Coisotropic submanifolds in b-symplectic geometry

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Coisotropic submanifolds in Poisson geometry

Let (M, π) be a Poisson manifold.

Definition

A submanifold $C \subset (M, \pi)$ is coisotropic
$\Leftrightarrow \sharp(TC^\circ) \subset TC$
$\Leftrightarrow \{\mathcal{I}, \mathcal{I}\} \subset \mathcal{I}$.

Properties:

(A) Given a map $f \colon M \to N$:
f is a Poisson map $\Leftrightarrow graph(f) \subset M \times \bar{N}$ is coisotropic.

(B) TC° is a Lie subalgebroid of T^*M

(C) Poisson reduction:
if $\underline{C} := C/\sharp TC^\circ$ is smooth, it inherits a Poisson structure.

b-geometry

Definition

- b-manifold: (M, Z), where Z codimension 1 submanifold.
- b-map: $f\colon (M_1, Z_1) \to (M_2, Z_2)$ transverse to Z_2, s.t. $f^{-1}(Z_2) = Z_1$.
- b-submanifold: submanifold N s.t. $N \pitchfork Z$.

(M, Z) a b-manifold $\rightsquigarrow$ Lie algebroid ${}^bTM$, with

$$\Gamma({}^bTM) \cong \{\text{vector fields on } M \text{ tangent to } Z\}.$$

Notation for anchor: $\rho : {}^bTM \to TM$.

Lemma

i) *Let D be a distribution on M that is tangent to Z.*
Then there exists a canonical splitting of the anchor ρ,

$$\sigma\colon D \to {}^bTM.$$

ii) *This induces a bijection between*

- *distributions on M tangent to Z*
- *subbundles of ${}^bTM$ intersecting trivially $\ker(\rho)$*

b-Poisson manifolds

Definition

A Poisson manifold (M^{2n}, π) is b-Poisson if $\pi^n \pitchfork 0$ in $\wedge^{2n} TM$

π is symplectic outside of a codimension 1 submanifold Z. Locally

$$\pi = x_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial y_2} + \dots.$$

π lifts to a non-degenerate $\Pi \in \Gamma(\wedge^2({}^bTM))$

$$\begin{array}{ccc} {}^bT^*M & \xrightarrow[\cong]{\Pi^\sharp} & {}^bTM \\ \uparrow{\scriptstyle \rho^*} & & \downarrow{\scriptstyle \rho} \\ T^*M & \xrightarrow{\pi^\sharp} & TM \end{array}$$

Proposition (Guillemin-Miranda-Pires)

b-Poisson structures $\longleftrightarrow$ b-symplectic structures.

A b-symplectic form on a b-manifold is a closed, non-degenerate b-2-form Ω.

b-coisotropic submanifolds

Let (M, Z, π) be a b-Poisson manifold.

Definition

A submanifold C is b-coisotropic
$\Leftrightarrow C \pitchfork Z$ and C is coisotropic
$\Leftrightarrow C \pitchfork Z$ and $({}^bTC)^\Omega$ is coisotropic in ${}^bTM|_C$.

Example (cf. (A)):

Let $f\colon (M_1, Z_1, \pi_1) \to (M_2, Z_2, \pi_2)$ be a Poisson map with $f(Z_1) \subset Z_2$.
The product $M_1 \times M_2$ is not b-Poisson, but[Polishchuk, Gualtieri-Li]

$$X := Bl_{Z_1 \times Z_2}(M_1 \times M_2) \setminus (\overline{\overline{M_1 \times Z_2}} \cup \overline{\overline{Z_1 \times M_2}})$$

is, and the blow-down map $p\colon X \to M_1 \times \bar{M}_2$ is Poisson.

$graph(f) \subset M_1 \times M_2$ "lifts" to a b-coisotropic submanifold of X,

$$\overline{\overline{graph(f)}}.$$

The b-Gotay theorem

If C is b-coisotropic, then $({}^b i)^* \Omega$ is b-presymplectic form (closed, const. rank). Conversely:

Theorem (b-Gotay theorem)

Let (C, Z_C, ω_C) be b-presymplectic.

i) *There is a b-coisotropic embedding of C into a b-symplectic manifold (M, Z, Ω).*
ii) *The embedding is unique up to b-symplectomorphism near C.*

"b-coisotropic" embedding ϕ means: $\phi(C)$ is b-coisotropic, $({}^b \phi)^* \Omega = \omega_C$.

Remark: This statement gives a normal form around b-coisotropic submanifolds. This is useful to study their deformations.

