Deformations of symplectic foliations

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joint work with
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in progress

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Outline

1. Introduction
2. Deformations of regular Poisson structures
3. Relation to deformations of foliations
4. Infinitesimal deformations
5. The proof: Background on Dirac geometry
6. The proof: Deformations of regular Poisson structures

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6. The proof: Deformations of regular Poisson structures

Symplectic Foliations

Definition

A symplectic foliation on M is a foliation $\mathcal{F}$ endowed with a leafwise symplectic structure, i.e. a non-degenerate $\omega \in \Omega^2(\mathcal{F}) := \Gamma(\wedge^2 T^*\mathcal{F})$ s. t. $d_\mathcal{F}\omega = 0$.

Definition

A Poisson structure $\Pi \in \mathfrak{X}^2(M)$ is regular if $\Pi^\sharp : T^*M \to TM$ has constant rank.

$$\text{Regular Poisson structures } \Pi \longleftrightarrow \text{Symplectic foliations } (\mathcal{F}, \omega)$$

$$\Pi \longmapsto (\mathcal{F} = \operatorname{im}\Pi^\sharp, \omega^\flat = -(\Pi|^\sharp_{T\mathcal{F}})^{-1})$$

Remark: (h-Principle for Symplectic Foliations) [Fernandes-Frejlich]
Let Π be a regular bivector field on an open manifold M. Then:
Π is homotopic, through regular bivector fields, to a regular Poisson structure
$\Leftrightarrow$ the distribution $\operatorname{im}\Pi^\sharp$ is homotopic to an involutive distribution.

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Deformations of Poisson structures

Often deformations are encoded by an algebraic structure.

Example:
Let $\pi \in \mathfrak{X}^2(M)$ be a Poisson structure.
A deformation of π is a Poisson structure $\pi + \tilde{\pi}$, where $\tilde{\pi} \in \mathfrak{X}^2(M)$.

$\pi + \tilde{\pi}$ Poisson

$$\Leftrightarrow 0 = [\pi + \tilde{\pi}, \pi + \tilde{\pi}] = 2[\pi, \tilde{\pi}] + [\tilde{\pi}, \tilde{\pi}] = 2(\underbrace{d_\pi \tilde{\pi} + \frac{1}{2}[\tilde{\pi}, \tilde{\pi}]}_{\text{Maurer-Cartan equation}})$$

$\Leftrightarrow$ $\tilde{\pi}$ is a MC element of the DGLA $(\mathfrak{X}^\bullet(M)[1], d_\pi, [\cdot, \cdot])$
(i.e. it is a degree 1 element satisfying the Maurer-Cartan equation).

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Goals

Let Π be a regular Poisson structure.

Goals:

- Find a DGLA whose Maurer Cartan elements parametrize regular Poisson structures at Π
- Use it to investigate the “smoothness at Π" of the space of regular Poisson structures
- Use it to relate deformations of regular Poisson structures to those of the underlying foliations

Immaterial choice: Choose a distribution G s.t. $T\mathcal{F} \oplus G = TM$.

Simplifying assumption: G is involutive.

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Good multivector fields

Problem:
{bivector fields of fixed constant rank} is not an affine subspace of $\mathfrak{X}^2(M)$.

$$\begin{aligned}\mathfrak{X}^\bullet_{good}(M) &:= \{W \in \mathfrak{X}^\bullet(M) : \iota_{T\mathcal{F}^\circ}\iota_{T\mathcal{F}^\circ}W = 0\} \\ &= \Gamma(\wedge^\bullet T\mathcal{F}) \oplus \Gamma(\wedge^{\bullet-1}T\mathcal{F}\otimes G) \\ &= C^\infty(M) \oplus \Gamma(TM) \oplus \Gamma(\wedge^2 T\mathcal{F} \oplus (T\mathcal{F}\otimes G)) \oplus \ldots\end{aligned}$$

Lemma

Let $\{\Pi_t\}$ be a smooth curve of regular Poisson structures, then

$$\frac{d}{dt}|_0 \Pi_t \in \mathfrak{X}^2_{good}(M)$$

and is d_Π-closed.

