of the specific form $(\alpha_1, 0)$ or $(0, f_1)$ are smoothly unobstructed, since $s(\alpha_1, 0)$ and $s(0, f_1)$ satisfy the Maurer-Cartan equation (2.25) for all $s \in \mathbb{R}$.

Obstructedness

We show that the above deformation problem is formally obstructed in general. The Kuranishi map of the DGLA $\left(\Gamma\left(\wedge^\bullet\left(T^*\mathcal{F}_L \times \mathbb{R}\right)\right), d, [\![\cdot,\cdot]\!]\right)$ described in Corollary 2.4.10 reads

$$\begin{aligned} Kr : H^1\left(\Gamma\left(\wedge^\bullet(T^*\mathcal{F}_L \times \mathbb{R})\right)\right) \to H^2\left(\Gamma\left(\wedge^\bullet(T^*\mathcal{F}_L \times \mathbb{R})\right)\right) : \\ [(\alpha, f)] \mapsto \left[(0, 2f\pounds_X\alpha)\right], \end{aligned} \tag{2.58}$$

and the following example shows that this map need not be identically zero.

Example 2.5.11 (An obstructed example). Consider the manifold $\mathbb{T}^2 \times \mathbb{R}^2$, regarded as a trivial vector bundle over $\mathbb{T}^2$. Denote its coordinates by $(\theta_1, \theta_2, \xi_1, \xi_2)$ and endow it with a log-symplectic structure Π given by

$$\Pi = \partial_{\theta_1} \wedge \xi_1\partial_{\xi_1} + \partial_{\theta_2} \wedge \partial_{\xi_2}.$$

Note that $L := \mathbb{T}^2 \times \{(0,0)\}$ is a Lagrangian submanifold contained in the singular locus $\mathbb{T}^2 \times \mathbb{R} = \{\xi_1 = 0\}$. It inherits a codimension-one foliation $\mathcal{F}_L$

with tangent distribution $T\mathcal{F}_L = \mathrm{Ker}(d\theta_1)$, so the cotangent bundle $T^*\mathcal{F}_L$ has a global frame given by $d\theta_2$. In the notation established earlier, we now have

$$\begin{cases} \gamma = 0 \\ X = \partial_{\theta_1} \end{cases},$$

and the differential d of the DGLA acts as

$$d : \Gamma\left(T^*\mathcal{F}_L \times \mathbb{R}\right) \to \Gamma\left(\wedge^2(T^*\mathcal{F}_L \times \mathbb{R})\right) : (g d\theta_2, k) \mapsto \left(0, -\frac{\partial k}{\partial \theta_2} d\theta_2\right). \quad (2.59)$$

Since the Kuranishi map (2.58) is given by

$$Kr\Big(\left[(g d\theta_2, f)\right]\Big) = \left[\left(0, 2f\frac{\partial g}{\partial \theta_1} d\theta_2\right)\right],$$

it is clear that

$$\begin{aligned} Kr\Big(\left[(g d\theta_2, f)\right]\Big) = 0 &\Leftrightarrow f\frac{\partial g}{\partial \theta_1} = \frac{\partial k}{\partial \theta_2} \text{ for some } k \in C^\infty(\mathbb{T}^2) \\ &\Leftrightarrow \int_{S^1} f\frac{\partial g}{\partial \theta_1} d\theta_2 = 0. \end{aligned} \quad (2.60)$$

The equation (2.60) is a non-trivial obstruction to the prolongation of infinitesimal deformations. For instance, $\left(\sin(\theta_1)d\theta_2, \cos(\theta_1)\right) \in \Gamma\left(T^*\mathcal{F}_L \times \mathbb{R}\right)$ is an infinitesimal deformation of L since it is closed with respect to the differential (2.59). But it cannot be prolonged to a path of deformations, since the integral (2.60) is nonzero.

Formally unobstructed deformations

It is well-known that a deformation problem is formally unobstructed whenever the second cohomology group of the DGLA governing it vanishes [OP, Theorem 11.2]. Specializing to our situation, say we have a first order deformation (α_1, f_1) as in eq. (2.57) and we wish to prolong it to a formal power series solution $\sum_{k\geq 1}(\alpha_k, f_k)\epsilon^k$ of the Maurer-Cartan equation. So we require that

$$d\left(\sum_{k\geq 1}(\alpha_k, f_k)\epsilon^k\right) + \frac{1}{2}\left[\left[\sum_{k\geq 1}(\alpha_k, f_k)\epsilon^k, \sum_{k\geq 1}(\alpha_k, f_k)\epsilon^k\right]\right] = 0.$$

Collecting all terms in ϵ^n gives

$$d(\alpha_n, f_n) + \frac{1}{2} \sum_{\substack{k,l \geq 1 \\ k+l=n}} [\![(\alpha_k, f_k), (\alpha_l, f_l)]\!] = 0$$

$$\Leftrightarrow (-d_{\mathcal{F}_L}\alpha_n, -d_{\mathcal{F}_L} f_n - f_n\gamma) + \frac{1}{2} \sum_{\substack{k,l \geq 1 \\ k+l=n}} (0, f_l \pounds_X \alpha_k + f_k \pounds_X \alpha_l) = 0$$

$$\Leftrightarrow \begin{cases} d_{\mathcal{F}_L}\alpha_n = 0 \\ d_{\mathcal{F}_L} f_n + f_n\gamma - \frac{1}{2}\sum_{\substack{k,l \geq 1 \\ k+l=n}} (f_l \pounds_X \alpha_k + f_k \pounds_X \alpha_l) = 0 \end{cases} .$$

We can always construct a formal power series solution if $H^1_\gamma(\mathcal{F}_L) = 0$. Concretely, constructing (α_k, f_k) inductively, we can set $\alpha_k = 0$ for $k \geq 2$ and choose f_k such that

$$d_{\mathcal{F}_L} f_k + f_k\gamma = f_{k-1}\pounds_X \alpha_1. \tag{2.61}$$

A quick proof by induction indeed shows that the right hand side of (2.61) is closed with respect to the differential $d^\gamma_{\mathcal{F}_L}$ for each $k \geq 2$. In conclusion, we have proved the following:

Corollary 2.5.12. *If $H^1_\gamma(\mathcal{F}_L) = 0$, then all first order deformations of the Lagrangian L are formally unobstructed.*

Note that this assumption is weaker than requiring that the second cohomology group of the DGLA is zero, since the latter is given by $H^2(\mathcal{F}_L) \oplus H^1_\gamma(\mathcal{F}_L)$.

We will see that the vanishing of $H^1_\gamma(\mathcal{F}_L)$ in fact ensures that the deformation problem is smoothly unobstructed, at least for Lagrangians that are compact and connected.

Smoothly unobstructed deformations: general results

We give a sufficient condition for smooth unobstructedness. When $\pounds_X\alpha$ is foliated exact, we have $H^0_\gamma(\mathcal{F}_L) \cong H^0_{\gamma - \pounds_X\alpha}(\mathcal{F}_L)$. Using this isomorphism, from a solution of the linearized Maurer-Cartan equation (2.57) we can construct a solution of the Maurer-Cartan equation (2.25). This leads to the following result, which we prove with a short direct computation.

Proposition 2.5.13. *If $(\alpha, f) \in \Gamma(T^*\mathcal{F}_L \times \mathbb{R})$ is a first order deformation such that $\pounds_X\alpha$ is foliated exact, then (α, f) is smoothly unobstructed.*

Proof. Let $\pounds_X\alpha = d_{\mathcal{F}_L}h$ for $h \in C^\infty(L)$. We claim that the path

$$s \mapsto (s\alpha, sfe^{sh}) \tag{2.62}$$

is a prolongation of (α, f) consisting of Lagrangian sections for all times s. Indeed, we have the following equivalences:

$$\begin{aligned} & d_{\mathcal{F}_L}(sfe^{sh}) + sfe^{sh}(\gamma - \pounds_X s\alpha) = 0 \\ \Leftrightarrow\ & d_{\mathcal{F}_L}(sfe^{sh}) + sfe^{sh}(\gamma - d_{\mathcal{F}_L}sh) = 0 \\ \Leftrightarrow\ & d_{\mathcal{F}_L}sf + sf\gamma = 0. \end{aligned} \tag{2.63}$$

In the last equivalence we use *i)* of Lemma 2.4.13, which says that the following map is an isomorphism of cochain complexes:

$$\left(\Omega^\bullet(\mathcal{F}_L), d_{\mathcal{F}_L}^{\gamma - d_{\mathcal{F}_L}sh}\right) \to \left(\Omega^\bullet(\mathcal{F}_L), d_{\mathcal{F}_L}^{\gamma}\right) : \beta \mapsto e^{-sh}\beta.$$

The equality (2.63) is satisfied, since (α, f) is a first order deformation. Hence, by Theorem 2.4.3, we know that $(s\alpha, sfe^{sh})$ is indeed a Lagrangian section for each time s. Clearly, the path passes through the zero section at $s = 0$ with velocity (α, f). This proves the claim. □

As a consistency check, we note that a first order deformation (α, f) as in Proposition 2.5.13 maps to zero under the Kuranishi map. By eq. (2.58), we have $Kr([(\alpha, f)]) = [(0, 2f\pounds_X\alpha)]$. If $\pounds_X\alpha = d_{\mathcal{F}_L}h$ for some $h \in C^\infty(L)$, then

$$d_{\mathcal{F}_L}^{\gamma}(f \cdot h) = d_{\mathcal{F}_L}^{\gamma}f \cdot h + f \cdot d_{\mathcal{F}_L}h = f\pounds_X\alpha.$$

Remark 2.5.14. We give a geometric interpretation of Proposition 2.5.13.

i) For a closed foliated one-form $\alpha \in \Omega^1(\mathcal{F}_L)$, exactness of $\pounds_X\alpha$ is equivalent with the existence of a closed one-form $\widetilde{\alpha} \in \Omega^1(L)$ that extends α. Indeed, if $\widetilde{\alpha} \in \Omega^1(L)$ is a closed extension of α and $r : \Omega^1(L) \to \Omega^1(\mathcal{F}_L)$ is the restriction map, then

$$0 = r(\iota_X d\widetilde{\alpha}) = r(\pounds_X\widetilde{\alpha} - d\iota_X\widetilde{\alpha}) = \pounds_X\alpha - d_{\mathcal{F}_L}(\iota_X\widetilde{\alpha}),$$

which shows that $\pounds_X\alpha = d_{\mathcal{F}_L}(\iota_X\widetilde{\alpha})$ is exact. Conversely, assume that $\pounds_X\alpha = d_{\mathcal{F}_L}h$ for some $h \in C^\infty(L)$. Let $\widetilde{\alpha} \in \Omega^1(L)$ be the unique extension of α satisfying $\widetilde{\alpha}(X) = h$. Then $\widetilde{\alpha}$ is closed, since

$$r(\iota_X d\widetilde{\alpha}) = r(\pounds_X\widetilde{\alpha} - d\iota_X\widetilde{\alpha}) = \pounds_X\alpha - d_{\mathcal{F}_L}h = 0.$$

ii) The smooth path[7] of Lagrangian deformations given by (2.62) is obtained by applying certain Poisson diffeomorphisms of $T^*\mathcal{F}_L \times \mathbb{R}$ to the smooth path of Lagrangian sections $s \mapsto (0, sf)$. More precisely, as in item i), assume that $\pounds_X \alpha = d_{\mathcal{F}_L} h$, and let $\widetilde{\alpha} \in \Omega^1(L)$ be the closed one-form extending α determined by $\widetilde{\alpha}(X) = h$. As before, denote by $pr\colon T^*\mathcal{F}_L \times \mathbb{R} \to L$ and $p : T^*\mathcal{F}_L \to \mathbb{R}$ the vector bundle projections. Since $\widetilde{\alpha}$ is a closed one-form, it gives rise to a Poisson vector field on $T^*\mathcal{F}_L \times \mathbb{R}$, namely

$$\widetilde{\Pi}^\sharp(pr^*\widetilde{\alpha}) = (pr^*h)t\partial_t + \Pi^\sharp_{can}(p^*\widetilde{\alpha}).$$

Notice that this vector field is tangent to the fibers of pr, and that the second summand is the constant vector field on the fibers of $T^*\mathcal{F}_L$ with value α. The flow at time s of $\widetilde{\Pi}^\sharp(pr^*\widetilde{\alpha})$ maps $graph(0, sf)$ to $graph(s\alpha, sfe^{sh})$.

In case $\alpha = d_{\mathcal{F}_L} g$ is exact, then we can interpret this construction in terms of the DGLA governing the deformation problem. Indeed, Remark 2.5.23 shows that the gauge action by the degree zero element $(g, 0)$ takes the Maurer-Cartan element $(0, sf)$ to $\big(sd_{\mathcal{F}_L} g, sfe^{sX(g)}\big)$. This is consistent with the above, since $X(g)$ is a primitive of $\pounds_X \alpha$.

Corollary 2.5.15. *If $H^1(\mathcal{F}_L) = 0$, then all first order deformations of the Lagrangian L are smoothly unobstructed.*

Proof. If (α, f) is a first order deformation, then α is closed. Since $H^1(\mathcal{F}_L) = 0$, it is exact. The same then holds for $\pounds_X \alpha$, so Prop. 2.5.13 gives the result. □

Corollary 2.5.15 shows in particular that obstructedness is a global issue, since the cohomology group $H^1(\mathcal{F}_L)$ always vanishes locally.

One may wonder if all first order deformations (α, f) that are smoothly unobstructed arise as in Prop. 2.5.13. The answer is negative, but it becomes positive if we restrict to first order deformations for which $f \in C^\infty(L)$ is nowhere vanishing. We spell this out in the following remark and lemma.

Remark 2.5.16. First order deformations of the form $(\alpha, 0)$, hence $d_{\mathcal{F}_L}\alpha = 0$, are smoothly unobstructed, but in general $\pounds_X \alpha$ is not foliated exact. For instance, consider the log-symplectic manifold $(\mathbb{T}^2 \times \mathbb{R}^2, \Pi)$ and Lagrangian submanifold $L := \mathbb{T}^2 \times \{(0,0)\}$ as in Example 2.5.11, for which the foliation $\mathcal{F}_L$ is one-dimensional. Any $\alpha = g(\theta_1, \theta_2)d\theta_2 \in \Omega^1(\mathcal{F}_L)$ is foliated closed, but in general the integral of α along the fibers of $L \to S^1 : (\theta_1, \theta_2) \to \theta_1$ is not independent of θ_1, implying that $\pounds_X \alpha$ is not foliated exact.

[7] This path can certainly not be obtained by applying Poisson diffeomorphisms to L itself, since the latter preserve the Poisson submanifold $T^*\mathcal{F}_L$.

Lemma 2.5.17. *Let (α, f) be a first order deformation of L that satisfies $Kr\big([(\alpha, f)]\big) = 0$. Assume moreover that $f \in C^\infty(L)$ is nowhere vanishing. Then $\pounds_X \alpha$ is foliated exact.*

Proof. The assumption $Kr\big([(\alpha, f)]\big) = 0$ is equivalent to $[f \pounds_X \alpha] = 0$ in $H^1_\gamma(\mathcal{F}_L)$ by (2.58), so it implies that there exists $g \in C^\infty(L)$ such that

$$f \pounds_X \alpha = d_{\mathcal{F}_L} g + g\gamma.$$

Since f is nowhere zero, we can divide by f and we obtain

$$\begin{aligned} \pounds_X \alpha &= \frac{1}{f} d_{\mathcal{F}_L} g + \frac{g}{f}\gamma \\ &= \frac{1}{f} d_{\mathcal{F}_L} g - \frac{g}{f^2} d_{\mathcal{F}_L} f \\ &= \frac{1}{f} d_{\mathcal{F}_L} g + g d_{\mathcal{F}_L}\left(\frac{1}{f}\right) \\ &= d_{\mathcal{F}_L}\left(\frac{g}{f}\right), \end{aligned} \tag{2.64}$$

using in the second equality that $d^\gamma_{\mathcal{F}_L} f = 0$. Hence $\pounds_X \alpha$ is foliated exact. □

Smoothly unobstructed deformations: the compact case

We now show that for compact connected Lagrangians $(L, \mathcal{F}_L)$, the condition $H^1_\gamma(\mathcal{F}_L) = 0$ from Corollary 2.5.12 in fact implies that the deformation problem is smoothly unobstructed. We actually prove more: one only needs that the Kuranishi map (2.58) is trivial.

Proposition 2.5.18. *Let $(L^n, \mathcal{F}_L)$ be a compact connected Lagrangian submanifold that is contained in the singular locus of a log-symplectic manifold (M^{2n}, Z, Π). A first order deformation $(\alpha, f) \in \Gamma\left(T^*\mathcal{F}_L \times \mathbb{R}\right)$ of L is smoothly unobstructed if and only if $Kr\big([(\alpha, f)]\big) = 0$.*

Proof. We only have to prove the backward implication. Let (α, f) be a first order deformation of L with $Kr\big([(\alpha, f)]\big) = 0$. We know that either the leaves of $\mathcal{F}_L$ are dense, or $(L, \mathcal{F}_L)$ is the foliation by fibers of a fiber bundle over S^1.

i) First assume that the leaves of $\mathcal{F}_L$ are dense.

- If γ is not exact, then $H^0_\gamma(\mathcal{F}_L) = \{0\}$ by Theorem 2.4.16. Since (α, f) is a first order deformation, we have that $f \in H^0_\gamma(\mathcal{F}_L) = \{0\}$. Therefore $(\alpha, f) = (\alpha, 0)$ and a path of Lagrangian sections that prolongs $(\alpha, 0)$ is simply given by $s \mapsto (s\alpha, 0)$.

- Now assume that $\gamma = d_{\mathcal{F}_L} k$ is exact. Thanks to (the proof of) Lemma 2.4.13 *i*), we know that $e^k f$ is constant on L. So either $f \equiv 0$, in which case we conclude that (α, f) is smoothly unobstructed as in the previous bullet point. Or f is nowhere zero, in which case we can use Lemma 2.5.17. There we showed that $\pounds_X \alpha$ is foliated exact, and Proposition 2.5.13 then implies that the first order deformation (α, f) is smoothly unobstructed.

ii) Now assume that $\mathcal{F}_L$ is the fiber foliation of a fiber bundle $p : L \to S^1$. The closed foliated one-form $\pounds_X \alpha$ defines a section $\sigma_{\pounds_X \alpha}$ of the vector bundle $\mathcal{H}^1 \to S^1$ via the correspondence (2.45). By Lemma 2.4.15, we can fix a smooth function $h \in C^\infty(L)$ satisfying

$$(\pounds_X \alpha)|_{p^{-1}(q)} = d\left(h|_{p^{-1}(q)}\right) \qquad \forall q \in \mathcal{Z}_{\pounds_X \alpha},$$

where we denote $\mathcal{Z}_{\pounds_X \alpha} := \sigma^{-1}_{\pounds_X \alpha}(0)$. Mimicking the proof of Proposition 2.5.13, we claim that the path $s \mapsto (s\alpha, sfe^{sh})$ is a prolongation of (α, f) by Lagrangian sections. So we have to show that

$$d_{\mathcal{F}_L}(sfe^{sh}) + sfe^{sh}(\gamma - \pounds_X s\alpha) = 0. \tag{2.65}$$

To do so, we denote $\mathcal{Z}_f := f^{-1}(0) \subset L$. Recall here that $f \in H^0_\gamma(\mathcal{F}_L)$, so that $\mathcal{Z}_f$ is a union of fibers of $p : L \to S^1$ (cf. the proof of Theorem 2.4.16). Clearly, the equality (2.65) holds on $\mathcal{Z}_f$. On the other hand, Lemma 2.5.17 implies that $\pounds_X \alpha$ is exact on $L \setminus \mathcal{Z}_f$. Therefore, $\pounds_X \alpha = d_{\mathcal{F}_L} h$ on $L \setminus \mathcal{Z}_f$, and the computation (2.63) in the proof of Proposition 2.5.13 shows that (2.65) holds on $L \setminus \mathcal{Z}_f$. This finishes the proof. □

Remark 2.5.19. A crucial point in the proof of Prop. 2.5.18 is that h is a smooth function defined on the whole of L. Its existence is guaranteed by Lemma 2.4.15, a statement about fiber bundles over S^1. Due to this, we do not expect the result of Prop. 2.5.18 to hold if one removes the compactness assumption on L.

We give an algorithmic overview of first order deformations and their obstructedness, for Lagrangians that are compact and connected.

i) Assume $(L, \mathcal{F}_L)$ is the foliation by fibers of a fiber bundle $p : L \to S^1$. Fix a smooth function $g \in C^\infty(L)$ that is a primitive of γ on $\mathcal{Z}_\gamma := \sigma_\gamma^{-1}(0)$, as constructed in Lemma 2.4.15. Thanks to Thm. 2.4.16 i) and its proof, we can characterize first order deformations (α, f) of L by the requirements

$$\begin{cases} d_{\mathcal{F}_L}\alpha = 0 \\ e^g f \text{ is constant on each } p\text{-fiber and vanishes on } S^1 \setminus \mathcal{Z}_\gamma \end{cases}.$$

By Prop. 2.5.18, a first order deformation (α, f) of L is smoothly unobstructed exactly when $Kr\big([(\alpha, f)]\big) = 0$. We claim that the latter condition is equivalent to the following:

$$[\pounds_X \alpha] = 0 \in H^1(\mathcal{F}_L) \text{ on } L \setminus \mathcal{Z}_f. \tag{2.66}$$

Here $\mathcal{Z}_f$ denotes the zero locus of f, as in the proof of Prop. 2.5.18.

To see that the two conditions are equivalent, recall that $Kr\big([(\alpha, f)]\big) = 0$ implies the condition (2.66), by Lemma 2.5.17. Conversely, assume that the condition (2.66) holds. As in the proof of Prop. 2.5.18, choose a smooth function $h \in C^\infty(L)$ such that $\pounds_X \alpha = d_{\mathcal{F}_L} h$ on $p^{-1}(\mathcal{Z}_{\pounds_X \alpha})$. In particular, this equality holds on $L \setminus \mathcal{Z}_f$. From this, we conclude that

$$f \pounds_X \alpha = d^{\gamma}_{\mathcal{F}_L}(fh).$$

Indeed, on $\mathcal{Z}_f$ this equation holds because both sides are zero; on the complement $L \setminus \mathcal{Z}_f$ it holds because $d^{\gamma}_{\mathcal{F}_L}(fh) = d^{\gamma}_{\mathcal{F}_L} f \cdot h + f \cdot d_{\mathcal{F}_L} h = f \pounds_X \alpha$. So $[f \pounds_X \alpha] = 0$ in $H^1_\gamma(\mathcal{F}_L)$, which by (2.58) implies that $Kr\big([(\alpha, f)]\big) = 0$.

ii) In case $\mathcal{F}_L$ has dense leaves, then we distinguish between two types of first order deformations. The first type are the ones of the form $(\alpha, 0)$ for closed $\alpha \in \Omega^1(\mathcal{F}_L)$. Clearly, these are smoothly unobstructed.

First order deformations of the second type, those with nonzero second component, can only occur if γ is foliated exact, by Thm. 2.4.16. They are characterized as the (α, f) for which

$$\begin{cases} d_{\mathcal{F}_L} \alpha = 0 \\ e^g f \text{ is a nonzero constant} \end{cases},$$

where $g \in C^\infty(L)$ is a primitive of γ. Such a first order deformation (α, f) is smoothly unobstructed exactly when $[\pounds_X \alpha] = 0$ in $H^1(\mathcal{F}_L)$: the forward implication holds by Lemma 2.5.17, the backward one by Prop. 2.5.13.

Notice that we now showed that the criterion (2.66) for unobstructedness in the fibration case also holds if $\mathcal{F}_L$ has dense leaves: the two types of infinitesimal deformations just described correspond with the extreme cases $L \setminus \mathcal{Z}_f = \emptyset$ and $L \setminus \mathcal{Z}_f = L$.

In conclusion, we have proved the following.

Corollary 2.5.20. *A first order deformation $(\alpha, f) \in \Gamma(T^*\mathcal{F}_L \times \mathbb{R})$ of a compact connected Lagrangian L is smoothly unobstructed exactly when*

$$[\pounds_X \alpha] = 0 \in H^1(\mathcal{F}_L) \text{ on } L \setminus \mathcal{Z}_f. \tag{2.67}$$

Here $\mathcal{Z}_f$ denotes the zero locus of f.

Given a first order deformation (α, f), the condition (2.67) is equivalent with α extending to a closed one-form on $L \setminus \mathcal{Z}_f$, by the argument of Remark 2.5.14 i). Therefore the condition (2.67) is independent of the data (X, γ) coming from the modular vector field.

Example 2.5.21. Consider the manifold $\mathbb{T}^2 \times \mathbb{R}^2$, regarded as a trivial vector bundle over $\mathbb{T}^2$, with coordinates $(\theta_1, \theta_2, \xi_1, \xi_2)$. Let $Z := \mathbb{T}^2 \times \mathbb{R} = \{\xi_1 = 0\}$ and $L := \mathbb{T}^2 \times \{(0,0)\}$.

i) Any orientable log-symplectic structure with singular locus Z so that L is Lagrangian with induced foliation $T\mathcal{F}_L = \mathrm{Ker}(d\theta_1)$, up to Poisson diffeomorphism, looks as follows nearby L:

$$\Pi = V \wedge \xi_1 \partial_{\xi_1} + \partial_{\theta_2} \wedge \partial_{\xi_2},$$

where

$$V = g_X(\theta_1)\partial_{\theta_1} + g_\gamma(\theta_1)\partial_{\xi_2}$$

for some function $g_\gamma \in C^\infty(S^1)$ and some nowhere vanishing function $g_X \in C^\infty(S^1)$. Here we use Corollary 2.2.18, Remark 2.2.19 and Corollary 2.3.5, along with Remark 2.3.7 ii).

We have $\gamma = g_\gamma(\theta_1)d\theta_2$, and a function on L satisfying the properties of Lemma 2.4.15 is the constant function zero. Hence first order deformations are given by pairs (α, f), with the condition that $f = f(\theta_1)$ and $f \cdot g_\gamma = 0$.

To see when such a first order deformation is unobstructed, we apply Corollary 2.5.20. In the case at hand, since the fibers of $p : L \to S^1$ are circles and thanks to Stokes' theorem, the condition (2.67) can be rephrased as:

$$\text{the function } q \mapsto \int_{p^{-1}(q)} \alpha \text{ is locally constant on } p(L \setminus \mathcal{Z}_f) \subset S^1.$$

For instance, in case $g_\gamma = 0$ (as in Ex. 2.5.11), any pair (α, f) with $f = f(\theta_1)$ is a first other deformation. Such a pair is unobstructed exactly when, writing $\alpha = a(\theta_1, \theta_2)d\theta_2$, the expression

$$\int_{\{\theta_1\} \times S^1} a(\theta_1, \theta_2) d\theta_2$$

is constant on connected components of $p(L \setminus \mathcal{Z}_f)$.

ii) Now let $\lambda \in \mathbb{R} \setminus \mathbb{Q}$ be a generic (i.e. not Liouville) irrational number. Any orientable log-symplectic structure with singular locus Z so that L is Lagrangian with induced foliation $T\mathcal{F}_L = \mathrm{Ker}(d\theta_1 - \lambda d\theta_2)$ is Poisson diffeomorphic around L with

$$\Pi = (C\partial_{\theta_1} + K\partial_{\xi_2}) \wedge \xi_1 \partial_{\xi_1} + (\lambda \partial_{\theta_1} + \partial_{\theta_2}) \wedge \partial_{\xi_2}, \tag{2.68}$$

for some $C, K \in \mathbb{R}$ with C nonzero. This follows from a similar reasoning as above, now using that

$$\mathfrak{X}(L)^{\mathcal{F}_L}/\Gamma(T\mathcal{F}_L) \cong H^0(\mathcal{F}_L) = \mathbb{R} \quad \text{and} \quad H^1(\mathcal{F}_L) = \mathbb{R}[d\theta_2].$$

Note that $\gamma = Kd\theta_2$ is exact if and only if $K = 0$. Therefore, first order deformations are given by $(\alpha, 0)$ if $K \neq 0$ and (α, c) if $K = 0$, with $c \in \mathbb{R}$. Clearly, the Lie derivative along $X = C\partial_{\theta_1}$ acts trivially in cohomology, since $H^1(\mathcal{F}_L) = \mathbb{R}[d\theta_2]$. Therefore, all first order deformations of L are smoothly unobstructed, by Corollary 2.5.20.

The situation is different when $\lambda \in \mathbb{R}\backslash\mathbb{Q}$ is a Liouville number. Disregarding trivially unobstructed first order deformations of the form $(\alpha, 0)$, Thm. 2.4.16 ii) implies that the ones with nonzero second component can only occur for log-symplectic structures that are isomorphic around L to the following model:

$$\Pi = C\partial_{\theta_1} \wedge \xi_1\partial_{\xi_1} + (\lambda\partial_{\theta_1} + \partial_{\theta_2}) \wedge \partial_{\xi_2},$$

where $C \in \mathbb{R}_0$. Notice that $H^1(\mathcal{F}_L)$ is now infinite dimensional, and that the Lie derivative along $X = C\partial_{\theta_1}$ no longer acts trivially in cohomology, which is a direct consequence of (the proof of) Lemma 2.5.8. This shows that there exist obstructed first order deformations.

2.5.3 Equivalences of deformations and rigidity

We now consider two natural equivalence relations on the space of Lagrangian deformations: equivalence by Hamiltonian diffeomorphisms and by Poisson isotopies. We show that the action by Hamiltonian diffeomorphisms agrees with the gauge action of the DGLA that governs the deformation problem. We also discuss rigidity of Lagrangians, both for Hamiltonian and Poisson equivalence.

