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Deformations of Lagrangian submanifolds in log-symplectic geometry

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Lagrangian submanifolds in symplectic geometry

Weinstein's Lagrangian neighborhood theorem

Around a Lagrangian L,

$$(M, \omega) \cong (T^*L, \omega_{can}).$$

In the local model (T^*L, ω_{can}):

- For $\alpha \in \Omega^1(L)$:

  $$\text{Gr}(\alpha) \text{ is Lagrangian } \Leftrightarrow \; d\alpha = 0$$

  A linear conditon

- $\text{Gr}(\alpha), \text{Gr}(\beta)$ related by Hamiltonian isotopy $\Leftrightarrow \; [\alpha] = [\beta]$ in $H^1(L)$. So

  $$\mathcal{M}^{Ham}(L) = \text{open in } H^1(L).$$

  Finite dimensional and smooth

Is the log-symplectic case also so nice?

Log-symplectic manifolds

Definition

(M^{2n}, π) is log-symplectic if $\wedge^n \pi : M \to \wedge^{2n} TM$ is transverse to the zero section.

π is symplectic away from hypersurface $Z := (\wedge^n \pi)^{-1}(0)$.

- $(Z, \pi|_Z)$ is corank-one Poisson structure.
- $(Z, \pi|_Z)$ has a Poisson vector field transverse to the symplectic leaves:

$$V_{mod}|_Z,$$

  where V_{mod} is a modular vector field of M.

Example: When $\dim(M) = 2$: Radko surfaces

Example: On $(\mathbb{R}^{2n}, x_1, y_1, \ldots, x_n, y_n)$:

$$\pi = \partial_{x_1} \wedge y_1 \partial_{y_1} + \sum_{i=2}^{n} \partial_{x_i} \wedge \partial_{y_i}.$$

Modular vector field is ∂_{x_1}. This is the local model around a point in Z.

Log-symplectic behaves like symplectic

Definition

${}^bTM$ is the Lie algebroid with sections $\{X \in \mathfrak{X}(M) : X \text{ tangent to } Z\}$

- [Guillemin-Miranda-Pires]

$$\pi \text{ log-symplectic} \quad \leftrightarrow \quad b\text{-symplectic form } \Omega \in {}^b\Omega^2(M)$$

- [Marcut-Osorno]

  If M is compact,

$$\frac{\{\text{Poisson structures nearby } \pi\}}{\text{isotopy}} \cong \text{open in } {}^bH^2(M)$$

- [Marcut-Osorno]

  If M is compact, there is

$$c \in {}^bH^2(M)$$

  s.t. $c^{n-1} \neq 0$, where $\dim(M) = 2n$.
  So non-zero ${}^bH^2(M), \ldots, {}^bH^{2n-2}(M)$.

Lagrangians in log-symplectic geometry

Definition

$L \subset (M, \pi)$ is Lagrangian if for all $p \in L$:

$$T_pL \cap T_pS \text{ is a Lagrangian subspace of } (T_pS, (\omega_S)_p).$$

Here (S, ω_S) is the symplectic leaf through $p \in L$.

Lagrangian submanifolds are in particular coisotropic.

Example:
Assume Z has a compact leaf. Then Z is a symplectic mapping torus

$$Z = ([0,1] \times S)/\sim$$

where S symplectic manifold, $\phi \colon S \to S$ symplectomorphism, and $(0, x) \sim (1, \phi(x))$.
Let $\ell \subset S$ Lagrangian such that $\phi(\ell) = \ell$. Then

$$L := ([0,1] \times \ell)/\sim$$

is Lagrangian in Z.

Lagrangians in log-symplectic geometry

Let $L \subset (M^{2n}, Z, \pi)$ Lagrangian.

- If $L \pitchfork Z$: use b-symplectic geometry[Kirchhoff-Lukat]
  Around L:
  $$(M, \omega) \cong ({}^{b}T^{*}L, \Omega_{can}).$$
  Hence $\mathcal{M}^{Ham}(L) = {}^{b}H^{1}(L) \cong H^{1}(L) \oplus H^{0}(L \cap Z)$.