Hence

$$T_\Pi\{\text{Regular Poisson str}\} = (\mathfrak{X}^2_{good}(M))_{closed}.$$

We will show that $\mathfrak{X}^\bullet_{good}(M)$ carries the desired DGLA structure.

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We will show that $\mathfrak{X}^{\bullet}_{good}(M)$ carries the desired DGLA structure.

Parametrizing regular bivector fields

Let $\gamma \in \Omega^2(M)$ be the unique extension of ω satisfying $\ker(\gamma) = G$.

Definition

Let $Z \in \mathfrak{X}^2(M)$ be "small". The gauge transformation of Z by γ is the unique bivector field Z^γ s.t.

$$Gr(Z^\gamma) = (Z^\sharp \xi, \xi + \iota_{Z^\sharp \xi}\gamma).$$

Proposition A

We have a bijection

$$(\mathfrak{X}^2_{good}(M))_{small} \longleftrightarrow \{W \in \mathfrak{X}^2(M) \text{ regular s.t. } \operatorname{im} W^\sharp \pitchfork G\}$$

$$Z \longmapsto \Pi + Z^\gamma$$

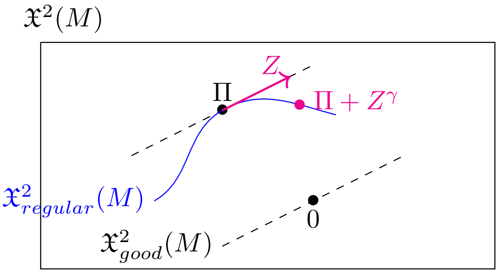

Figure: A “submanifold chart” for $\mathfrak{X}^2_{regular}(M)$

Theorem

The DGLA

$$(\mathfrak{X}^\bullet_{good}(M)[1], d_\Pi, [\ ,\ ]_\gamma)$$

controls the deformations of $(\mathcal{F}, \omega)$. Indeed, we have a bijection

$$MC(\mathfrak{X}^\bullet_{good}(M)[1])_{small} \longleftrightarrow \{\textit{Symplectic foliations } (\mathcal{F}', \omega') \textit{ with } T\mathcal{F}' \pitchfork G\}$$
$$Z \longmapsto \Pi + Z^\gamma$$

Here $[\ ,\ ]_\gamma$ is a graded Lie bracket, specified later.

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Deformations of foliations and foliated forms

Recall

There is a DGLA structure on

$$\Omega^\bullet(\mathcal{F};G) := \Gamma(\wedge T^*\mathcal{F} \otimes G),$$

with differential d_∇ the Bott-connection.
It controls the deformations of the foliation $\mathcal{F}$:[Huebschmann, Vitagliano, Ji]

$$\begin{aligned} MC(\Omega^\bullet(\mathcal{F};G)) &\longleftrightarrow \{\mathcal{F}' \text{ s.t. } T\mathcal{F}' \pitchfork G\} \\ \eta &\longmapsto Gr(\eta) \subset T\mathcal{F} \oplus G = TM. \end{aligned}$$

Remark:

The complex

$$(\mathfrak{X}^\bullet(\mathcal{F}) := \Gamma(\wedge T\mathcal{F}), d_\Pi)$$

is isomorphic to $(\Omega(\mathcal{F}), d_\mathcal{F})$, via $\omega^\flat$.

A short exact sequence

Proposition

There is a short exact sequence of DGLA's

$$\{0\} \to \mathfrak{X}^\bullet(\mathcal{F})[1] \hookrightarrow \mathfrak{X}^\bullet_{good}(M)[1] \to \Omega^\bullet(\mathcal{F}; G) \to \{0\}$$

Remark:
In degree 1, it reads

$$\Gamma(\wedge^2 T\mathcal{F}) \hookrightarrow \Gamma(\wedge^2 T\mathcal{F}) \oplus \Gamma(T\mathcal{F} \otimes G) \xrightarrow{\omega^\flat} \Gamma(T^*\mathcal{F} \otimes G).$$

Remark:
Given a MC element $Z \in \mathfrak{X}^2(\mathcal{F})$ (a cocycle), the corresponding deformation of $(\mathcal{F}, \omega)$ is

$$(\mathcal{F}, \omega + B),$$

where $B := (\wedge^2 \omega^\flat)(Z) \in \Omega^2_{closed}(\mathcal{F})$.
In other words, it is the gauge transformation $\Pi^{\widetilde{B}}$ for any extension of B.