Hamiltonian isotopies

We showed in §2.4.2 that the graph of $(\alpha, f) \in \Gamma(T^*\mathcal{F}_L \times \mathbb{R})$ defines a Lagrangian submanifold of $(U, \widetilde{\Pi})$ exactly when (α, f) is a Maurer-Cartan element of the DGLA $\big(\Gamma(\wedge^\bullet(T^*\mathcal{F}_L \times \mathbb{R})), d, [\![\cdot,\cdot]\!]\big)$. So if we write for short

$$\mathrm{Def}_U(L) := \left\{(\alpha, f) \in \Gamma(U) : \ \mathrm{gr}(\alpha, f) \text{ is Lagrangian inside } \big(U, \widetilde{\Pi}\big)\right\}$$

and

$$\mathrm{MC}_U\big(\Gamma(\wedge^\bullet(T^*\mathcal{F}_L \times \mathbb{R}))\big) := \left\{(\alpha, f) \in MC\big(\Gamma(\wedge^\bullet(T^*\mathcal{F}_L \times \mathbb{R}))\big) : \ \mathrm{gr}(\alpha, f) \subset U\right\},$$

then we have a correspondence

$$\mathrm{Def}_U(L) \overset{1:1}{\longleftrightarrow} \mathrm{MC}_U\big(\Gamma(\wedge^\bullet(T^*\mathcal{F}_L \times \mathbb{R}))\big). \tag{2.69}$$

We now define equivalence relations on both sides of (2.69) and we show that they agree under this correspondence. We closely follow the exposition in [SZ2]. There one considers equivalences of coisotropic submanifolds in symplectic geometry, but most of their results remain valid in the more general setting of fiberwise entire Poisson structures.

Definition 2.5.22. i) Two Lagrangian sections $(\alpha_0, f_0), (\alpha_1, f_1) \in \mathrm{Def}_U(L)$ are *Hamiltonian equivalent* if they are interpolated by a smooth family $\{(\alpha_s, f_s)\}_{s\in[0,1]}$ of Lagrangian sections in $\mathrm{Def}_U(L)$ that is generated by a (locally defined) Hamiltonian isotopy. In other words, there exists a time-dependent Hamiltonian vector field X_{H_s} on U such that the associated isotopy ϕ_s maps graph(α_0, f_0) to graph(α_s, f_s), for all $s \in [0, 1]$.

ii) Two MC elements $(\alpha_0, f_0), (\alpha_1, f_1) \in \mathrm{MC}_U\big(\Gamma(\wedge^\bullet(T^*\mathcal{F}_L \times \mathbb{R}))\big)$ are *gauge equivalent* if they are interpolated by a smooth family $\{(\alpha_s, f_s)\}_{s\in[0,1]}$ of sections whose graph lies inside U, and there exists a smooth family $\{g_s\}_{s\in[0,1]}$ of functions on L such that

$$\begin{aligned}\frac{d}{ds}(\alpha_s, f_s) &= [\![(g_s, 0), (\alpha_s, f_s)]\!] - d(g_s, 0)\\ &= \big(d_{\mathcal{F}_L} g_s, f_s \pounds_X g_s\big).\end{aligned} \tag{2.70}$$

Remark 2.5.23. By solving the flow equation (2.70), we obtain an explicit description for the gauge action of the DGLA. Namely, a path of degree zero elements $(g_s, 0)$ acts on a Maurer-Cartan element (α_0, f_0), which yields a path of Maurer-Cartan elements (α_s, f_s) given by

$$(\alpha_s, f_s) = \left(\alpha_0 + d_{\mathcal{F}_L}\left(\int_0^s g_u du\right), f_0 \exp\left(\pounds_X \int_0^s g_u du\right)\right). \tag{2.71}$$

We rewrite the gauge equivalence relation in more geometric terms.

Lemma 2.5.24. *Two MC elements* $(\alpha_0, f_0), (\alpha_1, f_1) \in MC_U\big(\Gamma(\wedge^\bullet(T^*\mathcal{F}_L \times \mathbb{R}))\big)$ *are gauge equivalent if and only if they are interpolated by a smooth family* $\{(\alpha_s, f_s)\}_{s\in[0,1]}$ *of sections whose graph lies inside* U*, and there exists a smooth family* $\{g_s\}_{s\in[0,1]}$ *of functions on* L *such that*

$$\frac{d}{ds}(\alpha_s, f_s) = X_{pr^* g_s}\big|_{graph(\alpha_s, f_s)}. \tag{2.72}$$

Here $pr : U \subset T^*\mathcal{F}_L \times \mathbb{R} \to L$ *denotes the bundle projection, and we see* (2.72) *as an equality of sections of the vertical bundle restricted to* $graph(\alpha_s, f_s)$*.*

Proof. We compute the Hamiltonian vector field $X_{pr^*g_s}$. As before, we denote by $p : T^*\mathcal{F}_L \to L$ the bundle projection. We obtain

$$\begin{aligned} X_{pr^*g_s} &= ((V_{vert} + V_{lift}) \wedge t\partial_t + \Pi_{can})^\sharp (dpr^*g_s) \\ &= pr^*(\pounds_X g_s)t\partial_t + \Pi^\sharp_{can}(p^*dg_s), \end{aligned} \tag{2.73}$$

and therefore

$$X_{pr^*g_s}|_{graph(\alpha_s,f_s)} = pr^*(f_s\pounds_X g_s)\partial_t + \Pi^\sharp_{can}(p^*dg_s).$$

The section of $T^*\mathcal{F}_L \times \mathbb{R}$ corresponding with this vertical fiberwise constant vector field is $\big(d_{\mathcal{F}_L}g_s, f_s\pounds_X g_s\big) \in \Gamma(T^*\mathcal{F}_L \times \mathbb{R})$, in agreement with (2.70). ☐

We need some technical results from [SZ2]. We state them here for convenience.

Lemma 2.5.25. *Let $A \to M$ be a vector bundle with vertical bundle V. Let X_s be one-parameter family of vector fields on A with flow ϕ_s, and let τ_0 be a section of A.*

i) If τ_s is a one-parameter family of sections of A such that the requirement $graph(\tau_s) = \phi_s(graph(\tau_0))$ holds for all $s \in [0,1]$, then τ_s satisfies

$$\frac{d}{ds}\tau_s = P_{\tau_s}X_s \qquad \forall s \in [0,1].$$

Here P_{τ_s} denotes the vertical projection with respect to the splitting $TA|_{graph(\tau_s)} = Tgraph(\tau_s) \oplus V|_{graph(\tau_s)}$.

ii) Conversely, assume that the integral curves of X_s starting at points of $graph(\tau_0)$ exist for all times $s \in [0,1]$, and suppose that τ_s is a one-parameter family of sections of A satisfying

$$\frac{d}{ds}\tau_s = P_{\tau_s}X_s \qquad \forall s \in [0,1].$$

Here P_{τ_s} is defined as above. Then the family of submanifolds $graph(\tau_s)$ coincides with $\phi_s(graph(\tau_0))$ for all $s \in [0,1]$.

Making some minor modifications to the proofs of [SZ2, Proposition 3.18] and [SZ2, Proposition 3.19], we can show that Hamiltonian equivalence coincides with gauge equivalence.

Proposition 2.5.26. *The bijection between Lagrangian sections and Maurer-Cartan elements*

$$Def_U(L) \to MC_U\big(\Gamma(\wedge^\bullet(T^*\mathcal{F}_L \times \mathbb{R}))\big) : (\alpha, f) \mapsto (\alpha, f)$$

descends to a bijection

$$Def_U(L)/\sim_{Ham} \to MC_U\big(\Gamma(\wedge^\bullet(T^*\mathcal{F}_L \times \mathbb{R}))\big)/\sim_{gauge}.$$

Proof. First assume that the Lagrangian sections $(\alpha_0, f_0), (\alpha_1, f_1) \in \mathrm{Def}_U(L)$ are Hamiltonian equivalent. Then they are interpolated by a smooth family of sections $(\alpha_s, f_s) \in \mathrm{Def}_U(L)$ generated by the flow ϕ_s of a time-dependent Hamiltonian vector field $X_{H_s} \in \mathfrak{X}(U)$. Part *i)* of Lemma 2.5.25 then implies that

$$\frac{d}{ds}(\alpha_s, f_s) = P_{(\alpha_s, f_s)} X_{H_s} \tag{2.74}$$

for all $s \in [0,1]$. Define $g_s := H_s \circ (\alpha_s, f_s) \in C^\infty(L)$ and observe that $H_s - pr^* g_s$ vanishes along $\mathrm{graph}(\alpha_s, f_s)$. Because $\mathrm{graph}(\alpha_s, f_s)$ is coisotropic, this implies that the Hamiltonian vector field $X_{H_s - pr^* g_s} = X_{H_s} - X_{pr^* g_s}$ is tangent to $\mathrm{graph}(\alpha_s, f_s)$. Consequently, the equality (2.74) becomes

$$\frac{d}{ds}(\alpha_s, f_s) = P_{(\alpha_s, f_s)} X_{pr^* g_s} = X_{pr^* g_s}|_{graph(\alpha_s, f_s)},$$

where we also used that $X_{pr^* g_s}$ is vertical (which is clear from the expression (2.73)). Lemma 2.5.24 implies that (α_0, f_0) and (α_1, f_1) are gauge equivalent.

Conversely, assume that $(\alpha_0, f_0), (\alpha_1, f_1) \in \mathrm{MC}_U\big(\Gamma(\wedge^\bullet(T^*\mathcal{F}_L \times \mathbb{R}))\big)$ are gauge equivalent. By Lemma 2.5.24, this means that they are interpolated by a smooth family of sections (α_s, f_s) inside U, such that

$$\frac{d}{ds}(\alpha_s, f_s) = X_{pr^* g_s}|_{graph(\alpha_s, f_s)} = P_{(\alpha_s, f_s)} X_{pr^* g_s} \qquad \forall s \in [0,1],$$

for a smooth family of functions $g_s \in C^\infty(L)$. In particular, the integral curve of $X_{pr^* g_s}$ starting at a point $(\alpha_0, f_0)(p) \in \mathrm{graph}(\alpha_0, f_0)$ is defined up to time 1, and is given by $(\alpha_s, f_s)(p)$ for $s \in [0,1]$. Part *ii)* of Lemma 2.5.25 gives $\phi_s(\mathrm{graph}(\alpha_0, f_0)) = \mathrm{graph}(\alpha_s, f_s)$ for all $s \in [0,1]$, where ϕ_s is the flow of $X_{pr^* g_s}$. This shows that (α_0, f_0) and (α_1, f_1) are Hamiltonian equivalent. □

Remark 2.5.27. The above proof is almost identical to the one presented in [SZ2]. The main difference is that in [SZ2], one needs to impose compactness on the coisotropic submanifold to obtain that gauge equivalence implies Hamiltonian equivalence, as otherwise the flow lines of $X_{pr^* g_s}$ need not be defined for long enough time. Since in our setting Hamiltonian vector fields of basic functions are vertical, we don't need this additional assumption.

As a consequence, we obtain that the formal tangent space at zero to the moduli space $\mathcal{M}_U^{Ham}(L) := \mathrm{Def}_U(L)/\sim_{Ham}$ can be identified with the first cohomology group of the differential graded Lie algebra $\big(\Gamma\left(\wedge^\bullet\left(T^*\mathcal{F}_L \times \mathbb{R}\right)\right), d, [\![\cdot,\cdot]\!]\big)$:

$$T_{[0]}\mathcal{M}_U^{Ham}(L) = H^1(\mathcal{F}_L) \oplus H^0_\gamma(\mathcal{F}_L). \tag{2.75}$$

Indeed, if (α_s, f_s) is a path of Lagrangian deformations of L, then $\frac{d}{ds}|_{s=0}(\alpha_s, f_s)$ is closed with respect to the differential d of the DGLA. Moreover, if the path

(α_s, f_s) is generated by the flow of a time-dependent Hamiltonian vector field, then (α_s, f_s) is obtained by gauge transforming the zero section, as we just proved. The expression (2.71) then shows that $\frac{d}{ds}|_{s=0}(\alpha_s, f_s)$ is of the form $(d_{\mathcal{F}_L} g, 0)$ for $g \in C^\infty(L)$. This proves the assertion (2.75).

Smoothness of the moduli space by Hamiltonian isotopies

In general, the moduli space $\mathcal{M}_U^{Ham}(L)$ is by no means smooth, since the formal tangent spaces at different points can be drastically different. For instance, let us look again at Example 2.5.11, where we considered $(\mathbb{T}^2 \times \mathbb{R}^2, \theta_1, \theta_2, \xi_1, \xi_2)$ with log-symplectic structure

$$\Pi = \partial_{\theta_1} \wedge \xi_1 \partial_{\xi_1} + \partial_{\theta_2} \wedge \xi_2$$

and Lagrangian $L = \mathbb{T}^2 \times \{(0,0)\}$. The induced foliation on L is the fiber foliation of $(L, \theta_1, \theta_2) \to (S^1, \theta_1)$. Since $\gamma = 0$, we get for any nonzero constant $c \in \mathbb{R}$ a Lagrangian section $(0, c) \in \Gamma(T^*\mathcal{F}_L \times \mathbb{R})$ whose graph lies outside the singular locus. Hence, by symplectic geometry, we have

$$T_{[(0,c)]}\mathcal{M}_U^{Ham}(L) \cong H^1(graph(0,c)) \cong H^1(L) = \mathbb{R}^2,$$

which is finite dimensional. On the other hand, we have

$$T_{[0]}\mathcal{M}_U^{Ham}(L) = H^1(\mathcal{F}_L) \oplus H^0_\gamma(\mathcal{F}_L) \cong H^1(\mathcal{F}_L) \oplus H^0(\mathcal{F}_L) \cong C^\infty(S^1) \oplus C^\infty(S^1),$$

which is infinite dimensional.

There are however instances in which the moduli space is locally smooth. Suppose a Lagrangian submanifold L^n contained in the singular locus Z has the property that $\mathcal{C}^1$-small Lagrangian deformations of L stay inside Z. This means that the $\mathcal{C}^1$-small deformations are precisely the graphs of $\mathcal{C}^1$-small elements of $\Omega^1_{cl}(\mathcal{F}_L)$. Then $\mathcal{M}_U^{Ham}(L)$ is naturally isomorphic to an open neighborhood of the origin in $H^1(\mathcal{F}_L)$, by Corollary 2.2.11. In particular, $\mathcal{M}_U^{Ham}(L)$ is smooth. We present two classes of examples.

i) A class of Lagrangians L as above are those satisfying the assumptions of Corollary 2.5.5. In that case $\mathcal{M}_U^{Ham}(L)$ is infinite-dimensional. Indeed, recall that $H^1(\mathcal{F}_L) \cong \Gamma(\mathcal{H}^1)$; if this was finite-dimensional, then $\mathcal{H}^1$ would be of rank zero, which implies that $H^1(\mathcal{F}_L) = 0$. Then γ would be exact, which is impossible under the assumptions of Corollary 2.5.5.

ii) Another class of Lagrangians L as above are those that are $\mathcal{C}^1$-rigid under Poisson equivalences (see later on), since Poisson diffeomorphisms of the ambient log-symplectic manifold necessarily preserve Z. In that case $\mathcal{M}_U^{Ham}(L)$ is finite-dimensional by Lemma 2.5.31, assuming L is compact and connected. We exhibit concrete examples of such L in Example 2.5.36. Notice that Proposition 2.5.34 as stated does not quite provide examples, since it makes a statement only about $\mathcal{C}^\infty$-small deformations.

Rigidity and Hamiltonian isotopies

At this point, we would like to address some rigidity phenomena. A Lagrangian L is called *infinitesimally rigid* under Hamiltonian equivalence if the formal tangent space $T_{[0]}\mathcal{M}_U^{Ham}(L)$ is zero. We call a Lagrangian L *rigid* under Hamiltonian equivalence if small deformations of L are Hamiltonian equivalent with L. It turns out however that Hamiltonian equivalence is too restrictive for rigidity purposes: there are no Lagrangians that are infinitesimally rigid. Indeed, if the formal tangent space $T_{[0]}\mathcal{M}_U^{Ham}(L) = H^1(\mathcal{F}_L) \oplus H^0_\gamma(\mathcal{F}_L)$ is zero, then the triviality of the first summand implies that γ is foliated exact. But then $H^0(\mathcal{F}_L) = H^0_\gamma(\mathcal{F}_L) = \{0\}$ by Lemma 2.4.13 *i*), which is impossible. This is a motivation to look at a more flexible notion of equivalence.

Poisson isotopies

We will use flows of Poisson vector fields instead of Hamiltonian vector fields to obtain a less restrictive equivalence relation on the space of Lagrangian deformations of L.

Definition 2.5.28. Two Lagrangian sections $(\alpha_0, f_0), (\alpha_1, f_1) \in \mathrm{Def}_U(L)$ are called *Poisson equivalent* if they are interpolated by a smooth family (α_s, f_s) of Lagrangian sections in $\mathrm{Def}_U(L)$ that is generated by a (locally defined) Poisson isotopy. In other words, there exists a time-dependent Poisson vector field Y_s on U such that the associated isotopy ϕ_s maps $\mathrm{graph}(\alpha_0, f_0)$ to $\mathrm{graph}(\alpha_s, f_s)$, for all $s \in [0, 1]$.

We denote the moduli space $\mathrm{Def}_U(L)/{\sim_{Poiss}}$ of Lagrangian deformations under Poisson equivalence by $\mathcal{M}_U^{Poiss}(L)$. In order to study rigidity under Poisson equivalence, we want to compute the formal tangent space $T_{[0]}\mathcal{M}_U^{Poiss}(L)$, as done in (2.75) for Hamiltonian equivalence. We now quotient first order deformations of L by elements of the form $\frac{d}{ds}|_{s=0}(\alpha_s, f_s)$, where (α_s, f_s) is generated by the flow of a time-dependent Poisson vector field $Y_s \in \mathfrak{X}(U)$. Lemma 2.5.25 *i*) implies that

$$\left.\frac{d}{ds}\right|_{s=0} (\alpha_s, f_s) = P(Y_0), \tag{2.76}$$

where $P : \mathfrak{X}(U) \to \Gamma(T^*\mathcal{F}_L \times \mathbb{R})$ is the restriction to L composed with the vertical projection induced by the splitting $\big(T(T^*\mathcal{F}_L \times \mathbb{R})\big)|_L = TL \oplus (T^*\mathcal{F}_L \times \mathbb{R})$.

So we have to take a closer look at (vertical components of) Poisson vector fields on $U \subset T^*\mathcal{F}_L \times \mathbb{R}$.

Lemma 2.5.29. *Given the Poisson structure $\widetilde{\Pi} = V \wedge t\partial_t + \Pi_{can}$ defined on the neighborhood $U \subset T^*\mathcal{F}_L \times \mathbb{R}$, the following map is an isomorphism:*

$$H^1(L) \oplus H^0(L) \to H^1_{\widetilde{\Pi}}(U) : \big([\xi], g\big) \mapsto \Big[\widetilde{\Pi}^\sharp(pr^*\xi) + (pr^*g)V\Big],$$

where $pr : U \to L$ is the projection.

We remark that the existence of the isomorphism follows from known facts: L is a deformation retract of U and of $U \cap T^*\mathcal{F}_L$, and both cohomology groups appearing above are isomorphic to the first b-cohomology group of the pair $(U, U \cap T^*\mathcal{F}_L)$, by [Me] and [MO, Prop. 1] respectively.

Proof. Clearly, the map is well-defined. To check injectivity, we assume that $\widetilde{\Pi}^\sharp(pr^*\xi) + (pr^*g)V = \widetilde{\Pi}^\sharp(dh)$ for some $h \in C^\infty(U)$. Taking the restriction to $W := U \cap \{t = 0\}$, this implies that $\Pi^\sharp_{can}(p^*\xi) + (p^*g)V$ is tangent to the symplectic leaves, where $p : W \to L$ is the projection. Since V is transverse to the leaves, we get that $p^*g = 0$, and therefore $g = 0$. This means that $\widetilde{\Pi}^\sharp(pr^*\xi) = \widetilde{\Pi}^\sharp(dh)$, and since $\widetilde{\Pi}$ is invertible away from $W \subset U$, we get that $pr^*\xi = dh$ on $U \setminus W$. By continuity, $pr^*\xi = dh$ on all of U, so that $\xi = d(i_L^*h)$ is exact. This shows that the map is injective.

To prove surjectivity, we use some b-symplectic geometry. The b-symplectic form ω on U obtained by inverting $\widetilde{\Pi}$ reads [O, Proposition 4.1.2]

$$\omega = -\widetilde{\Pi}^{-1} = q^*\theta \wedge \frac{dt}{t} + q^*\eta,$$

where $q : U \to W$ is the projection and $(\theta, \eta) \in \Omega^1(W) \times \Omega^2(W)$ is the cosymplectic structure corresponding with the pair (Π_{can}, V). If $Y \in \mathfrak{X}(U)$ is a Poisson vector field, then Y is tangent to W, so we can evaluate

$$\omega^\flat(Y) = q^*\langle\theta, Y|_W\rangle\frac{dt}{t} + \left[\left(\langle q^*\theta, Y\rangle - q^*\langle\theta, Y|_W\rangle\right)\frac{dt}{t} - \left\langle\frac{dt}{t}, Y\right\rangle q^*\theta + \iota_Y q^*\eta\right], \tag{2.77}$$

which is a closed b-one form on U. Note indeed that the summand between square brackets is a smooth de Rham form since $q^*\langle\theta, Y|_W\rangle - \langle q^*\theta, Y\rangle$ vanishes along the hypersurface $W \leftrightarrow t = 0$ and Y is tangent to it. Invoking the Mazzeo-Melrose isomorphism [GMP2], [MO]

$${}^bH^1(U) \to H^1(U) \oplus H^0(W) : \left[q^*(h)\frac{dt}{t} + \beta\right] \mapsto ([\beta], h),$$

we know that the one-form

$$\beta := \left(\langle q^*\theta, Y\rangle - q^*\langle\theta, Y|_W\rangle\right)\frac{dt}{t} - \left\langle\frac{dt}{t}, Y\right\rangle q^*\theta + \iota_Y q^*\eta$$

appearing in (2.77) is closed, and that $h := \langle \theta, Y|_W \rangle$ is locally constant. We now have

$$\begin{aligned} Y &= -\widetilde{\Pi}^\sharp(\omega^\flat(Y)) \\ &= q^*\langle \theta, Y|_W\rangle V + \widetilde{\Pi}^\sharp\left((q^*\langle \theta, Y|_W\rangle - \langle q^*\theta, Y\rangle)\frac{dt}{t} + \left\langle \frac{dt}{t}, Y\right\rangle q^*\theta - \iota_Y q^*\eta \right) \\ &= (q^*h)V + \widetilde{\Pi}^\sharp(-\beta) \end{aligned} \tag{2.78}$$

We make sure that the neighborhood U is such that the map $i_L \circ pr : U \to U$ induces the identity map in cohomology. This means that $q^*h = pr^*(i_L^* q^* h)$ and $\beta - pr^*(i_L^*\beta)$ is exact. So if we put $\xi := -i_L^*\beta$ and $g := i_L^* q^* h$, then it follows from (2.78) that

$$[Y] = \left[\widetilde{\Pi}^\sharp(pr^*\xi) + (pr^*g)V\right] \in H^1_{\widetilde{\Pi}}(U). \qquad \square$$

Proposition 2.5.30. *The formal tangent space $T_{[0]}\mathcal{M}_U^{Poiss}(L)$ is given by*

$$T_{[0]}\mathcal{M}_U^{Poiss}(L) = \frac{\Omega^1_{cl}(\mathcal{F}_L)}{Im\,(r : \Omega^1_{cl}(L) \to \Omega^1_{cl}(\mathcal{F}_L)) + H^0(L)\cdot\gamma} \oplus H^0_\gamma(\mathcal{F}_L), \tag{2.79}$$

where the map r is restriction of closed one-forms on L to the leaves of $\mathcal{F}_L$.

Proof. Throughout, for all vector bundles appearing, we denote by P the map that restricts vector fields to the zero section, and then takes their vertical component. Because of (2.76), we have to show that the denominator appearing in (2.79) is equal to

$$\{P(Y_0) : Y_s \in \mathfrak{X}(U) \text{ time-dependent Poisson vector field}\}.$$

Notice that the above set lies in $\Omega^1(\mathcal{F}_L)$, since all Poisson vector fields on U are tangent to $W := U \cap \{t = 0\}$. For one inclusion, let Y_0 be a Poisson vector field on U. Using the fact that Y_0 is tangent to W and Lemma 2.5.29, we have

$$\begin{aligned} P(Y_0) &= P\left(Y_0|_W\right) \\ &= P\left(\Pi^\sharp_{can}(p^*\xi) + (p^*g)V + \Pi^\sharp_{can}(dh)\right) \end{aligned} \tag{2.80}$$

for some $\xi \in \Omega^1_{cl}(L), g \in H^0(L)$ and $h \in C^\infty(W)$. Here $p : W \to L$ is the projection. Now note that

$$P\left(\Pi^\sharp_{can}(dh)\right) = P\left(\Pi^\sharp_{can}(p^* d i_L^* h)\right).$$

Indeed, since L is coisotropic and $h - p^* i_L^* h$ vanishes along L, we have that $\Pi^\sharp_{can}(d(h - p^* i_L^* h))$ is tangent to L. So (2.80) becomes

$$\begin{aligned} P(Y_0) &= P\left(\Pi^\sharp_{can}(p^*\xi) + (p^*g)V + \Pi^\sharp_{can}(p^* d i_L^* h)\right) \\ &= r(\xi + d i_L^* h) + g\gamma, \end{aligned}$$

using the correspondence (2.20) in the last equality. This proves one inclusion.

For the reverse inclusion, given $\xi \in \Omega^1_{cl}(L)$ and $g \in H^0(L)$, we get a Poisson vector field

$$\widetilde{\Pi}^\sharp(pr^*\xi) + (pr^*g)V \in \mathfrak{X}(U),$$

and its vertical component along L is

$$P\left(\widetilde{\Pi}^\sharp(pr^*\xi) + (pr^*g)V\right) = P\left(\Pi^\sharp_{can}(p^*\xi) + (p^*g)V\right) = r(\xi) + g\gamma. \qquad \square$$

Smoothness of the moduli space by Poisson isotopies

The moduli space $\mathcal{M}_U^{Poiss}(L)$ is not smooth in general, since its formal tangent space can change drastically from point to point. For instance, let us again look at $(\mathbb{T}^2 \times \mathbb{R}^2, \theta_1, \theta_2, \xi_1, \xi_2)$ with log-symplectic structure

$$\Pi = \partial_{\theta_1} \wedge \xi_1 \partial_{\xi_1} + \partial_{\theta_2} \wedge \xi_2$$

and Lagrangian $L = \mathbb{T}^2 \times \{(0,0)\}$. Consider again a Lagrangian section $(0, c) \in \Gamma(T^*\mathcal{F}_L \times \mathbb{R})$ for nonzero $c \in \mathbb{R}$; its graph lies outside the singular locus. By symplectic geometry, $[(0, c)]$ is an isolated point in the moduli space $\mathcal{M}_U^{Poiss}(L)$ and therefore

$$T_{[(0,c)]}\mathcal{M}_U^{Poiss}(L) = 0.$$

On the other hand, we have

$$\begin{aligned} T_{[0]}\mathcal{M}_U^{Poiss}(L) &= \frac{\Omega^1_{cl}(\mathcal{F}_L)}{\operatorname{Im}\left(r : \Omega^1_{cl}(L) \to \Omega^1_{cl}(\mathcal{F}_L)\right)} \oplus H^0(\mathcal{F}_L) \\ &\cong \frac{C^\infty(\mathbb{T}^2)}{\left\{f \in C^\infty(\mathbb{T}^2) : \frac{\partial}{\partial\theta_1}\left(\int_{S^1} f d\theta_2\right) = 0\right\}} \oplus C^\infty(S^1), \end{aligned}$$

which is infinite dimensional. In the computation, we used Remark 2.5.14 i).

Rigidity and Poisson isotopies

We now address rigidity of Lagrangians under the equivalence relation by Poisson isotopies. As in the case of Hamiltonian equivalence, we call a Lagrangian L *infinitesimally rigid* under Poisson equivalence if the formal tangent space $T_{[0]}\mathcal{M}_U^{Poiss}(L)$ is zero. A Lagrangian L is called *rigid* under Poisson equivalence if small deformations of L are Poisson equivalent with L. Rigidity is a very restrictive property: since Poisson diffeomorphisms fix the singular locus of the log-symplectic structure, a Lagrangian L can only be rigid if small deformations of it stay inside the singular locus.

We will restrict ourselves to Lagrangians L that are compact and connected. It turns out that asking for infinitesimal rigidity under Poisson equivalence is only a little weaker than asking for infinitesimal rigidity under Hamiltonian equivalence, as the next lemma shows.

Lemma 2.5.31. *Let L be a compact, connected Lagrangian that is infinitesimally rigid under Poisson equivalence. Then $H^1(\mathcal{F}_L)$ is finite dimensional.*

Proof. Since L is compact, we know that $H^1(L)$ is finite dimensional. Choose a basis $\{[\beta_1], \ldots, [\beta_k]\}$ of $H^1(L)$. If $\alpha \in \Omega^1_{cl}(\mathcal{F}_L)$ is a closed foliated one-form, then infinitesimal rigidity implies that $\alpha = r(\widetilde{\alpha}) + c\gamma$ for some $\widetilde{\alpha} \in \Omega^1_{cl}(L)$ and $c \in \mathbb{R}$. Since $\widetilde{\alpha}$ can be written as $\widetilde{\alpha} = c_1\beta_1 + \cdots + c_k\beta_k + dh$ for some $c_1, \ldots, c_k \in \mathbb{R}$ and $h \in C^\infty(L)$, we get

$$\alpha = c_1 r(\beta_1) + \cdots + c_k r(\beta_k) + d_{\mathcal{F}_L} h + c\gamma.$$

So $H^1(\mathcal{F}_L)$ is spanned by $\{[r(\beta_1)], \ldots, [r(\beta_k)], [\gamma]\}$, hence finite dimensional. □

This implies that Lagrangians L for which $\mathcal{F}_L$ is the foliation by fibers of a fiber bundle over S^1 are never rigid, not even infinitesimally.

