Graded geometry and generalized reduction

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based on joint work with
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Recall for a symplectic manifold (M, ω)

- Coisotropic reduction:
  Let $N \subset M$ is coisotropic (i.e. $TN^\omega \subset TN$).
  Then
  $$\underline{N} := N/TN^\omega$$
  is a symplectic manifold (if smooth).
  Algebraically:
  $$C^\infty(\underline{N}) \cong Nor(I_N)/I_N$$
  isomorphism of Poisson algebras, where
  $I_N =$ vanishing ideal of N
  $Nor(I_N)$ consists of $f \in C^\infty(M)$ with $\{f, I_N\} \subset I_N$.

- A special case is Hamiltonian reduction:
  If $G \curvearrowright (M, \omega) \xrightarrow{J} \mathfrak{g}^*$ hamiltonian action, then
  $$J^{-1}(0)/G$$
  is a symplectic manifold (if smooth).

Aim of the talk

- Carry out these reductions for symplectic degree 2 manifolds.
- Obtain a reduction scheme for Courant algebroids.

Courant algebroids:
For instance $TM \oplus T^*M$, with bracket

$$[\![X + \alpha, Y + \beta]\!] = [X, Y] + \mathcal{L}_X \beta - \iota_Y d\alpha.$$

Courant algebroids are where the following geometries "live":

- Dirac geometry
  (unifies presymplectic and Poisson geometry)
- Generalized complex geometry
  (unifies symplectic and complex geometry)

A characterization of degree 2 manifolds

Let $\mathcal{M} = (M, C_{\mathcal{M}})$ be a degree 2 manifold (so “coordinates” are in degrees $0, 1, 2$).

By Serre-Swan, there are vector bundles E_1 and $\widetilde{E}$ over M s.t.

- $(C_{\mathcal{M}})_1 \cong \Gamma_{E_1^*}$
- $(C_{\mathcal{M}})_2 \cong \Gamma_{\widetilde{E}^*}$

Have $i\colon \wedge^2 E_1^* \hookrightarrow \widetilde{E}^*$ since $(C_{\mathcal{M}})_1 \cdot (C_{\mathcal{M}})_1 \subset (C_{\mathcal{M}})_2$.

Proposition

There is a bijection (up to iso) between

- *degree 2 manifolds*
- *triples*

$$(E_1, \tilde{E}, \phi_E)$$

consisting of vector bundles and a surjective map $\tilde{E} \twoheadrightarrow \wedge^2 E_1$

Remark: Recover $\mathcal{M}$ by

$$C_{\mathcal{M}} = \Gamma_{\wedge E_1^*} \otimes \Gamma_{S\widetilde{E}^*} / < T \otimes 1 - 1 \otimes i(T) >$$

for $T \in \Gamma_{\wedge^2 E_1^*}$.

Submanifolds

Definition

A sheaf of ideals $\mathcal{I} \subset C_{\mathcal{M}}$ is regular if

a) $\mathcal{I}_0 =$ vanishing ideal of a subset N of M

b) around each $x \in N$ there are coordinates (x_i, e_μ, p_J) for $\mathcal{M}$ s.t. a subset of them generates $\mathcal{I}$.

Proposition

Fix a submanifold $N \subset M$. There is a bijection between

- *submanifold $\mathcal{N}$ of $\mathcal{M}$ with body N*
- *regular sheaves of ideals $\mathcal{I}$ s.t. $\mathcal{I}_0 =$ (vanishing ideal of N in M)*

Further

$$C_{\mathcal{N}} = \iota^{-1}(C_{\mathcal{M}}/\mathcal{I}).$$

Symplectic degree 2 manifolds

Let $\mathcal{M}$ be a degree 2 manifold.

