KU LEUVEN

Functions on quasi-Poisson and quasi-symplectic groupoids

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work in progress

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Goal of the talk

Recall

M is a symplectic (or more generally Poisson) manifold
$\Rightarrow (C^\infty(M), \{\,,\,\})$ is a Lie algebra.

Question

On a quasi-symplectic or quasi-Poisson groupoid,
do functions form a "good" Lie 2-algebra?

"Good" means: preserved by Morita equivalence.

Remark

We consider "quasi" because symplectic groupoids and Poisson groupoids are not as well behaved under Morita equivalence.

Quasi-Poisson groupoids

Definition (Iglesias-Ponte, Laurent-Gengoux, Xu 2012)

A quasi-Poisson groupoid is a Lie groupoid $\Sigma \rightrightarrows M$ with

- $\Pi \in \mathfrak{X}^2_{\text{mult}}(\Sigma)$ (a multiplicative bivector field)
- $\phi \in \Gamma(\wedge^3 A)$

satisfying

$$\frac{1}{2}[\Pi, \Pi] = \overrightarrow{\phi} - \overleftarrow{\phi} \qquad \text{and} \qquad [\Pi, \overrightarrow{\phi}] = 0.$$

Here:
Π is multiplicative if the graph of the product is coisotropic w.r.t. $\Pi + \Pi + (-\Pi)$,
A is Lie algebroid of Σ,
$\overrightarrow{\phi} \in \mathfrak{X}^3(\Sigma)$ the right-invariant extension of ϕ.

Gauge transformations

Remark [Bonechi, Ciccoli, Laurent-Gengoux, Xu 2022]

Given a Lie groupoid Σ,

$$\oplus_{\bullet \geq -2}\Gamma(\wedge^{\bullet+2} A) \oplus \mathfrak{X}^{\bullet+1}_{\text{mult}}(\Sigma)$$

is a DGLA (differential graded Lie algebra).
A Maurer-Cartan element in this DGLA is a quasi-Poisson groupoid structure on Σ.

Remark [Iglesias-Ponte, Laurent-Gengoux, Xu 2012]

Let (Σ, Π, ϕ) be a quasi-Poisson groupoid.
For any $T \in \Gamma(\wedge^2 A)$, we get a new quasi-Poisson groupoid (Σ, Π^T, ϕ^T) where

$$\Pi^T = \Pi + \overrightarrow{T} - \overleftarrow{T}$$

$$\phi^T = \phi - [\Pi, \overrightarrow{T}]|_M - \frac{1}{2}[T, T]_A.$$

Example [pair groupoid]

Let $P \in \mathfrak{X}^2(M)$, denote $\phi := \frac{1}{2}[P, P] \in \mathfrak{X}^3(M)$. Then

$$(M \times M,\ \underbrace{P_1 - P_2}_{=:\Pi},\ \phi)$$

is a quasi-Poisson groupoid.

The Lie 2-algebra $L_{\mathrm{Pois}}(\Sigma, \Pi, \phi)$

Proposition (ALSO ÁLVAREZ, CUECA 2024; AFTER CHEN, LANG, LIU 2023)

The complex

$$C^\infty(M) \xrightarrow{\partial} C^\infty_{\mathit{mult}}(\Sigma)$$

is canonically a Lie 2-algebra.
The differential is $\partial := \mathbf{t}^* - \mathbf{s}^*$, the brackets

$$\begin{aligned} l_2(\varphi, \varphi') &:= \{\varphi, \varphi'\}, \\ l_2(\varphi, f) &:= \{\varphi, \mathbf{t}^* f\}|_M, \\ l_3(\varphi, \varphi', \varphi'') &:= \langle \phi, d\varphi|_A \wedge d\varphi'|_A \wedge d\varphi''|_A \rangle. \end{aligned}$$

Here: a function φ is multiplicative if $\varphi(gh) = \varphi(g) + \varphi(h)$.

Proposition

Two quasi-Poisson groupoids are Morita equivalent $\Longrightarrow$
the respective Lie 2-algebras L_{Pois} are quasi-isomorphic.