Corollary 2.5.32. *If L is a compact Lagrangian for which $\mathcal{F}_L$ is the foliation by fibers of a fiber bundle $p : L \to S^1$, then L is not infinitesimally rigid under Poisson equivalence.*

Proof. Assume to the contrary that L is infinitesimally rigid. By Lemma 2.5.31, we know that $H^1(\mathcal{F}_L) \cong \Gamma(\mathcal{H}^1)$ is finite dimensional. So $\mathcal{H}^1$ has to be of rank zero, which implies that $H^1(\mathcal{F}_L) = 0$. Consequently, γ is exact, and Theorem 2.4.16 *i)* ensures that $H^0_\gamma(\mathcal{F}_L)$ is nonzero. This contradicts that the infinitesimal moduli space (2.79) is zero. So L cannot be infinitesimally rigid. □

Remark 2.5.33. Alternatively, one can obtain Corollary 2.5.32 by using the flat connection ∇ on $\mathcal{H}^1$, which was defined in (2.46). Assuming that L is infinitesimally rigid, fix an open $U \subset S^1$ and a frame $\{\sigma_{\eta_1}, \ldots, \sigma_{\eta_m}\}$ for $\mathcal{H}^1|_U$

consisting of flat sections. If $\alpha \in \Omega^1_{cl}(\mathcal{F}_L)$, then infinitesimal rigidity implies that $\alpha = r(\tilde{\alpha}) + c\gamma$ for some $\tilde{\alpha} \in \Omega^1_{cl}(L)$ and $c \in \mathbb{R}$. Note that the section $\sigma_{r(\tilde{\alpha})} \in \Gamma(\mathcal{H}^1)$ is flat, since for all $Y \in \mathfrak{X}(S^1)$ we have

$$\nabla_Y \sigma_{r(\tilde{\alpha})} = \sigma_{r\left(\pounds_{\overline{Y}}\tilde{\alpha}\right)} = \sigma_{d_{\mathcal{F}_L}\iota_{\overline{Y}}\tilde{\alpha}} = 0,$$

where we used Cartan's magic formula. It follows that

$$\sigma_\alpha|_U = c_1\sigma_{\eta_1} + \cdots + c_m\sigma_{\eta_m} + c\sigma_\gamma|_U$$

for constants $c_1, \ldots, c_k, c \in \mathbb{R}$. This means that necessarily $H^1(\mathcal{F}_L) = 0$, and we obtain a contradiction as in the proof of Corollary 2.5.32.

So fibrations over S^1 don't give examples of rigid Lagrangians. However, if the foliation $\mathcal{F}_L$ on L has dense leaves, then we do obtain an interesting rigidity statement: infinitesimal rigidity implies rigidity with respect to the $\mathcal{C}^\infty$-topology.

Proposition 2.5.34. *Let L be a compact, connected Lagrangian whose induced foliation $\mathcal{F}_L$ has dense leaves. Assume that L is infinitesimally rigid under Poisson equivalence. Then there exists a neighborhood $\mathcal{V} \subset (\Gamma(T^*\mathcal{F}_L \times \mathbb{R}), \mathcal{C}^\infty)$ of 0 such that if Graph(α, f) is Lagrangian for $(\alpha, f) \in \mathcal{V}$, then (α, f) is Poisson equivalent with the zero section of $T^*\mathcal{F}_L \times \mathbb{R}$.*

Proof. Infinitesimal rigidity implies that $H^0_\gamma(\mathcal{F}_L) = 0$, so γ is not foliated exact by ii) of Thm. 2.4.16. Moreover, $H^1(\mathcal{F}_L)$ is finite dimensional by Lemma 2.5.31. By Prop. 2.5.10, we obtain a neighborhood $\mathcal{V} \subset (\Gamma(T^*\mathcal{F}_L \times \mathbb{R}), \mathcal{C}^\infty)$ of 0 such that if Graph(α, f) is Lagrangian for $(\alpha, f) \in \mathcal{V}$, then $f \equiv 0$. To show that $\mathcal{V}$ satisfies the criteria, we distinguish between two cases.

Case 1: γ extends to a closed one-form on L.

The assumption of infinitesimal rigidity then implies that

$$\Omega^1_{cl}(\mathcal{F}_L) = \mathrm{Im}\big(r : \Omega^1_{cl}(L) \to \Omega^1_{cl}(\mathcal{F}_L)\big).$$

So if $(\alpha, f) = (\alpha, 0) \in \mathcal{V}$ is such that the graph of $(\alpha, 0) \in \Gamma(T^*\mathcal{F}_L \times \mathbb{R})$ is Lagrangian, then we have $\alpha = r(\widetilde{\alpha})$ for some $\widetilde{\alpha} \in \Omega^1_{cl}(L)$. The time 1-flow of the Poisson vector field $\widetilde{\Pi}^\sharp(pr^*\widetilde{\alpha})$ then takes L to Graph$(\alpha, 0)$.

Case 2: γ does not extend to a closed one-form on L.

In this case, infinitesimal rigidity implies that $\left(\Omega^1_{cl}(\mathcal{F}_L), \mathcal{C}^\infty\right)$ splits into an algebraic direct sum

$$\Omega^1_{cl}(\mathcal{F}_L) = \mathrm{Im}\big(r : \Omega^1_{cl}(L) \to \Omega^1_{cl}(\mathcal{F}_L)\big) \oplus \mathbb{R}\gamma. \tag{2.81}$$

Since r is $\mathcal{C}^\infty$-continuous, linear and $\mathrm{Im}(r) \subset \Omega^1_{cl}(\mathcal{F}_L)$ is of finite codimension, we get that $\mathrm{Im}(r) \subset \left(\Omega^1_{cl}(\mathcal{F}_L), \mathcal{C}^\infty\right)$ is closed. This implies that (2.81) is in fact a topological direct sum: $\mathbb{R}\gamma$ is an algebraic complement to a maximal closed subspace, and therefore a topological complement [NB, Theorem 4.9.5]. So the projection onto the second summand of (2.81) is continuous, and therefore we get a continuous map

$$p_2 : \left(\Omega^1_{cl}(\mathcal{F}_L), \mathcal{C}^\infty\right) \to \mathbb{R} : r(\widetilde{\alpha}) + c\gamma \mapsto c.$$

Therefore, shrinking the neighborhood $\mathcal{V}$ constructed above if necessary, we can assume that $p_2(\pounds_X \alpha) < 1$ when $(\alpha, f) = (\alpha, 0) \in \mathcal{V}$ is a Lagrangian section.

Now let $(\alpha, f) = (\alpha, 0) \in \mathcal{V}$ be such that the graph of $(\alpha, 0) \in \Gamma(T^*\mathcal{F}_L \times \mathbb{R})$ is Lagrangian. We decompose α and $\pounds_X \alpha$ in the direct sum (2.81):

$$\begin{cases} \alpha = r(\xi) + C\gamma \\ \pounds_X \alpha = r(\eta) + K\gamma \end{cases} \tag{2.82}$$

for $\xi, \eta \in \Omega^1_{cl}(L)$ and $C, K \in \mathbb{R}$ with $K < 1$. We define smooth families $\xi_s \in \Omega^1_{cl}(L)$ and $C_s \in \mathbb{R}$ for $s \in [0,1]$ by the formulas

$$\xi_s := \xi + \frac{C}{1 - sK} s\eta, \qquad C_s := \frac{C}{1 - sK}. \tag{2.83}$$

Note that the denominator $1 - sK$ occurring in these expressions is never zero for $s \in [0,1]$ since $K < 1$. We claim that the isotopy ϕ_s generated by the time-dependent Poisson vector field $\widetilde{\Pi}^\sharp(pr^*\xi_s) + C_s V$ takes the zero section of $T^*\mathcal{F}_L \times \mathbb{R}$ to $graph(\alpha, 0)$, or more precisely, that $\phi_s(L) = graph(s\alpha, 0)$ for $s \in [0,1]$. To prove this, by [SZ2, Lemma 3.15] it is enough to check that

$$\frac{d}{ds}(s\alpha, 0) = P_{(s\alpha,0)}\left(\widetilde{\Pi}^\sharp(pr^*\xi_s) + C_s V\right), \tag{2.84}$$

where $P_{(s\alpha,0)}$ denotes the vertical projection induced by the direct sum decomposition of $T(T^*\mathcal{F}_L \times \mathbb{R})|_{graph(s\alpha,0)}$ into $Tgraph(s\alpha, 0)$ and the vertical bundle along $graph(s\alpha, 0)$. The right hand side of (2.84) is equal to

$$\begin{aligned} P_{s\alpha}\left(\Pi^\sharp_{can}(p^*\xi_s) + C_s V\right) &= r(\xi_s) + C_s P_{s\alpha}(V_{vert} + V_{lift}) \\ &= r(\xi_s) + C_s(\gamma - \pounds_X(s\alpha)) \\ &= r\left(\xi + \frac{C}{1 - sK} s\eta\right) + \frac{C}{1 - sK}\left(\gamma - sr(\eta) - sK\gamma\right) \\ &= r(\xi) + \frac{Cs}{1 - sK} r(\eta) + \frac{C(1 - sK)}{1 - sK}\gamma - \frac{Cs}{1 - sK} r(\eta) \\ &= r(\xi) + C\gamma. \end{aligned}$$

Here we used the correspondence (2.20) in the first equality, Lemma 2.5.35 below in the second equality and the expressions (2.82), (2.83) in the third equality. Since $\alpha = r(\xi) + C\gamma$ by (2.82), we now showed that the equality (2.84) holds. This finishes the proof. □

By definition of the $\mathcal{C}^\infty$-topology, one can rephrase this proposition as follows: infinitesimal rigidity of L implies $\mathcal{C}^k$-rigidity of L for some $k \in \mathbb{N}$.

Lemma 2.5.35. *Let $\alpha \in \Gamma(T^*\mathcal{F}_L)$, and denote by P_α the vertical projection induced by the splitting of $T(T^*\mathcal{F}_L)|_{graph(\alpha)}$ into $Tgraph(\alpha)$ and the vertical bundle along $graph(\alpha)$. We then have*

$$P_\alpha(V_{lift}) = -\pounds_X \alpha.$$

Proof. Denote by $\phi^{-\alpha}$ the translation map

$$\phi^{-\alpha} : T^*\mathcal{F}_L \to T^*\mathcal{F}_L : (p, \xi) \mapsto (p, \xi - \alpha(p)),$$

and let $P := P_0$ be the vertical projection along the zero section. We then have a commutative diagram

$$\begin{array}{ccc} T(T^*\mathcal{F}_L)|_{graph(\alpha)} & \xrightarrow{(\phi^{-\alpha})_*} & T(T^*\mathcal{F}_L)|_L \\ & \searrow^{P_\alpha} & \downarrow P \\ & & \Gamma(T^*\mathcal{F}_L) \end{array},$$

so the lemma follows immediately from the equality (2.27). □

Example 2.5.36. Let $L = (\mathbb{T}^2, \theta_1, \theta_2)$, endowed with the Kronecker foliation $T\mathcal{F}_L = \mathrm{Ker}(d\theta_1 - \lambda d\theta_2)$ for generic (i.e. not Liouville) $\lambda \in \mathbb{R} \setminus \mathbb{Q}$. Let ξ be the fiber coordinate on $T^*\mathcal{F}_L$ corresponding with the frame $\{d\theta_2\}$. As in eq. (2.68), we take a log-symplectic structure

$$\left(T^*\mathcal{F}_L \times \mathbb{R}, \widetilde{\Pi} := (C\partial_{\theta_1} + K\partial_\xi) \wedge t\partial_t + (\lambda\partial_{\theta_1} + \partial_{\theta_2}) \wedge \partial_\xi\right),$$

where now $C, K \in \mathbb{R}$ are both nonzero. Since for generic $\lambda \in \mathbb{R} \setminus \mathbb{Q}$, we have $H^1(\mathcal{F}_L) = \mathbb{R}[d\theta_2]$, it is clear that every element of $\Omega^1_{cl}(\mathcal{F}_L)$ extends to a closed one-form on L. Moreover, since $\gamma = Kd\theta_2$ is not exact, we have that $H^0_\gamma(\mathcal{F}_L) = 0$ by Theorem 2.4.16 *ii*). So L is infinitesimally rigid:

$$T_{[0]}\mathcal{M}^{Poiss}_U(L) = \frac{\Omega^1_{cl}(\mathcal{F}_L)}{\mathrm{Im}\,(r : \Omega^1_{cl}(L) \to \Omega^1_{cl}(\mathcal{F}_L)) + \mathbb{R}\gamma} \oplus H^0_\gamma(\mathcal{F}_L) = 0,$$

and therefore L is $\mathcal{C}^\infty$-rigid, by Proposition 2.5.34.

In this particular example, we in fact know a bit more. We already noted in Remark 2.5.9 that $\mathcal{C}^1$-small deformations of L stay inside the singular locus, i.e. they are of the form $(\alpha, f) = (\alpha, 0) \in \Gamma(T^*\mathcal{F}_L \times \mathbb{R})$ for $\alpha \in \Omega^1_{cl}(\mathcal{F}_L)$. Along with the fact that foliated closed one-forms extend to closed one-forms on L, this implies that the Lagrangian L is $\mathcal{C}^1$-rigid under Poisson equivalence. For if $\widetilde{\alpha} \in \Omega^1_{cl}(L)$ is a closed extension of α, then the flow of the Poisson vector field $\widetilde{\Pi}^\sharp(pr^*\widetilde{\alpha})$ takes L to $graph(\alpha, 0)$.

If instead we take $\lambda \in \mathbb{R}\backslash\mathbb{Q}$ to be a Liouville number, then L is not infinitesimally rigid by Lemma 2.5.31, since in that case $H^1(\mathcal{F}_L)$ is infinite dimensional.

2.6 Appendix

We list some facts about Liouville numbers and Fréchet spaces. We also prove that a codimension-one foliation defined by a closed one-form on a compact, connected manifold either has dense leaves, or is given by a fibration over S^1.

2.6.1 Liouville numbers

We collect some facts about Liouville numbers that are used in §2.5.1.

Definition 2.6.1. A Liouville number is a real number $\alpha \in \mathbb{R}$ with the property that, for all integers $p \geq 1$, there exist integers $m_p, n_p \in \mathbb{Z}$ such that $n_p > 1$ and

$$0 < \left| \alpha - \frac{m_p}{n_p} \right| < \frac{1}{n_p^p}.$$

Liouville numbers are irrational (even transcendental, see [M, Theorem 4.5]).

Remark 2.6.2. For any sequence $(m_p, n_p)_{p\in\mathbb{N}}$ as in Definition 2.6.1, the set of denominators $\{n_p : p \in \mathbb{N}\}$ is unbounded. Indeed, assume to the contrary that this set is bounded by some constant M. Since $n_p > 1$, the sequence $(m_p/n_p)_{p\in\mathbb{N}}$ converges to α. As there are only finitely many fractions a/b such that $1 < b \leq M$ and a/b lies within distance 1 of α, the sequence $(m_p/n_p)_{p\in\mathbb{N}}$ must have a constant subsequence. This subsequence must also converge to α, which yields $\alpha \in \mathbb{Q}$. This contradiction shows that $\{n_p : p \in \mathbb{N}\}$ is unbounded.

The following statement is used in the proof of Lemma 2.5.8. It appears (without proof) in [B].

Lemma 2.6.3. *If α is a Liouville number, then for each integer $p \geq 1$, there exists a pair of integers $(m_p, n_p) \in \mathbb{Z}^2$ such that*

$$|m_p + \alpha n_p| \leq \frac{1}{(|m_p| + |n_p|)^p}.$$

Proof. Since α is Liouville, we can fix a sequence (M_p, N_p) for integers $p \geq 1$, satisfying

$$0 < \left|\alpha - \frac{M_p}{N_p}\right| < \frac{1}{N_p^p}, \qquad N_p \geq 2.$$

The sequence $(M_p/N_p)_{p\in\mathbb{N}}$ is convergent hence bounded, so there exists an integer $k \geq 1$ such that

$$|M_p| \leq 2^k N_p, \qquad \forall p \geq 1. \tag{2.85}$$

Notice that

$$\left|\alpha - \frac{M_{(k+2)p}}{N_{(k+2)p}}\right| < \frac{1}{N_{(k+2)p}^{(k+2)p}} = \frac{1}{N_{(k+2)p}^p \cdot N_{(k+2)p}^{(k+1)p}} \leq \frac{1}{N_{(k+2)p}^p \cdot 2^{(k+1)p}}. \tag{2.86}$$

Since the function $x \mapsto x^p$ is convex on $(0, \infty)$, we have

$$\left(\frac{|N_{(k+2)p}| + |M_{(k+2)p}|}{2}\right)^p \leq \frac{|N_{(k+2)p}|^p + |M_{(k+2)p}|^p}{2},$$

and therefore

$$\begin{aligned}\left(|N_{(k+2)p}| + |M_{(k+2)p}|\right)^p &\leq 2^{p-1}\left(|N_{(k+2)p}|^p + |M_{(k+2)p}|^p\right)\\ &\leq 2^p \max\left(|N_{(k+2)p}|^p, |M_{(k+2)p}|^p\right)\\ &\leq 2^p \cdot 2^{kp}|N_{(k+2)p}|^p\\ &= 2^{(k+1)p}|N_{(k+2)p}|^p,\end{aligned} \tag{2.87}$$

using (2.85) in the third inequality. Combining (2.86) with (2.87) gives

$$\left|\alpha - \frac{M_{(k+2)p}}{N_{(k+2)p}}\right| < \frac{1}{\left(|N_{(k+2)p}| + |M_{(k+2)p}|\right)^p}.$$

Replacing M_p by $-M_p$, this implies that

$$\left|M_{(k+2)p} + \alpha N_{(k+2)p}\right| < \frac{N_{(k+2)p}}{\left(|N_{(k+2)p}| + |M_{(k+2)p}|\right)^p} \leq \frac{1}{\left(|N_{(k+2)p}| + |M_{(k+2)p}|\right)^{p-1}}.$$

So if we set $(m_p, n_p) := \big(M_{(k+2)(p+1)}, N_{(k+2)(p+1)}\big)$, then we have

$$|m_p + \alpha n_p| < \frac{1}{(|m_p| + |n_p|)^p}.$$

□

Remark 2.6.4. The proof of Lemma 2.6.3 shows that we can make the additional assumptions $n_p \geq p$ and $(m_p, n_p) \neq (m_q, n_q)$ for $p \neq q$. Indeed, since the set of denominators $\{N_p : p \in \mathbb{N}\}$ of the sequence $(M_p, N_p)_{p\in\mathbb{N}}$ is unbounded, we can ensure that $N_p \geq p$. For if $N_p < p$, then we know that there exists $p' > p$ such that the element $(M_{p'}, N_{p'})$ satisfies $N_{p'} \geq p$. We then have

$$\left|\alpha - \frac{M_{p'}}{N_{p'}}\right| < \frac{1}{N_{p'}^{p'}} < \frac{1}{N_{p'}^{p}},$$

so we can just replace (M_p, N_p) by $(M_{p'}, N_{p'})$. It then follows that

$$n_p = N_{(k+2)(p+1)} \geq (k+2)(p+1) \geq p.$$

In the same spirit, we can make sure that $N_p \neq N_q$ for $q \neq p$, which implies that also $(m_p, n_p) \neq (m_q, n_q)$.

2.6.2 Fréchet spaces

We recall some basic facts about Fréchet spaces, which are used in §2.5.1 and §2.5.3. For more details, see for instance [Ham].

Definition 2.6.5. A Fréchet space is a topological vector space X that satisfies the following three properties:

i) X is Hausdorff.

ii) The topology on X is induced by a countable family of seminorms $\{\|\cdot\|_k\}_{k\geq 0}$.

iii) X is complete.

By item ii), a base of neighborhoods of $x \in X$ is given by subsets of the form

$$\mathcal{B}_r^{k_1}(x) \cap \cdots \cap \mathcal{B}_r^{k_n}(x)$$

for $n \in \mathbb{N}$ and $r > 0$, where $\mathcal{B}_r^{k_j}(x)$ denotes the open ball

$$\mathcal{B}_r^{k_j}(x) = \{y \in X : \|y - x\|_{k_j} < r\}.$$

A sequence x_n converges to x if and only if $\|x_n - x\|_k$ converges to zero $\forall k \geq 0$.

Example 2.6.6. If L is compact, the space of sections of any vector bundle over L becomes a Fréchet space when endowed with the $\mathcal{C}^\infty$-topology generated by $\mathcal{C}^k$-norms $\|\cdot\|_k$. We recall the construction of such norms in the situation that is of interest to us. Let $(L, \mathcal{F}_L)$ be a compact manifold endowed with a codimension-one foliation; we will define $\mathcal{C}^k$-norms on the space $\Omega^\bullet(\mathcal{F}_L) = \Gamma(\wedge^\bullet(T^*\mathcal{F}_L))$ of foliated forms of fixed degree. Fix a finite cover $\{U_1, \ldots, U_m\}$ of L consisting of foliated charts with coordinates $(x_1, \ldots, x_{n-1}, x_n)$, such that plaques of $\mathcal{F}_L$ are level sets of x_n. Choose open subsets V_i for $i = 1, \ldots, m$ that still cover L and have compact closures satisfying $\overline{V_i} \subset U_i$. The k-norm of a foliated form $\eta \in \Omega^l(\mathcal{F}_L)$ with coordinate representation

$$\eta|_{U_j} = \sum_{1 \leq i_1 < \cdots < i_l \leq n-1} g^j_{i_1 \cdots i_l} dx_{i_1} \wedge \cdots \wedge dx_{i_l}$$

is then

$$\|\eta\|_k = \sum_{1 \leq j \leq m} \sum_{1 \leq i_1 < \cdots < i_l \leq n-1} \sum_{|\alpha| \leq k} \sup_{p \in \overline{V_j}} \left| \frac{\partial^\alpha g^j_{i_1 \cdots i_l}}{\partial x^\alpha}(p) \right|.$$

Recall also that a closed subspace of a Fréchet space is itself a Fréchet space. Finally, it is useful to note that, if X and Y are vector spaces whose topologies are generated by families of seminorms $\{\|\cdot\|_k\}$ and $\{\|\cdot\|'_k\}$ respectively, then a linear map $L : X \to Y$ is continuous if and only if for every $k \in \mathbb{N}$, there exist $n_1, \ldots, n_l \in \mathbb{N}$ and $C \in \mathbb{R}$ such that

$$\|L(x)\|'_k \leq C \sum_{j=1}^{l} \|x\|_{n_j}.$$

2.6.3 On foliations defined by a closed one-form

This subsection is devoted to proving the following theorem.

Theorem 2.6.7. *Let M be a compact, connected manifold with a codimension-one foliation $\mathcal{F}$ defined by a closed one-form $\theta \in \Omega^1(M)$. Then either $(M, \mathcal{F})$ is the fiber foliation of a fiber bundle over S^1, or all leaves of $\mathcal{F}$ are dense.*

The proof presented below was made before we became aware of the reference [C, Thm. 9.3.13], which contains a proof of Theorem 2.6.7. We decided to keep our proof in the thesis, because our argument deviates from the proof of [C, Thm. 9.3.13] in some places (more precisely, Case 2 in our proof of Theorem 2.6.7 below is dealt with in a different way).

Assuming the setup of Theorem 2.6.7, fix a vector field $X \in \mathfrak{X}(M)$ satisfying $\theta(X) = 1$. Then X is an infinitesimal automorphism of $\mathcal{F}$, since for any

$Y \in \Gamma(T\mathcal{F})$ we have

$$\theta\left([X,Y]\right) = X(\theta(Y)) - Y(\theta(X)) - d\theta(X,Y) = 0.$$

So the flow $\phi : \mathbb{R} \times M \to M$ of X, which is globally defined by compactness of M, carries leaves of $\mathcal{F}$ to leaves of $\mathcal{F}$. We first show that this action is transitive.

Lemma 2.6.8. *For any leaf $\sigma \in \mathcal{F}$, the set $\mathcal{U}_\sigma := \bigcup_{t\in\mathbb{R}} \phi_t(\sigma)$ is open.*

Proof. It suffices to show that every point $p \in \sigma$ has a neighborhood U_p contained in $\mathcal{U}_\sigma$. Indeed, then any point $q \in \phi_t(\sigma)$ has a neighborhood $\phi_t\left(U_{\phi_{-t}(q)}\right)$ contained in $\mathcal{U}_\sigma$. So let $p \in \sigma$ and choose a foliated chart $(V, x_1, \ldots, x_n)$ centered at p so that plaques are given by $x_n = c$. Consider the map ψ given by

$$\psi : O \to \mathbb{R} : t \mapsto x_n(\phi_t(p)),$$

defined on an open neighborhood $O \subset \mathbb{R}$ of 0. Notice that

$$\left.\frac{d}{dt}\right|_{t=0} \psi(t) = d_p x_n\left(X(p)\right) \neq 0,$$

since X is transverse to the x_n-fibers. By the inverse function theorem, ψ maps a neighborhood of $0 \in O$ diffeomorphically onto a neighborhood $]-\epsilon, \epsilon[$ of $0 \in \mathbb{R}$. It follows that $x_n^{-1}\left(]-\epsilon,\epsilon[\right)$ is an open neighborhood of p lying inside $\mathcal{U}_\sigma$. ☐

Corollary 2.6.9. *For each leaf $\sigma \in \mathcal{F}$, we have $\mathcal{U}_\sigma = M$.*

Proof. By Lemma 2.6.8, we have a decomposition into open subsets

$$M = \bigcup_{\sigma\in\mathcal{F}} \mathcal{U}_\sigma. \tag{2.88}$$

Now notice that for leaves $\sigma, \sigma' \in \mathcal{F}$, one has either $\mathcal{U}_\sigma \cap \mathcal{U}_{\sigma'} = \emptyset$ or $\mathcal{U}_\sigma = \mathcal{U}_{\sigma'}$. If there would exist $\sigma, \sigma' \in \mathcal{F}$ for which $\mathcal{U}_\sigma \cap \mathcal{U}_{\sigma'} = \emptyset$, then out of (2.88) we would get a separation of M

$$M = \mathcal{U}_\sigma \bigsqcup \left(\bigcup_{\tau\in\mathcal{F}:\ \mathcal{U}_\tau \neq \mathcal{U}_\sigma} \mathcal{U}_\tau \right),$$

which contradicts that M is connected. Hence for all leaves $\sigma, \sigma' \in \mathcal{F}$ we have $\mathcal{U}_\sigma = \mathcal{U}_{\sigma'}$, so that the equality (2.88) yields the conclusion of the corollary. ☐

Proof of Theorem 2.6.7. Fix a leaf $\sigma \in \mathcal{F}$, and consider the set of periods

$$\Gamma := \{t \in \mathbb{R} : \phi_t(\sigma) = \sigma\}. \tag{2.89}$$

Since Γ is a subgroup of $(\mathbb{R},+)$, it is either discrete or dense.

<u>Case 1:</u> $\Gamma \subset \mathbb{R}$ is dense.

We first show that σ is dense. Pick $p \in M$ and let U be an open containing p. Thanks to Corollary 2.6.9, there exists $t' \in \mathbb{R}$ and $q \in \sigma$ such that $\phi_{t'}(q) = p$. Now consider the map

$$\Psi : \mathbb{R} \to M : t \mapsto \phi_t(q).$$

Since $\Gamma \subset \mathbb{R}$ is dense, the open neighborhood $\Psi^{-1}(U)$ of t' contains an element $s \in \Gamma$. It then follows that $\Psi(s) \in U \cap \sigma$. This shows that σ is dense in M. Then also any other leaf $\phi_t(\sigma)$ is dense, since

$$M = \phi_t(M) = \phi_t\left(\overline{\sigma}\right) \subset \overline{\phi_t(\sigma)}.$$

<u>Case 2:</u> $\Gamma \subset \mathbb{R}$ is discrete.

We first argue that $\Gamma \neq \{0\}$. Assuming to the contrary that $\Gamma = \{0\}$, the map

$$\psi : \sigma \times \mathbb{R} \to M : (p,t) \mapsto \phi_t(p) \tag{2.90}$$

is injective. Since ψ is surjective by Corollary 2.6.9, it is a bijection. Moreover, we claim that ψ is an immersion. To see this, assume $(v,w) \in T_p\sigma \times T_t\mathbb{R}$ lies in the kernel of $d_{(p,t)}\psi$. Take a curve $(\alpha(s), t+sw)$ in $\sigma \times \mathbb{R}$ passing through (p,t) at time $s=0$ with velocity (v,w). Then

$$\begin{aligned}
0 &= d_{(p,t)}\psi(v,w) \\
&= \left.\frac{d}{ds}\right|_{s=0} \phi_{t+sw}(\alpha(s)) \\
&= d_p\phi_t\left(\left.\frac{d}{ds}\right|_{s=0} \phi_{sw}(p) + \left.\frac{d}{ds}\right|_{s=0} \alpha(s)\right) \\
&= d_p\phi_t\left(wX(p)+v\right),
\end{aligned}$$

so that $wX(p)+v=0$. Since the second summand is tangent to σ and the first is transverse to it, both must be zero. Since $X(p) \neq 0$, we obtain that $v=w=0$, showing that ψ is indeed an immersion. It then follows that ψ is a diffeomorphism, being a bijective immersion. But this is impossible, since M is compact and $\sigma \times \mathbb{R}$ is not. We now proved that $\Gamma \neq \{0\}$.

Then Γ has a minimal positive generator a, i.e. $\Gamma = \{na : n \in \mathbb{Z}\}$. Reconsidering the surjection ψ defined in (2.90), we remark that

$$\phi_t(p) = \phi_{t'}(p') \Leftrightarrow \begin{cases} t' = t + na \\ p = (\phi_a)^n(p') \end{cases} \quad \text{for some } n \in \mathbb{Z}.$$

This makes us define a $\mathbb{Z}$-action on $\sigma \times \mathbb{R}$ by

$$n \cdot (p,t) = \big((\phi_a)^{-n}(p), t + na\big).$$

This action is free and proper, since the action on the second factor is free and proper. So the quotient is a smooth manifold, and the map ψ induces a well-defined bijection

$$\overline{\psi} : \frac{\sigma \times \mathbb{R}}{\mathbb{Z}} \to M : [(p,t)] \mapsto \phi_t(p). \tag{2.91}$$

The map $\overline{\psi}$ is smooth because its composition $\overline{\psi} \circ \pi = \psi$ with the surjective submersion $\pi : \sigma \times \mathbb{R} \to (\sigma \times \mathbb{R})/\mathbb{Z}$ is smooth. Even more, $\overline{\psi}$ is in fact a diffeomorphism, being a bijective submersion. To see that $\overline{\psi}$ is indeed a submersion, it suffices to remark that ψ is a submersion, or equivalently, that ψ is an immersion, which we already showed above.

