Poisson geometry and coisotropic submanifolds

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Plan of the talk

- Poisson manifolds
- Coisotropic submanifolds
- Other interesting submanifolds: Pre-Poisson submanifolds
- Application to generalized complex geometry

Poisson manifolds

Def. A Poisson algebra is

- an associative, commutative algebra A
- a Lie bracket $\{\cdot,\cdot\}$ on A

satisfying the Leibniz identity

$$\{a, bc\} = \{a, b\}c + b\{a, c\}$$

(i.e. $\{a,\cdot\}$ is a derivation of the product, $\forall a \in A$).

Def. M is a Poisson manifold if $C^\infty(M)$ is a Poisson algebra.

Example. Any manifold M is a Poisson manifold setting $\{\cdot,\cdot\} \equiv 0$.

Example. Let (M, ω) be a symplectic manifold (i.e. $\omega \in \Omega^2(M)$ is closed, non-degenerate).
Then M is a Poisson manifold, with

$$\{f, g\} := \omega(X_f, X_g),$$

where the vector field X_f is defined by $\omega(X_f, \cdot) = df$.
The Jacobi identity holds because $d\omega = 0$.

Example. $G \circlearrowleft (M, \omega)$ free and proper action $\rightsquigarrow M/G$ is a Poisson manifold.
Indeed $pr : M \longrightarrow M/G$ identifies $C^\infty(M)^G$ with $C^\infty(M/G)$, and $C^\infty(M)^G$ is closed under $\{\cdot, \cdot\}$.

Example. $\mathfrak{g}$ f.d. Lie algebra $\rightsquigarrow \mathfrak{g}^*$ (linear) Poisson manifold.
For all $v, w \in \mathfrak{g} \subset C^\infty(\mathfrak{g}^*)$, define

$$\{v, w\} := [v, w]$$

and extend to arbitrary elements of $C^\infty(\mathfrak{g}^*)$ by the Leibniz rule.

Geometrically:

1) A Poisson manifold is (M, π) where π is a bivector field (a section of $\wedge^2 TM$) s.t. $[\pi, \pi] = 0$. We have

$$\{f, g\} = \pi(df, dg).$$

2) The image of

$$\sharp : T^*M \to TM, \xi \mapsto \pi(\xi, \cdot)$$

is a singular integrable distribution on M, and π gives rise to a symplectic form on each leaf. The symplectic foliation encodes the Poisson bivector π.

Example. Any manifold M with bivector $\pi = 0$.
All symplectic leaves are points.

Example. The symplectic manifold $(\mathbb{R}^{2n}, \sum_{i=1}^n dq_i \wedge dp_i) \rightsquigarrow$ Poisson bivector $\pi = \sum_{i=1}^n \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i}$.
There is just one symplectic leaf.

Example. Let $\mathfrak{g}$ be a Lie algebra.
Basis $e_1, \ldots, e_n$ of $\mathfrak{g} \rightsquigarrow$ coordinates $x_1, \ldots, x_n$ on $\mathfrak{g}^*$. Then $\mathfrak{g}^*$ has Poisson bivectorfield

$$\pi = \frac{1}{2} \sum c_{ij}^k x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$$

where $[e_i, e_j] = \sum c_{ij}^k e_k$.
The symplectic leaves are the orbits of $Ad^* : G \circlearrowleft \mathfrak{g}^*$.

Sub-example. Take $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$. There is a basis with

$$[e_1, e_2] = -e_3, \quad [e_2, e_3] = e_1 \quad [e_3, e_1] = e_2.$$

The symplectic foliation on $\mathfrak{g}^*$ is

The definition of Poisson manifold arises naturally. For any manifold M consider the $\mathbb{R}[[\varepsilon]]$ -module

$$C^\infty(M)[[\varepsilon]] := \{\sum_{n=0}^{\infty} f_n \varepsilon^n : f_n \in C^\infty(M)\}.$$

Def. A star-product on a manifold M is an associative product on $C^\infty(M)[[\varepsilon]]$ s.t.

$$\begin{aligned} f \star g &= fg + \textstyle\sum_{n=1}^{\infty} B_n(f,g)\varepsilon^n && \forall f,g \in C^\infty(M) \\ 1 \star f &= f \star 1 = f && \forall f \in C^\infty(M), \end{aligned}$$

where the B_n are bidifferential operators.

