Singular subalgebroids and their integrations

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arXiv:2008.07976

The main question

Well-known:
A Lie group differentiates to a Lie algebra
A Lie algebra is integrated by certain Lie groups

Topic of this talk:
What are singular subalgebroids integrated by?
Our answer: diffeological groupoids with extra structures and properties

Special case: Integration of wide Lie subalgebroids

Let $A \to M$ be a Lie algebroid, and G an integrating Lie groupoid.

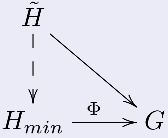
Theorem (Moerdijk-Mrčun)

Let B be a wide Lie subalgebroid of A.
There exists a unique map

$$\Phi\colon H_{min} \to G$$

where

1) H_{min} *is a Lie groupoid integrating* B*,*
2) Φ *is a Lie groupoid morphism integrating the inclusion* $\iota\colon B \hookrightarrow A$*,*
3) *minimality: any Lie groupoid morphism* $\tilde{H} \to G$ *integrating* ι *factors as*



Singular subalgebroids

Let $A \to M$ be a Lie algebroid.

Definition

A singular subalgebroid of A is a $C^\infty(M)$-submodule $\mathcal{B}$ of $\Gamma_c(A)$ which is

- involutive
- locally finitely generated.

Remark:

Singular subalgebroids

- are "relative" notion (need to fix the ambient Lie algebroid A).

- are Lie-Rinehart algebras with a geometric flavor.

Main Examples:

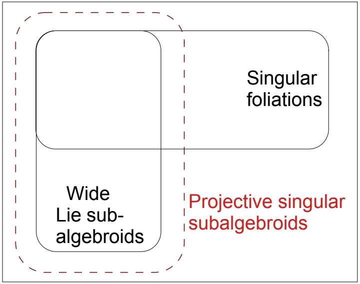
1) the case $A = TM$: singular foliations
2) wide Lie subalgebroids B of A (take $\mathcal{B} = \Gamma_c(B)$)

A hierarchy of examples

i) Images of Lie algebroid morphisms $\psi\colon E \to A$ over Id_M:
$$\mathcal{B} := \psi(\Gamma_c(E)).$$

ii) Special case: $\mathcal{B}$ projective, i.e.
$$\mathcal{B} \cong \Gamma_c(B)$$
for some vector bundle $B \to M$.
(Also: B is Lie algebroid, with an almost injective morphism $\tau\colon B \to A$.)

iii) Special case: wide Lie subalgebroids.



Singular subalgebroids

Diffeological groupoids

A diffeological groupoid $H \rightrightarrows M$ is a groupoid in the category of *diffeological spaces*. We assume: M is a manifold.

Lemma

Let $\{\chi : \mathcal{O}_\chi \to H\}$ be any generating set of plots.
Then for every $x \in M$ there is

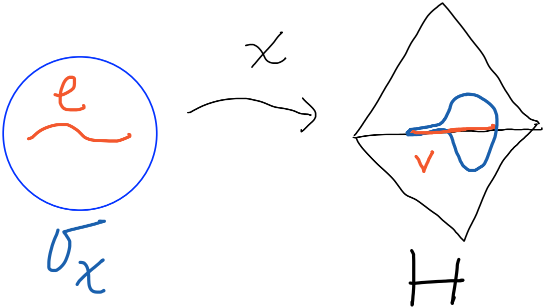
- *a plot χ in the generating set*
- *a submanifold $e \subset \mathcal{O}_\chi$*

such that

i)
$$\chi|_e : e \to V$$
is a diffeomorphism onto a neighbourhood of x in 1_M

ii) $\mathbf{s}_H \circ \chi$ *and* $\mathbf{t}_H \circ \chi$ *are submersions at points of e.*

The open map property (cont.)



Remark:

The image $\chi(\mathcal{O}_\chi)$ might not be open in H

The open map property

Open map property:
The diffeology of H is generated by open maps

Consequences:
$\Rightarrow$ through every point of H there is a local bisection
$\Rightarrow$ source s_H and target t_H are local subductions, in particular open maps

The holonomy groupoid of a singular subalgebroid

Let $A \to M$ be a Lie algebroid, and G an integrating Lie groupoid.

Theorem (Z, after [Androulidakis-Skandalis])

Let $\mathcal{B}$ be a singular subalgebroid of A.

1) There exists a canonical map

$$\Phi\colon H^G(\mathcal{B}) \to G,$$

- *where $H^G(\mathcal{B})$ is a diffeological groupoid over M*
- *Φ is a morphism of diffeological groupoids*

2) This specializes to:

- *When $\mathcal{B}$ comes from a wide Lie subalgebroid:*

$$H^G(\mathcal{B}) = H_{min}$$ *[Moerdijk-Mrčun]*

- *When $\mathcal{B}$ is a singular foliation, $G = M \times M$:*

$H^G(\mathcal{B}) =$ *the holonomy groupoid [Androulidakis-Skandalis]*

$H^G(\mathcal{B})$ is called the holonomy groupoid of $\mathcal{B}$.