Next, we get a well-defined projection

$$pr : \frac{\sigma \times \mathbb{R}}{\mathbb{Z}} \to \frac{\mathbb{R}}{a\mathbb{Z}} \cong S^1 : [(p,t)] \mapsto t \text{ mod } a\mathbb{Z}. \tag{2.92}$$

This is a smooth surjective submersion, since it is obtained by passing the smooth surjective submersion $\sigma \times \mathbb{R} \to S^1 : (p,t) \mapsto t$ mod $a\mathbb{Z}$ to the quotient $(\sigma \times \mathbb{R})/\mathbb{Z}$. Composing the diffeomorphism (2.91) with the projection (2.92) yields a surjective submersion

$$pr \circ \left(\overline{\psi}\right)^{-1} : M \to \frac{\mathbb{R}}{a\mathbb{Z}} \cong S^1, \tag{2.93}$$

which is proper since M is compact and S^1 is Hausdorff. Hence by Ehresmann's lemma, the map (2.93) is a locally trivial fibration, and the fiber over t mod $a\mathbb{Z}$ is the leaf $\phi_t(\sigma)$. This finishes the proof. □

Bibliography

[B] M. Bertelson, *Remarks on a Künneth formula for foliated de Rham cohomology*, Pacific J. Math. **252**(2), p. 257-274, 2011.

[CC] A. Candel and L. Conlon, *Foliations I*, Graduate Studies in Mathematics **23**, American Mathematical Society, Providence RI, 2000.

[CF] A.S. Cattaneo and G. Felder, *Relative formality theorem and quantisation of coisotropic submanifolds*, Adv. Math. **208**(2), p. 521-548, 2007.

[CZ] A.S. Cattaneo and M. Zambon, *Coisotropic embeddings in Poisson manifolds*, Trans. Amer. Math. Soc. **361**(7), p. 3721-3746, 2009.

[C] L. Conlon, *Differentiable manifolds: a first course*, Birkhäuser Advanced Texts (Basler Lehrbücher) **5**, Birkhäuser Boston, 1993.

[CM1] M. Crainic and I. Mărcuţ, *On the existence of symplectic realizations*, J. Symplectic Geom. **9**(4), p. 435-444, 2011.

[CM2] M. Crainic and I. Mărcuţ, *Reeb-Thurston stability for symplectic foliations*, Math. Ann. **363**(1-2), p. 217-235, 2015.

[CCS] M. Crampin, F. Cantrijn and W. Sarlet, *Lifting geometric objects to a cotangent bundle, and the geometry of the cotangent bundle of a tangent bundle*, J. Geom. Phys. **4**(4), p. 469-492, 1987.

[DI] M. Datta and M.R. Islam, *Smooth maps of a foliated manifold in a symplectic manifold*, Proc. Indian Acad. Sci. (Math. Sci.) **191**(3), p. 333-343, 2009.

[D] D.M. de Diego, *Lie-Poisson integrators*, preprint arXiv:1803.01427, 2018.

[DDFP] R. Donagi, B. Dubrovin, E. Frenkel and E. Previato, *Integrable Systems and Quantum Groups: Lectures given at the 1st Session of the Centro Internazionale Matematico Estivo (C.I.M.E.) held in Montecatini Terme, Italy, June 14-22, 1993*, Lecture Notes in Mathematics **1620**, Springer, 1996.

[DZ] J.-P. Dufour and N.T. Zung, *Poisson Structures and their Normal Forms*, Progress in Mathematics **242**, Birkhäuser Basel, 2005.

[FM] P. Frejlich and I. Mărcuţ, *The normal form theorem around Poisson transversals*, Pacific J. Math. **287**(2), p. 371-391, 2017.

[GLSW] M.J. Gotay, R. Lashof, J. Sniatycki and A. Weinstein, *Closed forms on symplectic fibre bundles*, Comment. Math. Helv. **58**, p. 617-621, 1983.

[GU] J. Grabowski and P. Urbanski, *Tangent lifts of Poisson and related structures*, J. Phys. A: Math. Gen. **28**(23), p. 6743-6777, 1995.

[GMP1] V. Guillemin, E. Miranda and A.R. Pires, *Codimension one symplectic foliations and regular Poisson structures*, Bull. Braz. Math. Soc. **42**(4), p. 607-623, 2011.

[GMP2] V. Guillemin, E. Miranda and A.R. Pires, *Symplectic and Poisson geometry on b-manifolds*, Adv. Math. **264**, p. 864-896, 2014.

[Hae] A. Haefliger, *Some remarks on foliations with minimal leaves*, J. Differential Geom. **15**(2), p. 269-284, 1980.

[HR] S. Haller and T. Rybicki, *On the Group of Diffeomorphisms Preserving a Locally Conformal Symplectic Structure*, Ann. Global Anal. Geom. **17**(5), p. 475-502, 1999.

[Ham] R. Hamilton, *The inverse function theorem of Nash and Moser*, Bull. Amer. Math. Soc. **7**(1), p. 65-222, 1982.

[Hi] M. W. Hirsch, *Differential Topology*, Graduate Texts in Mathematics **33**, Springer-Verlag, 1976.

[K] C. Kirchhoff-Lukat, *Aspects of Generalized Geometry: Branes with Boundary, Blow-ups, Brackets and Bundles*, Ph.D. Thesis, University of Cambridge, 2018.

[Ma] M. Manetti, *On some formality criteria for DG-Lie algebras*, J. Algebra **438**, p. 90-118, 2015.

[MO] I. Mărcuţ and B. Osorno Torres, *Deformations of log-symplectic structures*, J. London Math. Soc. (2) **90**(1), p. 197-212, 2014.

[Me] R.B. Melrose, *The Atiyah-Patodi-Singer index theorem*, Research Notes in Mathematics **4**, A.K. Peters, Wellesley, 1993.

[MM] I. Moerdijk and J. Mrčun, *Introduction to Foliations and Lie Groupoids*, Cambridge Studies in Advanced Mathematics **91**, Cambridge University Press, 2003.

[M] R.A. Molin, *Advanced Number Theory with Applications*, Discrete Mathematics and its Applications, CRC Press (Taylor and Francis Group), 2009.

[MS] C. Moore and C. Schochet, *Global analysis on foliated spaces*, Second Edition, Mathematical Sciences Research Institute Publications **9**, Cambridge University Press, 2006.

[NB] L. Narici and E. Beckenstein, *Topological Vector Spaces*, Second Edition, Pure and Applied Mathematics: a series of monographs and textbooks, CRC Press (Taylor and Francis Group), 2010.

[NT] R. Nest and B. Tsygan, *Formal deformations of symplectic manifolds with boundary*, J. Reine Angew. Math. **1996**(481), p. 27-54, 1996.

[OP] Y.-G. Oh and J.-S. Park, *Deformations of coisotropic submanifolds and strong homotopy Lie algebroids*, Invent. Math. **161**(2), p. 287-360, 2005.

[O] B. Osorno Torres, *Codimension-one symplectic foliations: constructions and examples*, Ph.D. thesis, Utrecht University, 2015.

[R] O. Radko, *A classification of topologically stable Poisson structures on a compact oriented surface*, J. Symplectic Geom. **1**(3), p. 523-542, 2001.

[S] F. Schätz, *Coisotropic submanifolds and the BFV-complex*, Ph.D. thesis, University of Zürich, 2009.

[SZ1] F. Schätz and M. Zambon, *Deformations of coisotropic submanifolds for fibrewise entire Poisson structures*, Lett. Math. Phys. **103**(7), p. 777-791, 2013.

[SZ2] F. Schätz and M. Zambon, *Equivalences of coisotropic submanifolds*, J. Symplectic Geom. **15**(1), p. 107-149, 2017.

[V] I. Vaisman, *Lectures on the Geometry of Poisson Manifolds*, Progress in Mathematics **118**, Birkhäuser Basel, 1994.

[W1] A. Weinstein, *Symplectic manifolds and their Lagrangian submanifolds*, Adv. Math. **6**(3), p. 329-346, 1971.

[W2] A. Weinstein, *Symplectic geometry*, Bull. Amer. Math. Soc. **5**(1), p. 1-13, 1981.

[W3] A. Weinstein, *The modular automorphism group of a Poisson manifold*, J. Geom. Phys. **23**(3-4), p. 379-394, 1997.

Chapter 3

The Poisson saturation of regular submanifolds

This chapter reports on a project that I carried out under supervision of Ioan Mărcuţ. It follows the preprint *"The Poisson saturation of regular submanifolds"*, which is the content of arXiv:2011.12650.

Abstract - This chapter is devoted to a certain class of submanifolds in Poisson geometry, which we call regular. We show that the local Poisson saturation of such a submanifold is an embedded Poisson submanifold, and we prove a normal form theorem for this Poisson submanifold around the regular submanifold. This result recovers the normal form around Poisson transversals, and it extends some normal form/rigidity results around constant rank submanifolds in symplectic geometry. As an application, we prove a uniqueness result concerning coisotropic embeddings of Dirac manifolds in Poisson manifolds. We also show how our results generalize to the setting of regular submanifolds in Dirac geometry.

3.1 Introduction

A well-known result in symplectic geometry states that symplectic manifolds (M, ω) are rigid around submanifolds $X \subset M$, in the sense that the restriction of ω to $TM|_X$ determines the symplectic form ω on a neighborhood of X [We2].

By contrast, given a Poisson manifold (M, Π) and any submanifold $X \subset M$, one should not expect Π to be determined, up to neighborhood equivalence, by its restriction $\Pi|_X$. For instance, the origin in $\mathbb{R}^2$ is a fixed point for both the zero Poisson structure and the Poisson structure $\Pi = (x^2 + y^2)\partial_x \wedge \partial_y$, which are clearly not diffeomorphic around $(0, 0)$.

In order for the restriction $\Pi|_X$ to determine Π around X, the ambient Poisson manifold needs to satisfy a minimality condition with respect to X. Since $\Pi|_X$ only contains information in the leafwise direction along X, we are led to consider the *saturation* of $X \subset (M, \Pi)$, i.e. the union of the symplectic leaves that intersect X. Clearly, the saturation of X fails to be a submanifold in general; the purpose of this chapter is to single out a class of submanifolds $X \subset (M, \Pi)$ whose saturation is smooth near X, in a sense that will be made precise later. Since the saturation $Sat(X)$ of $X \subset (M, \Pi)$ is traced out by following Hamiltonian flows starting at points of X in directions normal to $X \subset M$, it is natural to impose the following regularity condition on X.

Definition. We call an embedded submanifold X of a Poisson manifold (M, Π) *regular* if the map $pr \circ \Pi^\sharp : T^*M|_X \to TM|_X/TX$ has constant rank.

It is equivalent to ask that the Π-orthogonal $TX^{\perp_\Pi} := \Pi^\sharp(TX^0)$ has constant rank. Extreme examples are transversals and Poisson submanifolds, and we show that any regular submanifold $X \subset (M, \Pi)$ is the intersection of such submanifolds. If Π is symplectic, then any submanifold of (M, Π) is regular.

The main result of Section 3.2 is the fact that the saturation of a regular submanifold $X \subset (M, \Pi)$ is smooth around X, in the following sense.

Theorem 3A. *If $X \subset (M, \Pi)$ is a regular submanifold, then there exists a neighborhood V of X such that the saturation of X inside $(V, \Pi|_V)$ is an embedded Poisson submanifold.*

We will refer to this Poisson submanifold as the *local Poisson saturation* of X. The proof of Theorem 3A relies on some contravariant geometry and some results concerning dual pairs in Poisson geometry.

Sections 3.3 and 3.4 are devoted to the construction of a normal form for the local Poisson saturation of a regular submanifold. In Section 3.3, we introduce the local model; it is defined on the total space of the vector bundle $(TX^{\perp_\Pi})^*$, and it depends on two additional choices:

1. A choice of complement W to $TX^{\perp_\Pi}$ inside $TM|_X$. Such a choice yields an inclusion $j : (TX^{\perp_\Pi})^* \hookrightarrow T^*M|_X$.
2. A choice of closed two-form η on a neighborhood of X in $(TX^{\perp_\Pi})^*$, with prescribed restriction $\eta|_X = -\sigma - \tau$ along the zero section $X \subset (TX^{\perp_\Pi})^*$. Here $\sigma \in \Gamma(\wedge^2 TX^{\perp_\Pi})$ and $\tau \in \Gamma(T^*X \otimes TX^{\perp_\Pi})$ are bilinear forms defined by
$$\sigma(\xi_1, \xi_2) = \Pi\big(j(\xi_1), j(\xi_2)\big),$$
$$\tau\big((v_1, \xi_1), (v_2, \xi_2)\big) = \langle v_1, j(\xi_2)\rangle - \langle v_2, j(\xi_1)\rangle,$$
for $\xi_1, \xi_2 \in \big(T_x X^{\perp_\Pi}\big)^*$ and $v_1, v_2 \in T_x X$.

To such a complement W and closed extension η, we associate a Poisson structure $\big(U, \Pi(W, \eta)\big)$ on a neighborhood U of $X \subset (TX^{\perp_\Pi})^*$. It is defined as follows: pull back the Dirac structure L_Π defined by the Poisson structure Π under $i : X \hookrightarrow (M, L_\Pi)$, then pull back once more by the bundle projection $pr : (TX^{\perp_\Pi})^* \to (X, i^*L_\Pi)$ and gauge transform by the closed extension η. The obtained Dirac structure $\big(pr^*(i^*L_\Pi)\big)^\eta$ is Poisson on a neighborhood U of $X \subset (TX^{\perp_\Pi})^*$. This Poisson structure, denoted by $\big(U, \Pi(W, \eta)\big)$, is the local model for the local Poisson saturation of $X \subset (M, \Pi)$, as shown in Section 3.4.

Theorem 3B. *Let $X \subset (M, \Pi)$ be a regular submanifold. A neighborhood of X in its local Poisson saturation is Poisson diffeomorphic with the local model $\big(U, \Pi(W, \eta)\big)$.*

The proof of this result goes along the same lines as the proof of the normal form around Poisson transversals [FM1], using dual pairs in Dirac instead of Poisson geometry. The statement shows that the pullback Dirac structure i^*L_Π determines the local Poisson saturation in a neighborhood of X, up to diffeomorphisms and exact gauge transformations. Since X is a transversal in its local Poisson saturation, this result is consistent with the normal form around Dirac transversals, which was proved in [BLM] and [FM2]. Our argument has the advantage that it proves Theorem 3A and Theorem 3B at the same time.

In Section 3.5, we specialize our normal form result to some particular classes of regular submanifolds $X \subset (M, \Pi)$. These allow for a good choice of complement W and/or closed extension η, and as such our normal form becomes more explicit. Most notably, we obtain statements concerning the following types of submanifolds:

i) Poisson transversals: We recover the normal form theorem around Poisson transversals, which was established in [FM1], [BLM].

ii) Regular coisotropic submanifolds: We obtain a Poisson version of Gotay's theorem from symplectic geometry [G], which shows that the local Poisson saturation of a regular coisotropic submanifold $i : X \hookrightarrow (M,\Pi)$ is determined around X by the pullback Dirac structure i^*L_Π.

iii) Regular pre-Poisson submanifolds: We obtain a Poisson version of Marle's constant rank theorem from symplectic geometry [Ma]. The statement shows that the local Poisson saturation of a regular pre-Poisson submanifold $i : X \hookrightarrow (M,\Pi)$ is determined around X by the pullback Dirac structure i^*L_Π and the restriction of Π to $(TX^{\perp_\Pi})^*/(TX^{\perp_\Pi} \cap TX)^*$.[1]

In Section 3.6, we present an application of our normal form specialized to the case of regular coisotropic submanifolds. We address the problem of embedding a Dirac manifold (X,L) coisotropically into a Poisson manifold (M,Π), which was considered before in [CZ2] and [Wa]. Existence of coisotropic embeddings is settled in [CZ2], where it is proved that such an embedding exists exactly when $L \cap TX$ has constant rank. An explicit construction of the Poisson manifold (M,Π) is given in that case; another construction appears in [Wa]. The uniqueness of such embeddings has not been established yet in full generality. A partial uniqueness result appears in [CZ2], under additional regularity assumptions on the Dirac manifold (X,L). As a consequence of our normal form result, we obtain that any coisotropic embedding of (X,L) factors through the model (M,Π) constructed in [CZ2], which settles the uniqueness of coisotropic embeddings.

In Section 3.7, we discuss how our results can be generalized to the setting of regular submanifolds in Dirac geometry. The Appendix contains a result in differential topology for which we could not find a proof in the literature.

3.2 The saturation of a regular submanifold

In this section, we discuss the saturation of submanifolds X in a Poisson manifold (M,Π). Our aim is to give sufficient conditions on X that ensure smoothness of its saturation locally around X. We introduce a class of submanifolds $X \subset (M,\Pi)$, which we call *regular*, and we show that such a submanifold X has a neighborhood U in M such that the saturation of X in $(U,\Pi|_U)$ is an embedded Poisson submanifold.

Definition 3.2.1. The **saturation** of a submanifold X of a Poisson manifold (M,Π) is the union of all the leaves of (M,Π) that intersect X. We denote the saturation of X by $Sat(X)$.

[1]Suitable complements need to be chosen in order to make sense of this, see Lemma 3.5.3.

Note that $Sat(X)$ is the smallest complete Poisson submanifold of (M,Π) containing X, if it is smooth. Indeed, complete Poisson submanifolds $P \subset (M,\Pi)$ are saturated [CW, Prop. 6.1], so if $X \subset P$ then $Sat(X) \subset Sat(P) = P$.

The saturation of a submanifold can be very wild; in general it does not have a submanifold structure. For instance, consider the x-axis in the log-symplectic manifold $(\mathbb{R}^2, x\partial_x \wedge \partial_y)$; its saturation is $\{x<0\} \cup \{(0,0)\} \cup \{x>0\}$. Clearly, this saturation doesn't even contain a Poisson submanifold around the x-axis.

We now single out classes of submanifolds $X \subset (M,\Pi)$ that do satisfy this property, i.e. whose saturation contains a Poisson submanifold around X. Trivial examples of such submanifolds are transversals (whose saturation is open, and therefore a Poisson submanifold) and Poisson submanifolds. These are extreme cases of what we call *regular* submanifolds.

Definition 3.2.2. We call an embedded submanifold $X \subset (M,\Pi)$ **regular** if the map $pr \circ \Pi^\sharp : T^*M|_X \to TM|_X/TX$ has constant rank.

Note that transversals and Poisson submanifolds are exactly those submanifolds $X \subset (M,\Pi)$ for which the map $pr \circ \Pi^\sharp$ is of full rank resp. identically zero.

We will now list some more observations about regular submanifolds. For any submanifold $X \subset (M,\Pi)$, we denote its Π-orthogonal by $TX^{\perp_\Pi} := \Pi^\sharp(TX^0)$. If $x \in X$ and L is the symplectic leaf through x, then $T_xX^{\perp_\Pi}$ is the symplectic orthogonal of $T_xX \cap T_xL$ in the symplectic vector space $\left(T_xL, (\Pi|_L)_x^{-1}\right)$. Various types of submanifolds in Poisson geometry are defined in terms of their Π-orthogonal; see [CFM] and [Z] for a systematic overview.

a) Given a submanifold $X \subset (M,\Pi)$, we get an exact sequence at points $x \in X$:

$$0 \longrightarrow \left(T_xX^{\perp_\Pi}\right)^0 \hookrightarrow T_x^*M \xrightarrow{pr\circ\Pi^\sharp} T_xM/T_xX. \tag{3.1}$$

In particular, $X \subset (M,\Pi)$ is regular exactly when $TX^{\perp_\Pi}$ has constant rank.

b) We give an alternative characterization of regular submanifolds $X \subset (M,\Pi)$ in Dirac geometric terms. Denote by $L_\Pi := \{\Pi^\sharp(\alpha) + \alpha : \alpha \in T^*M\}$ the Dirac structure corresponding with Π, and let $i : X \hookrightarrow (M,\Pi)$ be any submanifold. Then X is regular exactly when $L_\Pi \cap \mathrm{Ker}(i^*)$ has constant rank. Indeed, $L_\Pi \cap \mathrm{Ker}(i^*) = \mathrm{Ker}(\Pi^\sharp) \cap TX^0$, so for any $x \in X$ we have

$$\dim\left(L_\Pi \cap \mathrm{Ker}(i^*)\right)_x = \dim\left(T_xX^0\right) - \dim\left(T_xX^{\perp_\Pi}\right).$$

In particular, if $X \subset (M,\Pi)$ is regular, then the Dirac structure L_Π pulls back to a smooth Dirac structure on X [B, Prop. 5.6].

We mention here that X being regular is not a necessary condition for $Sat(X)$ to contain a Poisson submanifold around X, as demonstrated by the following.

Examples 3.2.3. i) Consider $\left(\mathfrak{so}(3)^*, z\partial_x \wedge \partial_y + x\partial_y \wedge \partial_z + y\partial_z \wedge \partial_x\right)$ and let X be the plane defined by $z = 0$. The symplectic foliation of $\mathfrak{so}(3)^*$ consists of concentric spheres of radius $r \geq 0$ centered at the origin, so that $Sat(X) = \mathfrak{so}(3)^*$. However, $TX^{\perp_\Pi} = \text{Span}\{y\partial_x - x\partial_y\}$ vanishes at the origin, so X is not regular.

ii) Consider the regular Poisson manifold $(\mathbb{R}^3, \partial_x \wedge \partial_y)$ and let X be defined by the equation $z = x^3$. Then the saturation $Sat(X)$ is all of $\mathbb{R}^3$, but X is not regular. Indeed, we have $TX^{\perp_\Pi} = \text{Span}\{-3x^2\partial_y\}$, which drops rank at points of the form $(0, y, 0) \in X$.

To construct a Poisson submanifold around the regular submanifold $X \subset (M, \Pi)$, we use some contravariant geometry and some theory of dual pairs [CM],[FM1].

Definition 3.2.4. A **Poisson spray** on a Poisson manifold (M, Π) is a vector field χ on the cotangent bundle T^*M satisfying:

i) $dpr(\chi(\xi)) = \Pi^\sharp(\xi)$ for all $\xi \in T^*M$,

ii) $m_t^*\chi = t\chi$ for all $t > 0$,

where $pr : T^*M \to M$ is the projection map and $m_t : T^*M \to T^*M$ denotes fiberwise multiplication by t.

Poisson sprays $\chi \in \mathfrak{X}(T^*M)$ exist on any Poisson manifold. Since χ vanishes along the zero section $M \subset T^*M$, there exists a neighborhood $\Sigma \subset T^*M$ of M on which the flow ϕ_χ^t is defined for all times $t \in [0, 1]$. One can then define the **contravariant exponential map** $\exp_\chi$ of χ by

$$\exp_\chi : \Sigma \subset T^*M \to M : \xi \mapsto pr(\phi_\chi^1(\xi)).$$

This neighborhood $\Sigma \subset T^*M$ also supports a closed two-form Ω_χ, which is defined by averaging the canonical symplectic form ω_{can} with respect to the flow ϕ_χ^t of the Poisson spray $\chi \in \mathfrak{X}(T^*M)$:

$$\Omega_\chi := \int_0^1 \left(\phi_\chi^t\right)^* \omega_{can} dt.$$

As proved in [CM], Ω_χ is non-degenerate along the zero zection $M \subset T^*M$, so shrinking $\Sigma \subset T^*M$ if necessary, we can assume that Ω_χ is symplectic on Σ. By [FM1, Lemma 9], the symplectic manifold (Σ, Ω_χ) fits in a **full dual pair**

$$(M, \Pi) \xleftarrow{\;pr\;} (\Sigma, \Omega_\chi) \xrightarrow{\;\exp_\chi\;} (M, -\Pi). \tag{3.2}$$

That is, denoting by $\Pi_\chi := \Omega_\chi^{-1}$ the Poisson structure corresponding with Ω_χ, the maps $pr : \left(\Sigma, \Pi_\chi\right) \to (M, \Pi)$ and $\exp_\chi : \left(\Sigma, \Pi_\chi\right) \to (M, -\Pi)$ are surjective Poisson submersions with symplectically orthogonal fibers:

$$(\ker dpr)^{\perp_{\Omega_\chi}} = \ker d\exp_\chi.$$

Both legs in the diagram (3.2) are symplectic realizations. We will need the following lemma, which concerns the interplay between symplectic realizations and regular submanifolds of a Poisson manifold.

Lemma 3.2.5. *Let $X \subset (M, \Pi)$ be a regular submanifold and assume that $\mu : (\Sigma, \Omega) \to (M, \Pi)$ is a symplectic realization. Then $(\ker d\mu)^{\perp_\Omega} \cap T(\mu^{-1}(X))$ has constant rank, equal to the corank of $TX^{\perp_\Pi} \subset TM|_X$.*

Proof. Denote by $\Pi_\Omega := \Omega^{-1}$ the Poisson structure corresponding with Ω. For $\xi \in \mu^{-1}(X)$, we have

$$(\ker d\mu)_\xi^{\perp_\Omega} = (\Omega^\flat)^{-1}\big((\ker d\mu)_\xi^0\big) = \Pi_\Omega^\sharp\big((d\mu)_\xi^* T^*_{\mu(\xi)}M\big).$$

Since for any $\beta \in T^*_{\mu(\xi)}M$, we have

$$\begin{aligned}(d\mu)_\xi \Pi_\Omega^\sharp\big((d\mu)_\xi^*\beta\big) \in T_{\mu(\xi)}X &\Leftrightarrow \Pi^\sharp(\beta) \in T_{\mu(\xi)}X \\ &\Leftrightarrow \beta \in \big(T_{\mu(\xi)}X^{\perp_\Pi}\big)^0,\end{aligned}$$

we obtain

$$\begin{aligned}(\ker d\mu)_\xi^{\perp_\Omega} \cap T_\xi(\mu^{-1}(X)) &= (\ker d\mu)_\xi^{\perp_\Omega} \cap (d\mu)_\xi^{-1}(T_{\mu(\xi)}X) \\ &= \Pi_\Omega^\sharp\left((d\mu)_\xi^*\big(T_{\mu(\xi)}X^{\perp_\Pi}\big)^0\right). \qquad (3.3)\end{aligned}$$

So, the rank of $(\ker d\mu)_\xi^{\perp_\Omega} \cap T_\xi(\mu^{-1}(X))$ is equal to $\dim M - rk(TX^{\perp_\Pi})$. □

We now prove that for a regular submanifold $X \subset (M, \Pi)$, there exists an embedded Poisson submanifold of (M, Π) containing X that lies in the saturation $Sat(X)$. This Poisson submanifold is in fact the saturation of X in a neighborhood $(U, \Pi|_U)$ of X.

Theorem 3.2.6. *Let $X \subset (M, \Pi)$ be a regular submanifold.*

1. *There is an embedded Poisson submanifold $(P, \Pi_P) \subset (M, \Pi)$ containing X that lies inside the saturation $Sat(X)$.*
2. *Shrinking P if necessary, there is a neighborhood U of X in M such that (P, Π_P) is the saturation of X in $(U, \Pi|_U)$.*

Proof. We divide the proof into four steps.

Step 1: Construction of the embedded submanifold $P \subset M$.

Choose a Poisson spray $\chi \in \mathfrak{X}(T^*M)$ and denote by $\exp_\chi : \Sigma \subset T^*M \to M$ the corresponding contravariant exponential map. Note that the restriction $\exp_\chi : \Sigma|_X \to M$ takes values in $Sat(X)$. To prove this, it is enough to show that $pr(\xi)$ and $\exp_\chi(\xi)$ lie in the same symplectic leaf of (M,Π), for each $\xi \in \Sigma|_X$. This in turn follows if we prove that $pr(\xi) \in X$ and $\exp_\chi(\xi)$ are connected by a cotangent path [CF2, Section 1]. To this end, consider the curve $\gamma(t) := pr(\phi^t_\chi(\xi))$, which satisfies $\gamma(0) = pr(\xi)$ and $\gamma(1) = \exp_\chi(\xi)$. Using property i) in Definition 3.2.4, we have

$$\gamma'(t) = (dpr)_{\phi^t_\chi(\xi)}\left(\frac{d}{dt}\phi^t_\chi(\xi)\right) = (dpr)_{\phi^t_\chi(\xi)}\left(\chi(\phi^t_\chi(\xi))\right) = \Pi^\sharp_{\gamma(t)}(\phi^t_\chi(\xi)),$$

showing that $t \mapsto (\phi^t_\chi(\xi), \gamma(t))$ is a cotangent path. So $\exp_\chi(\Sigma|_X) \subset Sat(X)$.

Choosing a complement to $TX^{\perp_\Pi}$ in $TM|_X$, we get an inclusion of $(TX^{\perp_\Pi})^*$ into $T^*M|_X$. The restriction of the exponential map $\exp_\chi : (TX^{\perp_\Pi})^* \cap \Sigma|_X \to M$ fixes points of X, and its differential along X reads [FM1, Lemma 8]:

$$(d\exp_\chi)_x : T_xX \oplus (T_xX^{\perp_\Pi})^* \to T_xM : (v,\xi) \mapsto v + \Pi^\sharp_x(\xi).$$

This map is injective. Indeed, if we would have that $\Pi^\sharp_x(\xi) = -v \in T_xX$, then $\xi \in \left(\Pi^\sharp_x\right)^{-1}(T_xX) = (T_xX^{\perp_\Pi})^0$ and therefore $\xi \in (T_xX^{\perp_\Pi})^* \cap (T_xX^{\perp_\Pi})^0 = \{0\}$. Theorem 3.8.1 and Remark 3.8.2 in the Appendix now imply that the map $\exp_\chi : (TX^{\perp_\Pi})^* \cap \Sigma|_X \to M$ is an embedding, shrinking Σ if necessary. Setting

$$P := \exp_\chi\left((TX^{\perp_\Pi})^* \cap \Sigma|_X\right),$$

this is an embedded submanifold of M containing X that lies inside $Sat(X)$.

Step 2: Shrinking Σ if necessary, we have $P = \exp_\chi(\Sigma|_X)$.

To see this, denote for short $\Sigma_X := \Sigma|_X \subset T^*M|_X$ and $\widetilde{\Sigma_X} := (TX^{\perp_\Pi})^* \cap \Sigma|_X$. First, we claim that the restriction $\exp_\chi\big|_{\Sigma_X}$ has constant rank, equal to the rank of $\exp_\chi\big|_{\widetilde{\Sigma_X}}$. Indeed, using the self-dual pair (3.2), we have that

$$\ker\left(d\left(\exp_\chi|_{\Sigma_X}\right)\right) = \ker(d\exp_\chi) \cap T(pr^{-1}(X)) = \ker(dpr)^{\perp_{\Omega_\chi}} \cap T(pr^{-1}(X)),$$

which has constant rank equal to $corank(TX^{\perp_\Pi})$ by Lemma 3.2.5. Hence, the rank of $\exp_\chi\big|_{\Sigma_X}$ is equal to $\dim X + rk(TX^{\perp_\Pi})$, which is the rank of $\exp_\chi\big|_{\widetilde{\Sigma_X}}$.