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Infinitesimal deformations

Let $Z(t)$ be a curve of Maurer-Cartan elements in $\mathfrak{X}^2_{good}(M)$, with $Z(0) = 0$. Since

$$0 = d_\Pi(Z(t)) + \frac{1}{2}[Z(t), Z(t)]_\gamma$$

we have

$$d_\Pi(Z'(0)) = 0$$
$$[Z'(0), Z'(0)]_\gamma = d_\Pi(Z''(0)).$$

Definition

An infinitesimal deformation of Π is $W \in \mathfrak{X}^2_{good}(M)$ such that

$$\underbrace{d_\Pi W = 0}_{\text{linearized Maurer-Cartan equation}}$$

Corollary (Kuranishi criterion)

An infinitesimal deformation W can be extended to a curve of deformations

$\Rightarrow \quad [W, W]_\gamma \in \mathfrak{X}^3_{good}(M)$ *is exact.*

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Infinitesimal deformations and foliations (I)

There exist infinitesimal deformations of Π (as regular Poisson str.) which are obstructed, i.e. can't be prolonged to a smooth curve of MC elements.

Denote

$$\phi\colon \mathfrak{X}^\bullet_{good}(M)[1] \xrightarrow{\omega^\flat} \Omega^\bullet(\mathcal{F}; G).$$

Let $W \in \mathfrak{X}^2_{good}(M)$ be an infinitesimal deformation.

$$\begin{aligned} & W \text{ unobstructed} \\ \Rightarrow\ & \phi(W) \text{ unobstructed deformation of } \mathcal{F}. \end{aligned}$$

Proposition

The converse does not hold.

Intuition: Given a path of foliations $\{\mathcal{F}_t\}$ through $\mathcal{F}$, in general you don't know how to put a leaf-wise symplectic form on $\mathcal{F}_t$ for all t.
Further, you would want regular Poisson structures $\{Z_t\}$ with $\frac{d}{dt}|_0 Z_t = W$.

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Further, you would want regular Poisson structures $\{Z_t\}$ with $\frac{d}{dt}|_0 Z_t = W$.

Infinitesimal deformations and foliations (II)

Example: On $M = S^1 \times \mathbb{T}^4$,

$$\Pi = \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4},$$
$$W = f(\psi)\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial \psi}$$

where $f(\psi)$ non-constant.
(Apply Kuranishi criterion.)

Remark:
There are $\Pi \equiv (\mathcal{F}, \omega)$ with this property:

- the space of regular Poisson structures is "not smooth at Π" (there exists an obstructed infinitesimal deformation)
- the space of foliations is "smooth at $\mathcal{F}$" (all infinitesimal deformation are unobstructed)

Infinitesimal deformations and foliations (III)

Remark:
Infinitesimal deformations $W \in \mathfrak{X}^2(\mathcal{F})$ are unobstructed: a prolongation is given by the path of symplectic foliations

$$t \mapsto (\mathcal{F}, \omega + tB_W)$$

where $B_W := (\wedge^2 \omega^\flat)(W)$.

Corollary

Infinitesimal deformations $W \in \mathfrak{X}^2_{good}(M)$ with

$$[\phi(W)] = 0 \in H^1(\mathcal{F}; G)$$

are unobstructed, provided M is compact.