The holonomy groupoid as diffeological groupoid

Sketch of the construction:

- A bisubmersion for $\mathcal{B}$ is a manifold U with a smooth map
$$\varphi\colon U \to G$$
(plus properties). It desingularizes and encodes $\mathcal{B}$.
- The holonomy groupoid is a quotient of a union of bisumersions:
$$H^G(\mathcal{B}) := \coprod U_i / \sim$$
- The map $\Phi\colon H^G(\mathcal{B}) \to G$ is induced by the φ_i

Corollary

$H^G(\mathcal{B})$ is a diffeological groupoid.

Remark:

The quotient map $\coprod U_i \to H^G(\mathcal{B})$ is a local subduction.
In particular, the diffeological groupoid $H^G(\mathcal{B})$ has the open map property.

Remark:

Later: $\Phi\colon H^G(\mathcal{B}) \to G$ provides an “integration” of $\mathcal{B}$.

Other properties

Remark[Androulidakis-Skandalis]
There is a C^*-algebra associated to $H^G(\mathcal{B})$

Remark:
$H^G(\mathcal{B})|_L$ is a Lie groupoid for all leaves $L \subset M$ of $\mathcal{B}$[after Debord]
$H^G(\mathcal{B})$ is a Lie groupoid $\Leftrightarrow$ $\mathcal{B}$ is projective.

Examples of holonomy groupoids

Example:

Let $\mathcal{B}$ be the singular foliation on $M = \mathbb{R}^2$ of vector fields vanishing at 0.

$$H^{M\times M}(\mathcal{B}) = (\mathbb{R}^2 - \{0\}) \times (\mathbb{R}^2 - \{0\}) \coprod GL(2,\mathbb{R}).$$

Example:

Let B be a wide Lie subalgebroid of A, and $\mathcal{B} = \Gamma_c(B)$. Then

$$H^G(\mathcal{B}) = H_{min}$$

Example:

Let $\psi\colon E \to A$ be a Lie algebroid morphisms over Id_M, and $\mathcal{B} := \psi(\Gamma_c(E))$.
Let $\Psi\colon K \to G$ be an integrating morphism of Lie groupoids. Then

$$H^G(\mathcal{B}) = K/\mathcal{I},$$

where

$$\mathcal{I} := \{k \in K : \exists \text{ a local bisection } \mathbf{b} \text{ through } k \text{ such that } \Psi(\mathbf{b}) \subset 1_M\}$$

The holonomy-like property

This is a “uniqueness” requirement for “nice” plots near 1_M as in the Lemma.

Definition

A diffeological groupoid $H \rightrightarrows M$ is called holonomy-like if there exists a generating set of plots[a] $\mathcal{X} = \{\chi : \mathcal{O}_\chi \to \overset{\circ}{H}\}$ such that $\forall \chi, \chi' \in \mathcal{X}$:

- Let $e \subset \mathcal{O}_\chi, e' \subset \mathcal{O}_{\chi'}$ be submanifolds such that $\chi|_e$ and $\chi'|_{e'}$ are diffeomorphisms onto the same open subset of $1_M \subset H$.
  Then $\exists$ a smooth map
  $$k \colon \mathcal{O}_{\chi'} \to \mathcal{O}_\chi$$
  with $k(e') = e$ such that
  $$\begin{array}{ccc} & H & \\ {\scriptstyle \chi'} \nearrow & & \nwarrow {\scriptstyle \chi} \\ \mathcal{O}_{\chi'} & \overset{k}{\dashrightarrow} & \mathcal{O}_\chi \end{array} \tag{1}$$

[a] $\overset{\circ}{H}$ is an open neighborhood of 1_M in H.

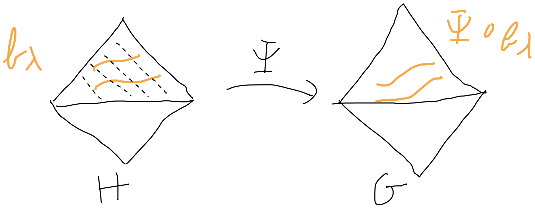
The induced module $\mathcal{S}$

- Let H a diffeological groupoid,
- $\Psi\colon H \to G$ a morphism covering Id_M, with G a Lie groupoid.