Definition

Let $\{\cdot,\cdot\}$ be a degree -2 Poisson bracket on $C_{\mathcal{M}}$.
We call it symplectic if at each $x \in M$ it induces an isomorphism

$$(T_x\mathcal{M})^* \to T_x\mathcal{M}, \;\; [f] \mapsto \{f,\cdot\}_x.$$

Recall:
$(T_x\mathcal{M})^* \cong \mathcal{I}_{(x)}/\mathcal{I}_{(x)}^2$ where

$$\mathcal{I}_{(x)} = \{\text{elements of the stalk of } C_{\mathcal{M}} \text{ at } x \text{ vanishing at } x\}.$$

Symplectic degree 2 manifolds (cont.)

Theorem (Roytenberg 2001)

There is a bijection (up to iso) between

- *Symplectic degree 2 manifolds* $(\mathcal{M}, \omega)$
- *Pseudo-euclidean vector bundles* $(E, \langle\cdot,\cdot\rangle)$.

Given $(\mathcal{M}, \omega)$, define E so that

$$(C_{\mathcal{M}})_1 \cong \Gamma_{E^*} \text{ and } \langle\cdot,\cdot\rangle = \{\cdot,\cdot\}|_{(C_{\mathcal{M}})_1 \times (C_{\mathcal{M}})_1}$$

Given $(E, \langle\cdot,\cdot\rangle)$, define $\mathcal{M}$ so that

$$E_1^* = E^* \text{ and } \tilde{E}^* = \mathbb{A}_E.$$

Recall:

The Atiyah Lie algebroid $\mathbb{A}_E \to M$ is characterized by

$$\Gamma(\mathbb{A}_E) = \{\phi \colon \Gamma(E) \to \Gamma(E) \text{ satisfying Leibniz rule and preserving } \langle\cdot,\cdot\rangle\}.$$

It fits in a short exact sequence of Lie algebroids

$$0 \longrightarrow \wedge^2 E^* \longrightarrow \mathbb{A}_E \longrightarrow TM \longrightarrow 0$$

Coisotropic submanifolds

Let $(\mathcal{M}, \omega)$ be a degree 2 symplectic manifold, corresponding to $(E, \langle\cdot,\cdot\rangle)$.

Definition

A submanifold $\mathcal{N}$ of $(\mathcal{M}, \omega)$ is coisotropic if $\{\mathcal{I}, \mathcal{I}\} \subset \mathcal{I}$

Theorem

Coisotropic submanifolds are in bijection with (N, K, F, ∇) *where*

- $N \subset M$ *is a submanifold*
- $K \to N$ *is an isotropic subbundle of* E
- F *is an involutive distribution on* N
- ∇ *is a flat, metric partial* F*-connection on the vector bundle* $K^\perp/K \to N$.

Given a quadruple, $\mathcal{I}$ is given by:

- $\mathcal{I}_0 = I_N$
- $\mathcal{I}_1 = \Gamma_{E,K} := \{s \in \Gamma_E : s|_N \subset K\}$
- $\mathcal{I}_2 = \{(D, X) \in \mathbb{A}_E : X|_N \subset F, D$ preserves $\Gamma_{E,K}$ and $[D|_N] = \nabla_{X|_N}\}$

Example[Ševera 2005]

Lagrangian submanifolds are equivalent to lagrangian subbundles $K \to N$.

Coisotropic reduction

Proposition (Coisotropic reduction)

The following are equivalent:

- N/F *is a smooth manifold and* ∇ *has trivial holonomy*
- *there is a reduced symplectic degree 2 manifold* $\underline{\mathcal{N}}$ *(s.t.* $C_{\underline{\mathcal{N}}} \cong p_* \iota^{-1}(Nor(\mathcal{I})/\mathcal{I})$ *as sheaves of Poisson algebras)*

Remark:

$\underline{\mathcal{N}}$ corresponds to the pseudo-euclidean vector bundle on the right, where $\sim$ is given by ∇.

$$\begin{array}{ccc} K^\perp/K & \longrightarrow & (K^\perp/K)/\sim \\ \downarrow & & \downarrow \\ N & \xrightarrow{\;p\;} & N/F \end{array}$$

Courant algebroids

Definition (Liu-Weinstein-Xu 1997)

A Courant algebroid consists of

- a pseudo-euclidean vector bundle $(E, \langle\ ,\ \rangle) \to M$
- a bundle map $\rho\colon E \to TM$
- a bilinear bracket $[\![\cdot,\cdot]\!]$ on $\Gamma(E)$

such that $[\![\cdot,\cdot]\!]$ satisfies the "Jacobi identity", plus other axioms.