Infinitesimal deformations and foliations (IV)

Proposition

Suppose $rank(\mathcal{F}) = 2$ *and* M *is compact.*
Then for all infinitesimal deformations $W \in \mathfrak{X}^2_{good}(M)$*:*

$$\phi(W) \text{ unobstructed deformation of } \mathcal{F}$$
$$\Rightarrow W \text{ unobstructed.}$$

Remark:

Given $W = W_1 + W_2 \in \Gamma(\wedge^2 T\mathcal{F}) \oplus \Gamma(T\mathcal{F} \otimes G)$, a desired path of symplectic foliations is

$$(\mathcal{F}_t, (\gamma + \widetilde{B_{W_1}})|_{\mathcal{F}_t})$$

where

- $\mathcal{F}_t$ is a path of foliations through $\mathcal{F}$,
- $B_{W_1} := (\wedge^2 \omega^\flat)(W_1) \in \Omega^2(\mathcal{F})$,
- $\widetilde{B_{W_1}}$ denotes an arbitrary extension to $\Omega^2(M)$.

Infinitesimal deformations and Poisson structures

Let $W \in \mathfrak{X}^2_{good}(M)$ be an infinitesimal deformation. Clearly:

W unobstructed as a deformation of regular Poisson structures
$\Rightarrow$ W unobstructed as a deformation of Poisson structures.

Proposition

The converse does not hold.

Example: On $M = \mathbb{T}^4$,

$$\Pi = h(x_3, x_4)\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}$$
$$W = \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_4}$$

where $h(x_3, x_4)$ no-where vanishing and non-constant.
($\Pi + tW$ is Poisson, non-regular. W is obstructed by the Kuranishi criterion.)

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A related question

Stability of symplectic foliations $(\mathcal{F}, \omega)$:
Does any nearby foliation $\mathcal{F}'$ carry a (nearby) leaf-wise symplectic structure?

Stability:
Surjectivity of

$$MC(\mathfrak{X}^\bullet_{good}(M)[1]) \ \stackrel{\phi}{\to} \ MC(\Omega^\bullet(\mathcal{F}; G)),$$

locally near 0.

Infinitesimal stability:
Surjectivity of

$$\text{Inf. deformations in } \mathfrak{X}^2_{good}(M) \ \stackrel{\phi}{\to} \ \text{Inf. deformations in } \Omega^1(\mathcal{F}; G).$$

Equivalently: surjectivity of the map in cohomology.

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Dirac structures (I)

Recall:

$\mathbb{T}M := TM \oplus T^*M$ is a Courant algebroid with

- non-degenerate symmetric pairing $\langle -, - \rangle$ given by

$$\langle X + \alpha, Y + \beta \rangle := \iota_X \beta + \iota_Y \alpha,$$

- Dorfman bracket $[\![-, -]\!]$ on $\Gamma(\mathbb{T}M)$ given by

$$[\![X + \alpha, Y + \beta]\!] := [X, Y] + \mathcal{L}_X \beta - \iota_Y d\alpha.$$

Remark:

- For all $Z \in \mathfrak{X}^2(M)$:

$$\mathfrak{t}_Z : \mathbb{T}M \to \mathbb{T}M, \ (X, \alpha) \mapsto (X + Z^\sharp \alpha, \alpha)$$

  preserves $\langle -, - \rangle$ but not $[\![-, -]\!]$.
- For all closed $B \in \Omega^2(M)$:

$$\mathfrak{t}_B : \mathbb{T}M \to \mathbb{T}M, \ (X, \alpha) \mapsto (X, \alpha + B^\flat X)$$

  preserves $\langle -, - \rangle$ and $[\![-, -]\!]$.

Dirac structures (II)

Definition

A Dirac structure is a vector subbundle $L \subset \mathbb{T}M$ which is Lagrangian w.r.t. $\langle -, - \rangle$ and involutive w.r.t. $[\![-, -]\!]$.