Quasi-symplectic groupoids

Definition (Xu 2004; Bursztyn, Crainic, Weinstein, Zhu 2004)

A quasi-symplectic groupoid is a Lie groupoid $\Sigma \rightrightarrows M$ with

- $\omega \in \Omega^2_{\text{mult}}(\Sigma)$ (a multiplicative 2-form)
- $\Omega \in \Omega^3(M)$ a closed 3-form

such that writing $\partial := \mathbf{t}^* - \mathbf{s}^*$

$$d\omega = \partial\Omega$$

$$\ker\omega \cap \ker \mathbf{t}_* \cap \ker \mathbf{s}_* = \{0\}.$$

Here: ω is multiplicative if the graph of the product is isotropic w.r.t. $\omega + \omega + (-\omega)$

Remark

We have $\dim(\Sigma) = 2\dim(M)$.

The Bott-Shulman complex

Given a Lie groupoid $\Sigma \rightrightarrows M$, there is a double complex of differential forms on the nerve.

$$
\begin{array}{ccccccccc}
C^\infty(\Sigma^{(2)}) & \xrightarrow{d} & \Omega^1(\Sigma^{(2)}) & \xrightarrow{d} & \dots & & & & \\
\partial\uparrow & & \partial\uparrow & & \partial\uparrow & & & & \\
C^\infty(\Sigma) & \xrightarrow{d} & \Omega^1(\Sigma) & \xrightarrow{d} & \Omega^2(\Sigma) & \xrightarrow{d} & \dots & & \\
\partial\uparrow & & \partial\uparrow & & \partial\uparrow & & \partial\uparrow & & \\
C^\infty(M) & \xrightarrow{d} & \Omega^1(M) & \xrightarrow{d} & \Omega^2(M) & \xrightarrow{d} & \Omega^3(M) & \xrightarrow{d} & \dots
\end{array}
$$

The Bott-Shulman complex is the total complex, with differential $D = d \pm \partial$.

Remark

Given forms $\omega \in \Omega^2(\Sigma)$ and $\Omega \in \Omega^3(M)$:

$\omega + \Omega$ is D-closed $\Leftrightarrow$

(Σ, ω, Ω) is a quasi-symplectic groupoid, except for non-degeneracy.

Gauge transformations

Remark
Let (Σ, ω, Ω) be a quasi-symplectic groupoid.
For any $\sigma \in \Omega^2(M)$, we get a new quasi-symplectic groupoid $(\Sigma, \omega^\sigma, \Omega^\sigma)$ where

$$\omega^\sigma = \omega + \partial\sigma$$
$$\Omega^\sigma = \Omega + d\sigma.$$

In other words:

$$(\omega^\sigma + \Omega^\sigma) = (\omega + \Omega) + D\sigma.$$

Example [pair groupoid]
Let $\sigma \in \Omega^2(M)$, denote $\Omega := d\sigma$. Then

$$(M \times M, \underbrace{pr_1^*\sigma - pr_2^*\sigma}_{=:\omega}, \Omega)$$

is a quasi-Poisson groupoid.

Example[Alekseev, Malkin, Meinrenken 1998]

Let $\mathfrak{d}$ be a Lie algebra with invariant inner product.

The transformation groupoid of the action of D on itself by conjugation

$$D \rtimes D \rightrightarrows D$$

is a quasi-symplectic groupoid, with the Cartan 3-form Ω on D.

"Hamiltonian" vector fields

Let $(\Sigma \rightrightarrows M, \omega, \Omega)$ be a quasi-symplectic groupoid.
The following chain map $\omega^\flat$ induces[Del Hoyo, Ortiz 2020] an isomorphism in cohomology:

$$\begin{array}{ccc} \mathfrak{X}_{\text{mult}}(\Sigma) & \xrightarrow{\ \omega\ } & \Omega^1_{\text{mult}}(\Sigma) \\ \partial \big\uparrow & & \big\uparrow \partial \\ \Gamma(A) & \xrightarrow{\ \iota_\bullet \omega|_{TM}\ } & \Omega^1(M) \end{array}$$

Corollary

For any $\varphi \in C^\infty_{mult}(\Sigma)$ *there are* $\alpha \in \Omega^1(M)$ *and* $\xi \in \mathfrak{X}_{mult}(\Sigma)$ *s.t.*

$$d\varphi - \partial\alpha = \iota_\xi \omega.$$

We view ξ as a “Hamiltonian vector field” for φ.

The graded Lie algebra $\mathfrak{g}(\Sigma, \omega, \Omega)$

Proposition

There is a canonical graded Lie algebra structure on

$$C^\infty_{\text{basic}}(M)[1] \oplus \frac{C^\infty_{\text{mult}}(\Sigma)}{\partial C^\infty(M)}.$$

The brackets are

$$[\underline{\varphi}, \underline{\varphi'}] := \underline{\iota_\xi \iota_{\xi'} \omega}, \qquad [\underline{\varphi}, f] := \xi|_M(f),$$

where the "Hamiltonian" vector field ξ of φ.