- If $L \subset Z$ then
  - if $\dim L = n-1$: L lies inside a leaf of Z.
  - if $\dim L = n$: L is transverse to the leaves of Z
    L inherits a foliation $\mathcal{F}_L$ of corank 1, unimodular.

We focus on $L^n \subset Z \subset M^{2n}$.

Assumption: M is orientable

Normal form around L

Construct normal form in two steps: $L \subset (Z, \pi|_Z)$ and $Z \subset (M, \pi)$.

Step 1: General fact

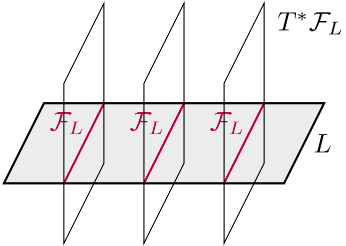
Let $L \subset (N, \pi)$ Lagrangian transverse to leaves. Then

- π is regular nearby L $\rightsquigarrow$ foliation $\mathcal{F}_L$ on L.
- there is a canonical Poisson structure π_{can} on $T^*\mathcal{F}_L$ s.t.

$$T^*\mathcal{F}_L = \coprod_{B \in \mathcal{F}_L} (T^*B, \omega_{T^*B}).$$

- around L,

$$(N, \pi) \cong (T^*\mathcal{F}_L, \pi_{can}).$$



Normal form around a Lagrangian

Step 2: $Z \subset (M, \pi)$ [Guillemin-Miranda-Pires]

Let (M, Z, π) be an orientable log-symplectic manifold.
The local model for (M, π) around Z is $Z \times \mathbb{R}$ with

$$V_{mod}|_Z \wedge t\partial_t + \pi|_Z.$$

Corollary (Normal form ad interim)

The local model around $L^n \subset Z \subset (M^{2n}, \pi)$ is $T^*\mathcal{F}_L \times \mathbb{R}$ with

$$V \wedge t\partial_t + \pi_{can}.$$

Here V is image of $V_{mod}|_Z$ under $(Z, \pi|_Z) \xrightarrow{\sim} (T^*\mathcal{F}_L, \pi_{can})$.

We can choose any representative of $[V] \in H^1_{\pi_{can}}(T^*\mathcal{F}_L)$.
Next: find a representative "compatible" with L.

Intermezzo: Symplectic vector fields on T^*N

Lemma

Let Y be a symplectic vector field on a cotangent bundle (T^*N, ω_{can}).

(i) *There is $h \in C^\infty(T^*N)$ such that $Y + X_h$ is a vertical vector field.*

(ii) *Suppose Y is vertical.*
Then Y must be constant on each fiber.
When viewed as an element of $\Gamma(T^*N) = \Omega^1(N)$, it lies in

$$\Omega^1_{closed}(N)$$

Intermezzo: Poisson vector fields on $T^*\mathcal{F}_L$

Let $(L, \mathcal{F}_L)$ be a foliated manifold. Denote

$$\mathfrak{X}(L)^{\mathcal{F}_L} := \{X \in \mathfrak{X}(L) : [X, \ ] \text{ preserves } \Gamma(T\mathcal{F}_L)\}.$$

$X \in \mathfrak{X}(L)^{\mathcal{F}_L} \rightsquigarrow$ cotangent lift $\rightsquigarrow$ Poisson vector field $\widetilde{X}$ on $(T^*\mathcal{F}_L, \pi_{can})$.