Using the previous claim, we now assert that $\exp_\chi(\widetilde{\Sigma_X}) = \exp_\chi(\Sigma_X)$, shrinking Σ if necessary. It is enough to prove that every point $\xi \in \widetilde{\Sigma_X}$ has a neighborhood $V^\xi \subset \Sigma_X$ such that $\exp_\chi(V^\xi) \subset \exp_\chi(\widetilde{\Sigma_X})$. We keep in mind the diagram

$$\begin{array}{ccc} \widetilde{\Sigma_X} \subset \left(TX^{\perp_\Pi}\right)^* & \hookrightarrow & \Sigma_X \subset T^*M|_X \\ {\scriptstyle \exp_\chi|_{\widetilde{\Sigma_X}}}\downarrow \wr & & \downarrow {\scriptstyle \exp_\chi|_{\Sigma_X}} \\ \exp_\chi(\widetilde{\Sigma_X}) & \hookrightarrow & M \end{array}.$$

Pick $\xi \in \widetilde{\Sigma_X} \subset \Sigma_X$. Since $\exp_\chi|_{\Sigma_X}$ has constant rank, there is an open neighborhood $U^\xi \subset \Sigma_X$ of ξ such that $\exp_\chi(U^\xi) \subset M$ is an embedded submanifold. Since $\exp_\chi$ is an embedding on $\widetilde{\Sigma_X}$, also $\exp_\chi(U^\xi \cap \widetilde{\Sigma_X}) \subset M$ is an embedded submanifold. Since $\dim \exp_\chi(U^\xi \cap \widetilde{\Sigma_X}) = \dim \exp_\chi(U^\xi)$ by the previous claim, the inverse function theorem implies that $\exp_\chi(U^\xi \cap \widetilde{\Sigma_X})$ is open in $\exp_\chi(U^\xi)$. Since the map $\exp_\chi|_{U^\xi} : U^\xi \to \exp_\chi(U^\xi)$ is continuous, we get that $\exp_\chi|_{U^\xi}^{-1}\big(\exp_\chi(U^\xi \cap \widetilde{\Sigma_X})\big)$ is open in U^ξ, and therefore in Σ_X. Setting $V^\xi := \exp_\chi|_{U^\xi}^{-1}\big(\exp_\chi(U^\xi \cap \widetilde{\Sigma_X})\big)$ proves the assertion. This finishes Step 2.

Step 3: P is a Poisson submanifold of (M, Π).

We use the previous step, which states that $P = \exp_\chi(\Sigma_X)$. Pick a point $x \in P$ and let $\xi \in \Sigma_X$ be such that $\exp_\chi(\xi) = x$. We have to show that $\Pi^\sharp(T_x P^0) = \{0\}$. Making use of the dual pair (3.2), we have

$$\Pi^\sharp(T_x P^0) = \left[(d\exp_\chi)_\xi \circ \left(\Pi^\sharp_\chi\right)_\xi \circ (d\exp_\chi)^*_\xi\right] \left(T_x P^0\right),$$

so it is enough to show that

$$\left(\Pi^\sharp_\chi\right)_\xi \left((d\exp_\chi)^*_\xi(T_x P^0)\right) \subset \ker(d\exp_\chi)_\xi = \left(\Pi^\sharp_\chi\right)_\xi (\ker dpr)^0_\xi.$$

To see that this inclusion holds, note that $(\ker dpr)_\xi \subset T_\xi \Sigma_X$ and $(d\exp_\chi)_\xi(T_\xi \Sigma_X) \subset T_x P$, which implies that

$$(d\exp_\chi)^*_\xi(T_x P^0) \subset T_\xi \Sigma_X^0 \subset (\ker dpr)^0_\xi.$$

So $P \subset (M, \Pi)$ is a Poisson submanifold, and Step 3 is done.

Step 4: Construction of the neighborhood U of X.

The idea is to extend $\exp_\chi : (TX^{\perp_\Pi})^* \cap \Sigma|_X \to M$ to a local diffeomorphism, using the same reasoning as in the proof of Proposition 3.8.1 in the Appendix. Choosing a complement

$$TM|_X = TX \oplus \Pi^\sharp\left(TX^{\perp_\Pi}\right)^* \oplus C,$$

and a linear connection ∇ on TM, we obtain a map

$$\psi : V \subset \left((TX^{\perp_\Pi})^* \oplus C\right) \to M : (\xi, c) \mapsto \exp_\nabla\left(Tr_{\exp_\chi(t\xi)} c\right),$$

which is a diffeomorphism onto an open neighborhood of X. Here V is a suitable convex neighborhood of the zero section, and $Tr_{\exp_\chi(t\xi)}$ denotes parallel transport along the curve $t \mapsto \exp_\chi(t\xi)$ for $t \in [0,1]$. Note that ψ satisfies $\psi(\xi, 0) = \exp_\chi(\xi)$. Consequently, shrinking P if necessary, we can assume that

$$P = \psi\left(V \cap \left(\left(TX^{\perp_\Pi}\right)^* \oplus \{0\}\right)\right).$$

Setting $U := \psi(V)$, we check that P is the Poisson saturation of X in $(U, \Pi|_U)$. On one hand, since $(TX^{\perp_\Pi})^*$ is closed in $(TX^{\perp_\Pi})^* \oplus C$, also P is closed in U. Since properly embedded Poisson submanifolds are saturated, it follows that the saturation of X in $(U, \Pi|_U)$ is contained in P. On the other hand, if $\exp_\chi(\xi) = \psi(\xi, 0) \in P \subset U$, then also $\exp_\chi(t\xi) \in U$ for $t \in [0,1]$ since V is convex. Consequently, the path $t \mapsto (\phi^t_\chi(\xi), \exp_\chi(t\xi))$ is a cotangent path covering a path in U that connects $\exp_\chi(\xi)$ with a point in X. This shows that $\exp_\chi(\xi)$ is contained in the Poisson saturation of X in $(U, \Pi|_U)$. □

Theorem 3.2.6 shows that the saturation of a regular submanifold $X \subset (M, \Pi)$ in some neighborhood $(U, \Pi|_U)$ of X is an embedded Poisson submanifold. Clearly, one cannot take U to be all of M in general, see for instance Example 3.2.9 below. In this respect, we have the following sufficient condition.

Corollary 3.2.7. *Let $X \subset (M, \Pi)$ be a regular submanifold. If the submanifold P constructed in Theorem 3.2.6 is open in $Sat(X)$ for the induced topology, then $Sat(X)$ is an embedded submanifold of M.*

Proof. Recall the following fact [CFM]: if $\{N_i\}_{i\in\mathcal{I}}$ is a collection of embedded submanifolds of M, all of the same dimension, such that $N_i \cap N_j$ is open in N_i for all $i, j \in \mathcal{I}$, then $N := \cup_{i\in\mathcal{I}} N_i$ has a natural smooth structure for which the inclusion $N \hookrightarrow M$ is an immersion. The smooth structure is uniquely determined by the condition that the maps $N_i \hookrightarrow N$ are smooth open embeddings.

We want to apply this fact to the collection $\{\phi^1_{X_f}(P) : f \in C^\infty_c([0,1] \times M)\}$, where $\phi^1_{X_f}$ denotes the time 1-flow of the Hamiltonian vector field associated with the compactly supported function $f \in C^\infty_c([0,1] \times M)$. We have to check that $\phi^1_{X_f}(P) \cap \phi^1_{X_g}(P)$ is open in $\phi^1_{X_f}(P)$. To this end, note that both $\phi^1_{X_f}(P)$ and $\phi^1_{X_g}(P)$ are open in $Sat(X)$, since P is open in $Sat(X)$ and $\phi^1_{X_f}, \phi^1_{X_g}$ are diffeomorphisms preserving $Sat(X)$. Hence, also $\phi^1_{X_f}(P) \cap \phi^1_{X_g}(P)$ is open in $Sat(X)$, so there exists an open $V \subset M$ such that

$$\phi^1_{X_f}(P) \cap \phi^1_{X_g}(P) = V \cap Sat(X).$$

Since also $\phi^1_{X_f}(P) = U \cap Sat(X)$ for some open $U \subset M$, we obtain

$$\phi^1_{X_f}(P) \cap \phi^1_{X_g}(P) = \phi^1_{X_f}(P) \cap \phi^1_{X_g}(P) \cap U = V \cap (U \cap Sat(X)) = V \cap \phi^1_{X_f}(P),$$

which shows that $\phi^1_{X_f}(P) \cap \phi^1_{X_g}(P)$ is open in $\phi^1_{X_f}(P)$.

So we can apply the fact mentioned above, which gives $Sat(X)$ a smooth structure for which $Sat(X) \hookrightarrow M$ is an immersion. But since the topology of this smooth structure is generated by open subsets of the submanifolds $\phi^1_{X_f}(P)$, it coincides with the induced topology on $Sat(X)$. Consequently, $Sat(X)$ is an embedded submanifold of M. □

In general, one cannot apply the argument in the proof of Corollary 3.2.7 to obtain a smooth structure on $Sat(X)$. See for instance Example 3.2.9 below.

Remark 3.2.8. We comment on the condition in Corollary 3.2.7 stating that $P = \exp_\chi(\Sigma|_X)$ needs to be open in $Sat(X)$ for the induced topology. This occurs exactly when we are able to find a small transversal $\tau \subset (M, \Pi)$ to the leaves such that $\tau \cap Sat(X) = X$.
To see that then $\exp_\chi(\Sigma|_X)$ is indeed open in $Sat(X)$ with respect to the induced topology, we note that $\exp_\chi : \Sigma|_\tau \to M$ is a submersion, shrinking Σ if necessary. Indeed, at points $p \in \tau$, the differential

$$(d\exp_\chi)_p : T_p\tau \oplus T_p^*M \to T_pM : (v, \xi) \mapsto v + \Pi_p^\sharp(\xi)$$

is surjective since $\tau \subset (M, \Pi)$ is a transversal. Hence $\exp_\chi$ is of maximal rank in a neighborhood of $\tau \subset \Sigma|_\tau$. In particular, shrinking Σ if needed, we have that $\exp_\chi\left(\Sigma|_\tau\right) \subset M$ is open. It now suffices to remark that $\exp_\chi(\Sigma|_X) = \exp_\chi(\Sigma|_\tau) \cap Sat(X)$. The forward inclusion is clear, since $X \subset \tau$ and $\exp_\chi(\Sigma|_X) \subset Sat(X)$. For the backward inclusion, let $(p, \xi) \in \Sigma|_\tau$ be such that $\exp_\chi(\xi) \in Sat(X)$. Since p lies in the same leaf as $\exp_\chi(\xi) \in Sat(X)$ and $Sat(X)$ is saturated, it follows that $p \in Sat(X)$. Hence, $p \in \tau \cap Sat(X) = X$. This shows that $\exp_\chi(\Sigma|_X) = \exp_\chi(\Sigma|_\tau) \cap Sat(X)$ is open in $Sat(X)$ for the induced topology.

In the particular case where X is a point, then $Sat(X)$ is just the leaf through X, which is well-known to possess a natural smooth structure. Indeed, each leaf of a Poisson manifold is an initial submanifold, so in particular it possesses a unique smooth structure that turns it into an immersed submanifold. For an arbitrary regular submanifold X, its saturation does not have a natural smooth structure, as illustrated in the following example.

Example 3.2.9. We look at the manifold $(\mathbb{R}^3 \times S^1, x, y, z, \theta)$ with Poisson structure $\Pi = \partial_z \wedge \partial_\theta$. Consider the curve $\beta : \mathbb{R} \to \mathbb{R}^3 : t \mapsto (\sin(2t), \sin(t), t)$, which is a "figure eight" coming out of the xy-plane. Denote its image by $\mathcal{C} \subset \mathbb{R}^3$, and let $\mathcal{C}_{base}$ be the projection of $\mathcal{C}$ onto the xy-plane. The submanifold $X := \mathcal{C} \times S^1 \subset \mathbb{R}^3 \times S^1$ is embedded, and we claim that it is regular. To see this, we only have to check that $\dim(T_pX \cap T_pL)$ is constant for $p \in X$, where L denotes the leaf through p. Since at a point $p = (\beta(t_0), \theta_0)$ we have

$$T_pX = \mathrm{Span}\{\, \partial_\theta|_p \,, 2\cos(2t_0)\, \partial_x|_p + \cos(t_0)\, \partial_y|_p + \partial_z|_p\},$$

it is clear that $T_pX \cap T_pL = \mathrm{Span}\{\, \partial_\theta|_p\}$, since $\cos(t_0)$ and $\cos(2t_0)$ cannot be zero simultaneously. This confirms that $X \subset (\mathbb{R}^3 \times S^1, \Pi)$ is regular. Its saturation is given by $Sat(X) = \mathcal{C}_{base} \times \mathbb{R} \times S^1$, and this doensn't have a natural smooth structure for which the inclusion $X \hookrightarrow Sat(X)$ is smooth. Indeed, for the two obvious smooth structures on $Sat(X)$ induced by those on the "figure eight", the inclusion $X \hookrightarrow Sat(X)$ is not even continuous.

Coming back to the proof of Corollary 3.2.7, let's look at the figure below. We removed the S^1-factor, which is not essential to the spirit of the example. The embedded submanifold P in this case is obtained by slightly thickening the curve in vertical direction. One can take a Hamiltonian flow $\phi^1_{X_f}$ such that $\phi^1_{X_f}(P) \cap P$ consists of vertical segments of the line in which the surface intersects itself, which is not an open subset of P. So we cannot apply the general fact mentioned in the proof of Corollary 3.2.7.

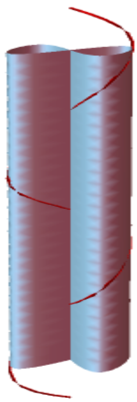

Figure 3.1: The regular submanifold X and its saturation $Sat(X)$. This is the picture in $\mathbb{R}^3$; the S^1-factor is omitted for the sake of depiction.

As a consequence of Theorem 3.2.6, we obtain an alternative characterization of regular submanifolds. The two extreme examples – Poisson submanifolds and transversals – turn out to be the building blocks of any regular submanifold.

Proposition 3.2.10. *A submanifold $X \subset (M, \Pi)$ is regular iff. X is the intersection of a Poisson submanifold $P \subset (M, \Pi)$ with a transversal $\tau \subset (M, \Pi)$.*

A transversal $\tau \subset (M, \Pi)$ is also transverse to Poisson submanifolds $P \subset (M, \Pi)$, since the intersection of P with any leaf of (M, Π) is open in the leaf. Indeed, if $p \in P$ and L is the leaf through P, then

$$T_pM = T_p\tau + T_pL = T_p\tau + T_p(P \cap L) \subset T_p\tau + T_pP,$$

which shows that $\tau \pitchfork P$. In particular, the intersection $\tau \cap P$ is smooth.

Proof of Prop. 3.2.10. First assume that $X \subset (M, \Pi)$ is a regular submanifold. Theorem 3.2.6 then gives a Poisson submanifold $P \subset (M, \Pi)$ containing X, and the proof shows that

$$TP|_X = TX \oplus \Pi^\sharp\big(TX^{\perp_\Pi}\big)^*. \tag{3.4}$$

Choose a complement $TM|_X = TX \oplus \Pi^\sharp\big(TX^{\perp_\Pi}\big)^* \oplus E$. We now define a transversal $\tau \subset (M,\Pi)$ by thickening X in direction of E; using an exponential map for instance, we can define $\tau \subset M$ to be a submanifold containing X, such that $T\tau|_X = TX \oplus E$. For small enough τ, we have $\tau \cap P = X$. Moreover,

$$TM|_X = TX \oplus \Pi^\sharp\big(TX^{\perp_\Pi}\big)^* \oplus E = \big(TX + \mathrm{Im}(\Pi^\sharp|_X)\big) \oplus E = \mathrm{Im}(\Pi^\sharp|_X)\big) + T\tau|_X,$$

which shows that τ is a transversal along X. Shrinking τ if necessary, this implies that τ is a transversal in (M,Π). This proves the forward implication.

For the converse, assume that $X = \tau \cap P$ is a submanifold of M, where $P \subset (M,\Pi)$ is a Poisson submanifold and $\tau \subset (M,\Pi)$ is a transversal. Then $TX = T\tau|_X \cap TP|_X$, so that $TX^0 = T\tau|_X^0 + TP|_X^0$. Using that P is a Poisson submanifold, we get $TX^{\perp_\Pi} = \Pi^\sharp(T\tau|_X^0)$. Since τ is a transversal, the restriction $\Pi^\sharp|_{T\tau^0}$ is injective, which shows that X is regular. □

In what follows, we denote by (P,Π_P) the Poisson submanifold containing X that was constructed in Theorem 3.2.6. We refer to (P,Π_P) as the **local Poisson saturation** of X. Since X is transverse to the leaves of (P,Π_P), the work of Bursztyn-Lima-Meinrenken [BLM] gives a normal form for (P,Π_P) around X. We will recover this normal form, continuing our argument from Theorem 3.2.6.

3.3 The local model

This section introduces the local model for the local Poisson saturation (P,Π_P) of a regular submanifold $X \subset (M,\Pi)$. The local model is defined on the vector bundle $(TX^{\perp_\Pi})^*$, which is indeed isomorphic with the normal bundle of X in P. An explicit isomorphism is obtained by choosing an embedding $(TX^{\perp_\Pi})^* \hookrightarrow T^*M|_X$ and then applying the bundle map $\Pi^\sharp$, see equation (3.4). The local model involves some extra choices, which we now explain.

Let $X \subset (M,\Pi)$ be a regular submanifold, and choose a **complement** W to $TX^{\perp_\Pi}$ inside $TM|_X$. It yields an inclusion map $j : \big(TX^{\perp_\Pi}\big)^* \hookrightarrow T^*M|_X$. Define skew-symmetric bilinear forms $\sigma \in \Gamma(\wedge^2 TX^{\perp_\Pi})$ and $\tau \in \Gamma(T^*X \otimes TX^{\perp_\Pi})$ on the restricted tangent bundle $T\big((TX^{\perp_\Pi})^*\big)|_X = TX \oplus (TX^{\perp_\Pi})^*$ by the formulas

$$\begin{aligned} &\sigma(\xi_1,\xi_2) = \Pi\big(j(\xi_1), j(\xi_2)\big),\\ &\tau\big((v_1,\xi_1),(v_2,\xi_2)\big) = \langle v_1, j(\xi_2)\rangle - \langle v_2, j(\xi_1)\rangle, \end{aligned} \tag{3.5}$$

for $\xi_1,\xi_2 \in \big(T_xX^{\perp_\Pi}\big)^*$ and $v_1,v_2 \in T_xX$. Denote by $\mathcal{E}_W(-\sigma-\tau)$ the set of all closed two-forms η, defined on a neighborhood of $X \subset (TX^{\perp_\Pi})^*$, whose restriction to the zero section $X \subset (TX^{\perp_\Pi})^*$ equals

$$\eta|_X = -\sigma \oplus -\tau \oplus 0 \in \Gamma(\wedge^2 TX^{\perp_\Pi}) \oplus \Gamma(T^*X \otimes TX^{\perp_\Pi}) \oplus \Gamma(\wedge^2 T^*X). \quad (3.6)$$

We refer to a two-form $\eta \in \mathcal{E}_W(-\sigma - \tau)$ as a **closed extension** of $-\sigma - \tau$. Closed extensions of $-\sigma - \tau$ exist, see for instance [We1, Extension Theorem].

The local model for the local Poisson saturation of the regular submanifold $X \overset{i}{\hookrightarrow} (M, \Pi)$ is now defined as follows: pull back the Dirac structure i^*L_Π on X to $(TX^{\perp_\Pi})^*$ under $pr : (TX^{\perp_\Pi})^* \to X$ and gauge transform by a closed extension $\eta \in \mathcal{E}_W(-\sigma - \tau)$. The obtained Dirac structure $(pr^*(i^*L_\Pi))^\eta$ indeed defines a Poisson stucture in a neighborhood of $X \subset (TX^{\perp_\Pi})^*$, as we now show.

Proposition 3.3.1. *Let $X \subset (M, \Pi)$ be a regular submanifold. Fix a complement W to $TX^{\perp_\Pi}$, define $\sigma \in \Gamma(\wedge^2 TX^{\perp_\Pi})$ and $\tau \in \Gamma(T^*X \otimes TX^{\perp_\Pi})$ by the formulas* (3.5) *and let $\eta \in \mathcal{E}_W(-\sigma - \tau)$ be any closed extension. The Dirac structure $(pr^*(i^*L_\Pi))^\eta$ is Poisson on a neighborhood U of $X \subset \big(TX^{\perp_\Pi}\big)^*$.*

Proof. It suffices to show that $(pr^*(i^*L_\Pi))^\eta$ is transverse to $T\big(TX^{\perp_\Pi}\big)^*$ along X. Because of (3.1), we have $i^*L_\Pi = \big\{\Pi^\sharp(\alpha) + i^*\alpha : \ \alpha \in (TX^{\perp_\Pi})^0\big\}$, so that

$$(pr^*(i^*L_\Pi))^\eta|_X = \big\{\Pi^\sharp(\alpha) + \xi + pr^*(i^*\alpha) + \iota_{\Pi^\sharp(\alpha)+\xi}\eta : \ \alpha \in (TX^{\perp_\Pi})^0, \xi \in (TX^{\perp_\Pi})^*\big\}.$$

Assume that $\Pi^\sharp(\alpha) + \xi \in T(TX^{\perp_\Pi})^*|_X \cap (pr^*(i^*L_\Pi))^\eta|_X$ for some $\alpha \in (TX^{\perp_\Pi})^0$, $\xi \in (TX^{\perp_\Pi})^*$. Then $pr^*(i^*\alpha) + \iota_{\Pi^\sharp(\alpha)+\xi}\eta = 0$, which implies the following:

- For all $v \in TX$, we get

  $$\alpha(v) + \eta\big(\Pi^\sharp(\alpha) + \xi, v\big) = 0 \Rightarrow \alpha(v) + \langle j(\xi), v\rangle = 0.$$

  So $\alpha + j(\xi) \in TX^0$, and therefore $\Pi^\sharp(\alpha + j(\xi)) \in TX^{\perp_\Pi}$.
- For all $\beta \in (TX^{\perp_\Pi})^*$, we get

  $$\eta\big(\Pi^\sharp(\alpha) + \xi, \beta\big) = 0 \Rightarrow \Pi\big(j(\xi), j(\beta)\big) + \langle \Pi^\sharp(\alpha), j(\beta)\rangle = 0$$
  $$\Rightarrow \big\langle \Pi^\sharp(\alpha + j(\xi)), j(\beta)\big\rangle = 0.$$

  Since $j\big((TX^\perp)^*\big) = W^0$, this shows that $\Pi^\sharp(\alpha + j(\xi))$ lies in W.

We now have $\Pi^\sharp(\alpha + j(\xi)) \in TX^{\perp_\Pi} \cap W = \{0\}$. So $\Pi^\sharp\big(j(\xi)\big) = -\Pi^\sharp(\alpha) \in TX$, which implies that $j(\xi) \in (TX^{\perp_\Pi})^0$, again using exactness of the sequence (3.1). But then $j(\xi) \in W^0 \cap (TX^{\perp_\Pi})^0 = \{0\}$, so that $\xi = 0$, which in turn implies that also $\Pi^\sharp(\alpha) = 0$. This finishes the proof. □

We denote the Poisson manifold from Proposition 3.3.1 by $\big(U,\Pi(W,\eta)\big)$, and we refer to it as the **local model corresponding with W and $\boldsymbol{\eta}$**. A priori, the construction depends on a choice of complement W and closed extension η. We now show that different choices produce isomorphic local models.

Proposition 3.3.2. *Any two local models $\big(U,\Pi(W_0,\eta_0)\big)$ and $\big(V,\Pi(W_1,\eta_1)\big)$ for the local Poisson saturation of a regular submanifold $X\subset(M,\Pi)$ are isomorphic around X, through a diffeomorphism that restricts to the identity along X.*

Proof. The idea of the proof is to construct this diffeomorphism in two stages, where each stage relies on a Moser argument. We first map the local model $\big(U,\Pi(W_0,\eta_0)\big)$ to an intermediate local model $\big(V',\Pi(W_1,\eta_1')\big)$, which is defined in terms of the complement W_1. Then we pull $\big(V',\Pi(W_1,\eta_1')\big)$ to the second local model $\big(V,\Pi(W_1,\eta_1)\big)$. Throughout, we shrink the neighorhoods on which the models are defined, whenever necessary.

We interpolate smoothly between the complements W_0,W_1 to $TX^{\perp_\Pi}$ in $TM|_X$, as follows. Decomposing W_1 in the direct sum $TM|_X=TX^{\perp_\Pi}\oplus W_0$, we find $A\in\Gamma\big(\mathrm{Hom}(W_0,TX^{\perp_\Pi})\big)$ such that $W_1=\mathrm{Graph}(A)$. Setting $W_t:=\mathrm{Graph}(tA)$ for $t\in[0,1]$, the family $\{W_t\}_{t\in[0,1]}$ consists of complements to $TX^{\perp_\Pi}$, i.e. $TM|_X=TX^{\perp_\Pi}\oplus W_t$, and it interpolates between W_0 and W_1. Denote by $q_t:TM|_X\twoheadrightarrow TX^{\perp_\Pi}$ and $j_t:\big(TX^{\perp_\Pi}\big)^*\hookrightarrow T^*M|_X$ the projection and inclusion, respectively, induced by the complement W_t. We first determine the bilinear forms $\sigma_t\in\Gamma(\wedge^2TX^{\perp_\Pi})$ and $\tau_t\in\Gamma(T^*X\otimes TX^{\perp_\Pi})$, which are defined by the formulas (3.5) using the inclusion j_t, in terms of σ_0 and τ_0.

<u>Step 1:</u> We compute σ_t and τ_t.

For $e+w\in TX^{\perp_\Pi}\oplus W_0=TM|_X$, we have

$$\begin{aligned}q_t(e+w)&=q_t\big(e-tA(w)+w+tA(w)\big)\\&=e-tA(w)\\&=q_0(e+w)-tA(Id-q_0)(e+w).\end{aligned}$$

This shows that $q_t=q_0-tA(Id-q_0)$ and therefore $j_t=j_0-t(Id-j_0)A^*$. We now compute for $v_1,v_2\in T_xX$ and $\xi_1,\xi_2\in(T_xX^{\perp_\Pi})^*$:

$$\begin{aligned}\tau_t\big((v_1,\xi_1),(v_2,\xi_2)\big)&=\langle v_1,j_t(\xi_2)\rangle-\langle v_2,j_t(\xi_1)\rangle\\&=\langle v_1,j_0(\xi_2)\rangle-t\langle v_1,(Id-j_0)A^*\xi_2\rangle\\&\qquad-\langle v_2,j_0(\xi_1)\rangle+t\langle v_2,(Id-j_0)A^*\xi_1\rangle\\&=\tau_0\big((v_1,\xi_1),(v_2,\xi_2)\big)+t\langle A(Id-q_0)v_2,\xi_1\rangle\\&\qquad-t\langle A(Id-q_0)v_1,\xi_2\rangle.\end{aligned}\tag{3.7}$$

Similarly, we obtain

$$\begin{aligned}\sigma_t(\xi_1,\xi_2) &= \langle q_t(\Pi^\sharp(j_t(\xi_1))),\xi_2\rangle \\ &= \left\langle \left[\left(q_0 - tA(Id-q_0)\right)\Pi^\sharp\left(j_0 - t(Id-j_0)A^*\right)\right](\xi_1),\xi_2\right\rangle \\ &= \sigma_0(\xi_1,\xi_2) - t\left\langle\left(q_0\Pi^\sharp(Id-j_0)A^*\right)(\xi_1),\xi_2\right\rangle \\ &\qquad - t\left\langle\left(A(Id-q_0)\Pi^\sharp j_0\right)(\xi_1),\xi_2\right\rangle \\ &\qquad + t^2\left\langle\left(A(Id-q_0)\Pi^\sharp(Id-j_0)A^*\right)(\xi_1),\xi_2\right\rangle. \end{aligned} \tag{3.8}$$

Step 2: Get closed extensions, smoothly varying in $t\in[0,1]$, for

$$-\sigma_t \oplus -\tau_t \oplus 0 \in \Gamma(\wedge^2 TX^{\perp_\Pi}) \oplus \Gamma(T^*X\otimes TX^{\perp_\Pi}) \oplus \Gamma(\wedge^2 T^*X).$$

Thanks to [We1, Extension Theorem] and [We1, Relative Poincaré Lemma], we find a one-form β_1, defined on a neighborhood of $X\subset (TX^{\perp_\Pi})^*$, such that

$$\begin{cases}\beta_1|_X = 0,\\ d\beta_1|_X \in \Gamma(T^*X\otimes TX^{\perp_\Pi}),\\ d\beta_1|_X\left((v_1,\xi_1),(v_2,\xi_2)\right) = \langle A(Id-q_0)v_1,\xi_2\rangle - \langle A(Id-q_0)v_2,\xi_1\rangle,\end{cases}$$

for $(v_1,\xi_1),(v_2,\xi_2)\in T_xX\oplus\left(T_xX^{\perp_\Pi}\right)^*$. Similarly, we find one-forms β_2,β_3 defined around $X\subset (TX^{\perp_\Pi})^*$ satisfying

$$\begin{cases}\beta_2|_X = 0,\\ d\beta_2|_X \in \Gamma(\wedge^2 TX^{\perp_\Pi}),\\ d\beta_2|_X(\xi_1,\xi_2) = \left\langle\left(q_0\Pi^\sharp(Id-j_0)A^*\right)(\xi_1),\xi_2\right\rangle + \left\langle\left(A(Id-q_0)\Pi^\sharp j_0\right)(\xi_1),\xi_2\right\rangle,\end{cases}$$

and

$$\begin{cases}\beta_3|_X = 0,\\ d\beta_3|_X \in \Gamma(\wedge^2 TX^{\perp_\Pi}),\\ d\beta_3|_X(\xi_1,\xi_2) = \left\langle\left(A(Id-q_0)\Pi^\sharp(Id-j_0)A^*\right)(\xi_1),\xi_2\right\rangle,\end{cases}$$

for $\xi_1,\xi_2\in\left(T_xX^{\perp_\Pi}\right)^*$. Using (3.7) and (3.8), we see that

$$\left(\eta_0 + td\beta_1 + td\beta_2 - t^2 d\beta_3\right)|_X = -\sigma_t \oplus -\tau_t \oplus 0. \tag{3.9}$$

Step 3: The Moser trick pulls $\left(U,\Pi(W_0,\eta_0)\right)$ to $\left(V',\Pi(W_1,\eta_0+d\beta_1+d\beta_2-d\beta_3)\right)$.