Definition

A Courant function on $(\mathcal{M}, \omega)$ is $\Theta \in C_3(\mathcal{M})$ s.t. $\{\Theta, \Theta\} = 0$.

Remark:
The hamiltonian vector field $\{\Theta, \cdot\}$ makes $\mathcal{M}$ into a dg-manifold.

Theorem (Roytenberg 2001)

There is a bijection (up to iso) between

- *Courant algebroids* E
- *symplectic degree 2 manifolds* $(\mathcal{M}, \omega)$ *with a Courant function.*

Coisotropic reduction of Courant algebroids

Let $\mathcal{N} \subset \mathcal{M}$ be coisotropic, corresponding to (N, K, F, ∇).

Proposition

A Courant function $\Theta \in C(\mathcal{M})_3$ is reducible w.r.t. $\mathcal{N}$ (i.e., $\{\Theta, \mathcal{I}\} \subseteq \mathcal{I}$) iff

(R1) $\rho(K^\perp) \subseteq TN$,

(R2) $\rho(K) \subseteq F$,

(R3) $[\rho(\Gamma^{flat}_{E,K^\perp}), \cdot\,]$ *preserves* $\Gamma_{TM,F}$,

(R4) $\Gamma^{flat}_{E,K^\perp}$ *is involutive w.r.t. Courant bracket.*

Example[Ševera 2005]

If $\mathcal{N}$ Lagrangian: $K \to N$ lagr. subbundle, $\rho(K) \subseteq TN$, and $\Gamma_{E,K}$ involutive.

Theorem (Coisotropic reduction of Courant algebroids)

Suppose that

a) F *is simple and* ∇ *has trivial holonomy*

b) *conditions (R1)–(R4) above hold.*

Then $(K^\perp/K)/\sim$ *is a Courant algebroid.*

Coisotropic reduction of generalized complex structures

Definition (Hitchin 2003)

A generalized complex (GC) structure on a Courant algebroid E is an endomorphism

$$J\colon E \to E$$

such that J is orthogonal, $J^2 = -\mathrm{Id}$ and the Nijenhuis torsion vanishes.

Remark[Grabowski 2006]

GC structures are equivalent to quadratic functions $\mathcal{J}$ on $\mathcal{M}$ satisfying

$$\{\{\Theta, \mathcal{J}\}, \mathcal{J}\} = -\Theta.$$

Corollary

In the previous theorem: $\mathcal{J}$ *is reducible* $\Leftrightarrow$ J *preserves* $\Gamma^{flat}_{E,K^\perp}$.
In that case J *induces a GC structure on the reduced Courant algebroid.*

Moment maps

Let $(\mathcal{M}, \omega)$ be a degree 2 symplectic manifold.
Let

$$\tilde{\mathfrak{g}} := \mathfrak{h}[2] \oplus \mathfrak{a}[1] \oplus \mathfrak{g}$$

be a graded Lie algebra concentrated in degrees $-2, -1, 0$.

Definition

A moment map is a morphism of degree 2 manifolds $\tilde{\mu} \colon \mathcal{M} \to \tilde{\mathfrak{g}}^*[2]$, so that the induced map

$$\tilde{\mu}^\sharp \colon \tilde{\mathfrak{g}} \to C(\mathcal{M})[2]$$

is a morphism of graded Lie algebras.