Remark:

$$\text{Bivector fields } Z \stackrel{1-1}{\longleftrightarrow} \text{Lagrangian subbundles } L \text{ s. t. } L \pitchfork TM$$

$$Z \longmapsto Gr(Z) = \{Z^\sharp \xi + \xi : \xi \in T^*M\}$$

$$Z \text{ is Poisson } \iff Gr(Z) \text{ is Dirac}$$

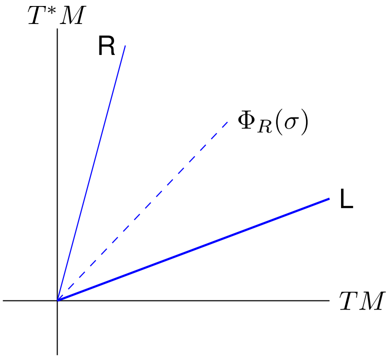
Deformations of Dirac structures (I)

Fact

Let $L, R \subset \mathbb{T}M$ be transverse Lagrangian subbundles.
There is a bijection

$$\Phi_R \colon \Gamma(\wedge^2 L^*) \cong \{\text{Lagrangian subbundles transverse to } R\}$$

$$\sigma \mapsto \text{ (graph of the map } L \overset{\sigma^\flat}{\to} L^* \cong R).$$



Deformations of Dirac structures (II)

When is $\Phi_R(\sigma)$ Dirac?

Proposition ([Liu-Weinstein-Xu])

Let L be a Dirac structure, and R a complementary Dirac structure.

1. *There is a DGLA*

$$(\Gamma(\wedge L^*)[1],\ d_L,\ [\ ,\ ]_{L^*})$$

   where
   - d_L *is the differential of Lie algebroid* L,
   - $[\ ,\ ]_{L^*}$ *is the bracket of the Lie algebroid* $R \cong L^*$*, extended by Leibniz rule.*

2. *Let* $\sigma \in \Gamma(\wedge^2 L^*)[1]$.

σ *is a Maurer-Cartan element*

$$\Leftrightarrow$$

the graph $\Phi_R(\sigma)$ *is a Dirac structure.*

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Parametrizing regular bivector fields (I)

A complement to $Gr(\Pi)$ is TM, but ill-behaved w.r.t. regular Poisson str.

Idea: Deform the Dirac structure $Gr(\Pi)$ using $G \oplus G^\circ$ as a complement.

Using $Gr(\Pi) \cong T^*M$, Fact $\Rightarrow$

$$\mathfrak{X}^2(M) \longleftrightarrow \{\text{Lagrangian subbundles} \pitchfork G \oplus G^\circ\}$$
$$Z \mapsto \mathrm{t}_\Pi \mathrm{t}_\gamma Gr(Z)$$

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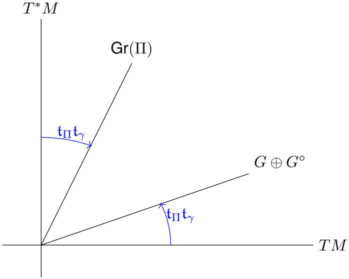

Parametrizing regular bivector fields (II)

Restring the above bijection to "small" bivector fields yields

$$\mathfrak{X}^2(M)_{small} \longleftrightarrow \{W \in \mathfrak{X}^2(M) \text{ s.t. } Gr(W) \pitchfork G \oplus G^\circ\}$$
$$Z \longmapsto \Pi + Z^\gamma$$

where Z^γ =(gauge transformation of Z by γ), i.e. $Gr(Z^\gamma) = \mathfrak{t}_\gamma(Gr(Z))$.

Lemma

For all $Z \in \mathfrak{X}^2(M)_{small}$:

$$Z \in \mathfrak{X}^2_{good}(M) \Leftrightarrow \Pi + Z^\gamma \text{ has constant rank.}$$

Proposition A

We have a bijection

$$(\mathfrak{X}^2_{good}(M))_{small} \longleftrightarrow \{W \in \mathfrak{X}^2(M) \text{ regular s.t. } \operatorname{im} W^\sharp \pitchfork G\}$$
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$\mathfrak{X}^2(M)$

