Deformations of coisotropic submanifolds in symplectic geometry

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Symplectic manifolds

Definition

Let M be a manifold. A symplectic form is a two-form $\omega \in \Omega^2(M)$ which is

- closed ($d\omega = 0$)
- non-degenerate.

At every $x \in M$, have a non-degenerate, skew-symmetric bilinear form

$$\omega_x \colon T_xM \times T_xM \to \mathbb{R}.$$

In a suitable basis of T_xM, it is $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.

Examples:

1. All orientable surfaces, with $\omega =$ volume form.
2. On $\mathbb{R}^{2n}$ with coordinates $q_1, \ldots, q_n, p_1, \ldots, p_n$:

$$\omega = \sum_{i=1}^{n} dq_i \wedge dp_i.$$

   Darboux's theorem: locally, all symplectic manifolds look like this!
3. More generally: for any manifold B, the cotangent bundle T^*B.

Lagrangian submanifolds

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Given a submanifold $L \subset M$,

$$TL^{\omega} := \{v \in TM : \omega(v, \cdot)|_{TL} = 0\}$$

is the symplectic orthogonal to TL.

Definition

A submanifold $L \subset M$ is Lagrangian iff

$$TL = TL^{\omega}.$$

Equivalently: if the pullback of ω to L vanishes and $dim(L) = \frac{1}{2}dim(M)$.

Examples:

1. Any curve in an orientable surface is Lagrangian.
2. $\mathbb{R}^n \times \{0\}$ is Lagrangian in $\mathbb{R}^{2n}$.
   The torus $S^1 \times \cdots \times S^1$ is Lagrangian in $\mathbb{R}^2 \times \cdots \times \mathbb{R}^2 = \mathbb{R}^{2n}$.
3. The zero section B of T^*B is Lagrangian. Any fiber T_x^*B is Lagrangian.
4. Graphs of symplectomorphisms are Lagrangian.

A normal form for Lagrangian submanifolds

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Theorem (Weinstein, 1971)

Let $L \subset (M, \omega)$ be Lagrangian. There is a symplectomorphism

$$\text{(Neighbourhood of } L \text{ in } M) \cong \text{(Neighbourhood of } L \text{ in } T^*L).$$

Deformations of Lagrangian submanifolds

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Proposition

For all $\alpha \in \Gamma(T^*L) = \Omega^1(L)$*:*

$$graph(\alpha) \text{ is Lagrangian in } T^*L \quad \Leftrightarrow \quad d\alpha = 0.$$

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Remark

This means: the deformations of the Lagrangian submanifold L are governed by the chain complex $(\Omega(L), d)$.

Example

$L := \mathbb{R}^2 \times \{0\}$ is Lagrangian in $\mathbb{R}^4 \cong T^*L$.
A 1-form on L is $\alpha = f dq_1 + g dq_2$.

$$graph(\alpha) := \{(q_1, q_2, f(q), g(q)) : q \in \mathbb{R}^2\} \text{ is Lagrangian}$$

$$\Updownarrow$$

$$d\alpha = 0$$

i.e., $-\partial_2 f + \partial_1 g = 0$

The space of deformations

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Remark

- By the last Proposition:

$$\{\text{Lagrangian submanifolds near } L\} \cong \{\text{“small” closed 1-forms on } L\},$$

  an open subset of the vector space $\Omega^1_{closed}(L)$.
- One can show:

$$\frac{\{\text{Lagrangian submanifolds near } L\}}{\text{Hamiltonian diffeomorphisms}}$$

  is an open subset of $H^1(L)$.
  Hence it is finite dimensional (if L compact) and smooth!

Coisotropic submanifolds

Definition

A submanifold $C \subset M$ is coisotropic iff

$$TC^{\omega} \subset TC.$$

Necessarily $dim(C) \geq \frac{1}{2} dim(M)$.