Remark: Get a graded Lie algebra morphism (an infinitesimal action)

$$\tilde{\mathfrak{g}} \to \mathfrak{X}(\mathcal{M}), \quad \xi \mapsto \{\tilde{\mu}^\sharp \xi, \cdot\}.$$

Hamiltonian reduction

Corollary

i) *Suppose* $0 \in \tilde{\mathfrak{g}}^*[2]$ *is a regular value.*
Then $\mathcal{N} := \tilde{\mu}^{-1}(0)$ *is a coisotropic submanifold of* $(\mathcal{M}, \omega)$.

ii) *Under regularity conditions:*
$(\underline{\mathcal{N}}, \underline{\omega})$ *is a degree 2 symplectic manifold.*

Assume now further that

- $\mathcal{M}$ has a Courant function Θ
- $\tilde{\mathfrak{g}}$ is a DGLA
- $\tilde{\mu}^\sharp : \tilde{\mathfrak{g}} \to (C(\mathcal{M})[2], \{\cdot, \cdot\}, \{\Theta, \cdot\})$ is a DGLA morphism

Corollary

Then $(\underline{\mathcal{N}}, \underline{\omega})$ *inherits a Courant function.*

Reason: $\Theta \in C(\mathcal{M})_3$ is reducible for $\mathcal{N}$ (i.e., $\{\Theta, \mathcal{I}\} \subseteq \mathcal{I}$), since all $\xi \in \tilde{\mathfrak{g}}$

$$\{\Theta, \tilde{\mu}^\sharp \xi\} = \tilde{\mu}^\sharp(\delta\xi).$$

Moment maps in classical terms

Symplectic degree 2 manifold $(\mathcal{M}, \omega)$ $\leftrightarrow$

pseudo-euclidean vector bundle $(E, \langle\ ,\ \rangle)$

Graded Lie algebra $\tilde{\mathfrak{g}}$ $\leftrightarrow$

- $\mathfrak{g}$ Lie algebra
- $\mathfrak{a}$ and $\mathfrak{h}$ modules
- $\varpi\colon \mathfrak{a} \otimes \mathfrak{a} \to \mathfrak{h}$ symmetric and $\mathfrak{g}$-equivariant

Morphism of graded Lie algebras $\tilde{\mu}^\sharp\colon \tilde{\mathfrak{g}} \to C(\mathcal{M})[2]$ $\leftrightarrow$

- $\mu\colon M \to \mathfrak{h}^*$ which is $\mathfrak{g}$-equivariant
- $\varrho\colon \mathfrak{a} \to \Gamma(E)$ which is $\mathfrak{g}$-equivariant
- $\varphi\colon \mathfrak{g} \to \Gamma(\mathbb{A}_E)$ Lie algebra map $\rightsquigarrow$ action $\mathfrak{g} \curvearrowright E$

s.t. $\mu^*(\varpi(a,a)) = \langle \varrho(a), \varrho(a) \rangle$.

Hamiltonian reduction in classical terms

$0 \in \tilde{\mathfrak{g}}^*[2]$ is a regular value $\quad \leftrightarrow$ ....

Result: Coisotropic $\mathcal{N} := \tilde{\mu}^{-1}(0) \quad \leftrightarrow$

(N, K, F, ∇) where

- $N = \mu^{-1}(0)$
- $K = \varrho(\mathfrak{a})|_N$
- $F =$ tangent to the $\mathfrak{g}$-orbits on N
- $\nabla_{u_N} = [\varphi(u)|_N]$ (acting on $K^\perp/K \to N$)

Regularity conditions: have $G \curvearrowright E|_N$, free and proper on N.

Result: Reduced space $(\underline{\mathcal{N}}, \underline{\omega}) \quad \leftrightarrow$

pseudo-euclidean vector bundle $\frac{K^\perp}{K}/G \to \mu^{-1}(0)/G$

Courant algebroid reduction

Courant function $\Theta \quad \leftrightarrow$

E Courant algebroid

DGLA morphism $\tilde{\mu}^\sharp \colon \tilde{\mathfrak{g}} \to (C(\mathcal{M})[2], \{\cdot,\cdot\}, \{\Theta,\cdot\}) \quad \leftrightarrow$