Lemma

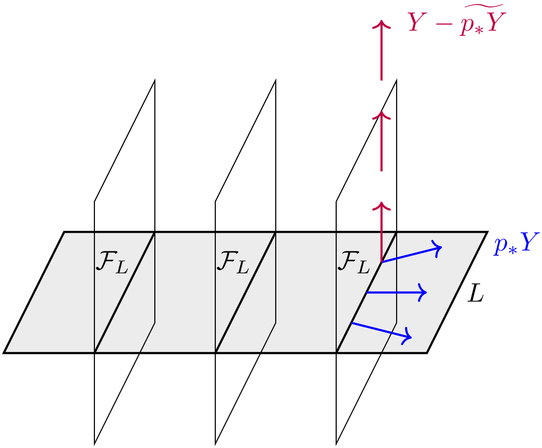
Let $(L, \mathcal{F}_L)$ be a foliated manifold. Fix a class in $H^1_{\pi_{can}}(T^*\mathcal{F}_L)$.
Then there exists a representative $Y \in \mathfrak{X}(T^*\mathcal{F}_L)$ such that

(i) $p_*Y \in \mathfrak{X}(L)^{\mathcal{F}_L}$

(ii) *the vector field $Y - \widetilde{p_*Y}$ is vertical and constant on each fiber of p. It corresponds to an element of*

$$\Omega^1_{closed}(\mathcal{F}_L)$$

Intermezzo: Poisson vector fields on $T^*\mathcal{F}_L$



Normal form for log-symplectic structure around L

Proposition

The first Poisson cohomology of $(T^*\mathcal{F}_L, \pi_{can})$ is:

$$\begin{aligned} H^1_{\pi_{can}}(T^*\mathcal{F}_L) &\cong \mathfrak{X}(L)^{\mathcal{F}_L}/\Gamma(T\mathcal{F}_L) \times H^1(\mathcal{F}_L) : \\ [\widetilde{X} + \pi^\sharp_{can}(p^*\gamma)] &\longleftarrow ([X], [\gamma]). \end{aligned}$$

Take

$$X \in \mathfrak{X}(L)^{\mathcal{F}_L}$$
$$\gamma \in \Omega^1_{cl}(\mathcal{F}_L)$$

such that $[V] \longleftrightarrow ([X], [\gamma])$

Corollary (Normal form)

The local model around $L^n \subset Z \subset (M^{2n}, \pi)$ is $T^*\mathcal{F}_L \times \mathbb{R}$ with log-symplectic structure

$$\underbrace{(\widetilde{X} + \pi^\sharp_{can}(p^*\gamma))}_{=:V} \wedge t\partial_t + \pi_{can}.$$

Lagrangian deformations

$T^*\mathcal{F}_L \times \mathbb{R}$ is a vector bundle over L.
Look at Lagrangian sections $(\alpha, f) \in \Gamma(T^*\mathcal{F}_L \times \mathbb{R})$ in the local model

$$\Big(T^*\mathcal{F}_L \times \mathbb{R},\ \underbrace{(\widetilde{X} + \pi_{can}^{\sharp}(p^*\gamma))}_{=V} \wedge t\partial_t + \pi_{can}\Big).$$

Proposition

The graph of a section $(\alpha, f) \in \Omega^1(\mathcal{F}_L) \times C^\infty(L)$ is Lagrangian exactly when

$$\begin{cases} d_{\mathcal{F}_L}\alpha = 0 \\ d_{\mathcal{F}_L} f + f(\gamma - \mathcal{L}_X \alpha) = 0 \end{cases}$$

Connectedness

Remark:
Deforming L into Gr(α, f) can be done in two steps:

1. Deform L inside singular locus along $\alpha \in \Omega^1_{cl}(\mathcal{F}_L)$.
2. Push Gr(α) out of singular locus along f.

I.e., concatenate these paths defined on $[0, 1]$:

1. $t \mapsto (t\alpha, 0)$ and
2. $s \mapsto (\alpha, sf)$.

Corollary:
$\{$Lagrangian sections of $T^*\mathcal{F} \times \mathbb{R}\}$ is connected.