By Proposition 3.3.1, we get a path of Dirac structures

$$\Pi_t := \left(pr^*(i^*L_\Pi)\right)^{\eta_0+td\beta_1+td\beta_2-t^2d\beta_3}$$

for $t \in [0,1]$, where Π_t is Poisson on a neighborhood U_t of X in $\big(TX^{\perp_\Pi}\big)^*$. Note that the set $\bigcup_{t\in[0,1]}\{t\}\times U_t$ is open, since it consists of the points (t,x) for which $(\Pi_t)_x$ is Poisson. The Tube Lemma [Mun] implies that $U' := \bigcap_{t\in[0,1]} U_t$ is an open neighborhood of X on which Π_t is Poisson for all $t \in [0,1]$. Now, these Poisson structures are related by gauge transformations:

$$\Pi_t = \Pi_0^{td\beta_1 + td\beta_2 - t^2 d\beta_3},$$

where

$$\frac{d}{dt}(td\beta_1 + td\beta_2 - t^2 d\beta_3) = -d(2t\beta_3 - \beta_2 - \beta_1).$$

A Poisson version of Moser's theorem (e.g. [Me, Theorem 2.11]) shows that the flow Φ_t of the time-dependent vector field $\Pi_t^\sharp(2t\beta_3 - \beta_2 - \beta_1)$ satisfies $(\Phi_t)_*\Pi_t = \Pi_0$, whenever it is defined. Moreover, since the primitive $2t\beta_3 - \beta_2 - \beta_1$ vanishes along X, the flow Φ_t fixes all points in X. Now set $\phi := \Phi_1^{-1}$. Shrinking U if necessary, we can assume that $\phi : U \to V'$ where $V' := \phi(U)$. We then have

$$\phi : \big(U, \Pi(W_0, \eta_0)\big) \xrightarrow{\sim} \big(V', \Pi(W_1, \eta_0 + d\beta_1 + d\beta_2 - d\beta_3)\big), \qquad \phi|_X = \mathrm{Id}.$$

<u>Step 4:</u> Another Moser argument pulls $\big(V', \Pi(W_1, \eta_0 + d\beta_1 + d\beta_2 - d\beta_3)\big)$ to the model $\big(V, \Pi(W_1, \eta_1)\big)$.

Both η_1 and $\eta_0 + d\beta_1 + d\beta_2 - d\beta_3$ are closed extensions of

$$-\sigma_1 \oplus -\tau_1 \oplus 0 \in \Gamma(\wedge^2 TX^{\perp_\Pi}) \oplus \Gamma(T^*X \otimes TX^{\perp_\Pi}) \oplus \Gamma(\wedge^2 T^*X),$$

see equation (3.9). So their difference $\eta_1 - (\eta_0 + d\beta_1 + d\beta_2 - d\beta_3)$ is exact around X with a primitive γ that vanishes along X, by the Relative Poincaré Lemma. Denote

$$\Pi_0' := (pr^*(i^*L_\Pi))^{\eta_0 + d\beta_1 + d\beta_2 - d\beta_3}, \qquad \Pi_t' := (\Pi_0')^{td\gamma},$$

for $t \in [0,1]$. Since Π_0' is Poisson on V' and $d\gamma|_X = 0$, we see that Π_t' is Poisson on a neighborhood V_t' of X in $(TX^{\perp_\Pi})^*$. Using the Tube Lemma as in Step 3, we find a neighborhood O of X in $(TX^{\perp_\Pi})^*$ such that Π_t' is Poisson on O for all $t \in [0,1]$. The Moser Theorem [Me, Theorem 2.11] implies that the flow Ψ_t of the time-dependent vector field $-(\Pi_t')^\sharp(\gamma)$ satisfies $(\Psi_t)_*\Pi_t' = \Pi_0'$, whenever it is defined. Moreover, since $\gamma|_X = 0$, the flow Ψ_t fixes all points of X. Now set $\psi := \Psi_1^{-1}$. Shrinking both V' and V if necessary, we can assume that $\psi : V' \to V$. We then have

$$\psi : \big(V', \Pi(W_1, \eta_0 + d\beta_1 + d\beta_2 - d\beta_3)\big) \xrightarrow{\sim} \big(V, \Pi(W_1, \eta_1)\big), \qquad \psi|_X = \mathrm{Id}.$$

The diffeomorphism $\psi \circ \phi$ now satisfies the criteria: it fixes points in X and

$$\psi \circ \phi : \big(U, \Pi(W_0, \eta_0)\big) \xrightarrow{\sim} \big(V, \Pi(W_1, \eta_1)\big). \qquad \square$$

It is now justified to call $\big(U, \Pi(W,\eta)\big)$ **the local model** for the local Poisson saturation of the regular submanifold $X \subset (M, \Pi)$.

3.4 The normal form

We now proceed by showing that the local Poisson saturation of a regular submanifold $X \subset (M, \Pi)$ is isomorphic around X to the local model $\big(U, \Pi(W,\eta)\big)$ constructed in Proposition 3.3.1. We will use the theory of dual pairs in Dirac geometry, as developed in [FM2]. We first need a lemma, which describes how to obtain a weak Dirac dual pair out of the self-dual pair (3.2)

$$(M,\Pi) \xleftarrow{pr} (\Sigma, \Omega_\chi) \xrightarrow{\exp_\chi} (M, -\Pi),$$

given a regular submanifold $X \subset (M, \Pi)$. Recall from the proof of Theorem 3.2.6 that the local Poisson saturation (P, Π_P) of $X \subset (M, \Pi)$ is given by $\exp_\chi(\Sigma|_X)$.

Lemma 3.4.1. *Let $i : X \hookrightarrow (M, \Pi)$ be a regular submanifold with local Poisson saturation (P, Π_P). Then the following is a weak Dirac dual pair, in the sense of [FM2]:*

$$(X, i^*L_\Pi) \xleftarrow{pr} \big(\Sigma|_X, Gr(\Omega_\chi|_X)\big) \xrightarrow{\exp_\chi} (P, -L_{\Pi_P}).$$

This means that $\Omega_\chi|_X$ is a closed two-form on $\Sigma|_X$, that pr and $\exp_\chi$ are surjective forward Dirac submersions and

$$\big(\Omega_\chi|_X\big)(S_1, S_2) = 0, \tag{3.10}$$

$$rk(S_1 \cap K \cap S_2) = \dim \Sigma|_X - \dim X - \dim P, \tag{3.11}$$

where $S_1 := \ker dpr, S_2 := \ker d\exp_\chi$ and $K := \ker\big(\Omega_\chi|_X\big)$.

Proof. It is clear that pr is a surjective submersion. The fact that $\exp_\chi$ is a surjective submersion follows from the proof of Theorem 3.2.6. Also the property (3.10) is automatic, since (3.2) is a dual pair. To see that $pr : \big(\Sigma|_X, \mathrm{Gr}(\Omega_\chi|_X)\big) \to (X, i^*L_\Pi)$ is forward Dirac, consider the following commutative diagram of Dirac manifolds and smooth maps:

$$\begin{array}{ccc} \big(\Sigma|_X, \mathrm{Gr}(\Omega_\chi|_X)\big) & \xrightarrow{pr} & (X, i^*L_\Pi) \\ \Big\downarrow{\scriptstyle i'} & & \Big\downarrow{\scriptstyle i} \\ (\Sigma, \mathrm{Gr}(\Omega_\chi)) & \xrightarrow{pr} & (M, L_\Pi) \end{array}.$$

The maps i' on the left and i on the right are backward Dirac by definition, and the bottom map pr is forward Dirac because of the dual pair (3.2). Since the bottom map pr is a submersion, we can apply [FM3, Lemma 3] to obtain that also the map at the top $pr : \big(\Sigma|_X, \mathrm{Gr}(\Omega_\chi|_X)\big) \to (X, i^* L_\Pi)$ is forward Dirac.

Similarly, we get that $\exp_\chi : \big(\Sigma|_X, \mathrm{Gr}(\Omega_\chi|_X)\big) \to (P, -L_{\Pi_P})$ is forward Dirac considering the diagram

$$\begin{array}{ccc} \big(\Sigma|_X, \mathrm{Gr}(\Omega_\chi|_X)\big) & \xrightarrow{\exp_\chi} & (P, -L_{\Pi_P}) \\ \Big\downarrow{\scriptstyle i'} & & \Big\downarrow{\scriptstyle i} \\ (\Sigma, \mathrm{Gr}(\Omega_\chi)) & \xrightarrow{\exp_\chi} & (M, -L_\Pi) \end{array}.$$

Here the map i' is backward Dirac, the map i is backward (and forward) Dirac, and the bottom map $\exp_\chi$ is forward Dirac because of the dual pair (3.2). Again, the map $\exp_\chi$ on the bottom is a submersion, so we can apply [FM3, Lemma 3] to obtain that also $\exp_\chi : \big(\Sigma|_X, \mathrm{Gr}(\Omega_\chi|_X)\big) \to (P, -L_{\Pi_P})$ is forward Dirac.

It remains to check that the property (3.11) holds. For $(x,\xi) \in \Sigma|_X$, we have

$$\begin{aligned} K_{(x,\xi)} &= \big(\Omega_\chi^\flat\big)^{-1}\left(T_{(x,\xi)}(T^*M|_X)\right)^0 \cap T_{(x,\xi)}(T^*M|_X) \\ &= \Pi_\chi^\sharp\left((dpr)^*_{(x,\xi)} T_x X^0\right) \cap T_{(x,\xi)}(T^*M|_X). \end{aligned}$$

Consequently, we obtain

$$(S_1)_{(x,\xi)} \cap K_{(x,\xi)} = \Pi_\chi^\sharp\left((dpr)^*_{(x,\xi)}\left(T_x X^0 \cap \ker \Pi_x^\sharp\right)\right),$$

using that the left leg of the dual pair (3.2) is a Poisson map. The equality (3.3) in the proof of Lemma 3.2.5 shows that

$$(S_2)_{(x,\xi)} = \Pi_\chi^\sharp\left((dpr)^*_{(x,\xi)}\left(T_x X^{\perp_\Pi}\right)^0\right),$$

so we obtain

$$(S_1)_{(x,\xi)} \cap K_{(x,\xi)} \cap (S_2)_{(x,\xi)} = \Pi_\chi^\sharp\left((dpr)^*_{(x,\xi)}\left(T_x X^0 \cap \ker \Pi_x^\sharp\right)\right).$$

Consequently,

$$\begin{aligned} rk(S_1 \cap K \cap S_2)_{(x,\xi)} &= \dim\left(T_x X^0 \cap \ker \Pi_x^\sharp\right) \\ &= \dim(T_x X^0) - \dim(T_x X^{\perp_\Pi}) \\ &= (\dim M - \dim X) - (\dim P - \dim X) \\ &= \dim \Sigma|_X - \dim X - \dim P. \end{aligned}$$

So also the property (3.11) holds, and this finishes the proof. □

We are now ready to state the main result of this section.

Theorem 3.4.2. *Let $X \subset (M, \Pi)$ be a regular submanifold with local Poisson saturation (P, Π_P). Choose a complement W to $TX^{\perp_\Pi}$ in $TM|_X$ and denote by $j : (TX^{\perp_\Pi})^* \hookrightarrow T^*M|_X$ the inclusion. Then $-j^*(\Omega_\chi|_X) \in \mathcal{E}_W(-\sigma - \tau)$. Moreover, the corresponding local model $\big(U, \Pi\big(W, -j^*(\Omega_\chi|_X)\big)\big)$ is isomorphic with (P, Π_P) around X. Explicitly, a Poisson diffeomorphism onto an open neighborhood of X is given by*

$$\exp_\chi \circ j : \big(U, \Pi\big(W, -j^*(\Omega_\chi|_X)\big)\big) \xrightarrow{\sim} (P, \Pi_P).$$

We will denote by pr_M and pr_X the bundle projections $T^*M|_X \to X$ and $\big(TX^{\perp_\Pi}\big)^* \to X$, respectively. So $pr_M \circ j = pr_X$.

Proof. We first check that $-j^*(\Omega_\chi|_X) \in \mathcal{E}_W(-\sigma - \tau)$. The fact that $-j^*(\Omega_\chi|_X)$ restricts along $X \subset (TX^{\perp_\Pi})^*$ as required in (3.6) is an immediate consequence of the following equality [FM1, Lemma 8]:

$$\Omega_\chi\big((v_1, \xi_1), (v_2, \xi_2)\big) = \langle v_1, \xi_2 \rangle - \langle v_2, \xi_1 \rangle + \Pi(\xi_1, \xi_2),$$

where $(v_1, \xi_1), (v_2, \xi_2) \in T_x(T^*M) = T_xM \oplus T_x^*M$ for $x \in M$.

To prove the second statement, we apply [FM2, Proposition 6] to the weak dual pair constructed in Lemma 3.4.1:

$$(X, i^*L_\Pi) \xleftarrow{pr_M} \big(\Sigma|_X, \mathrm{Gr}(\Omega_\chi|_X)\big) \xrightarrow{\exp_\chi} (P, -L_{\Pi_P}),$$

and we get the following equality of Dirac structures on $\Sigma|_X$:

$$(pr_M^*(i^*L_\Pi))^{-\Omega_\chi|_X} = \exp_\chi^* L_{\Pi_P}.$$

Since $j : (TX^{\perp_\Pi})^* \hookrightarrow T^*M|_X$ is transverse to the leaves of this Dirac structure, we can pull it back to $j^{-1}(\Sigma|_X) \cong j(TX^{\perp_\Pi})^* \cap \Sigma|_X$, and we obtain

$$\begin{aligned}(\exp_\chi \circ j)^* L_{\Pi_P} &= j^*\big[(pr_M^*(i^*L_\Pi))^{-\Omega_\chi|_X}\big] \\ &= \big[(pr_M \circ j)^*(i^*L_\Pi)\big]^{-j^*(\Omega_\chi|_X)} \\ &= (pr_X^*(i^*L_\Pi))^{-j^*(\Omega_\chi|_X)}. \end{aligned} \tag{3.12}$$

The Dirac structure (3.12) corresponds with a Poisson structure on a neighborhood $U \subset j^{-1}(\Sigma|_X) \subset (TX^{\perp_\Pi})^*$, where it defines the local model $\big(U, \Pi\big(W, -j^*(\Omega_\chi|_X)\big)\big)$. Moreover, by the proof of Theorem 3.2.6, we know that $\exp_\chi \circ j$ takes $j^{-1}(\Sigma|_X)$ diffeomorphically onto P. So we obtain that

$$\exp_\chi \circ j : \big(U, \Pi\big(W, -j^*(\Omega_\chi|_X)\big)\big) \to (P, \Pi_P)$$

is a Poisson diffeomorphism onto an open neighborhood of $X \subset (P, \Pi_P)$. □

Remark 3.4.3. As we mentioned before, our normal form is consistent with another normal form result that already appeared in the literature. In [BLM], [FM2] one proves that, given a Dirac manifold (M, L) and a transversal $\tau \subset M$, the pullback Dirac structure on τ determines L in a neighborhood of τ, up to diffeomorphisms and exact gauge transformations. In our case, a regular submanifold $X \subset (M, \Pi)$ is a transversal in its local Poisson saturation, and our normal form for (P, Π_P) around X agrees with the above result.

3.5 Some particular cases

We proved that the local model $\big(U, \Pi(W, \eta)\big)$ described in Proposition 3.3.1 does not depend on the choice of complement W to $TX^{\perp_\Pi}$ in $TM|_X$, nor on the choice of closed extension η. We now show that, for certain classes of regular submanifolds $X \subset (M, \Pi)$, a good choice of complement and/or closed extension simplifies the normal form considerably. Some of our results recover well-known normal form and rigidity statements around distinguished submanifolds in symplectic and Poisson geometry.

3.5.1 Submanifolds in symplectic geometry

Recall that, if (M, ω) is a symplectic manifold and $N \subset M$ is any submanifold, then the restriction of ω to $TM|_N$ determines the symplectic form ω on a neighborhood of N (see [We2, Theorem 4.1]). We can recover this result from our normal form, as follows.

In case $\Pi = \omega^{-1}$ is symplectic, any submanifold $X \subset (M, \Pi)$ is regular since $TX^{\perp_\Pi} = TX^{\perp_\omega}$, where $TX^{\perp_\omega} = \{v \in TM|_X : \omega(v, w) = 0 \ \forall w \in TX\}$ denotes the symplectic orthogonal of X. Next, the local Poisson saturation (P, Π_P) of X is an embedded submanifold of M of dimension $\dim X + rk\big(\Pi^\sharp(TX^{\perp_\Pi})^*\big)$, by the equality (3.4). So if Π is symplectic, then $P \subset M$ is an open neighborhood of X. At last, the Poisson structure $\Pi(W, \eta) = \big(pr^*(i^* L_\Pi)\big)^\eta$ from the local model is determined by the restriction $\Pi|_X$, as the pullback $i^* L_\Pi$ and $\eta|_X$ only depend on $\Pi^\sharp|_{(TX^{\perp_\Pi})^0}$ and $\Pi^\sharp|_{(TX^{\perp_\Pi})^*}$, respectively.

In conclusion, our normal form shows that, for any submanifold X of the symplectic manifold (M, Π), the restriction $\Pi|_X$ determines Π on a neighborhood of $X \subset M$, which recovers the aforementioned result in symplectic geometry.

3.5.2 Poisson transversals

A submanifold X of a Poisson manifold (M, Π) is called a Poisson transversal if it meets each symplectic leaf transversally and symplectically, that is

$$TX \oplus TX^{\perp_\Pi} = TM|_X.$$

In the local model of Proposition (3.3.1), we can take TX as a canonical complement to $TX^{\perp_\Pi}$ in $TM|_X$. The inclusion $j : (TX^{\perp_\Pi})^* \hookrightarrow T^*M|_X$ induced by it identifies $(TX^{\perp_\Pi})^*$ with TX^0. The following simplifications occur in the local model:

- The pullback i^*L_Π of the Dirac structure L_Π to X defines a Poisson structure on X [FM1, Lemma 1], which we denote by $\Pi_X \in \Gamma(\wedge^2 TX)$.
- Consider $\sigma \in \Gamma(\wedge^2 TX^{\perp_\Pi})$ and $\tau \in \Gamma(T^*X \otimes TX^{\perp_\Pi})$ defined in (3.5):

$$\sigma(\xi_1, \xi_2) = \Pi\big(j(\xi_1), j(\xi_2)\big),$$
$$\tau\big((v_1, \xi_1), (v_2, \xi_2)\big) = \langle v_1, j(\xi_2)\rangle - \langle v_2, j(\xi_1)\rangle,$$

  for $\xi_1, \xi_2 \in (T_xX^{\perp_\Pi})^*$ and $v_1, v_2 \in T_xX$. Since $j\left((TX^{\perp_\Pi})^*\right) = TX^0$, we have $\tau \equiv 0$, and since the restriction of Π to the conormal bundle TX^0 is fiberwise non-degenerate, we get a symplectic vector bundle $\big((TX^{\perp_\Pi})^*, \sigma\big)$.

Moreover, since X is a transversal, its local Poisson saturation (P, Π_P) is in fact an open neighborhood of X in M. In conclusion, our normal form shows that an open neighborhood of X in (M, Π) is Poisson diffeomorphic with a neighborhood of X in $(TX^{\perp_\Pi})^*$, endowed with the Poisson structure

$$\big(pr^*(L_{\Pi_X})\big)^\eta,$$

where η is a closed extension of $-\sigma$. This is the normal form proved in [FM1].

3.5.3 Regular coisotropic submanifolds

Recall that a submanifold N of a symplectic manifold (M, ω) is called coisotropic if its symplectic orthogonal $TN^{\perp_\omega}$ is contained in TN. Gotay's theorem [G] provides a normal form for ω around N, which is obtained as follows. Choose a complement to $TN^{\perp_\omega}$ inside TN, and denote by $j : (TN^{\perp_\omega})^* \hookrightarrow T^*N$ the induced inclusion. On the total space of the vector bundle $pr : (TN^{\perp_\omega})^* \to N$, one gets a closed two-form

$$pr^*(i^*\omega) + j^*\omega_{can},$$

where $i^*\omega$ is the pullback of ω to N and ω_{can} is the canonical symplectic form on T^*N. This two-form is non-degenerate on a neighborhood of the zero section

$N \subset (TN^{\perp_\omega})^*$, and (M, ω) is isomorphic with $\big((TN^{\perp_\omega})^*, pr^*(i^*\omega) + j^*\omega_{can}\big)$ around N. In particular, the pullback $i^*\omega \in \Gamma(\wedge^2 T^*N)$ determines ω on a neighborhood of $N \subset M$.

More generally, recall that a submanifold X of a Poisson manifold (M, Π) is coisotropic if $TX^{\perp_\Pi} \subset TX$. In this subsection, we prove a Poisson version of Gotay's theorem by specializing our normal form to regular submanifolds $X \subset (M, \Pi)$ that are coisotropic. First, we want to find a convenient complement to $TX^{\perp_\Pi}$ in $TM|_X$.

Lemma 3.5.1. *Let $X \subset (M, \Pi)$ be a regular coisotropic submanifold. For any choice of splitting $TX = TX^{\perp_\Pi} \oplus G$, there is a splitting $TM|_X = TX^{\perp_\Pi} \oplus W_G$ such that*

$$\Pi^\sharp(W_G^0) \subset W_G \quad \text{and} \quad W_G \cap TX = G.$$

Proof. We divide the proof into four steps.

Step 1: $\Pi^\sharp(G^0)$ has constant rank, equal to twice the rank of $TX^{\perp_\Pi}$.

Note that $\ker \Pi^\sharp \subset (TX^{\perp_\Pi})^0$ and therefore

$$\ker \Pi^\sharp \cap G^0 = \ker \Pi^\sharp \cap (TX^{\perp_\Pi})^0 \cap G^0 = \ker \Pi^\sharp \cap (TX^{\perp_\Pi} + G)^0 = \ker \Pi^\sharp \cap TX^0.$$

Since X is regular, the latter has constant rank, which shows that also $\Pi^\sharp(G^0)$ has constant rank. Explicitly,

$$\begin{aligned} rk(\Pi^\sharp(G^0)) &= \dim M - rk(G) - rk(\ker \Pi^\sharp \cap G^0) \\ &= \dim M - rk(G) - rk(\ker \Pi^\sharp \cap TX^0) \\ &= \dim M - rk(G) - \big(\dim M - \dim X - rk(TX^{\perp_\Pi})\big) \\ &= 2rk(TX^{\perp_\Pi}). \end{aligned}$$

Step 2: $\big(\Pi^\sharp(G^0), \omega\big)$ is a symplectic vector bundle, where

$$\omega(\Pi^\sharp(\alpha), \Pi^\sharp(\beta)) := \Pi(\alpha, \beta).$$

We first check that $\Pi^\sharp(G^0) \cap G = \{0\}$. If $\alpha \in G^0$ and $\Pi^\sharp(\alpha) \in G \subset TX$, then $\alpha \in (TX^{\perp_\Pi})^0$ because of the exact sequence (3.1). But we then have that $\alpha \in G^0 \cap (TX^{\perp_\Pi})^0 = (G + TX^{\perp_\Pi})^0 = TX^0$, so that $\Pi^\sharp(\alpha) \in TX^{\perp_\Pi}$. Consequently, $\Pi^\sharp(\alpha) \in G \cap TX^{\perp_\Pi} = \{0\}$.

It now follows that ω is non-degenerate: if $\Pi^\sharp(\alpha) \in \ker \omega$ for $\alpha \in G^0$, then for all $\beta \in G^0$, we get $\langle \Pi^\sharp(\alpha), \beta \rangle = 0$, which implies that $\Pi^\sharp(\alpha) \in G$. So we obtain $\Pi^\sharp(\alpha) \in \Pi^\sharp(G^0) \cap G = \{0\}$, which shows that ω is non-degenerate.

Step 3: $TX^{\perp_\Pi} \subset \big(\Pi^\sharp(G^0), \omega\big)$ is a Lagrangian subbundle.

Since $G \subset TX$, we have $TX^0 \subset G^0$ and therefore $TX^{\perp_\Pi} \subset \Pi^\sharp(G^0)$. By Step 1, the rank of $TX^{\perp_\Pi}$ is half the rank of $\Pi^\sharp(G^0)$, so we only have to show that $TX^{\perp_\Pi} \subset \big(\Pi^\sharp(G^0), \omega\big)$ is an isotropic subbundle. This is clearly the case, since for $\alpha, \beta \in TX^0$ we have

$$\omega(\Pi^\sharp(\alpha), \Pi^\sharp(\beta)) = \langle \Pi^\sharp(\alpha), \beta \rangle = 0,$$

using that $\Pi^\sharp(\alpha) \in TX^{\perp_\Pi} \subset TX$.

Step 4: Let $V \subset \big(\Pi^\sharp(G^0), \omega\big)$ be a Lagrangian complement of $TX^{\perp_\Pi}$, and choose a subbundle $H \subset TM|_X$ such that $TM|_X = TX^{\perp_\Pi} \oplus (V \oplus G \oplus H)$. Then $W_G := V \oplus G \oplus H$ satisfies the criteria.

We check that $\Pi^\sharp(W_G^0) \subset W_G$. If $\alpha \in W_G^0$, then in particular $\alpha \in V^0 \cap G^0$, so for all $v \in V$ we get

$$0 = \langle \alpha, v \rangle = \omega(v, \Pi^\sharp(\alpha)).$$

This shows that $\Pi^\sharp(\alpha)$ lies in the symplectic orthogonal $V^{\perp_\omega}$, which is equal to V because V is Lagrangian. So $\Pi^\sharp(\alpha) \in V \subset W_G$. At last, the fact that $W_G \cap TX = G$ follows immediately from the decomposition

$$TM|_X = TX^{\perp_\Pi} \oplus (V \oplus G \oplus H) = TX \oplus V \oplus H. \qquad \square$$

Corollary 3.5.2 (Poisson version of Gotay's Theorem)**.** *Let $i : X \hookrightarrow (M, \Pi)$ be a regular coisotropic submanifold. Pick a complement $TX = TX^{\perp_\Pi} \oplus G$, and let $j : (TX^{\perp_\Pi})^* \hookrightarrow T^*X$ be the induced inclusion. The local Poisson saturation of X is Poisson diffeomorphic around X with the model*

$$\Big(U, \big(pr^*(i^*L_\Pi)\big)^{j^*\omega_{can}}\Big), \tag{3.13}$$

where $U \subset (TX^{\perp_\Pi})^*$ is an open neighborhood of X and ω_{can} denotes the canonical symplectic form on T^*X.

Proof. It suffices to show that the expression (3.13) is diffeomorphic with a local model for the local Poisson saturation of X. By Lemma 3.5.1, we know that the splitting $TX = TX^{\perp_\Pi} \oplus G$ induces a splitting $TM|_X = TX^{\perp_\Pi} \oplus W_G$, where

$$\Pi^\sharp(W_G^0) \subset W_G \quad \text{and} \quad W_G \cap TX = G.$$

Denote by $\tilde{j} : (TX^{\perp_\Pi})^* \hookrightarrow T^*M|_X$ the inclusion induced by the complement W_G; it embeds $(TX^{\perp_\Pi})^*$ into $T^*M|_X$ as $(W_G)^0$. Consider $\sigma \in \Gamma(\wedge^2 TX^{\perp_\Pi})$ and $\tau \in \Gamma(T^*X \otimes TX^{\perp_\Pi})$ as defined in (3.5):

$$\sigma(\xi_1, \xi_2) = \Pi\big(\tilde{j}(\xi_1), \tilde{j}(\xi_2)\big),$$
$$\tau\big((v_1, \xi_1), (v_2, \xi_2)\big) = \langle v_1, \tilde{j}(\xi_2)\rangle - \langle v_2, \tilde{j}(\xi_1)\rangle,$$

for $\xi_1, \xi_2 \in \big(T_xX^{\perp_\Pi}\big)^*$ and $v_1, v_2 \in T_xX$. Since $\Pi^\sharp(W_G^0) \subset W_G$, we have $\sigma \equiv 0$, and since $W_G \cap TX = G$, we have

$$\begin{aligned}\tau\big((v_1,\xi_1),(v_2,\xi_2)\big) &= \langle v_1, \tilde{j}(\xi_2)\rangle - \langle v_2, \tilde{j}(\xi_1)\rangle \\ &= \langle v_1, j(\xi_2)\rangle - \langle v_2, j(\xi_1)\rangle \\ &= (j^*\omega_{can})|_X \big((v_1,\xi_1),(v_2,\xi_2)\big)\end{aligned}$$

for $\xi_1, \xi_2 \in (T_xX^{\perp_\Pi})^*$ and $v_1, v_2 \in T_xX$. This shows that $\big(U, \Pi(W_G, -j^*\omega_{can})\big)$ is a local model for the local Poisson saturation of X, where $U \subset (TX^{\perp_\Pi})^*$ is a suitable neighborhood of X. Note that the Dirac structure (3.13) still differs by a sign from this model; we now show that changing the sign produces a diffeomorphic Dirac structure. Shrinking U if necessary, we can assume that U is invariant under fiberwise multiplication by -1. Denoting this map by m_{-1}, we have

$$m_{-1}^*\left((pr^*(i^*L_\Pi))^{j^*\omega_{can}}\right) = \big((pr{\circ}m_{-1})^* i^* L_\Pi\big)^{(j\circ m_{-1})^*\omega_{can}} = (pr^*(i^*L_\Pi))^{-j^*\omega_{can}}.$$

Since the latter is the Poisson structure $\big(U, \Pi(W_G, -j^*\omega_{can})\big)$, we are done. □

In particular, the pullback Dirac structure i^*L_Π determines a neighborhood of X in its local Poisson saturation. If (M,Π) is symplectic, then the above corollary indeed recovers Gotay's theorem.