Definition

We denote by $\mathcal{S}$ this $C^\infty(M)$-submodule of $\Gamma_c(AG)$:

$$\left\{ \frac{d}{d\lambda}|_0(\Psi \circ \mathbf{b}_\lambda)\colon \{\mathbf{b}_\lambda\}_{\lambda\in I} \text{ smooth family of global bisections of } H \text{ s.t. } \mathbf{b}_0 = Id_M \right\}_c$$



Assumption ($\mathcal{S}$ arises from 1-parameter groups):
Locally, every element of $\mathcal{S}$ arises from a smooth 1-parameter group $\{\mathbf{b}_\lambda\}_{\lambda\in I}$ of global bisections of H.

Differentiation

Theorem

Assume that

i) H *is holonomy-like*

ii) $\mathcal{S}$ *arises from 1-parameter groups.*

Then $\mathcal{S}$ *is a singular subalgebroid of* AG*.*

We say that (H, Ψ) *differentiates to* $\mathcal{S}$*.*

Remark:

i) $\Rightarrow \mathcal{S}$ is locally finitely generated

ii) $\Rightarrow \mathcal{S}$ is involutive.

N.B. Involutivity also holds when:
if $\{\alpha_\lambda\} \subset \Gamma_c(A)$ smooth 1-parameter family in $\mathcal{S}$ with $\alpha_0 = 1_M$, then

$$\frac{d}{d\lambda}|_{\lambda=0}\alpha_\lambda \in \mathcal{S}.$$

Definition of integration

- Let H a diffeological groupoid,
- $\Psi\colon H \to G$ a morphism covering Id_M, with G a Lie groupoid.

Definition

Let $\mathcal{B}$ be a singular subalgebroid of AG.

(H, Ψ) is an integral of $\mathcal{B}$ over G if:

1. (H, Ψ) differentiates to $\mathcal{B}$
2. The diffeology of H is generated by open maps
3. The morphism Ψ is almost injective.

Almost injective means that H has the “right size":

$$\text{nearby the units } 1_{M,H},$$
$$(\mathbf{b} \text{ local bisection with } \Psi(\mathbf{b}) \subset 1_{M,G}) \;\Rightarrow\; \mathbf{b} \subset 1_{M,H}.$$

Remark:

We saw: 2) $\Rightarrow$ through every point of H there is a local bisection.
Hence 3) is a meaningful requirement.

Holonomy groupoids are integrals

Example

For any singular subalgebroid $\mathcal{B}$ of AG:
The holonomy groupoid $(H^G(\mathcal{B}), \Phi)$ is an integral of $\mathcal{B}$ over G.

Remark:
The graph of $\mathcal{B}$ over G is the image of the groupoid morphism $\Phi : H^G(\mathcal{B}) \to G$.
It is a diffeological groupoid, with the quotient diffeology.

- Fact: the graph of $\mathcal{B}$ is not an integral of $\mathcal{B}$.
- Reason: its diffeology is not generated by open maps.

Example: When $\mathcal{B}$ is a singular foliation and $G = M \times M$,

graph of $\mathcal{B} = \{(x, y) : x$ and y belong to the same leaf of $\mathcal{F}\}$.

Lie subalgebras

Example

Let $\mathfrak{g}$ a Lie algebra with Lie group G,
let $\mathfrak{k}$ a Lie subalgebra.

1) All integrals of $\mathfrak{k}$ over G are smooth
2) They are exactly the Lie groups covering K, the connected Lie subgroup of G corresponding to $\mathfrak{k}$.

The proof uses:

Lemma (Souriau)

On a Lie group, the diffeology is the smallest diffeology

- *containing all 1-parameter groups* $t \mapsto \exp(tv)$, *and*
- *so that the group multiplication and inversion are smooth.*

Wide Lie subalgebroids

Example

Let B be a wide Lie subalgebroid of A. The following coincides:

- The smooth integrals of $\mathcal{B} = \Gamma_c(B)$
- The Lie groupoid morphism $H \to G$ integrating the inclusion $\iota\colon B \hookrightarrow A$ (i.e. the integrations of B as in [Moerdijk-Mrčun])

Open question:

Are there non-smooth integrals of $\mathcal{B}$?

Projective Lie subalgebroids

Let $\mathcal{B}$ be a singular subalgebroid of AG.

Proposition

$\mathcal{B}$ admits a smooth integral $\Leftrightarrow$ $\mathcal{B}$ is projective.

Proposition

The holonomy groupoid $H^G(\mathcal{B})$ is the minimal smooth integral:
Any smooth integral (H, Ψ) of $\mathcal{B}$ factors as

$$\begin{array}{ccc} H & & \\ \downarrow & \searrow^{\Psi} & \\ H^G(\mathcal{B}) & \xrightarrow{\Phi} & G \end{array}$$

This generalizes the Theorem for wide Lie subalgebroids by [Moerdijk-Mrčun].

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Thank you for your attention