The DGLA behind the deformation problem

Proposition

The deformation problem of the Lagrangian L is governed by a DGLA structure on

$$\Gamma\big(\wedge^\bullet(T^*\mathcal{F}_L \times \mathbb{R})\big) = \Gamma(\wedge^\bullet T^*\mathcal{F}_L \oplus \wedge^{\bullet-1}T^*\mathcal{F}_L) = \Omega^\bullet(\mathcal{F}_L) \oplus \Omega^{\bullet-1}(\mathcal{F}_L)$$

The structure maps d and $[\![\cdot,\cdot]\!]$ are defined by

$$(\alpha,\beta) \mapsto (-d_{\mathcal{F}_L}\alpha, -d_{\mathcal{F}_L}\beta - \gamma\wedge\beta)\,,$$
$$(\alpha,\beta)\otimes(\alpha',\beta') \mapsto \Big(0, \mathcal{L}_X\alpha\wedge\beta' - (-1)^{kl}\mathcal{L}_X\alpha'\wedge\beta\Big)\,.$$

The equations for Lagrangian sections $(\alpha, f) \in \Gamma(T^*\mathcal{F}_L \times \mathbb{R})$ are the Maurer-Cartan equation of this DGLA:

$$\text{Gr}(\alpha, f) \text{ is Lagrangian } \Leftrightarrow\ d(\alpha, f) + \frac{1}{2}[\![(\alpha, f), (\alpha, f)]\!] = 0.$$

The DGLA behind the deformation problem

This DGLA arises from the Cattaneo-Felder construction.

Theorem ([Cattaneo-Felder])

Let $E \to C$ be a vector bundle,
Π a fiberwise entire Poisson structure on E such that C is coisotropic.

- *There is an L_∞-algebra structure on $\Gamma\left(\wedge^\bullet E\right)$ with multibrackets*

$$\xi_1 \otimes \cdots \otimes \xi_k \mapsto \pm P\left([\ldots[[\Pi, \xi_1], \xi_2]\ldots, \xi_k]\right) \quad (1)$$

  where $P = pr_{vert} \circ |_C$. Here the ξ_i are seen as vertical fiberwise constant multivector fields on E

- *For all $\alpha \in \Gamma(E)$:*

$$\text{Graph}(\alpha) \text{ is a coisotropic submanifold}$$
$$\Leftrightarrow \alpha \text{ satisfies the Maurer-Cartan equation.}$$

Some geometric aspects of the deformation problem

1. When do small deformations stay inside the singular locus?
2. Is the deformation problem obstructed?

Remark:

When L is compact, there are two options:

1. $(L, \mathcal{F}_L)$ is the foliation of a fibration $L \to S^1$
2. All leaves of $(L, \mathcal{F}_L)$ are dense

Interlude: Morse-Novikov cohomology

Let M be a connected manifold, $\eta \in \Omega^1_{cl}(M)$.
Get differential

$$d^\eta := d \quad + \eta\wedge$$

- The cohomology $H^\bullet_\eta(M)$ depends only on $[\eta]$.
- $$H^0_\eta(M) \cong \begin{cases} \mathbb{R} & \text{if } \eta \text{ exact} \\ 0 & \text{otherwise} \end{cases}$$

Recall:
The graph of $(\alpha, f) \in \Gamma(T^*\mathcal{F}_L \times \mathbb{R})$ is Lagrangian exactly when

$$\begin{cases} d_{\mathcal{F}_L}\alpha = 0 \\ d^{\gamma - \mathcal{L}_X \alpha}_{\mathcal{F}_L} f = 0 \end{cases} .$$

Deformations constrained to the singular locus

1. When do small deformations stay inside the singular locus?

Require that $H^0_{\gamma - \mathcal{L}_X \alpha}(\mathcal{F}_L) = 0$ for all small $\alpha \in \Omega^1_{cl}(\mathcal{F}_L)$.

Lemma

Assume L is compact. Let $\eta \in \Omega^1(\mathcal{F}_L)$ be leafwise closed.

