A super-geometric approach to the reduction of Poisson manifolds

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based on work in progress
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Plan of the talk

–The reduction problem in Poisson geometry

–The problem in super-geometric terms

–Translating back to Poisson geometry

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Poisson manifolds

Definition

M is a Poisson manifold if $C^\infty(M)$ is endowed with a Lie bracket $\{\bullet,\bullet\}$ satisfying $\{f,gh\} = \{f,g\}h + g\{f,h\}$.

Examples

- $\mathfrak{g}^*$ the dual of a Lie algebra:
  for $v,w \in \mathfrak{g} \subset C^\infty(\mathfrak{g}^*)$ define $\{v,w\} := [v,w]$.
- symplectic manifolds
  (i.e. ω is a non-degenerate two-form with $d\omega = 0$).
  We have $\{f,g\} := \omega(X_f, X_g)$.
  Cotangent bundles T^*N are symplectic.

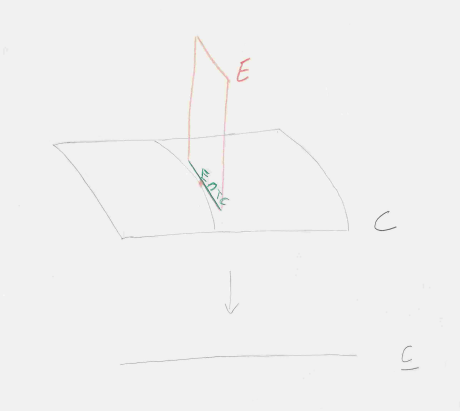
Are quotients of submanifolds Poisson?

Problem

Let M be a Poisson manifold. Given

- *a submanifold C*
- *a subbundle $E \subset TM|_C$,*

is there an induced Poisson bracket on $\underline{C} := C/(E \cap TC)$?



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Graded manifolds

Definition

Let $U_0 \subset \mathbb{R}^n$ open subset and $V = \oplus_i V_i$ a $\mathbb{Z}$-graded vector space.

The local model for a graded manifold consists of a pair

- U_0 (the "body")
- $C^\infty(U_0) \otimes S^\bullet(V^*)$ (the graded comm. algebra of functions)

Fact

$E = \oplus_i E_i \to M$ a graded vector bundle $\rightsquigarrow$ graded manifold with body M and functions $\Gamma(S^\bullet E^*)$.

Graded manifolds

Example

W a usual vector space $\rightsquigarrow$
$W[1]$ concentrated in degree -1 $\rightsquigarrow$
graded manifold with body $\{pt\}$ and functions

$$S^\bullet(W[1])^* = S^\bullet(W^*[-1]) = \wedge^\bullet W^*.$$

Graded manifolds

Example

$T^*[1]M \rightsquigarrow$
graded manifold with body M and functions

$$\Gamma(S^\bullet(T[-1]M)) = \Gamma(\wedge^\bullet TM) = \{\text{multivector fields on } M\}.$$

x_j coordinates on $M \rightsquigarrow p_j$ coordinates on fibers of $T^*M \rightsquigarrow$
θ_j degree 1 coordinates of fibers of $T^*[1]M$.
Examples of functions on $T^*[1]M$ are $g(x)\theta_1, \;\theta_1\theta_2 = -\theta_2\theta_1$.

Remark

$T^*[1]M$ has a symplectic form $\omega = dx_j \wedge d\theta_j$
$\rightsquigarrow$ Poisson bracket of degree -1: $\{\theta_j, x_k\} = \delta_{jk}$.
It is just the Schouten bracket on multivector fields.

The problem in super-geometric terms

Fact

Poisson bracket $\{\bullet, \bullet\}$ *on* $M \leftrightarrow$
bivector field $\pi \in \Gamma(\wedge^2 TM)$ *satisfying* $[\pi, \pi] = 0 \leftrightarrow$
degree 2 function $\mathcal{S}$ *on* $T^*[1]M$ *satisfying* $\{\mathcal{S}, \mathcal{S}\} = 0$*.*

$$\pi = \pi_{ij}(x)\partial_{x_i} \wedge \partial_{x_j} \leftrightarrow \mathcal{S} = \pi_{ij}(x)\theta_i\theta_j.$$

Idea

Submanifold $\mathcal{C}$ of $(T^*[1]M, \omega, \mathcal{S})$ $\rightsquigarrow$ quotient $\underline{\mathcal{C}}$ to which ω and $\mathcal{S}$ descend.