Examples

1. the coisotropic submanifolds of dimension $\frac{1}{2} dim(M)$ are the Lagrangian ones
2. all submanifolds of dimension $dim(M) - 1$ are coisotropic.
3. for any submanifold $S \subset B$, $T^*B|_S$ is coisotropic in T^*B.
4. if $J \colon M \to \mathfrak{g}^*$ is a moment map, then $J^{-1}(0)$ is coisotropic.

Remark

Denote by $\{\cdot, \cdot\}$ the Poisson bracket on $C^{\infty}(M)$. Let

$$I_C := \{\text{functions on } M \text{ which vanish on } C\}.$$

Then: C is a coisotropic submanifold $\Leftrightarrow \{I_C, I_C\} \subset I_C$.

A normal form for coisotropic submanifolds

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Theorem (Gotay 1982)

Let $C \subset (M, \omega)$ be coisotropic.
Denote $K := TC^{\omega}$. There is a symplectomorphism

$$\text{(Neighbourhood of } C \text{ in } M\text{)} \cong \text{(Neighbourhood of } C \text{ in } K^*\text{)}.$$

Remark

1) The symplectic structure on K^* depends on a choice of distribution G s.t. $K \oplus G = TC$.
2) When C is Lagrangian, we have $K = TC$, so we recover Weinstein's theorem.

The instrinsic structure of a coisotropic submanifold

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Definition

A presymplectic form on a manifold is a two-form Ω s.t.

- Ω is closed
- $ker(\Omega)$ has constant rank.

Let $C \subset (M, \omega)$ be coisotropic.
Then $i^*\omega$ is a presymplectic form on C, where i is the inclusion.

- There is a foliation $\mathcal{F}$ tangent to $ker(i^*\omega) = TC^{\omega}$.
- $$C^{\infty}(C)_{bas} := \{\text{functions on } C \text{ constant along } \mathcal{F}\}$$

  is a Poisson algebra.

Interlude: Relation to deformation quantization

Theorem (Cattaneo-Felder, 2005)

If the first and second foliated cohomology groups vanish, then $C^\infty(C)_{bas}$ admits a deformation quantization.

Remark

When the space of leaves $C/\mathcal{F}$ is smooth, it is a symplectic manifold, and

$$C^\infty(C)_{bas} \cong C^\infty(C/\mathcal{F}).$$

It was already known that $C/\mathcal{F}$ admits a deformation quantization[Fedosov 1996, Kontsevich 1997].

An example

Example

Let

- $M = \underbrace{\mathbb{T}^4}_{x_1,x_2,x_3,x_4} \times \underbrace{\mathbb{R}^2}_{\xi_3,\xi_4}$
- $\omega = dx_1 \wedge dx_2 + dx_3 \wedge d\xi_3 + dx_4 \wedge d\xi_4$.

Then $C = \mathbb{T}^4 \times \{0\}$ is a coisotropic submanifold of (M, ω).
(TC^ω is spanned by $\frac{\partial}{\partial x_3}$ and $\frac{\partial}{\partial x_4}$.)

For any $(f, g) : \mathbb{T}^4 \to \mathbb{R}^2$:

$$graph(f,g) \text{ is coisotropic} \quad \Leftrightarrow \quad \partial_4 f - \partial_3 g = \partial_1 f \; \partial_2 g - \partial_2 f \; \partial_1 g.$$

Remark

This equation is non-linear.
So the deformations of the coisotropic submanifold C are *not* governed by a chain complex.

L_∞-algebras

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Definition (Stasheff, 1990s)

A L_∞-algebra consists of a graded vector space $V = \oplus_{i\in\mathbb{Z}} V_i$ and

$$[\cdot, \cdots, \cdot]_n : \otimes^n V \longrightarrow V \qquad (n \geq 1)$$

graded skewsymmetric, of degree $2-n$, satisfying “higher Jacobi identities”:

- $d^2 = 0$, where $d := [\cdot]_1$
- $d[a,b]_2 = [da,b]_2 + (-1)^{|a|}[b,da]_2$
- $[[a,b]_2,c]_2 \pm c.p. = \pm d[a,b,c]_3 \pm ([da,b,c]_3 \pm c.p.)$
- ...