3.5.4 Regular pre-Poisson submanifolds

Recall that, given a symplectic manifold (M,ω), a submanifold $i : N \hookrightarrow (M,\omega)$ is said to be of constant rank if the pullback $i^*\omega$ has constant rank. Marle's constant rank theorem [Ma] states that a neighborhood of a constant rank submanifold $i : N \hookrightarrow (M,\omega)$ is determined by the pullback $i^*\omega$ together with the restriction of ω to the symplectic vector bundle $TN^{\perp_\omega}/\big(TN^{\perp_\omega} \cap TN\big)$.

Generalizing this notion to Poisson geometry, a submanifold X of a Poisson manifold (M,Π) is called pre-Poisson if $TX + TX^{\perp_\Pi}$ has constant rank [CZ1]. It is equivalent to ask that the bundle map $pr \circ \Pi^\sharp : TX^0 \to TX^{\perp_\Pi} \to TM|_X/TX$ has constant rank. Examples include Poisson transversals (in which case $pr \circ \Pi^\sharp$ is an isomorphism) and coisotropic submanifolds (in which case $pr \circ \Pi^\sharp$ is the zero map). If X is regular pre-Poisson, i.e. $TX^{\perp_\Pi}$ has constant rank, then its characteristic distribution $TX^{\perp_\Pi} \cap TX$ also has constant rank.

In this subsection, we prove a Poisson version of Marle's theorem by specializing our normal form to regular pre-Poisson submanifolds $X \subset (M,\Pi)$. We will need the following result, which generalizes Lemma 3.5.1.

Lemma 3.5.3. *Let $X \subset (M, \Pi)$ be a regular pre-Poisson submanifold. For any choice of splittings $TX = (TX^{\perp_\Pi} \cap TX) \oplus G$ and $TX^{\perp_\Pi} = (TX^{\perp_\Pi} \cap TX) \oplus H$, there exists a complement $TM|_X = (TX^{\perp_\Pi} \cap TX) \oplus H \oplus W_{G,H}$ such that*

$$\Pi^\sharp\left(\left(H + W_{G,H}\right)^0\right) \subset W_{G,H} \quad \text{and} \quad W_{G,H} \cap TX = G.$$

Proof. We have in particular

$$TX + TX^{\perp_\Pi} = (TX^{\perp_\Pi} \cap TX) \oplus G \oplus H. \tag{3.14}$$

The proof is divided into four steps, which generalize those of Lemma 3.5.1.

Step 1: $\Pi^\sharp\big((G+H)^0\big)$ has constant rank, equal to twice the rank of $TX^{\perp_\Pi} \cap TX$.

Since $\ker \Pi^\sharp \subset (TX^{\perp_\Pi})^0 \subset (TX^{\perp_\Pi} \cap TX)^0$, we have

$$\begin{aligned}
\ker \Pi^\sharp \cap (G+H)^0 &= \ker \Pi^\sharp \cap (TX^{\perp_\Pi} \cap TX)^0 \cap (G+H)^0 \\
&= \ker \Pi^\sharp \cap \big((TX^{\perp_\Pi} \cap TX) + G + H\big)^0 \\
&= \ker \Pi^\sharp \cap (TX + TX^{\perp_\Pi})^0 \\
&= \ker \Pi^\sharp \cap TX^0 \cap (TX^{\perp_\Pi})^0 \\
&= \ker \Pi^\sharp \cap TX^0.
\end{aligned}$$

Since X is regular, the latter has constant rank, showing that also $\Pi^\sharp\big((G+H)^0\big)$ has constant rank. Explicitly,

$$\begin{aligned}
rk\left(\Pi^\sharp\big((G+H)^0\big)\right) &= \dim M - rk(G+H) - rk\Big(\ker \Pi^\sharp \cap (G+H)^0\Big) \\
&= \dim M - rk(TX + TX^{\perp_\Pi}) + rk(TX^{\perp_\Pi} \cap TX) \\
&\qquad - rk\Big(\ker \Pi^\sharp \cap TX^0\Big) \\
&= \dim M - rk(TX + TX^{\perp_\Pi}) + rk(TX^{\perp_\Pi} \cap TX) \\
&\qquad - \Big(\dim M - \dim X - rk(TX^{\perp_\Pi})\Big) \\
&= rk(TX) + rk(TX^{\perp_\Pi}) - rk(TX + TX^{\perp_\Pi}) \\
&\qquad + rk(TX^{\perp_\Pi} \cap TX) \\
&= 2rk(TX^{\perp_\Pi} \cap TX).
\end{aligned}$$

Step 2: $\left(\Pi^\sharp\big((G+H)^0\big), \omega\right)$ is a symplectic vector bundle, where

$$\omega(\Pi^\sharp(\alpha), \Pi^\sharp(\beta)) := \Pi(\alpha, \beta).$$

We first show that $\Pi^\sharp\big((G+H)^0\big)\cap(G+H)=\{0\}$. Assume that $\gamma\in(G+H)^0$ is such that $\Pi^\sharp(\gamma)=g+h\in G+H$. Since $h\in TX^{\perp_\Pi}$, we can write $h=\Pi^\sharp(\beta)$ for some $\beta\in TX^0$, and we obtain that $\Pi^\sharp(\gamma-\beta)=g\in TX$. By (3.1), we then get that $\gamma-\beta\in(TX^{\perp_\Pi})^0$, and therefore $\gamma\in TX^0+(TX^{\perp_\Pi})^0=(TX\cap TX^{\perp_\Pi})^0$. Hence,

$$\gamma\in(TX\cap TX^{\perp_\Pi})^0\cap(G+H)^0=(TX+TX^{\perp_\Pi})^0=TX^0\cap(TX^{\perp_\Pi})^0,$$

using (3.14) in the first equality. This implies that $\Pi^\sharp(\gamma)\in TX^{\perp_\Pi}\cap TX$, so we obtain that $\Pi^\sharp(\gamma)\in(TX^{\perp_\Pi}\cap TX)\cap(G+H)=\{0\}$. This shows that $\Pi^\sharp\big((G+H)^0\big)\cap(G+H)=\{0\}$.

It now follows that ω is non-degenerate: if $\Pi^\sharp(\alpha)\in\ker\omega$ for $\alpha\in(G+H)^0$, then for all $\beta\in(G+H)^0$ we get $\langle\Pi^\sharp(\alpha),\beta\rangle=0$, which shows that $\Pi^\sharp(\alpha)\in G+H$. By what we just proved, we then get $\Pi^\sharp(\alpha)\in\Pi^\sharp\big((G+H)^0\big)\cap(G+H)=\{0\}$, which shows that ω is non-degenerate.

Step 3: $TX^{\perp_\Pi}\cap TX\subset\big(\Pi^\sharp\big((G+H)^0\big),\omega\big)$ is a Lagrangian subbundle.

Since $G+H\subset TX+TX^{\perp_\Pi}$, we have $(TX+TX^{\perp_\Pi})^0\subset(G+H)^0$ and therefore

$$TX^{\perp_\Pi}\cap TX=\Pi^\sharp\big(TX^0\cap(TX^{\perp_\Pi})^0\big)=\Pi^\sharp\big((TX+TX^{\perp_\Pi})^0\big)\subset\Pi^\sharp\big((G+H)^0\big).$$

By Step 1, we know that the rank of $\Pi^\sharp\big((G+H)^0\big)$ is twice the rank of $TX^{\perp_\Pi}\cap TX$, so we only have to check that $TX^{\perp_\Pi}\cap TX\subset\big(\Pi^\sharp\big((G+H)^0\big),\omega\big)$ is an isotropic subbundle. This is clearly the case, for if $\alpha,\beta\in TX^0\cap(TX^{\perp_\Pi})^0$ then

$$\omega\big(\Pi^\sharp(\alpha),\Pi^\sharp(\beta)\big)=\langle\Pi^\sharp(\alpha),\beta\rangle=0.$$

Here we use that $\Pi^\sharp(\alpha)\in TX$ since $\alpha\in\big(TX^{\perp_\Pi}\big)^0$, and that $\beta\in TX^0$.

Step 4: Let $C\subset\big(\Pi^\sharp\big((G+H)^0\big),\omega\big)$ be a Lagrangian complement of $TX^{\perp_\Pi}\cap TX$, and choose any subbundle $Y\subset TM|_X$ such that

$$TM|_X=(TX^{\perp_\Pi}\cap TX)\oplus(H\oplus G\oplus C\oplus Y).$$

Then the subbundle $W_{G,H}:=G\oplus C\oplus Y$ satisfies the criteria.

If $\alpha\in(H+G+C+Y)^0$, then $\alpha\in(G+H)^0$ and $\alpha\in C^0$. So for all $c\in C$,

$$0=\langle\alpha,c\rangle=\omega\big(c,\Pi^\sharp(\alpha)\big),$$

so that $\Pi^\sharp(\alpha)\in C^{\perp_\omega}=C\subset G+C+Y$. Hence, $\Pi^\sharp\big((H+W_{G,H})^0\big)\subset W_{G,H}$. The fact that $W_{G,H}\cap TX=G$ follows immediately from the decomposition

$$TM|_X=(TX^{\perp_\Pi}\cap TX)\oplus H\oplus W_{G,H}=TX\oplus H\oplus C\oplus Y. \qquad \square$$

Corollary 3.5.4 (Poisson version of Marle's theorem)**.** *If $i : X \hookrightarrow (M,\Pi)$ is a regular pre-Poisson submanifold, then its local Poisson saturation is completely determined around X by the pullback Dirac structure i^*L_Π and the restriction of Π to $(TX^{\perp_\Pi})^*/(TX^{\perp_\Pi}\cap TX)^*$.*

Proof. We show that any local model for the local Poisson saturation of X defined in terms of a complement as specified in Lemma 3.5.3 only depends on the data mentioned in the statement. Lemma 3.5.3 implies that there is a splitting $TM|_X = (TX^{\perp_\Pi}\cap TX)\oplus V\oplus W$, satisfying

$$TX^{\perp_\Pi} = (TX^{\perp_\Pi}\cap TX)\oplus V \quad \text{and} \quad \Pi^\sharp\big((V+W)^0\big)\subset W. \tag{3.15}$$

We get inclusion maps $j_1 : (TX^{\perp_\Pi}\cap TX)^* \hookrightarrow T^*M|_X$, $j_2 : (TX^{\perp_\Pi})^* \hookrightarrow T^*M|_X$ and $j : (TX^{\perp_\Pi}\cap TX)^* \hookrightarrow (TX^{\perp_\Pi})^*$ satisfying

$$j_1 = j_2\circ j, \quad j_1\big((TX^{\perp_\Pi}\cap TX)^*\big) = (V+W)^0 \quad \text{and} \quad j_2\big((TX^{\perp_\Pi})^*\big) = W^0.$$

This implies that, for $\xi_1\in(TX^{\perp_\Pi}\cap TX)^*$ and $\xi_2\in(TX^{\perp_\Pi})^*$:

$$\Pi\big(j_2(j(\xi_1)), j_2(\xi_2)\big) = \Pi\big(j_1(\xi_1), j_2(\xi_2)\big) = 0,$$

using the inclusion (3.15). So the local model in Proposition 3.3.1 only depends on the pullback i^*L_Π and the restriction of Π to $(TX^{\perp_\Pi})^*/(TX^{\perp_\Pi}\cap TX)^*$. □

The corollary shows that the local Poisson saturation of a regular pre-Poisson submanifold is determined by less data than that of a general regular submanifold, since it uses a quotient of $(TX^{\perp_\Pi})^*$ rather than all of $(TX^{\perp_\Pi})^*$. The exception are those pre-Poisson submanifolds X for which $TX^{\perp_\Pi}\cap TX = 0$; these are exactly the regular Poisson-Dirac submanifolds of (M,Π) (see [CF1]). In [BFM], they are called *coregular*.

Corollary 3.5.4 indeed recovers Marle's constant rank theorem when $\Pi=\omega^{-1}$ is symplectic, since then the following map is – up to sign – an isomorphism of symplectic vector bundles

$$\Pi^\sharp : \big((TX^{\perp_\omega})^*/(TX^{\perp_\omega}\cap TX)^*, \Pi\big) \xrightarrow{\sim} \big(TX^{\perp_\omega}/(TX^{\perp_\omega}\cap TX), \omega\big).$$

In some detail, fix a decomposition $TM|_X = (TX^{\perp_\omega}\cap TX)\oplus H\oplus W$ as in Lemma 3.5.3. Since $\Pi^\sharp\big((H+W)^0\big)\subset W$, we have $\Pi^\sharp(W^0)\subset H+W$. Moreover, for $\beta\in W^0$ we have that $\Pi^\sharp(\beta)\in H$ exactly when $\Pi^\sharp(\beta)\in TX^{\perp_\omega}$, which in turn is equivalent with $\beta\in TX^0$. So there is an induced map $\Pi^\sharp : W^0\cap TX^0\to H$. At last, since $W+TX = W+(TX^{\perp_\omega}\cap TX)$, we have $W^0\cap TX^0\cong H^*$, so that $\Pi^\sharp$ induces an isomorphism $H^*\xrightarrow{\sim} H$:

$$\Pi^\sharp : \left(\frac{TX^{\perp_\omega}}{TX^{\perp_\omega}\cap TX}\right)^* \xrightarrow{\sim} \frac{TX^{\perp_\omega}}{TX^{\perp_\omega}\cap TX}.$$

Clearly, this map intertwines $-\Pi$ and ω since $\omega = \Pi^{-1}$, so it becomes an isomorphism of symplectic vector bundles, up to sign.

Remark 3.5.5. For the different classes of regular submanifolds $X \subset (M, \Pi)$ considered in this section, we summarize the data that determine the local Poisson saturation (P, Π_P) near X.

Type of submanifold	(P, Π_P) locally determined by
$X \subset (M, \Pi)$ Poisson transversal	i^*L_Π and $\Pi\|_{(TX^{\perp_\Pi})^*}$
$X \subset (M, \Pi)$ regular coisotropic	i^*L_Π
$X \subset (M, \Pi)$ regular pre-Poisson	i^*L_Π and $\Pi\|_{(TX^{\perp_\Pi})^*/(TX^{\perp_\Pi}\cap TX)^*}$

3.6 Coisotropic embeddings of Dirac manifolds in Poisson manifolds

As an application of Corollary 3.5.2, we look at the following question, which was considered by Cattaneo and Zambon [CZ2] and by Wade [Wa]: Given a Dirac manifold (X, L), when can it be embedded coisotropically into a Poisson manifold (M, Π)? That is, when does there exist an embedding $i : X \hookrightarrow (M, \Pi)$ such that $i^*L_\Pi = L$ and $i(X)$ is coisotropic in (M, Π)? Moreover, to what extent is such an embedding unique?

The question on the existence of coisotropic embeddings $(X, L) \hookrightarrow (M, \Pi)$ is settled in [CZ2, Theorem 8.1]: such an embedding exists exactly when L is co-regular, i.e. $L \cap TX$ has constant rank. The construction of (M, Π) in that case is carried out as follows: a choice of complement V to $L \cap TX$ in TX gives an inclusion $j : (L \cap TX)^* \hookrightarrow T^*X$, one takes M to be the total space of $pr : (L \cap TX)^* \to X$ and one shows that the Dirac structure $(pr^*L)^{j^*\omega_{can}}$ on M is in fact Poisson on a neighborhood of $X \subset M$. A different proof of the existence result is given in [Wa, Theorem 4.1].

The question on the uniqueness of coisotropic embeddings $(X, L) \hookrightarrow (M, \Pi)$ is still open. In [Wa], one claims (without proof) that uniqueness can be obtained if $L \cap TX$ defines a simple foliation on X. In [CZ2] one conjectures that, if (X, L) is embedded coisotropically in two different Poisson manifolds, then these must be neighborhood equivalent around X, provided that they are of minimal dimension $\dim X + rk(L \cap TX)$. However, a proof of this uniqueness statement is only given under the additional regularity assumption that the presymplectic leaves of (X, L) have constant dimension [CZ2, Proposition 9.4].

We now show that this extra assumption can be dropped. Using Corollary 3.5.2, we prove that the model $\big(U,(pr^*L)^{j^*\omega_{can}}\big)$ constructed in [CZ2] is minimal, thereby obtaining the uniqueness result in full generality. In the proof below, given an embedding $i : X \hookrightarrow (M,\Pi)$, we may assume that it is the inclusion map by identifying X with $i(X)$.

Proposition 3.6.1. *Let (X,L) be a Dirac manifold for which $L \cap TX$ has constant rank, and denote by $pr : (L\cap TX)^* \to X$ the bundle projection.*

i) Any coisotropic embedding $i : (X,L) \hookrightarrow (M,\Pi)$ into a Poisson manifold (M,Π) factors through the local model $\big(U,(pr^*L)^{j^*\omega_{can}}\big)$. That is, we have a diagram

$$\begin{array}{ccc} (X,L) & \xrightarrow{\quad i \quad} & (M,\Pi) \\ \big\downarrow & \nearrow_{\psi} & \\ \big(U,(pr^*L)^{j^*\omega_{can}}\big) & & \end{array} ,$$

where $\psi : \big(U,(pr^*L)^{j^*\omega_{can}}\big) \hookrightarrow (M,\Pi)$ is a Poisson embedding.

ii) In particular, if (M_1,Π_1) and (M_2,Π_2) are Poisson manifolds of minimal dimension $\dim X + rk(L\cap TX)$ in which (X,L) embeds coisotropically, then (M_1,Π_1) and (M_2,Π_2) are Poisson diffeomorphic around X.

Proof. i) The assumptions imply that $X \subset (M,\Pi)$ is a regular coisotropic submanifold, since

$$TX^{\perp_\Pi} = \Pi^\sharp(TX^0) = (i^*L_\Pi)\cap TX = L\cap TX. \tag{3.16}$$

Denote by (P,Π_P) the local Poisson saturation of $X \subset (M,\Pi)$. By Corollary 3.5.2, there is a neighborhood $U \subset (L\cap TX)^*$ of X and a Poisson embedding

$$\phi : \big(U,(pr^*L)^{j^*\omega_{can}}\big) \to (P,\Pi_P).$$

Since (P,Π_P) is an embedded submanifold of (M,Π), we are done.

ii) By what we just proved, there exist a neighborhood $U \subset (L\cap TX)^*$ of X and two Poisson embeddings

$$\phi_1 : \Big(U,(pr^*L)^{j^*\omega_{can}}\Big) \to (P_1,\Pi_{P_1}),$$
$$\phi_2 : \Big(U,(pr^*L)^{j^*\omega_{can}}\Big) \to (P_2,\Pi_{P_2}),$$

where (P_1,Π_{P_1}) and (P_2,Π_{P_2}) denote the local Poisson saturations of X in (M_1,Π_1) and (M_2,Π_2), respectively. The assumption implies that, for $l = 1,2$:

$$\dim P_l = \dim TX^{\perp_{\Pi_l}} = \dim X + rk(L\cap TX) = \dim M_l,$$

where we used the equality (3.16). Since $P_l \subset M_l$ is an embedded submanifold, this shows that $P_l \subset M_l$ is an open neighborhood of X, for $l = 1, 2$. So the composition $\phi_2 \circ \phi_1^{-1}$ is a Poisson diffeomorphism between open neighborhoods of X in (M_1, Π_1) and (M_2, Π_2), respectively. □

3.7 Regular submanifolds in Dirac geometry

We now discuss how the results that we obtained in Sections 3.2, 3.3 and 3.4 can be generalized to the setting of Dirac manifolds. The relevant tools are developed in [FM2], from which we adopt the terminology and notation. For background on Dirac geometry, see e.g. [B].

Definition 3.7.1. We call an embedded submanifold X of a Dirac manifold (M, L) **regular** if the map $\overline{pr_T} : L|_X \to TM|_X/TX$, which is obtained composing the anchor $pr_T : L \to TM$ with the projection to the normal bundle, has constant rank.

Given any submanifold $i : X \hookrightarrow (M, L)$, we have at points $x \in X$ that

$$\overline{pr_T}(L_x) = \frac{pr_T(L_x) + T_xX}{T_xX},$$

and therefore

$$\begin{aligned} X \subset (M, L) \text{ is regular} &\Leftrightarrow pr_T(L) + TX \text{ has constant rank} \\ &\Leftrightarrow \ker(i^*) \cap L \text{ has constant rank,} \end{aligned}$$

where the last equivalence holds since $\ker(i^*) \cap L = (pr_T(L) + TX)^0$. In particular, the Dirac structure L can be pulled back to a regular submanifold $X \subset (M, L)$ [B, Prop. 5.6].

We recall some results about sprays and dual pairs in Dirac geometry [FM2].

Definition 3.7.2. Let $L \subset TM \oplus T^*M$ be a Dirac structure on M, and let $\mathbf{s} : L \to M$ denote the bundle projection. A **spray** for L is a vector field $\mathcal{V} \in \mathfrak{X}(L)$ satisfying

i) $\mathbf{s}_*(\mathcal{V}_a) = pr_T(a)$ for all $a \in L$,

ii) $m_t^*\mathcal{V} = t\mathcal{V}$, where $m_t : L \to L$ denotes fiberwise multiplication by $t \neq 0$.

Sprays exist on any Dirac structure. The condition ii) implies that the spray $\mathcal{V}$ vanishes along the zero section $M \subset L$, so that there exists a neighborhood $\Sigma \subset L$ of M on which the flow φ_ϵ of $\mathcal{V}$ is defined for all times $\epsilon \in [0, 1]$. We can then define the **Dirac exponential map** associated with the spray $\mathcal{V}$ as

$$\exp_{\mathcal{V}} : \Sigma \to M : a \mapsto \mathbf{s}(\varphi_1(a)).$$

Moreover, this neighborhood $\Sigma \subset L$ supports a two-form ω defined by

$$\omega := \int_0^1 \varphi_\epsilon^* \big((pr_{T^*})^* \omega_{can}\big) d\epsilon,$$

where $pr_{T^*} : L \to T^*M$ is the projection and ω_{can} is the canonical symplectic form on T^*M. It is proved in [FM2] that, shrinking $\Sigma \subset L$ if necessary, these data fit into a **Dirac dual pair**:

$$(M, L) \xleftarrow{\ \mathbf{s}\ } (\Sigma, \mathrm{Gr}(\omega)) \xrightarrow{\ \exp_{\mathcal{V}}\ } (M, -L). \tag{3.17}$$

This means that both legs in the diagram (3.17) are surjective, forward Dirac submersions, and we have the additional requirements that $\omega(V, W) = 0$ and $V \cap K \cap W = 0$, where $V = \ker \mathbf{s}_*$, $W = \ker(\exp_{\mathcal{V}})_*$ and $K = \ker \omega$.

We need the following lemma, which serves as a substitute for Lemma 3.2.5 in the Dirac setting. The statement is not exactly the Dirac analog of Lemma 3.2.5; we address this in Remark 3.7.4 below.

Lemma 3.7.3. *Consider a Dirac dual pair*

$$(M_0, L_0) \xleftarrow{\ \boldsymbol{s}\ } (\Sigma, Gr(\omega)) \xrightarrow{\ \boldsymbol{t}\ } (M_1, -L_1),$$

and let $X \subset (M_0, L_0)$ be a regular submanifold. As before, we denote $V := \ker \boldsymbol{s}_*$, $W := \ker \boldsymbol{t}_*$ and $K := \ker \omega$. Then $W \cap \boldsymbol{s}_*^{-1}(TX)$ has constant rank, equal to the rank of $pr_T^{-1}(TX) \subset L_0|_X$.

Proof. Consider the following diagram of vector bundle maps:

$$\begin{array}{ccc} W|_{\mathbf{s}^{-1}(X)} & \xrightarrow{\ \overline{\mathbf{s}_*}\ } & TM_0|_X/TX \\ {\scriptstyle R_\omega}\big\downarrow & & \big\uparrow{\scriptstyle \overline{pr_T}} \\ R_\omega(W|_{\mathbf{s}^{-1}(X)}) & \xrightarrow[\ \psi\ ]{} & L_0|_X \end{array} . \tag{3.18}$$

Here R_ω is an injective bundle map defined by

$$R_\omega : W \to T\Sigma \oplus T^*\Sigma : w \mapsto w + \iota_w \omega.$$

The map $\psi : R_\omega(W) \to L_0$ is defined by setting $\psi(w + \iota_w\omega) := \mathbf{s}_*(w) + \beta$, where β is uniquely determined by the relation $\mathbf{s}^*(\beta) = \iota_w\omega$. Note that ψ is well-defined: existence of β follows from the fact that $\omega(V, W) = 0$, and β is unique since $\mathbf{s}$ is a submersion. Since the map $\mathbf{s} : (\Sigma, \mathrm{Gr}(\omega)) \to (M_0, L_0)$ is forward Dirac, we see that $\psi(w + \iota_w\omega) = \mathbf{s}_*(w) + \beta$ is indeed contained in L_0.

Moreover, we claim that the map ψ is an isomorphism. To see that ψ is injective, assume that $\psi(w + \iota_w\omega) = \mathbf{s}_*(w) + \beta = 0$ for some $w \in W$. Then $\beta = 0$, and therefore $\iota_w\omega = \mathbf{s}^*(\beta) = 0$, so that $w \in W \cap K$. But also $\mathbf{s}_*(w) = 0$, so that $w \in V$. Hence $w \in V \cap K \cap W = 0$, which shows that ψ is injective. Since the rank of $R_\omega(W)$ is given by

$$rk(R_\omega(W)) = rk(W) = \dim \Sigma - \dim M_1 = \dim M_0 = rk(L_0),$$

it follows that $\psi : R_\omega(W) \to L_0$ is a vector bundle isomorphism. Since the diagram (3.18) commutes, it follows that

$$\begin{aligned} rk\big(\overline{\mathbf{s}_*} : W|_{\mathbf{s}^{-1}(X)} \to TM_0|_X/TX\big) &= rk\big(\overline{pr_T} : L_0|_X \to TM_0|_X/TX\big) \\ &= \dim M_0 - rk\big(pr_T^{-1}(TX)\big). \end{aligned}$$

This gives the conclusion of the lemma:

$$rk\big(W \cap \mathbf{s}_*^{-1}(TX)\big) = rk(W) - \dim M_0 + rk\big(pr_T^{-1}(TX)\big) = rk\big(pr_T^{-1}(TX)\big). \;\square$$

Remark 3.7.4. For completeness, we state here the Dirac geometric analog of Lemma 3.2.5. Recall that a forward Dirac map $\varphi : (M_0, L_0) \to (M_1, L_1)$ is **strong** if $L_0 \cap \ker \varphi_* = 0$. When L_0 is the graph of a closed 2-form, then the map φ is called a **presymplectic realization** of (M_1, L_1). One can show that the following is true:

"*Let $\boldsymbol{s} : (\Sigma, Gr(\omega)) \to (M, L)$ be a strong forward Dirac submersion, and assume that $X \subset (M, L)$ is a regular submanifold. If $V := \ker \boldsymbol{s}_*$, then $V^{\perp_\omega} \cap \boldsymbol{s}_*^{-1}(TX)$ has constant rank, equal to the rank of $pr_T^{-1}(TX)$.*"

We won't address this in more detail, since we want to use the legs of the diagram (3.17) and these are in general not presymplectic realizations. Indeed, using expressions for $\omega|_M$ that appear in [FM2], one can check that

$$\begin{aligned} (\mathrm{Gr}(\omega) \cap \ker \mathbf{s}_*)|_M &= 0 \oplus L \cap TM \subset TM \oplus L, \\ (\mathrm{Gr}(\omega) \cap \ker(\exp_{\mathcal{V}})_*|_M &= \{(-v, v) : v \in L \cap TM\} \subset TM \oplus L, \end{aligned}$$

so that both legs are presymplectic realizations only when the Dirac structure L is Poisson. In that case, ω is non-degenerate along $M \subset \Sigma$, so that shrinking Σ if necessary, the diagram (3.17) is a full dual pair. In particular, the legs of the diagram (3.17) are symplectic realizations.

We obtain the following generalization of Theorem 3.2.6.

Theorem 3.7.5. *Let $X \subset (M, L)$ be a regular submanifold.*

1. *There is an embedded invariant submanifold $(P, L_P) \subset (M, L)$ containing X that lies inside the saturation $Sat(X)$.*

2. Shrinking P if necessary, there is a neighborhood U of X in M such that (P, L_P) is the saturation of X in $(U, L|_U)$.

Proof. The proof is divided into four steps, like the proof of Theorem 3.2.6.

Step 1: Construction of the submanifold $P \subset M$.

Choose a spray $\mathcal{V} \in \mathfrak{X}(L)$ and denote by $\exp_{\mathcal{V}} : \Sigma \subset L \to M$ the corresponding Dirac exponential map. Let $\mathbf{s} : L \to M$ denote the bundle projection. Note that $\exp_{\mathcal{V}}(a)$ and $\mathbf{s}(a)$ lie in the same presymplectic leaf of (M, L), for all $a \in L$. Indeed, the path $t \mapsto \varphi_t(a)$ is an A-path for the Lie algebroid $A = \big(L, [\![\cdot,\cdot]\!], pr_T\big)$, covering the path $t \mapsto \mathbf{s}(\varphi_t(a))$ which connects $\mathbf{s}(a)$ with $\exp_{\mathcal{V}}(a)$. In particular, we have that $\exp_{\mathcal{V}}(\Sigma|_X) \subset Sat(X)$.

Since $X \subset (M, L)$ is regular, we have that $pr_T^{-1}(TX)$ is a subbundle of $L|_X$, being the kernel of the constant rank bundle map $\overline{pr_T} : L|_X \to TM|_X/TX$. Choose a complement $L|_X = pr_T^{-1}(TX) \oplus C$ and consider the restriction $\exp_{\mathcal{V}} : C \cap \Sigma|_X \to M$. It fixes points of X, and its differential along X reads [FM2, Lemma 7]:

$$(d\exp_{\mathcal{V}})_x : T_xX \oplus C_x \to T_xM : (u, a) \mapsto u + pr_T(a).$$

This map is injective, and therefore the map $\exp_{\mathcal{V}} : C \cap \Sigma|_X \to M$ is an embedding by Prop. 3.8.1, shrinking Σ if necessary. Set $P := \exp_{\mathcal{V}}(C \cap \Sigma|_X)$.

Step 2: Shrinking Σ if necessary, we have that $P = \exp_{\mathcal{V}}(\Sigma|_X)$.