1. If $\mathcal{F}_L$ is given by fibration $p : L \to S^1$ then $H^1(\mathcal{F}_L) \cong \Gamma(\mathcal{H}^1)$, where

$$\mathcal{H}^1 \to S^1 \text{ vector bundle with fibers } \mathcal{H}^1_q = H^1(p^{-1}(q)).$$

   Then

$$H^0_\eta(\mathcal{F}_L) \cong \{f \in C^\infty(S^1) : f \cdot [\eta] = 0\}.$$

2. If the leaves of $\mathcal{F}_L$ are dense, then

$$H^0_\eta(\mathcal{F}_L) = \begin{cases} \mathbb{R} & \text{if } \eta \text{ is exact} \\ 0 & \text{otherwise} \end{cases}$$

Theorem

Suppose

$$[\gamma] = 0 \in H^1(\mathcal{F}_L)$$

(i.e. V can be chosen tangent to L).
Then there is a path of Lagrangian submanifolds L_s not contained in the singular locus for $s > 0$.

Example: The local model around a point of L.

At the opposite extreme, assuming L is compact:

Theorem

1. Suppose $\mathcal{F}_L$ is given by a fibration $L \to S^1$. If for each leaf B of $\mathcal{F}_L$

   $$[\gamma|_B] \neq 0 \in H^1(B)$$

   then $\mathcal{C}^1$-small deformations of L stay inside the singular locus.
2. Suppose $\mathcal{F}_L$ has dense leaves, and $H^1(\mathcal{F}_L)$ is finite dimensional. If

   $$[\gamma] \neq 0 \in H^1(\mathcal{F}_L)$$

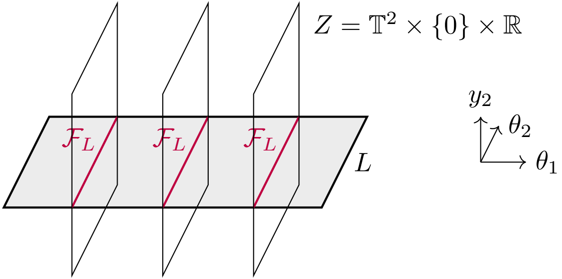
   then $\mathcal{C}^\infty$-small deformations of L stay inside the singular locus.

Example

Consider $(\mathbb{T}^2 \times \mathbb{R}^2, \theta_1, \theta_2, y_1, y_2)$ with log-symplectic structure

$$\pi = \Big(g_X(\theta_1)\partial_{\theta_1} + g_\gamma(\theta_1)\partial_{y_2}\Big) \wedge y_1\partial_{y_1} + \partial_{\theta_2} \wedge \partial_{y_2}$$

Let $L = \mathbb{T}^2 \times \{(0,0)\}$.
So $\gamma = -g_\gamma(\theta_1)d\theta_2 \in \Omega^1(\mathcal{F}_L)$.



If g_γ has no zeros $\Rightarrow$ $\mathcal{C}^1$-small deformations of L stay inside the singular locus.

The Kuranishi criterion

Let $(W, d, [\![\cdot,\cdot]\!])$ be a DGLA.
Let $z(t)$ be a curve of Maurer-Cartan elements with $z(0) = 0$. Since

$$0 = d(z(t)) + \frac{1}{2}[\![z(t), z(t)]\!]$$

we have

$$d(z'(0)) = 0$$
$$[\![z'(0), z'(0)]\!] = d(z''(0)).$$

Definition

Let $w \in W_1$ with $dw = 0$ (infinitesimal deformation).
w is unobstructed if it is tangent to a curve of Maurer-Cartan elements.

The Kuranishi map is

$$Kr : H^1(W) \to H^2(W), \ \ [w] \mapsto \big[[\![w, w]\!]\big].$$

Corollary (Kuranishi criterion)

$$w \textit{ unobstructed} \Rightarrow Kr[w] = 0.$$

Obstructedness of infinitesimal deformations

2. Is the deformation problem obstructed?

Recall:

The graph of $(\alpha', f') \in \Gamma(T^*\mathcal{F}_L \times \mathbb{R}) = \Omega^1(\mathcal{F}_L) \times C^\infty(L)$ is Lagrangian iff

$$\begin{cases} d_{\mathcal{F}_L}\alpha' = 0 \\ d_{\mathcal{F}_L}f' + f'(\gamma - \mathcal{L}_X\alpha') = 0 \end{cases}$$