Remark

The above space is

$$H^0_d(\Sigma)[1] \oplus H^1_d(\Sigma)$$

(differentiable cohomology of the Lie groupoid Σ).

Remark

Two quasi-symplectic groupoids are Morita equivalent $\Longrightarrow$ the respective graded Lie algebras $\mathfrak{g}$ are isomorphic.

Compatibility with product

Remark
$(H^0_d(\Sigma)[1] \oplus H^1_d(\Sigma))[-1]$ is actually a graded Poisson algebra of degree 1, w.r.t. the product

$$f_1 * f_2 := f_1 \cdot f_2, \qquad f * \underline{\varphi} := \underline{\mathbf{t}^* f \cdot \varphi}.$$

First interpretation of $\mathfrak{g}$: from Dirac structures

Remark[Also Gualtieri, Matviichuk, Scott 2020]

Let L be a Dirac structure of some Courant algebroid $E \to M$.

- $C^\infty_{\text{basic}}(M)$ is an abelian subalgebra of $(C^\infty_{\text{admissible}}(M, L), \{\ ,\ \})$ (Casimirs)
- Every element of $\Gamma(L^*)$ is of the form

$$\lambda = \langle e, \cdot \rangle|_L \qquad \text{for some } e \in \Gamma(E).$$

  λ is d_L-closed $\Leftrightarrow$ $[e, \cdot]_c$ preserves $\Gamma(L)$ ("symmetry of L")
- $\rightsquigarrow$ The Courant bracket induces a Lie bracket on $H^1(L)$.
- $\rightsquigarrow$ The anchor induces a representation of $H^1(L)$ on $C^\infty_{\text{basic}}(M)$.

First interpretation of $\mathfrak{g}$: from Dirac structures

Quasi-symplectic groupoid $(\Sigma \rightrightarrows M, \omega, \Omega)$
$\rightsquigarrow$ Dirac structure $L \subset (TM \oplus T^*M)_\Omega$.

If Σ is s.s.c., have Van Est isomorphism

$$H^1_d(\Sigma) \cong H^1(L).$$

It preserves the Lie algebra structure and representation on $C^\infty_{\text{basic}}(M)$.

2nd interpretation of $\mathfrak{g}$: from quasi-Poisson groupoids

Let $(\Sigma \rightrightarrows M, \omega, \Omega)$ be a quasi-symplectic groupoid.

Choice of Lagrangian complement to the Dirac structure L
$\rightsquigarrow$ Quasi-Poisson groupoid structure [Bursztyn, Iglesias, Ševera 2009]

$$(\Sigma, \Pi, \phi).$$

$\Pi^\sharp$ and $\omega^\flat$ induce inverse maps in cohomology.

Remark
Two different choices of Lagrangian complement $\Rightarrow$
Morita equivalent quasi-Poisson groupoid structures.

2nd interpretation of $\mathfrak{g}$: from quasi-Poisson groupoids

Proposition

The following graded Lie algebra structures on $C^\infty_{basic}(M)[1] \oplus \frac{C^\infty_{mult}(\Sigma)}{\partial C^\infty(M)}$ *agree:*

- $\mathfrak{g}(\Sigma, \omega, \Omega)$
- $H(L_{\mathrm{Pois}}(\Sigma, \Pi, \phi))$, *the cohomology of the Lie 2-algebra* $L_{\mathrm{Pois}}(\Sigma, \Pi, \phi)$.

(Σ, ω, Ω) quasi-sympl $\underset{(choice)}{\overset{invert}{\rightsquigarrow}}$ (Σ, Π, ϕ) quasi-Poisson

(Σ, Π, ϕ) quasi-Poisson $\rightsquigarrow$ $L_{\mathrm{Pois}}(\Sigma, \Pi, \phi)$ $\rightsquigarrow$ cohomology of $L_{\mathrm{Pois}}(\Sigma, \Pi, \phi)$

(Σ, ω, Ω) quasi-sympl $\rightsquigarrow$ $\mathfrak{g}(\Sigma, \omega, \Omega)$

$\mathfrak{g}(\Sigma, \omega, \Omega)$ = cohomology of $L_{\mathrm{Pois}}(\Sigma, \Pi, \phi)$

Conclusion

To a quasi-symplectic groupoid we can associate various Lie 2-algebras of functions. They are all quasi-isomorphic, and the cohomology (a graded Lie algebra) is canonical.