The problem in super-geometric terms

More precisely:

1) $\mathcal{C}$ presymplectic submanifold of $T^*[1]M \rightsquigarrow$
$\underline{\mathcal{C}} := \mathcal{C}/ker(i^*\omega)$ is a degree 1 graded symplectic manifold, hence symplectomorphic to $T^*[1]X$ for some X.

Algebraically: Let

$$\mathcal{I}_{\mathcal{C}} = \{F : F|_{\mathcal{C}} = 0\} \subset C^{\infty}(T^*[1]M)$$

and

$$\mathcal{N}(\mathcal{I}_{\mathcal{C}}) = \{F : \{F, \mathcal{I}_{\mathcal{C}}\} \subset \mathcal{I}_{\mathcal{C}}\}.$$

It is clear that $\mathcal{N}(\mathcal{I}_{\mathcal{C}})/\mathcal{N}(\mathcal{I}_{\mathcal{C}}) \cap \mathcal{I}_{\mathcal{C}}$ has an induced Poisson bracket. Under regularity assumptions it is $C^{\infty}(\underline{\mathcal{C}})$ for some graded manifold $\underline{\mathcal{C}}$.

The problem in super-geometric terms

2) $\mathcal{S}|_{\mathcal{C}}$ is invariant along the distribution $ker(i^*\omega) \rightsquigarrow$ degree 2 function $\underline{\mathcal{S}}$ on $\underline{\mathcal{C}} \cong T^*[1]X$.
If $\{\underline{\mathcal{S}}, \underline{\mathcal{S}}\} = 0$ then $\underline{\mathcal{S}}$ corresponds to a Poisson structure on X.

Algebraically: $\mathcal{S}|_{\mathcal{C}}$ is invariant $\Leftrightarrow \mathcal{S} \in \mathcal{N}(\mathcal{I}_{\mathcal{C}}) + \mathcal{I}_{\mathcal{C}}$.
A *sufficient* condition for $\{\underline{\mathcal{S}}, \underline{\mathcal{S}}\} = 0$ is clearly $\mathcal{S} \in \mathcal{N}(\mathcal{I}_{\mathcal{C}})$.
It turns out: the weaker condition $\{\mathcal{S}, (\mathcal{I}_{\mathcal{C}})_0\} \subset \mathcal{I}_{\mathcal{C}}$ is also sufficient.

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Statement 1

Lemma

- $\mathcal{C}$ *is presymplectic* $\Leftrightarrow \mathcal{C} = E^{\circ}[1]$
  where $E \to C$ *is a subbundle of* $TM \to M$ *s.t.* $E \cap TC$ *is a constant rank, involutive distribution.*
- *In that case*
  $$\underline{\mathcal{C}} \cong T^*[1]\underline{C}$$
  where $\underline{C} := C/(E \cap TC)$.

Statement 1

Proposition

Suppose

1. $(\mathcal{L}_{\Gamma(E\cap TC)}\pi)|_C \subset E \wedge TM|_C$.
2. $\sharp E^\circ \subset TC$

   where $\sharp$ *denotes contraction with* $\pi \in \Gamma(\wedge^2 TM)$.

Then $\underline{C}$ *inherits a Poisson structure.*

Statement 2

To obtain a statement with weaker assumptions we apply *reduction in stages*: let $\mathcal{A}$ be a coisotropic submanifold of $T^*[1]M$ containing $\mathcal{C}$.

- Take the image of $\mathcal{C}$ under the projection $\mathcal{A} \to \mathcal{A}/T\mathcal{A}^\omega$. Assuming that $T\mathcal{C} \cap T\mathcal{A}^\omega$ has constant rank, it is a presymplectic submanifold.
- Take its presympletic quotient. It is (locally) symplectomorphic to $\underline{\mathcal{C}}$.

Statement 2

Theorem

Let $D|_C$ *be a subbundle of* $TM|_C$ *with*

$$(E \cap TC) \subset D|_C \subset E$$
$$\sharp E^\circ \subset TC + D|_C.$$

Extend C *to a submanifold* A *with* $TA|_C = TC + D|_C$ *and* $D|_C$ *to an integrable distribution* D *on* A*. Assume*

$$(\mathcal{L}_{\Gamma(D)}\pi)|_C \subset E \wedge TM|_C.$$

Then $\underline{C}$ *is a Poisson manifold.*

Statement 2