It is enough to show that the restriction of $\exp_{\mathcal{V}}$ to $\Sigma|_X$ has constant rank, equal to the rank of $\exp_{\mathcal{V}}|_{(C\cap\Sigma|_X)}$. To see this, we apply Lemma 3.7.3 to the self-dual pair (3.17), and we obtain that

$$\ker\big(d(\exp_{\mathcal{V}}|_{\Sigma|_X})\big) = \ker(d\exp_{\mathcal{V}}) \cap \mathbf{s}_*^{-1}(TX)$$

has constant rank, equal to the rank of $pr_T^{-1}(TX) \subset L|_X$. This implies that the rank of $\exp_{\mathcal{V}}|_{\Sigma|_X}$ is constant, equal to

$$\begin{aligned} rk\big(\exp_{\mathcal{V}}|_{\Sigma|_X}\big) &= \dim X + rk(L) - rk\big(pr_T^{-1}(TX)\big) \\ &= \dim X + rk(C) \\ &= rk\big(\exp_{\mathcal{V}}|_{(C\cap\Sigma|_X)}\big). \end{aligned}$$

Step 3: The submanifold $P \subset (M, L)$ is invariant.

We have to check that the characteristic distribution $pr_T(L)$ of L is tangent to P, i.e. that $pr_T\big(L_{\exp_{\mathcal{V}}(a)}\big) \subset (d\exp_{\mathcal{V}})_a(T_a\Sigma|_X)$ for all $a \in \Sigma|_X$. We will first show that

$$pr_T\big(L_{\exp_{\mathcal{V}}(a)}\big) = (d\exp_{\mathcal{V}})_a(W^{\perp_\omega}),$$

where W denotes $\ker(\exp_{\mathcal{V}})_*$ as before. To see this, first pick $u+\xi \in L_{\exp_{\mathcal{V}}(a)}$. Then $u-\xi \in -L$, and since the map $\exp_{\mathcal{V}} : (\Sigma, \mathrm{Gr}(\omega)) \to (M, -L)$ is forward Dirac, there exists $v \in T_a\Sigma$ such that $v + \iota_v\omega$ is $\exp_{\mathcal{V}}$-related with $u-\xi$, i.e.

$$\begin{cases} \iota_v\omega = \exp_{\mathcal{V}}^*(-\xi), \\ u = (d\exp_{\mathcal{V}})_a(v). \end{cases}$$

This implies that $v \in W^{\perp_\omega}$, so $pr_T(u+\xi) = u = (d\exp_{\mathcal{V}})_a(v)$ is contained in $(d\exp_{\mathcal{V}})_a(W^{\perp_\omega})$. Conversely, assume $v \in T_a\Sigma$ lies in $W^{\perp_\omega}$. Then $\iota_v\omega = \exp_{\mathcal{V}}^*(\xi)$ for some $\xi \in T^*_{\exp_{\mathcal{V}}(a)}M$. This implies that $(d\exp_{\mathcal{V}})_a(v) + \xi$ is $\exp_{\mathcal{V}}$-related with $v + \iota_v\omega \in \mathrm{Gr}(\omega)$, and since the map $\exp_{\mathcal{V}} : (\Sigma, \mathrm{Gr}(\omega)) \to (M, -L)$ is forward Dirac, we get that $(d\exp_{\mathcal{V}})_a(v) + \xi \in -L$, i.e. $(d\exp_{\mathcal{V}})_a(v) - \xi \in L$. It follows that $(d\exp_{\mathcal{V}})_a(v) \in pr_T\big(L_{\exp_{\mathcal{V}}(a)}\big)$.

Consequently, we obtain that

$$\begin{aligned} pr_T\big(L_{\exp_{\mathcal{V}}(a)}\big) &= (d\exp_{\mathcal{V}})_a\big(W^{\perp_\omega}\big) \\ &= (d\exp_{\mathcal{V}})_a\big(V + W \cap K\big) \\ &\subset (d\exp_{\mathcal{V}})_a\big(\mathbf{s}_*^{-1}(TX)\big) \\ &= (d\exp_{\mathcal{V}})_a(T_a\Sigma|_X), \end{aligned} \tag{3.19}$$

where the second equality uses [FM2, Lemma 3], and the third equality holds because $W = \ker(\exp_{\mathcal{V}})_*$ and $V = \ker \mathbf{s}_* \subset \mathbf{s}_*^{-1}(TX)$. This proves Step 3.

Step 4: Construction of the neighborhood U of X.

The proof is completely analogous to the proof of Step 4 in Theorem 3.2.6. We want to extend the map $\exp_{\mathcal{V}} : C \cap \Sigma|_X \to M$ to a local diffeomorphism. To do so, we choose a complement

$$TM|_X = TX \oplus (pr_T(C) \oplus E)$$

and a linear connection ∇ on TM. We obtain a map

$$\psi : O \subset (C \oplus E) \to M : (a, e) \mapsto \exp_\nabla\left(Tr_{\exp_{\mathcal{V}}(ta)}e\right),$$

which is a diffeomorphism onto an open neighborhood of X. Here O is a suitable convex neighborhood of the zero section, and $Tr_{\exp_{\mathcal{V}}(ta)}$ denotes parallel transport along the curve $t \mapsto \exp_{\mathcal{V}}(ta)$ for $t \in [0,1]$. Since $\psi(a,0) = \exp_{\mathcal{V}}(a)$, shrinking P if necessary, we can assume that $P = \psi(O \cap (C \oplus \{0\}))$. Setting $U := \psi(O)$ finishes the proof. □

In the following, we denote by (P, L_P) the Dirac manifold constructed in Theorem 3.7.5; we refer to it as the **local Dirac saturation** of X. By the normal form around Dirac transversals [BLM], [FM2], this Dirac manifold is determined around X by the pullback of L to $X \subset (M, L)$, up to diffeomorphisms and exact gauge transformations. We will reprove this result, continuing the argument of Theorem 3.7.5. We first obtain a Dirac version of Lemma 3.4.1.

Lemma 3.7.6. *Let $i : X \hookrightarrow (M, L)$ be a regular submanifold with local Dirac saturation (P, L_P). Then the following is a weak Dirac dual pair, in the sense of [FM2]:*

$$(X, i^*L) \xleftarrow{\ \mathbf{s}\ } \big(\Sigma|_X, Gr(\omega|_X)\big) \xrightarrow{\exp_{\mathcal{V}}} (P, -L_P). \tag{3.20}$$

This means that $\omega|_X$ is a closed two-form on $\Sigma|_X$, that $\mathbf{s}$ and $\exp_{\mathcal{V}}$ are surjective forward Dirac submersions and

$$\omega|_X(S_1, S_2) = 0, \tag{3.21}$$

$$rk(S_1 \cap \widetilde{K} \cap S_2) = \dim \Sigma|_X - \dim X - \dim P, \tag{3.22}$$

where $S_1 := \ker \mathbf{s}_*, S_2 := \ker(\exp_{\mathcal{V}})_*$ and $\widetilde{K} := \ker(\omega|_X)$.

Proof. The only non-trivial part is that the equality (3.22) holds. The other claims are proved exactly like in Lemma 3.4.1, so we don't address them here.

To prove (3.22), note that $S_1 \cap \widetilde{K} \cap S_2 = V \cap (\mathbf{s}_*^{-1}(TX))^{\perp_\omega} \cap W$, where V, W are the vertical distributions of the original dual pair (3.17). Note that for any subspace $U_a \subset (T_a\Sigma, \omega_a)$, we have

$$\dim(U_a^{\perp_\omega}) = \dim(T_a\Sigma) - \dim(U_a) + \dim(U_a \cap K_a),$$

where $K := \ker \omega$. It follows that a family $U \subset T\Sigma$ of linear subspaces has constant rank if both $U^{\perp_\omega}$ and $U \cap K$ have constant rank. On one hand, we have

$$\big(V \cap (\mathbf{s}_*^{-1}(TX))^{\perp_\omega} \cap W\big) \cap K = 0,$$

since $V \cap K \cap W = 0$. On the other hand, we have

$$\begin{aligned}
\big(V \cap (\mathbf{s}_*^{-1}(TX))^{\perp_\omega} \cap W\big)^{\perp_\omega} &= V^{\perp_\omega} + \big((\mathbf{s}_*^{-1}(TX))^{\perp_\omega}\big)^{\perp_\omega} + W^{\perp_\omega} \\
&= V^{\perp_\omega} + \mathbf{s}_*^{-1}(TX) + K + W^{\perp_\omega} \\
&= V^{\perp_\omega} + \mathbf{s}_*^{-1}(TX) + W^{\perp_\omega} \\
&= W + V \cap K + \mathbf{s}_*^{-1}(TX) + V + W \cap K \\
&= W + \mathbf{s}_*^{-1}(TX) + V \\
&= W + \mathbf{s}_*^{-1}(TX).
\end{aligned}$$

In the fourth equality, we use [FM2, Lemma 3]. Using Lemma 3.7.3, we have now proved that $S_1 \cap \widetilde{K} \cap S_2 = V \cap (\mathbf{s}_*^{-1}(TX))^{\perp_\omega} \cap W$ has constant rank. The rank is given by

$$\begin{aligned} rk\big(V \cap (\mathbf{s}_*^{-1}(TX))^{\perp_\omega} \cap W\big) &= rk(T\Sigma) - rk(W + \mathbf{s}_*^{-1}(TX)) \\ &= rk(T\Sigma) - rk(W) - rk(\mathbf{s}_*^{-1}(TX)) \\ &\qquad + rk(W \cap \mathbf{s}_*^{-1}(TX)) \\ &= \dim(L) - rk(W) - \dim(X) - rk(V) \\ &\qquad + rk(pr_T^{-1}(TX)) \\ &= \dim(M) - \dim(\exp_{\mathcal{V}}(\Sigma|_X)) + rk(L) - rk(V) \\ &= \dim(\Sigma) - rk(V) - \dim(\exp_{\mathcal{V}}(\Sigma|_X)) \\ &= \dim(\Sigma|_X) - \dim(X) - \dim(\exp_{\mathcal{V}}(\Sigma|_X)). \end{aligned}$$

This is exactly the rank condition (3.22), so the proof is finished. □

Corollary 3.7.7. *Let $i : X \hookrightarrow (M, L)$ be a regular submanifold, choose a complement $L|_X = pr_T^{-1}(TX) \oplus C$ and let $j : C \hookrightarrow L|_X$ denote the inclusion. The local Dirac saturation (P, L_P) of X is diffeomorphic with*

$$\big(C \cap \Sigma|_X, (\boldsymbol{s}^*(i^*L))^{-j^*\omega|_X}\big).$$

In particular, (P, L_P) is determined by the pullback Dirac structure i^*L, up to diffeomorphisms and exact gauge transformations.

Proof. Applying [FM2, Prop.6] to the diagram (3.20), we have the following equality of Dirac structures on $\Sigma|_X$:

$$(\mathbf{s}^*(i^*L))^{-\omega|_X} = (\exp_{\mathcal{V}})^* L_P.$$

Since j is transverse to the leaves of this Dirac structure, we can pull it back to $C \cap \Sigma|_X$, and we obtain

$$(\mathbf{s}^*(i^*L))^{-j^*\omega|_X} = (\exp_{\mathcal{V}} \circ j)^* L_P,$$

which is an equality of Dirac structures on $C \cap \Sigma|_X$. We showed in Theorem 3.7.5 that $\exp_{\mathcal{V}} \circ j$ is a diffeomorphism from $C \cap \Sigma|_X$ onto P, which proves the first statement. Moreover, since $-j^*\omega|_X$ is closed and its pullback to $X \subset C \cap \Sigma|_X$ vanishes, it is exact on a neighborhood of X, by the relative Poincaré lemma. This implies the second statement of the corollary. □

Remark 3.7.8. As mentioned before, Corollary 3.7.7 agrees with the normal form around Dirac transversals [BLM],[FM2]. Note indeed that X is a transversal in (P, L_P) since

$$TP|_X = TX + pr_T(C) = TX + pr_T(L|_X) = TX + pr_T(L_P|_X).$$

Here the last equality holds since $pr_T(L|_X) \subset TP|_X$. This implies that, if $u + \xi \in L|_X$, then $u + \iota^*\xi \in L_P|_X$ where $\iota : P \hookrightarrow M$ is the inclusion. So indeed, $pr_T(L|_X) = pr_T(L_P|_X)$.

3.8 Appendix

3.8.1 Some differential topology

We prove a result in differential topology that may be of independent interest. It should be standard, but we could not find a reference in the literature. The statement is well-known under the stronger assumption that the derivative of the map is an isomorphism along the zero section [Muk, Lemma 6.1.3]. Our strategy is to reduce the proof to this case.

Proposition 3.8.1. *Let $E \to N$ be a vector bundle, and let $\varphi : E \to M$ be a smooth map satisfying*

$$\begin{cases} \varphi|_N \text{ is an embedding} \\ (d\varphi)_p \text{ is injective } \forall p \in N \end{cases}. \tag{3.23}$$

Then there is a neighborhood $U \subset E$ of N such that $\varphi|_U$ is an embedding.

Proof. We get a vector subbundle $d\varphi|_N(E) \subset TM|_{\varphi(N)}$ which has trivial intersection with $T\varphi(N)$. Choose a complement C to $d\varphi|_N(E) \oplus T\varphi(N)$ in $TM|_{\varphi(N)}$, i.e.

$$TM|_{\varphi(N)} = T\varphi(N) \oplus d\varphi|_N(E) \oplus C.$$

Fix a linear connection ∇ on TM, and define a map

$$\psi : E \oplus (\varphi|_N)^*C \to M : (e, c) \to \exp_\nabla\left(Tr_{\varphi(te)}c\right),$$

where $Tr_{\varphi(te)}$ denotes parallel transport along the curve $t \mapsto \varphi(te)$ for $t \in [0, 1]$. We slightly abuse notation, since the map ψ is only defined on a small enough neighborhood of the zero section N. Clearly, ψ satisfies the following properties:

- ψ restricts to $\varphi|_N$ along the zero section N.

- For $p \in N$ and a vertical tangent vector $(e,c) \in T_p\big(E \oplus (\varphi|_N)^*C\big)$, we have

$$\begin{aligned}(d\psi)_p(e,c) &= \frac{d}{ds}\bigg|_{s=0} \psi(se,0) + \frac{d}{ds}\bigg|_{s=0} \psi(0,sc) \\ &= \frac{d}{ds}\bigg|_{s=0} \exp_\nabla\left(0_{\varphi(se)}\right) + \frac{d}{ds}\bigg|_{s=0} \exp_\nabla(sc) \\ &= \frac{d}{ds}\bigg|_{s=0} \varphi(se) + c \\ &= (d\varphi)_p(e) + c,\end{aligned}$$

  which shows that $d\psi$ is an isomorphism at points of the zero section.
- We have that $\psi(e,0) = \varphi(e)$, i.e. the following diagram commutes:

$$\begin{array}{ccc} & & E \oplus (\varphi|_N)^*C \\ & \nearrow & \Big\downarrow \psi \\ E & \xrightarrow{\ \varphi\ } & M \end{array} \tag{3.24}$$

Using the first and second bullet point above, the inverse function theorem for submanifolds (e.g. [Muk, Lemma 6.1.3]) shows that ψ is an embedding on a neighborhood of N. Also the inclusion $E \hookrightarrow E \oplus (\varphi|_N)^*C$ on the left in (3.24) is an embedding, so that φ is an embedding on a neighborhood of N in E. □

Remark 3.8.2. If a map $\varphi : U \subset E \to M$ satisfying the assumptions (3.23) of Proposition 3.8.1 is only defined on a neighborhood $U \subset E$ of N, then the conclusion of the proposition still holds. This can be obtained, for instance, by constructing a smooth map $\mu : E \to E$ such that $\mu(E) \subset U$ and $\mu = \mathrm{Id}$ near N (see [H, Chapter 4, §5]). By Proposition 3.8.1, the composition $\varphi \circ \mu : E \to M$ is an embedding on a neighborhood of N, hence the same holds for φ.

3.8.2 Relation with other normal forms

In Corollary 3.5.2, we found a normal form for the local Poisson saturation of a regular coisotropic submanifold X of any Poisson manifold (M,Π). This result relates with other normal forms in the previous chapters. We show that Corollary 3.5.2 recovers the normal form around Lagrangians transverse to the symplectic leaves (Proposition 2.2.9), as well as the b-Gotay theorem (Proposition 1.3.15) in case the b-coisotropic submanifold is *strong* (see Definition 1.4.1).

Lagrangians transverse to the leaves

If $L \subset (M, \Pi)$ is a Lagrangian submanifold transverse to the symplectic leaves, then L is in particular regular coisotropic. Corollary 3.5.2 implies that a neighborhood of L in (M, Π) is Poisson diffeomorphic with the model

$$\Big(U \subset (TL^{\perp_\Pi})^*, (pr^*(i^*L_\Pi))^{j^*\omega_{can}}\Big), \tag{3.25}$$

where U is a neighborhood of the zero section of $pr : (TL^{\perp_\Pi})^* \to L$ and the inclusion $j : (TL^{\perp_\Pi})^* \hookrightarrow T^*L$ comes from a complement $TL = TL^{\perp_\Pi} \oplus G$. We show that this local model is nothing else but

$$\big(U \subset T^*\mathcal{F}_L, \Pi_{can}\big),$$

so that we recover the normal form of Proposition 2.2.9.

- Since L is Lagrangian, we have $TL^{\perp_\Pi} = T\mathcal{F}_L$, where $\mathcal{F}_L$ is the foliation induced on L. So the model (3.25) lives on the vector bundle $T^*\mathcal{F}_L$.
- The pullback Dirac structure i^*L_Π is just $\mathcal{F}_L$, which is clear when thinking of Dirac structures in terms of their presymplectic foliations. The underlying foliation of i^*L_Π is $\mathcal{F}_L$ and the presymplectic forms, which are obtained pulling back the symplectic forms on the leaves of Π, vanish since L is Lagrangian. Hence $pr^*(i^*L_\Pi) = pr^*\mathcal{F}_L$.
- To conclude that the model (3.25) coincides with the canonical Poisson structure Π_{can} on $T^*\mathcal{F}_L$, it remains to check that the pullback of $j^*\omega_{can}$ to the leaves of $pr^*\mathcal{F}_L$ reads

  $$\sum_{i=1}^{k} dx_i \wedge dy_i, \tag{3.26}$$

  in cotangent coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ coming from a foliated chart $(x_1, \ldots, x_n)$ in which plaques of $\mathcal{F}_L$ are level sets of $(x_{k+1}, \ldots, x_n)$.

  To see that this is the case, we decompose

  $$TL = \text{Span}\{\partial_{x_1}, \ldots, \partial_{x_k}\} \oplus \text{Span}\{\partial_{x_{k+1}}, \ldots, \partial_{x_n}\}$$

  and since the complement G is transverse to the first summand, there exists a fiberwise linear map Φ such that

  $$\begin{aligned} G &= \text{Graph}\big(\Phi : \text{Span}\{\partial_{x_{k+1}}, \ldots, \partial_{x_n}\} \to \text{Span}\{\partial_{x_1}, \ldots, \partial_{x_k}\}\big) \\ &= \text{Span}\{\partial_{x_{k+1}} + \Phi(\partial_{x_{k+1}}), \ldots, \partial_{x_n} + \Phi(\partial_{x_n})\}. \end{aligned}$$

Let us write for $j = k+1, \dots, n$:

$$\Phi(\partial_{x_j}) = \sum_{i=1}^{k} f_j^i(x)\partial_{x_i}.$$

We then obtain for $i = 1, \dots, k$ that

$$\begin{aligned}\Big\langle j(dx_i), \sum_{l=1}^{n} g_l(x)\partial_{x_l}\Big\rangle &= \Big\langle dx_i, \sum_{l=1}^{k} g_l(x)\partial_{x_l} - \sum_{l=k+1}^{n} g_l(x)\Phi(\partial_{x_l})\Big\rangle \\ &= \Big\langle dx_i, \sum_{l=1}^{k} g_l(x)\partial_{x_l} - \sum_{l=k+1}^{n}\sum_{\alpha=1}^{k} f_l^\alpha(x)g_l(x)\partial_{x_\alpha}\Big\rangle \\ &= g_i(x) - \sum_{l=k+1}^{n} f_l^i(x)g_l(x).\end{aligned}$$

This shows that

$$j(dx_i) = dx_i - \sum_{l=k+1}^{n} f_l^i(x)dx_l.$$

In particular, expressing the map $j : T^*\mathcal{F}_L \to T^*L$ in the cotangent coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$, we have

$$\begin{cases} x_i \circ j = x_i & \text{for } i = 1, \dots, n \\ y_i \circ j = y_i & \text{for } i = 1, \dots, k \end{cases}.$$

Therefore,

$$j^*\omega_{can} = \sum_{i=1}^{k} dx_i \wedge dy_i + \sum_{i=k+1}^{n} dx_i \wedge d(y_i \circ j).$$

Pulling back to leaves of $pr^*\mathcal{F}_L$, we indeed obtain the expression (3.26).

b-Gotay for strong b-coisotropic submanifolds

Assume that (M, Z, Π) is a log-symplectic manifold, and let $C \subset (M, Z, \Pi)$ be a strong b-coisotropic submanifold. In particular, C is regular since $TC^{\perp_\Pi}$ has constant rank equal to $codim(C)$ (see Proposition 1.4.2). Below, we denote by $\rho : {}^bTM \to TM$ the anchor map of the b-tangent bundle ${}^bTM$.

- On one hand, Corollary 3.5.2 implies that a neighborhood of $C \subset (M, Z, \Pi)$ is Poisson diffeomorphic with

$$\Big(U \subset (TC^{\perp_\Pi})^*, (pr_1^*(i^*L_\Pi))^{j_1^*\omega_{T^*C}}\Big), \tag{3.27}$$

where $pr_1 : (TC^{\perp_\Pi})^* \to C$ is the projection and $j_1 : (TC^{\perp_\Pi})^* \hookrightarrow T^*C$ is the inclusion corresponding with a choice of complement $TC = TC^{\perp_\Pi} \oplus V$. This model is log-symplectic, being Poisson diffeomorphic with Π, and we claim that its singular locus is exactly $pr_1^{-1}(C \cap Z)$. To see why, note that by (3.1),

$$pr_T(i^* L_\Pi) = \Pi^\sharp\big(TC^{\perp_\Pi}\big)^0,$$

and

$$\mathrm{Ker}(\Pi^\sharp) \cap \big(TC^{\perp_\Pi}\big)^0 = \big(\mathrm{Im}(\Pi^\sharp) + TC^{\perp_\Pi}\big)^0 = \big(\mathrm{Im}(\Pi^\sharp)\big)^0.$$

So $pr_T(i^* L_\Pi)$ drops rank exactly along $C \cap Z$, which proves the claim.

- On the other hand, first notice that the splitting $TC = TC^{\perp_\Pi} \oplus V$ gives a canonical splitting

$$^bTC = {}^bTC^\omega \oplus G, \tag{3.28}$$

where $G := \rho^{-1}(V)$. It is clear that $G \subset {}^bTC$; the fact that it is indeed a smooth subbundle follows from ρ being transverse to V:

$$\begin{aligned} TM|_{C\cap Z} &= TZ|_{C\cap Z} + TC|_{C\cap Z} \\ &= TZ|_{C\cap Z} + TC^{\perp_\Pi}|_{C\cap Z} + V|_{C\cap Z} \\ &= \mathrm{Im}(\rho|_{C\cap Z}) + \rho\big(({}^bTC)^\omega\big)|_{C\cap Z} + V|_{C\cap Z} \\ &= \mathrm{Im}(\rho|_{C\cap Z}) + V|_{C\cap Z}. \end{aligned}$$

Note that G intersects $({}^bTC)^\omega$ trivially because $\rho\big(({}^bTC)^\omega\big) = TC^{\perp_\Pi}$. We now apply the b-Gotay theorem (Proposition 1.3.15) for the choice of splitting (3.28). Denoting by $\omega \in {}^b\Omega^2(M)$ the b-symplectic form corresponding with Π, we obtain that a neighborhood of C in (M, Z, ω) is b-symplectomorphic with

$$\Big(V \subset ({}^bTC^\omega)^*, {}^bpr_2^*(\omega_C) + {}^bj_2^*(\omega_{({}^bT^*C)})\Big). \tag{3.29}$$

Here $pr_2 : ({}^bTC^\omega)^* \to C$ is the projection and $j_2 : ({}^bTC^\omega)^* \hookrightarrow {}^bT^*C$ is the inclusion induced by the splitting (3.28).

The models (3.27) and (3.29) are canonically isomorphic around C. In fact, we claim that the map

$$\Phi : (TC^{\perp_\Pi})^* \to ({}^bTC^\omega)^*$$

dual to the map $\rho : {}^bTC^\omega \to TC^{\perp_\Pi}$ is a b-symplectomorphism between them.

To see why, first notice that

$$p_1 \circ \rho = \rho \circ p_2 : {}^bTC \to TC^{\perp_\Pi}, \tag{3.30}$$

where $p_1 : TC \to TC^{\perp_\Pi}$ and $p_2 : {}^bTC \to {}^bTC^\omega$ are the projections in the direct sums $TC = TC^{\perp_\Pi} \oplus V$ and ${}^bTC = {}^bTC^\omega \oplus G$, respectively. Indeed, for all $v \in {}^bTC$, we have

$$\begin{aligned} p_1(\rho(v)) &= p_1(\rho(p_2(v)) + \rho(v - p_2(v))) \\ &= p_1(\rho(p_2(v))) \\ &= \rho(p_2(v)), \end{aligned}$$

using in the second equality that $\rho(v - p_2(v)) \in V$ and in the third equality that $\rho(p_2(v)) \in TC^{\perp_\Pi}$. This proves that (3.30) holds. Taking duals in that equality, we obtain a commutative diagram

$$\begin{array}{ccc} (TC^{\perp_\Pi})^* & \xrightarrow[\simeq]{\Phi} & ({}^bTC^\omega)^* \\ \downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle j_2} \\ T^*C & \xrightarrow{\rho^*} & {}^bT^*C \end{array}.$$

Consequently, pulling back (3.29) along Φ gives

$${}^b\Phi^*\left({}^bpr_2^*(\omega_C) + {}^bj_2^*(\omega_{({}^bT^*C)})\right) = {}^bpr_1^*(\omega_C) + {}^b(\rho^* \circ j_1)^*(\omega_{({}^bT^*C)}).$$

This is a b-symplectic form with singular locus $pr_1^{-1}(C \cap Z)$, just like the one in (3.27). To see that they are equal, it is enough to check that their symplectic forms on the complement of $pr_1^{-1}(C \cap Z)$ agree. Clearly they do, because they are both given by

$$pr_1^*(\omega_C|_{C \setminus (C \cap Z)}) + j_1^*(\omega_{T^*C}|_{(T^*C \setminus T^*C|_{C \cap Z})}).$$

This proves that the models (3.27) and (3.29) are isomorphic around C.

Bibliography

[BFM] L. Brambila, P. Frejlich and D. Martinez Torres, *Coregular submanifolds and Poisson submersions*, preprint arXiv:2010.09058, 2020.

[B] H. Bursztyn, *A brief introduction to Dirac manifolds*, Geometric and topological methods for quantum field theory, Cambridge University Press, p. 4-38, 2013.

[BLM] H. Bursztyn, H. Lima and E. Meinrenken, *Splitting theorems for Poisson and related structures*, J. Reine Angew. Math. **2019**(754), p. 281-312, 2019.

[CW] A. Cannas da Silva and A. Weinstein, *Geometric Models for Noncommutative Algebras*, Berkeley Mathematics Lecture Notes series **10**, American Mathematical Society, 1999.

[CZ1] A.S. Cattaneo and M. Zambon, *Pre-Poisson submanifolds*, Trav. Math. **XVII**, p. 61-74, 2007.

[CZ2] A.S. Cattaneo and M. Zambon, *Coisotropic embeddings in Poisson manifolds*, Trans. Amer. Math. Soc. **361**(7), p. 3721-3746, 2009.

[CF1] M. Crainic and R.L. Fernandes, *Integrability of Poisson brackets*, J. Differential Geom. **66**(1), p. 71-137, 2004.

[CF2] M. Crainic and R.L. Fernandes, *Stability of symplectic leaves*, Invent. Math. **180**(3), p. 481-533, 2010.

[CFM] M. Crainic, R.L. Fernandes and I. Mărcuţ, *Lectures on Poisson Geometry*, In preparation.

[CM] M. Crainic and I. Mărcuţ, *On the existence of symplectic realizations*, J. Symplectic Geom. **9**(4), p. 435-444, 2011.

[FM1] P. Frejlich and I. Mărcuţ, *The normal form theorem around Poisson transversals*, Pacific J. Math. **287**(2), p. 371-391, 2017.

[FM2] P. Frejlich and I. Mărcuţ, *On dual pairs in Dirac geometry*, Math. Z. **289** (1-2), p. 171-200, 2018.

[FM3] P. Frejlich and I. Mărcuţ, *Normal forms for Poisson maps and symplectic groupoids around Poisson transverals*, Lett. Math. Phys. **108**(3), p. 711-735, 2018.

[G] M. J. Gotay, *On coisotropic imbeddings of presymplectic manifolds*, Proc. Amer. Math. Soc. **84**(1), p. 111-114, 1982.

[H] M. W. Hirsch, *Differential Topology*, Graduate Texts in Mathematics **33**, Springer-Verlag, 1976.

[Ma] C.-M. Marle, *Sous-variétés de rang constant d'une variété symplectique*, Astérisque **107-108**, p. 69-86, 1983.

[Me] E. Meinrenken, *Poisson geometry from a Dirac perspective*, Lett. Math. Phys. **108**(3), p. 447-498, 2018.

[Muk] A. Mukherjee, *Differential Topology*, Second edition, Birkhäuser, 2015.

[Mun] J. Munkres, *Topology*, Second edition, Prentice Hall, 2000.

[Wa] A. Wade, *On the geometry of coisotropic submanifolds of Poisson manifolds*, Contemp. Math. **467**, p. 63-72, 2008.

[We1] A. Weinstein, *Lectures on symplectic manifolds*, Regional Conference Series in Mathematics No. 29, American Mathematical Society, 1977.

[We2] A. Weinstein, *Symplectic manifolds and their Lagrangian submanifolds*, Adv. Math. **6**(3), p. 329-346, 1971.

[Z] M. Zambon, *Submanifolds in Poisson geometry: a survey*, Complex and Differential Geometry, Springer, p. 403-420, 2011.

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