The geometry of foliations with singularities

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Inaugurale lezingen
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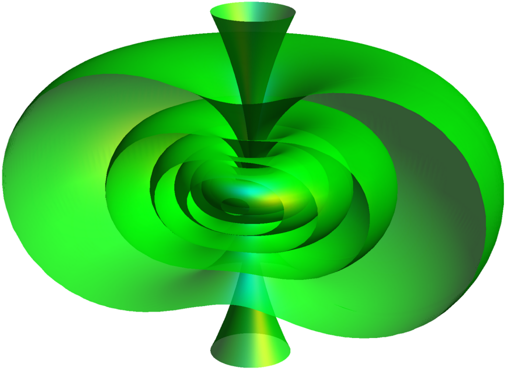
What are foliations?

This is a picture of a (regular) foliation:



As a field, foliation theory arose in the 1950s through the work of Ehresmann and Reeb.

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Foliations are common in nature

Definition

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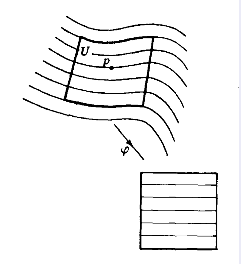
Let M be a manifold (=smooth space) of dimension n.

Definition

A foliation is a partition of M into disjoint connected subsets (called leaves), which locally look like "copies of $\mathbb{R}^k$ piled on top of each other":



Examples of foliations

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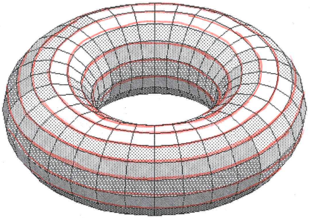
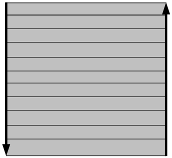
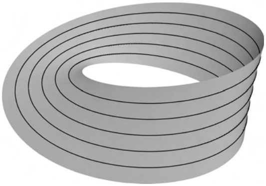
1. On the torus:

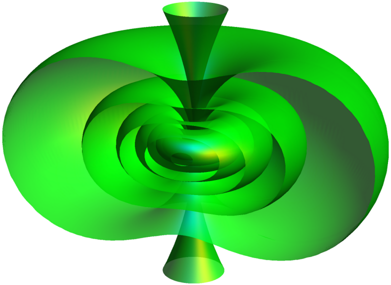


2. On the Möbius band:

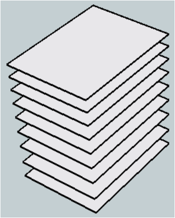




3. On $\mathbb{R}^3 - \{\text{horizontal circle}\} - \{z\text{-axis}\}$:



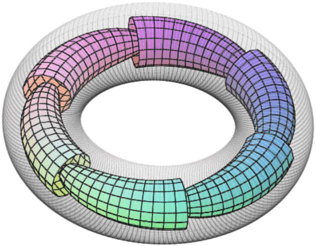
locally looks like



On S^2 there is no foliation by 1-dimensonal leaves.

Reason: there is no nowhere-vanishing vector field, by the Poincaré-Hopf theorem and since the Euler characteristic is $\chi(S^2) = 2 \neq 0$.

4. This foliation on the solid torus there has exactly one compact leaf (the gray torus)



The Reeb foliation on S^3 is obtained taking 2 copies of the above foliation, and gluing the 2 gray tori to each other (exchanging meridians and parallels).

Remark: Hopf (1935) asked:

On S^3, is there a no-where vanishing vector field X with $X \perp curl(X)$?

Equivalently: is there a foliation of S^3 by surfaces? Reeb (1948): yes.

The Frobenius theorem

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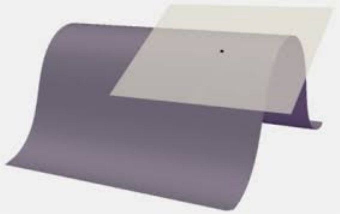
Definition

A rank-k distribution is a field of k-dimensional "planes" on M.