Definition

A Maurer-Cartan element of a L_∞-algebra V is an element $Q \in V_1$ satisfying

$$\sum_{n=1}^{\infty} (-1)^{\frac{n(n-1)}{2}} \frac{1}{n!}[Q,\ldots,Q]_n = 0.$$

L_∞-algebras (cont.)

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Examples

1. If V is concentrated in degree 0, i.e. $V = V_0$, then V is a Lie algebra.
2. If only $d := [\cdot]_1$ is non-zero, then V is a chain complex.
   The Maurer-Cartan equation reads
   $$dQ = 0.$$
3. If only $d := [\cdot]_1$ and $[\cdot, \cdot]_2$ are non-zero, then W is a differential graded Lie algebra (DGLA).
   The Maurer-Cartan equation reads
   $$dQ - \frac{1}{2}[Q, Q]_2 = 0.$$

Remark

The above notion is equivalent to the one of $L_\infty[1]$-algebra, which is conceptually cleaner. The correspondence is given by a degree shift.

The L_∞-algebra of Oh-Park

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Let C be a coisotropic submanifold of (M, ω).

Recall that C has a foliation $\mathcal{F}$, tangent to $K = TC^\omega$. We denote

$$\Omega_{\mathcal{F}}(C) := \Gamma(\wedge K^*)$$

the differential forms along the leaves.

Theorem (Oh-Park, 2003)

$\Omega_{\mathcal{F}}(C)$ is endowed with an L_∞-algebra structure.

Remark:

1) $[\cdot]_1$ is the leaf-wise de Rham differential $d_{\mathcal{F}}$
2)
$$H^0(\Omega_{\mathcal{F}}(C), d_{\mathcal{F}}) = C^\infty(C)_{bas},$$
and the Lie bracket induced by $[\cdot,\cdot]_2$ is the Poisson bracket on $C^\infty(C)_{bas}$.

Remark:

The L_∞-algebra structure is “invisible to the naked eye”, since it depends on a choice of distribution G s.t. $K \oplus G = TC$.
Different choices give quasi-isomorphic L_∞-algebras.

Deformations

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Theorem (Oh-Park 2003)

Let $\alpha \in \Omega^1_{\mathcal{F}}(C)$ be a foliated 1-form.

α is a Maurer-Cartan element for the L_∞-algebra $\Omega_{\mathcal{F}}(C)$ $\Leftrightarrow$
$graph(\alpha)$ is a coisotropic submanifold of K^*.

Remark

This means: the deformations of the coisotropic submanifold C are governed by the L_∞-algebra $\Omega_{\mathcal{F}}(C)$.

Example

Again: $C = \mathbb{T}^4 \times \{0\}$ is coisotropic in the symplectic manifold $M = \mathbb{T}^4 \times \mathbb{R}^2$.
One can show: the L_∞-algebra structure on $\Omega_{\mathcal{F}}(C)$ is a DGLA.

The foliation $\mathcal{F}$ on C is tangent to $\frac{\partial}{\partial x_3}$ and $\frac{\partial}{\partial x_4}$.
A foliated 1-form is $\alpha = f(x)dx_3 + g(x)dx_4$.

$graph(\alpha)$ is coisotropic $\Leftrightarrow$

$$d_{\mathcal{F}}\alpha - \frac{1}{2}[\alpha,\alpha]_2 = 0, \text{ i.e., } \partial_4 f - \partial_3 g = \partial_1 f\,\partial_2 g - \partial_2 f\,\partial_1 g.$$

Infinitesimal deformations

Let $\alpha(t)$ be a curve in $\Omega^1_{\mathcal{F}}(C)$ with $\alpha(0) = 0$.
Assume that $graph(\alpha(t))$ is coisotropic, i.e.

$$0 = d_{\mathcal{F}}(\alpha(t)) - \frac{1}{2}[\alpha(t), \alpha(t)]_2 - \frac{1}{3!}\dots$$

Then

$$d_{\mathcal{F}}(\alpha'(0)) = 0$$
$$[\alpha'(0), \alpha'(0)]_2 = d_{\mathcal{F}}(\alpha''(0))$$

Definition

An infinitesimal deformation of the coisotropic submanifold C is

$$A \in \Omega^1_{\mathcal{F}}(C) \text{ s.t. } \underbrace{d_{\mathcal{F}} A = 0}_{\text{linearized Maurer-Cartan equation}} .$$

Corollary

An infinitesimal deformation A can be extended to a curve of deformations $\Rightarrow [A, A]_2 \in \Omega^2_{\mathcal{F}}(C)$ is $d_{\mathcal{F}}$-exact.