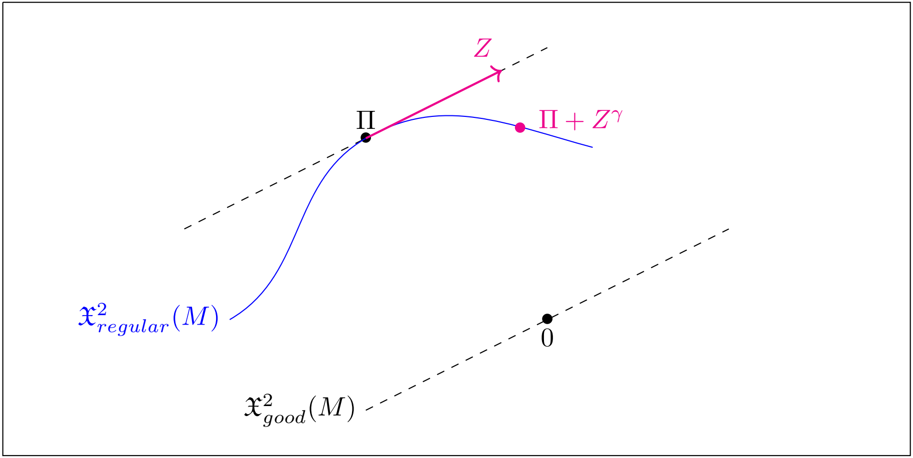

Figure: A "submanifold chart" for $\mathfrak{X}^2_{regular}(M)$

Deformations of regular Poisson structures (I)

Proposition [LWX] $\Rightarrow$
a DGLA structure on $\Gamma(\wedge Gr(\Pi)^*)[1]$ governing deformations of the Dirac structure $Gr(\Pi)$.
We now describe this using $Gr(\Pi) \cong T^*M$.

The DGLA becomes

$$(\mathfrak{X}^\bullet(M)[1], d_\Pi, [\ ,\ ]_\gamma)$$

where

$$[X,Y]_\gamma = [X_G, Y_G] - \Pi^\sharp(\mathcal{L}_{X_G}\iota_Y\gamma - \mathcal{L}_{Y_G}\iota_X\gamma)$$

for vector fields X, Y (extend to $\mathfrak{X}^\bullet(M)$ by the Leibniz rule).

Remark:
$pr_G : TM \to TM$ is a Nijenhuis endomorphism.
Further $[X,Y]_\gamma = [X,Y]_{pr_G} - \Pi^\sharp(\iota_Y\iota_X d\gamma)$.

Deformations of regular Poisson structures (I)

Proposition [LWX] $\Rightarrow$
a DGLA structure on $\Gamma(\wedge Gr(\Pi)^*)[1]$ governing deformations of the Dirac structure $Gr(\Pi)$.
We now describe this using $Gr(\Pi) \cong T^*M$.

The DGLA becomes

$$(\mathfrak{X}^\bullet(M)[1], d_\Pi, [\ ,\ ]_\gamma)$$

where

$$[X,Y]_\gamma = [X_G, Y_G] - \Pi^\sharp(\mathcal{L}_{X_G}\iota_Y\gamma - \mathcal{L}_{Y_G}\iota_X\gamma)$$

for vector fields X, Y (extend to $\mathfrak{X}^\bullet(M)$ by the Leibniz rule).

Remark:
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Deformations of regular Poisson structures (II)

Proposition B

The map $Z \longmapsto \mathfrak{t}_\Pi \mathfrak{t}_\gamma Gr(Z)$ induces a bijection

$$MC(\mathfrak{X}^\bullet(M)[1], d_\Pi, [\ ,\ ]_\gamma) \longleftrightarrow \{\text{Dirac structures transverse to } G \oplus G^\circ\}.$$

Lemma

$\mathfrak{X}^\bullet_{good}(M)[1]$ is a sub-DGLA.

Propositions A and B together give:

Theorem

The DGLA

$$(\mathfrak{X}^\bullet_{good}(M)[1], d_\Pi, [\ ,\ ]_\gamma)$$

controls the deformations of $(\mathcal{F}, \omega)$.
Indeed, we have a bijection

$$MC(\mathfrak{X}^\bullet_{good}(M)[1])_{small} \longleftrightarrow \{\text{Symplectic foliations } (\mathcal{F}', \omega') \text{ with } T\mathcal{F}' \pitchfork G\}$$
$$Z \longmapsto \Pi + Z^\gamma$$

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Thank you for your attention