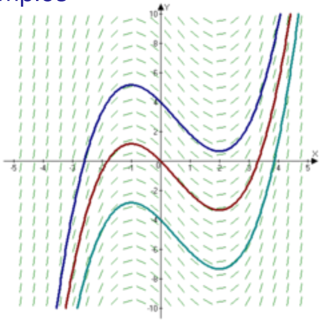
Given a foliation on M by leaves of dimension k, by taking the tangent spaces to the leaves we obtain a rank-k distribution.



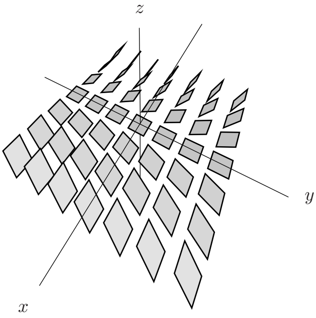
Theorem (Frobenius theorem DEAHNA 1840, CLEBSCH 1860)

Let D be a distribution on M.
D comes from a foliation $\Leftrightarrow$
for all vector fields X, Y lying in D, their Lie bracket $[X, Y]$ lies in D.

Examples



A rank-1 distribution on $\mathbb{R}^2$. It gives rise to a foliation of $\mathbb{R}^2$ by 1-dimensional leaves.



$D = Span\{\partial_x, \partial_y - x\partial_z\}$ does not come from a foliation.
It is the kernel of the contact 1-form $xdy + dz$.

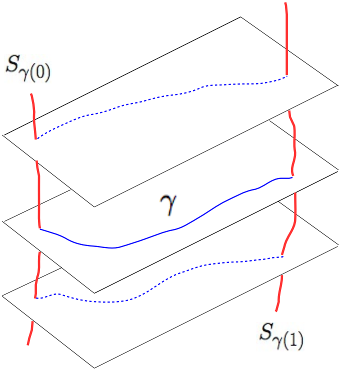
Holonomy

Definition (EHRESMANN, 1950)

Let $\gamma\colon [0,1] \to M$ be a path lying in a leaf, and $S_{\gamma(0)}, S_{\gamma(1)}$ slices transverse to the foliation. The holonomy of γ is the germ of the diffeomorphism

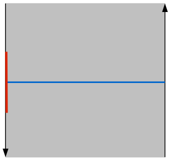
$$S_{\gamma(0)} \to S_{\gamma(1)}$$

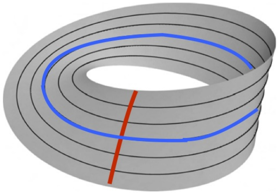
obtained "following nearby paths lying in leaves".



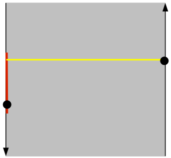
Example

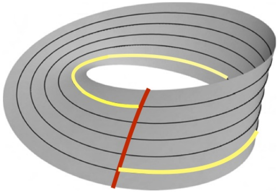
The foliation on the Möbius band has one special circle.





The holonomy around the special circle is "$-Id$".





A motivation: Reeb's local stability theorem

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Homotopic paths have the same the holonomy. So, for any leaf L and $x \in L$, get a surjective map

$$\pi_1(L, x) \to H_x^x := \{\text{holonomy of loops based at } x\}.$$

The local model of $\mathcal{F}$ near L is

$$(\hat{L} \times S_x)/H_x^x$$

with the foliation induced by $\hat{L} \times \{point\}$. Here $\hat{L}$ be the covering space of L such that $\hat{L}/H_x^x = L$.

Theorem (Reeb's local stability theorem Reeb, 1952)

Suppose L is a compact leaf and H_x^x is finite.
Then, nearby L, the foliation F is isomorphic to the local model.

In particular, all leaves nearby L are also compact.

Example: the Möbius band as above.

Groupoids

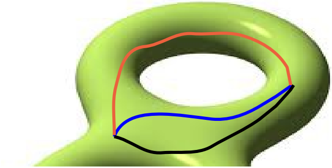
A groupoid is a set with a *partially defined*, associative composition law.

Example:

1. Let M be a topological space. Then

$$\{\text{continuous paths } [0,1] \to M\}/(\text{homotopy of paths})$$

   is a groupoid over M, with composition law=composition of paths.



2. Let M be a set. Then

$$M \times M$$

   is a groupoid over M, with composition $(x,y)(y,z) = (x,z)$.