The space of deformations

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Remark

In general, not all infinitesimal deformations can be extended to a curve of deformations. (See the example of $\mathbb{T}^4 \times \{0\}$ in $\mathbb{T}^4 \times \mathbb{R}^2$.)

Heuristically this means: in general,

$$\{\text{coisotropic submanifolds near } C\}$$

is not “smooth”.

Equivalences of deformations

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Notions of equivalence on $\{$coisotropic submanifolds near $C\}$:

- Geometric:
  Given by Hamiltonian diffeomorphisms of (M, ω).
  It is generated by $C^\infty(M)$.
- Algebraic:
  Given by the "gauge equivalences" of the L_∞-algebra.
  It is generated by $C^\infty(C)$.

Theorem (Lê-Oh-Tortorella-Vitagliano 2014, Schätz-Zambon 2014)

The two equivalence relations agree, provided C is compact.

Remark: Denote

$$\mathcal{M} := \frac{\{\text{coisotropic submanifolds near } C\}}{\text{Hamiltonian diffeomorphisms}}.$$

The formal tangent space to $\mathcal{M}$ at $[C]$ is $H^1_{\mathcal{F}}(C)$.
Remark: One can also consider the quotient by symplectomorphisms. Its formal tangent space at $[C]$ is

$$H^1_{\mathcal{F}}(C) \,/\, r(H^1(C))$$

where $r : \Omega^1(C) \to \Omega^1_{\mathcal{F}}(C)$ is the restriction to the leaves.

Explicit formulae via Poisson geometry

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Poisson geometry is an extension of symplectic geometry.

Example: The symplectic manifold $(\mathbb{R}^2, dx \wedge dy)$ can be regarded as the Poisson manifold $(\mathbb{R}^2, \Pi = \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y})$.

Let C be a coisotropic submanifold in a Poisson manifold.
Identify a tubular neighbourhood of C with a vector bundle E (this is a choice).

Theorem (Cattaneo-Felder 2005)

$\Gamma(\wedge E)[1]$ *has an* $L_\infty[1]$*-algebra structure with multibrackets* $(n \geq 1)$*:*

$$\{s_1, \ldots, s_n\}_n = P([[\ldots[\Pi, s_1], s_2], \ldots, s_n]),$$

where $P : \chi^{multi}(E) \to \Gamma(\wedge E)$ *is the canonical projection.*

Remark

1) In the symplectic case, upon a degree shift, this recovers the L_∞-algebra of Oh-Park.
2) If Π is "analytic in fiber directions", the Maurer-Cartan elements correspond to coisotropic submanifolds.[Schätz-Zambon 2012]

References

- Y.-G. Oh and J.-S. Park.
  Deformations of coisotropic submanifolds and strong homotopy Lie algebroids.
  Invent. Math., 161(2):287-360, 2005.

- A. S. Cattaneo and G. Felder.
  Relative formality theorem and quantisation of coisotropic submanifolds.
  Adv. in Math., 208:521-548, 2007.

- F. Schätz and M. Zambon.
  Deformations of coisotropic submanifolds for fibrewise entire Poisson struc tures.
  Lett. Math. Phys., 103(7):777-791, 2013.

- Lê, Oh, Tortorella, Vitagliano
  Deformations of Coisotropic Submanifolds in Abstract Jacobi Manifolds
  ArXiv:1410.8446.

F. Schätz and M. Zambon.
Equivalences of coisotropic submanifolds.
ArXiv:1411.3227

Thank you for your attention!