A star-product $\star$ on M induces a Poisson structure on M by

$$\{f,g\} := \frac{1}{2} \cdot \frac{f \star g - g \star f}{\varepsilon} \quad mod(\varepsilon) \qquad \forall f,g \in C^\infty(M).$$

Interpretation: $\star$ is a "*deformation of the commutative product on $C^\infty(M)$ in direction of the Poisson bracket*".
Actually $\star$ is equivalent to star product $\star'$ of the form $f \star' g = fg + \{f,g\}\varepsilon + O(\varepsilon^2)$.

Question (deformation quantization): given a Poisson manifold (M,π), does there exist a star-product inducing the given Poisson bracket on M?

Theorem (Kontsevich 1997). *Yes.*

Coisotropic submanifolds

If M is any manifold and C a submanifold,

$$I_C := \{\text{functions on } M \text{ which vanish on } C\}$$

is a multiplicative ideal.

Def. $C \subset (M, \pi)$ is a coisotropic submanifold if I_C is a subalgebra of $(C^\infty(M), \{\cdot, \cdot\})$.

Geometrically: Denote for each $x \in C$

$$(T_xC)^\circ := \{\xi \in T_x^*M : \xi|_{T_xC} \equiv 0\}.$$

C is coisotropic iff $\sharp TC^\circ \subset TC$.
The singular distribution $\sharp TC^\circ$ integrates to a singular foliation $\mathcal{F}$, the characteristic foliation.

Example. If (M, ω) is a symplectic manifold, the $C \subset M$ is coisotropic iff $TC^\omega \subset TC$.

Example. $\mathfrak{h} \subset \mathfrak{g}$ a Lie subalgebra $\rightsquigarrow \mathfrak{h}^\circ$ coisotropic submanifold of $\mathfrak{g}^*$ (clear for linear functions).

A map $\phi : (M_1, \pi_1) \longrightarrow (M_2, \pi_2)$ is called Poisson map if the pullback of functions $\phi^* : C^\infty(M_2) \longrightarrow C^\infty(M_1)$ is a morphism of Poisson algebras, or equivalently if $\phi_*\pi_1 = \pi_2$.
Example. A map $\phi : (M_1, \pi_1) \longrightarrow (M_2, \pi_2)$ is a Poisson map if and only if $graph(\phi) \subset (M_1 \times M_2, \pi_1 - \pi_2)$ is coisotropic.

Let $C \subset (M, \pi)$ be coisotropic, denote by

$$N(I_C) := \{f \in C^\infty(M) : \{f, I_C\} \subset I_C\}$$

the Poisson-normalizer of I_C in $C^\infty(M)$. Then

$$N(I_C)/I_C \cong C^\infty(C)^{inv} := \{\text{functions on } C \text{ constant along } \mathcal{F}\}$$

is a Poisson algebra.
If smooth, $C/\mathcal{F}$ is a Poisson manifold! And it admits a deformation quantization by Kontsevich's theorem.

In general:
Theorem (Cattaneo-Felder). *If the first and second Lie algebroid cohomologies of TC° vanish, $C^\infty(C)^{inv}$ admits a deformation quantization.*

Example. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra. Then

$$Pol(\mathfrak{h}^\circ)^{inv} \cong S(\mathfrak{g}/\mathfrak{h})^{\mathfrak{h}},$$

the functions invariant under the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}/\mathfrak{h}$.
Suppose $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$ where $\mathfrak{k}$ is $ad_{\mathfrak{h}}$-invariant. Then $S(\mathfrak{k})^{\mathfrak{h}}$ admits a deformation quantization.

Pre-Poisson submanifolds

Question: Are there submanifolds of (M, π) which are not coisotropic but can be threated as such?

Def. $C \subset (M, \pi)$ is called pre-Poisson submanifold if $TC + \sharp TC^\circ$ has constant rank along C.

Proposition (Cattaneo-Z). *Let $C \subset (M, \pi)$ be a pre-Poisson submanifold.*

Then there exists a submanifold $\tilde{M}$ such that

- *$\tilde{M}$ inherits a Poisson structure $\tilde{\pi}$ from (M, π)*
- *C is a coisotropic submanifold of $(\tilde{M}, \tilde{\pi})$.*

Further $(\tilde{M}, \tilde{\pi})$ is unique up to Poisson diffeomorphism.