3. a groupoid over a point is a group.

Lie groupoid=smooth groupoid.

The holonomy groupoid

Consider a foliation on M.

Definition (Winkelnkemper, 1983)

The holonomy groupoid is

$$H = \{\text{paths in leaves of the foliation}\}/(\text{holonomy of paths}).$$

It is a Lie groupoid!

Examples of holonomy groupoids

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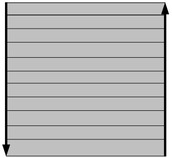
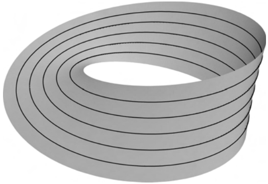
1. The one-leaf foliation on M: its holonomy groupoid is

$$M \times M \rightrightarrows M,$$

   with composition $(x, y)(y, z) = (x, z)$.

2. On the Möbius band M





This foliation "comes" from an action of S^1 on M which "wraps around M twice". Notice that the action is not free.
The holonomy groupoid is the transformation groupoid of the action, i.e.

$$S^1 \times M \rightrightarrows M,$$

with composition

$$(g, hy)(h, y) = (ghy, y).$$

Motivation for the holonomy groupoid

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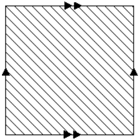
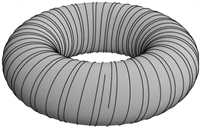
1) A foliation on M is an equivalence relation on M. The graph

$$\{(p,q) : p, q \text{ lie in the same leaf of the foliation}\} \subset M \times M$$

is usually not smooth.
However the holonomy groupoid H is always smooth.

2) The leaf space of a foliation is a topological space. It can be very non-smooth, as for the Kronecker foliation on the torus:



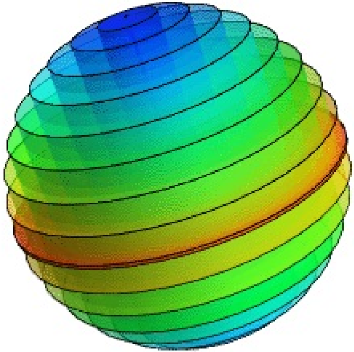


The holonomy groupoid H, for many purposes, replaces the leaf space.
(When the leaf space is a smooth manifold, the Lie groupoids H and the leaf space are Morita equivalent.)

3) To the holonomy groupoid H one can associate a C^*-algebra and do non-commutative geometry (Connes, 1970s).

What are singular foliations?

In part of the literature, a singular foliation is a suitable partition of a manifold into leaves of variying dimension.



We will use a more refined notion.

Let M be a manifold.

Definition (Stefan and Sussman, 1970s)

A singular foliation $\mathcal{F}$ is a $C^\infty(M)$-module of vector fields such that:

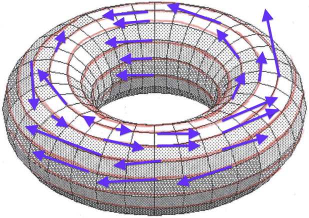
- $\mathcal{F}$ is locally finitely generated,
- $[\mathcal{F}, \mathcal{F}] \subset \mathcal{F}$.

Theorem (Stefan and Sussman, 1970s)

$(M, \mathcal{F})$ *is partitioned into leaves, of varying dimension.*

Remark: A (regular) foliation can be viewed as a singular foliation, namely

$$\mathcal{F} := \{\text{vector fields tangent to the leaves}\}.$$



Examples of singular foliations

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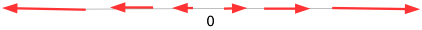
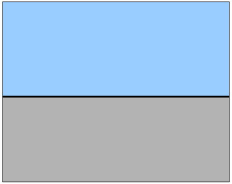
1. On $M = \mathbb{R}$ take $\mathcal{F} = \langle x\partial_x \rangle$, the singular foliation generated by $x\partial_x$.



$\mathcal{F}$ has three leaves: $\mathbb{R}_-, \{0\}, \mathbb{R}_+$.