A Lie 2-algebra on quasi-symplectic groupoids

Let (Σ, ω, Ω) be a quasi-symplectic groupoid.

Proposition

There is a Lie 2-algebra structure on the complex

$$C^\infty(M) \xrightarrow{(d,\partial)} \underbrace{L_0}_{\subset \, C^\infty_{mult}(\Sigma) \oplus \Omega^1(M)}$$

where

$$L_0 := \left\{ (\varphi, \alpha) : \exists \xi \in \mathfrak{X}_{mult}(\Sigma) \text{ s.t. } D(\varphi + \alpha) = \iota_\xi(\omega + \Omega) \right\}.$$

Remark

The above holds also for quasi-pre-symplectic groupoid, i.e. when the non-degeneracy condition on ω is omitted.

Remark

Two quasi-symplectic groupoids are Morita equivalent $\not\Longrightarrow$ the respective Lie 2-algebras are quasi-isomorphic.

Our original motivation: Path space (heuristic)

Let M be a manifold, let $PM = \{[0,1] \to M\}$ the path space.

- The diagram

$$\begin{array}{c} PM \\ \downarrow\downarrow ev_0 \\ M \end{array}$$

  resembles a groupoid (but the concatenation of paths is not associative).

- The transgression map is

$$\tau \colon \Omega^k(M) \to \Omega^{k-1}(PM),$$

$$(\tau\alpha)_{\gamma}(v_1, \dots, v_{k-1}) = \int_0^1 \alpha(\dot{\gamma}(t), v_1(t), \dots, v_{k-1}(t))\, dt.$$

  It is not a chain map:

$$d \circ \tau + \tau \circ d = \partial := ev_1^* - ev_0^*.$$

Our original motivation: Path space (heuristic)

Let $\Omega \in \Omega^3_{\text{closed}}(M)$. Get

$$\omega := \tau(\Omega) \in \Omega^2(PM),$$

which satisfies

$$d\omega = \partial\Omega.$$

Further ω is multiplicative.

Conclusion:

$$\begin{array}{c}(PM, \omega)\\ \downdownarrows\\ (M, \Omega)\end{array}$$

resembles a quasi-pre-symplectic groupoid.

The BHR Lie 2-algebra

Let $\Omega \in \Omega^3_{\text{closed}}(M)$ be non-degenerate (i.e., 2-plectic).
The Hamiltonian 1-forms

$$\Omega^1_{\text{ham}}(M,\Omega) := \{\alpha \in \Omega^1(M) \mid \exists X_\alpha \in \mathfrak{X}(M) \text{ s.t. } d\alpha = \iota_{X_\alpha}\Omega\}$$

carry a skew-symmetric bracket.

Proposition (Baez, Hoffnung, Rogers 2010)

There is a Lie 2-algebra structure on the complex

$$C^\infty(M) \xrightarrow{d} \Omega^1_{Ham}(M,\Omega)$$

with higher brackets

- $l_2(\alpha,\beta) = \iota_{X_\alpha}\iota_{X_\beta}\Omega$
- $l_2(\alpha,f) = 0$
- $l_3(\alpha,\beta,\gamma) = \Omega(X_\alpha,X_\beta,X_\gamma)$,

for $f \in C^\infty(M)$ *and* $\alpha,\beta,\gamma \in \Omega^1_{ham}(M,\Omega)$.

Relation of the BHR Lie 2-algebra and path space

The transgression τ gives an injective cochain map

$$\begin{array}{ccc} C^\infty(M) & \xrightarrow{d} & \Omega^1_{\text{ham}}(M,\Omega) \\ \downarrow & & \downarrow \tau \\ C^\infty(M) & \xrightarrow{\partial} & C^\infty_{\text{mult}}(PM) \end{array}$$

This suggests: Look for a Lie 2-algebra structure on $C^\infty(M) \xrightarrow{\partial} C^\infty_{\text{mult}}(PM)$, and similarly for quasi-pre-symplectic groupoids.

There is probably none, however:

Proposition

$$\begin{array}{ccc} C^\infty(M) & \xrightarrow{d} & \Omega^1_{ham}(M,\Omega) \\ \downarrow & & \downarrow (\tau, Id) \\ C^\infty(M) & \xrightarrow{(\partial, d)} & L_0(PM) \end{array}$$

is a strict Lie 2-algebra morphism.

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Thank you for your attention