Proof. Write $R \oplus (TC + \sharp TC^\circ) = TM$, and extend C "along R" to obtain $\tilde{M}$. Then $\tilde{M}$ intersects transversely the symplectic leaves of (M, π), and the intersections are symplectic submanifolds of $\tilde{M}$, so $\tilde{M}$ is Poisson.
Uniqueness: construction of explicit Poisson diffeomorphism from $\tilde{M}$ to $\tilde{M}'$. □

Example. Let C be coisotropic in (M, π).
Then $TC + \sharp TC^\circ = TC$ obviously has constant rank, and $\tilde{M} =$ open set in M

Example. Let C be a point $x \in (M, \pi)$. Then the $\tilde{M}$ as above are slices transvese to the symplectic leaf through x.

Example. Let (M, ω) be symplectic and $i : C \hookrightarrow M$ a submanifold.
C is pre-Poisson iff $ker(i^*\omega)$ has constant rank.

Example. $\mathfrak{h} \subset \mathfrak{g}$ Lie subalgebra and $\lambda \in \mathfrak{g}^* \rightsquigarrow \mathfrak{h}^\circ + \lambda$ pre-Poisson submanifold of $\mathfrak{g}^*$.
Indeed one can show that if $\phi : (M_1, \pi_1) \to (M_2, \pi_2)$ is a submersive Poisson map then ϕ^{-1}(Pre-Poisson) $=$ Pre-Poisson. Now take $\phi : \mathfrak{g}^* \to \mathfrak{h}^*$ and pull back $\lambda|_{\mathfrak{h}}$.

Corollary. *If C is a pre-Poisson submanifold and the first and second Lie algebroid cohomologies of $TC^\circ \cap \sharp^{-1}TC$ vanish, then the Poisson algebra $N(I_C)/N(I_C) \cap I_C$ admits a deformation quantization.*

Coisotropic submanifolds in generalized complex geometry

Def. A generalized complex structure on a manifold M is

$$J : TM \oplus T^*M \to TM \oplus T^*M$$

with $J^2 = -Id$ and satisfying a certain integrability condition.

Example. Let $I : TM \to TM$ be a complex structure.
Then $J = \begin{pmatrix} I & 0 \\ 0 & -I^* \end{pmatrix}$ is a generalized complex structure.

Example. Let $\omega \in \Omega^2(M)$ be a symplectic form.
Then $J = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$ is a generalized complex structure.

In general

$$J = \begin{pmatrix} A & \pi \\ \omega & -A^* \end{pmatrix}$$

where $A \in End(TM), \omega \in \Omega^2(M)$ and π is a bivector.
J integrable $\rightsquigarrow$ (M, π) is a *Poisson manifold*!

Question: Are there submanifolds $C \subset (M, J)$ such that some quotient of C inherits a generalized complex structure?

Def. A brane is a submanifold C together with a $F \in \Omega^2_{closed}(C)$ s.t. the following subbundle of $(TM \oplus T^*M)|_C$ is J-invariant:

$$graph(F) := \{(X, \xi) \in TM \oplus T^*M : X \in TC, \xi|_{TC} = i_X F\}.$$

Since for $(0, \xi) \in 0 \oplus TC^\circ \subset graph(F)$ we have

$$\begin{pmatrix} A & \pi \\ \omega & -A^* \end{pmatrix} \begin{pmatrix} 0 \\ \xi \end{pmatrix} = \begin{pmatrix} \pi\xi \\ -A^*\xi \end{pmatrix} \in graph(F)$$

it follows that C is a coisotropic submanifold of (M, π).

We know that π descends to $C/\mathcal{F}$, however A and ω do *not* descend, not even pointwise
(i.e. $A_x : T_xM \to T_xM$ does not preserve T_xC and $\mathcal{F}_x$, so it does *not* induce a map $T_xC/\mathcal{F}_x \to T_xC/\mathcal{F}_x$).
Nevertheless:
Theorem (Z). *When smooth, $C/\mathcal{F}$ inherits a generalized complex structure.*