Notice: for $k \in \mathbb{N}_{>0}$, the singular foliations $\langle x^k \partial_x \rangle$ are all different, but have the same partition into leaves.

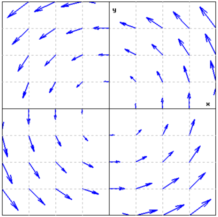
2. On $M = \mathbb{R}^2$ take $\mathcal{F} = \langle \partial_x, y\partial_y \rangle$.

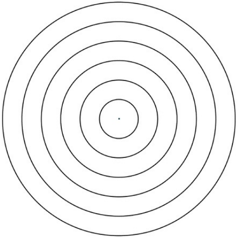


Remark: Any singular foliation, locally near a point p, is a product

(leaf through p) $\times$ (singular foliation vanishing at p).

3. On $M = \mathbb{R}^2$ take $\mathcal{F} = \langle x\partial_y - y\partial_x \rangle$.





4. Let G be a Lie group acting on M. The infinitesimal action is

$$\mathfrak{g} := (\text{Lie algebra of } G) \to \{\text{vector fields}\}, \quad v \mapsto v_M.$$

Take

$$\mathcal{F} = \langle v_M : v \in \mathfrak{g} \rangle.$$

Its leaves are the orbits of the action.
(For the action of S^1 on $\mathbb{R}^2$ by rotations, $\mathcal{F}$ is as in the example above.)

5. A Poisson structure on M induces a singular foliation, by even-dimensional leaves.

A Lie algebra at every point

A Lie algebra is a vector space with a suitable bracket.
It is the infinitesimal counterpart of a Lie group.

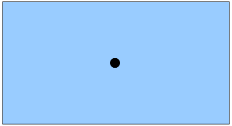
At any point p, we get a Lie algebra

$$\mathfrak{g}_p := \frac{\{X \in \mathcal{F} : X(p) = 0\}}{I_p \mathcal{F}}$$

where $I_p = \{\text{functions on } M \text{ vanishing at } p\}$.

Example

$$\mathcal{F} = \{\text{Vector fields on } \mathbb{R}^2 \text{ vanishing at the origin}\}.$$



$\mathcal{F}$ is generated by $x\partial_x,\ y\partial_x,\ x\partial_y,\ y\partial_y$. At $p = 0$ we have

$$\mathfrak{g}_p \cong \{2 \times 2 \text{ matrices}\}$$

$$x\partial_x \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, etc$$

The holonomy groupoid

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Definition

Let $X_1, \ldots, X_n \in \mathcal{F}$ be local generators of $\mathcal{F}$.
A path holonomy bi-submersion is $(U, \mathbf{s}, \mathbf{t})$ where

$$U \subset M \times \mathbb{R}^n \underset{\mathbf{t}}{\overset{\mathbf{s}}{\rightrightarrows}} M$$

and the maps $\mathbf{s}$ and $\mathbf{t}$ are

$\mathbf{s}(y, \xi) = y$

$\mathbf{t}(y, \xi) = exp_y(\sum_{i=1}^n \xi_i X_i)$, the time-1 flow of $\sum_{i=1}^n \xi_i X_i$ starting at y.

There is a notion of composition and inversion of path holonomy bi-submersions.

Take a family of path holonomy bi-submersions $\{U_i\}_{i \in I}$ covering M. Let $\mathcal{U}$ be the family of all finite compositions of elements of $\{U_i\}_{i \in I}$ and of their inverses.

Definition (Androulidakis-Skandalis, 2005)

The holonomy groupoid of the singular foliation $\mathcal{F}$ is

$$H := \coprod_{U \in \mathcal{U}} U / \sim$$

where $\sim$ is a suitable equivalence relation.

Remark: H is a topological groupoid over M, usually not smooth.

Remark: This extends the construction of the holonomy groupoid of a (regular) foliation.

Examples of holonomy groupoids

1. Consider $\mathcal{F} = \langle x\partial_y - y\partial_x \rangle$. It “comes” from the action of S^1 on $\mathbb{R}^2$ by rotations. Then H is the transformation groupoid of the action, i.e.