Remark: Statement was known for b-Lagrangian submanifolds[Kirchhoff-Lukat].

The proof

i) Let (C, Z_C, ω_C) be b-presymplectic. Let

$$E := \ker(\omega_C).$$

Choose a complement G; this gives $j\colon E^* \hookrightarrow E^* \oplus G^* = {}^bT^*C$. Get a b-symplectic form on E^* (near C)

$$({}^b pr_C)^*\Omega_C + ({}^b j)^*\Omega_{can}$$

ii) Apply the Relative b-Moser theorem.

Proposition (Relative b-Moser theorem)

Let (M, Z) be a b-manifold, C a b-submanifold. Suppose Ω_0 and Ω_1 are b-symplectic forms on (M, Z) such that

$$\Omega_0|_C = \Omega_1|_C.$$

Then there exists a b-diffeomorphism φ between neighborhoods of C such that $\varphi|_C =$ Id and

$${}^b\varphi^*\Omega_1 = \Omega_0.$$

Lie subalgebroids

Lemma (cf. (B))

Let C be b-coisotropic. Then $({}^bTC)^\circ$ is a Lie subalgebroid of ${}^bT^*M$.

$$\begin{array}{ccc} ({}^bTC)^\circ & \xrightarrow[\cong]{\Pi^\sharp} & ({}^bTC)^\Omega \\ {\scriptstyle \rho^*}\uparrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \rho} \\ TC^\circ & \xrightarrow{\pi^\sharp} & TC \end{array}$$

Remark:
In general, the characteristic "distribution"

$$D := \pi^\sharp(TC^\circ) = \rho(({}^bTC)^\Omega)$$

does not have constant rank.

Definition

Let (M, Z, π) be a b-Poisson manifold.

Definition

A submanifold C is strong b-coisotropic

$\Leftrightarrow C$ is coisotropic and $C \pitchfork$ (symplectic leaves)

$\Leftrightarrow C$ is coisotropic and $\pi^\sharp|_{TC^\circ}$ injective.

It follows: $D = \pi^\sharp(TC^\circ)$ has constant rank.

Remark: $\dim(C) \geq n + 1$, i.e. C can not be Lagrangian.

Example: On $\mathbb{R}^4$ take $\pi = x_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial y_2}$.
Then

- $\{y_2 = 0\}$ is strong b-coisotropic,
- $\{y_1 = 0\}$ is not (but is b-coisotropic).

Coisotropic reduction

Proposition (Coisotropic reduction, cf. (C))

Let C be strong b-coisotropic. Assume that $\underline{C} := C/D$ is smooth.

i) *The usual coisotropic reduction yields a b-Poisson structure on $\underline{C}$.*
ii) *The corresponding b-symplectic structure $\underline{\Omega}$ satisfies, for $q : C \to \underline{C}$,*

$$({}^b q^*)\underline{\Omega} = ({}^b i^*)\Omega.$$

Corollary

Let $G \circlearrowright (M, Z, \pi) \xrightarrow{J} \mathfrak{g}^*$ an equivariant moment map, such that the G-action is free on $J^{-1}(0)$. Then

- *$J^{-1}(0)$ is strong b-coisotropic,*
- *$J^{-1}(0)/G$ is b-Poisson.*

Example: $G \circlearrowright B \rightsquigarrow G \circlearrowright ({}^bT^*B, \Omega_{can})$, with canonical moment map.
The coisotropic quotient is

$$({}^bT^*(B/G), \Omega_{can}).$$

Reverse engineering

Corollary

Let h be any smooth function on $\mathbb{C}P^1$ that vanishes transversely. Consider $P\colon \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}P^1$ and $\alpha := \bar{z}_1 dz_1 + \bar{z}_2 dz_2$.

(i) *In a neighborhood of the unit sphere S^3, there is a b-symplectic form:*

$$\Omega = \frac{1}{r^2}\left(-1 + \frac{1}{P^* h}\right)\left(-\frac{i}{r^2}(\alpha \wedge \bar{\alpha}) + 2\sigma_{can}\right) + 2\sigma_{can}.$$

(ii) *The unit sphere S^3 is a strong b-coisotropic submanifold, and the reduced b-symplectic manifold is $(\mathbb{C}P^1, \frac{1}{h}2\sigma_{FS})$.*

Example: Let h be the function on $\mathbb{C}P^1$ induced by $Im(\bar{z}_1 z_2)$ on S^3. Then h vanishes transversely on a circle. The coefficient reads

$$\left(-\frac{1}{r^2} + \frac{1}{Im(\bar{z}_1 z_2)}\right).$$

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Thank you for your attention