(α, f) is an infinitesimal deformation when

$$\begin{cases} d_{\mathcal{F}_L}\alpha = 0 \\ d_{\mathcal{F}_L}f + f\gamma = 0 \quad \leftarrow \text{depends on } \gamma \end{cases}$$

Remark:

- There exist obstructed infinitesimal deformations
- $(\alpha, 0)$ and $(0, f)$ are unobstructed:
  $t \mapsto (t\alpha, 0)$ and $t \mapsto (0, tf)$ are prolongations.

Criteria for unobstructedness

Lemma

If $[\mathcal{L}_X \alpha] = 0 \in H^1(\mathcal{F}_L)$ *then* (α, f) *is unobstructed.*

Reason:

$$f \in \{\text{cocycles for } d^{\gamma}_{\mathcal{F}_L}\} \leftrightarrow \{\text{cocycles for } d^{\gamma - \mathcal{L}_X \alpha}_{\mathcal{F}_L}\}$$

Example: The local model around a point of L, since $H^1(\mathcal{F}_L) = 0$ there.

Theorem

Let $(\alpha, f) \in \Omega^1(\mathcal{F}_L) \times C^\infty(L)$ be an infinitesimal deformation, where L compact. The following are equivalent:

1. (α, f) is smoothly unobstructed
2. $Kr[(\alpha, f)] = 0$
3. $\mathcal{L}_X \alpha$ is exact on $L \setminus \mathcal{Z}_f$
4. α extends to a closed one-form on $L \setminus \mathcal{Z}_f$ $\leftarrow$ independent of X and γ

Example

Consider $(\mathbb{T}^2 \times \mathbb{R}^2, \theta_1, \theta_2, y_1, y_2)$ with log-symplectic structure

$$\pi = \Big(g_X(\theta_1)\partial_{\theta_1} + g_\gamma(\theta_1)\partial_{y_2} \Big) \wedge y_1 \partial_{y_1} + \partial_{\theta_2} \wedge \partial_{y_2}$$

Let $L = \mathbb{T}^2 \times \{(0,0)\}$.
So $\gamma = -g_\gamma(\theta_1) d\theta_2 \in \Omega^1(\mathcal{F}_L)$.

Infinitesimal deformations are $(\alpha, f) \in \Omega^1(\mathcal{F}_L) \times C^\infty(L)$ such that

$$f = f(\theta_1) \text{ and } f g_\gamma = 0.$$

The following are equivalent:

- (α, f) is unobstructed
- $\mathcal{L}_X \alpha$ is exact on $L \setminus \mathcal{Z}_f$
- On $L \setminus \mathcal{Z}_f$,

  $$[\alpha] \in H^1(\mathcal{F}_L) \cong \Gamma(\mathcal{H}^1)$$

  is a locally constant section of the vector bundle $\mathcal{H}^1 = \underline{\mathbb{R}} \to S^1$ (i.e. a flat section w.r.t. the Gauss-Manin connection).

Conclusions

Are deformations of Lagrangians $L^n \subset Z$ in log-symplectic geometry as nice as in the symplectic case?

Deformations: not as nice, but quite nice

- The Lagrangian condition is not linear, but quadratic.
- The space of nearby Lagrangians is connected.
- When the Kuranishi obstruction vanishes, infinitesimal deformations are unobstructed. Formality of the DGLA?

Moduli space $\mathcal{M}^{Ham}$ smooth at L?

- Not smooth in general:
  - $T_{[L]}\mathcal{M}^{Ham}$ typically infinite dimensional.
  - $T_{[L']}\mathcal{M}^{Ham}$ finite dimensional for Lagrangians L' in $M \setminus Z$.
- Smooth when $\mathcal{C}^1$-small deformations of L stay inside Z: $\mathcal{M}^{Ham}$ is open in $H^1(\mathcal{F}_L)$.

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Thank you for your attention