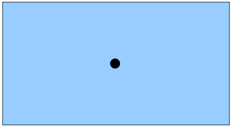
$$S^1 \times \mathbb{R}^2 \rightrightarrows \mathbb{R}^2,$$

with composition

$$(g, hy)(h, y) = (ghy, y).$$

2.

$$\mathcal{F} = \{ \text{ Vector fields on } \mathbb{R}^2 \text{ vanishing at the origin} \}.$$



Then

$$H = (\mathbb{R}^2 - \{0\}) \times (\mathbb{R}^2 - \{0\}) \quad \coprod \quad GL(2, \mathbb{R}).$$

Smoothness

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Given a singular foliation $(M, \mathcal{F})$, H is a topological groupoid over M, usually not smooth. However:

Theorem (DEBORD 2013)

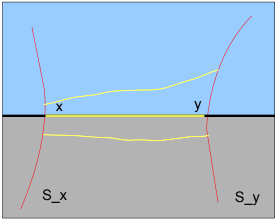
Let L be a leaf. The restriction of H to L is a Lie groupoid.

Remark: For any $p \in L$: the restriction of H to $\{p\}$ is a Lie group integrating the Lie algebra $\mathfrak{g}_p$.

Holonomy

Recall: For a (regular) foliation, we associated holonomy to a *path* γ in a leaf, by "following nearby paths in the leaves".

For singular foliations this fails.



Question: How to extend the notion of holonomy to singular foliations?

Let $x, y \in (M, \mathcal{F})$ be points in the same leaf L.
Fix slices S_x and S_y transversal to L.

Theorem (Androulidakis-Z 2014)

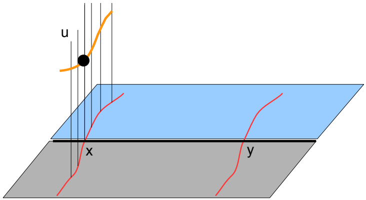
There is a well defined map

$$\Phi_x^y \colon H_x^y \to \frac{GermDiff(S_x, S_y)}{exp(I_x\mathcal{F}|_{S_x})}.$$

Remark: The map sends $h \in H_x^y$ to $[\tau]$, where τ is defined as follows:

- take any bi-submersion $(U, \mathbf{t}, \mathbf{s})$ and $u \in U$ satisfying $[u] = h$,
- take any section $\bar{b} \colon S_x \to U$ through u of $\mathbf{s}$ such that $(\mathbf{t} \circ \bar{b})(S_x) \subset S_y$,

and define $\tau := \mathbf{t} \circ \bar{b} \colon S_x \to S_y$.

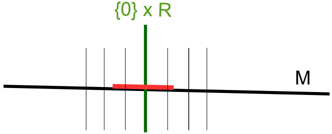


Remark: $\Phi_x^y(h)$ is just an equivalence class of diffeomorphisms, but its derivative is a well-defined map $T_x S_x \to T_y S_y$.

Example:

Let $M = \mathbb{R}$ and $\mathcal{F} = \langle x\partial_x \rangle$.

We have $H = \mathbb{R} \times M \rightrightarrows M$. So $H_0^0 = \{0\} \times \mathbb{R}$, and a transversal S_0 at 0 is a neighbourhood of 0 in M.



For all $\lambda \in H_0^0$ we have:

$$\Phi_0^0(\lambda) = [y \ \mapsto \ e^{\lambda} y] \in \frac{GermDiff(S_0, S_0)}{exp(I_0 x\partial_x)}.$$

Here we quotient by time-one flows of vector fields lying in $I_0 x\partial_x = \langle x^2 \partial_x \rangle$.

Interpretation of the holonomy groupoid

We obtain a groupoid morphism

$$\Phi\colon H \to \cup_{x,y} \frac{GermDiff(S_x, S_y)}{exp(I_x\mathcal{F}|_{S_x})}.$$

Proposition

Φ *is injective.*

Remark: If $\mathcal{F}$ is a regular foliation, then $exp(I_x\mathcal{F}|_{S_x}) = \{Id_{S_x}\}$, hence the map Φ recovers the usual notion of holonomy for regular foliations.

This provides a geometric justifications for calling H *holonomy groupoid*.

References

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  Adv. Math. **256** *(May 2014)*

Thank you for your attention!