μ, ϱ, φ also satisfy

$$\begin{array}{ccccc} \mathfrak{h} & \xrightarrow{\delta} & \mathfrak{a} & \xrightarrow{\delta} & \mathfrak{g} \\ \downarrow{\scriptstyle \mu^*} & & \downarrow{\scriptstyle \varrho} & & \downarrow{\scriptstyle \varphi} \\ C^\infty(M) & \xrightarrow{\rho^* d} & \Gamma(E) & \xrightarrow{\mathrm{ad}} & \Gamma_{CA}(\mathbb{A}_E). \end{array}$$

Result: Reduced space $(\underline{\mathcal{N}}, \underline{\omega}, \underline{\Theta}) \quad \leftrightarrow$

Courant algebroid $\quad \dfrac{K^\perp}{K}/G \to \mu^{-1}(0)/G$

A special case: exact DGLAs

Exact DGLA $\tilde{\mathfrak{g}} := \mathfrak{h}[2] \oplus \mathfrak{a}[1] \oplus \mathfrak{g} \quad \leftrightarrow$

Exact Courant algebra: surjective, bracket-preserving map

$$p : \mathfrak{a} \to \mathfrak{g},$$

where $\mathfrak{a}$ Leibniz algebra, $\mathfrak{g}$ Lie algebra, s.t. $\mathfrak{h} := \ker p$ left-central.

Simplification: DGLA morphism $\tilde{\mu}^\sharp \colon \tilde{\mathfrak{g}} \to (C(\mathcal{M})[2], \{\cdot,\cdot\}, \{\Theta, \cdot\}) \quad \leftrightarrow$

- $\mu \colon M \to \mathfrak{h}^*$ equivariant
- $\varrho \colon \mathfrak{a} \to \Gamma(E)$ bracket-preserving s.t. $[\![\varrho(\mathfrak{h}), \cdot]\!] = 0$

s.t. $\mu^*(\varpi(a,a)) = \langle \varrho(a), \varrho(a) \rangle$ and

$$\begin{array}{ccc} \mathfrak{h} & \xrightarrow{\delta} & \mathfrak{a} \\ \downarrow{\scriptstyle \mu^*} & & \downarrow{\scriptstyle \varrho} \\ C^\infty(M) & \xrightarrow{\rho^* d} & \Gamma(E) \end{array}$$

Remark: When E is an exact Courant algebroid, this recovers the extended actions of [Bursztyn-Cavalcanti-Gualtieri 2007]

Getting examples

Let $A \to M$ be a Lie algebroid. Then $E = A \oplus A^*$ is a Courant algebroid with the standard Courant bracket.
The corresponding degree 2 symplectic manifold is $\mathcal{M} = T^*[2]A[1]$.

Remark:
Graded analogue of the following $\rightsquigarrow$
exact DGLA and DGLA morphism $\tilde{\mu}^\sharp$ into $C(\mathcal{M})[2]$.

Let N be a manifold, $\mathfrak{g}$ a Lie algebra, $\mathfrak{h}$ a module. Consider

(i) $\eta\colon \mathfrak{g} \to \mathfrak{X}(N) = C^\infty_{lin}(T^*N)$ Lie algebra map

(ii) $\nu\colon \mathfrak{g} \to C^\infty(N)$ satisfying

$$\nu([u_1, u_2]) = \pounds_{\eta(u_1)}(\nu(u_2)) - \pounds_{\eta(u_2)}(\nu(u_1)), \quad \forall u_1, u_2 \in \mathfrak{g}.$$

(iii) $\mu\colon N \to \mathfrak{h}^*$ which is $\mathfrak{g}$-equivariant

$\rightsquigarrow$ hamiltonian $\mathfrak{g} \ltimes \mathfrak{h}$-action on (T^*N, ω_{can}), with comoment map

$$\begin{aligned} \tilde{\mu}\colon \mathfrak{g} \ltimes \mathfrak{h} &\to C^\infty(T^*N) \\ (u, w) &\to \big(\eta(u) + \pi^*(\nu(u))\big) + \pi^*(\mu^* w). \end{aligned}$$

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Thank you for your attention