KU LEUVEN

Deformations of presymplectic forms via Dirac geometry

Marco Zambon

based on joint work with
Florian Schätz

Pre-symplectic forms

Definition

Let M be a manifold. A pre-symplectic form is $\eta \in \Omega^2(M)$ for which

- $\ker(\eta)$ has constant rank
- $d\eta = 0$.

Example: If M is a coisotropic submanifold of a symplectic manifold (X, Ω), then $(M, \iota^*\Omega)$ is pre-symplectic.

Remark: $K := \ker(\eta)$ is an involutive distribution, hence tangent to a foliation.

- If M/K is smooth, η induces a symplectic form there.
- In general, η can be viewed as a “transversely symplectic form” on the foliated manifold M.

We look at deformations of η, i.e. pre-symplectic forms nearby.
For instance, on $(\mathbb{R}^4, dx_1 \wedge dx_2)$, a nearby pre-symplectic form is

$$dx_1 \wedge dx_2 + f(x_1, x_3) dx_1 \wedge dx_3.$$

Deformations

Often deformations are encoded by an algebraic structure.

Example: Let $\pi \in \Gamma(\wedge^2 TM)$ be a Poisson structure.
A deformation of π is a Poisson structure $\pi + \tilde{\pi}$, where $\tilde{\pi} \in \Gamma(\wedge^2 TM)$.

$$
\begin{aligned}
& \pi + \tilde{\pi} \text{ Poisson} \\
\Leftrightarrow\ & 0 = [\pi + \tilde{\pi}, \pi + \tilde{\pi}] = 2[\pi, \tilde{\pi}] + [\tilde{\pi}, \tilde{\pi}] = 2\underbrace{(d_\pi \tilde{\pi} + \frac{1}{2}[\tilde{\pi}, \tilde{\pi}])} \\
\Leftrightarrow\ & \tilde{\pi} \text{ satisfies the Maurer-Cartan equation} \\
& \text{of the DGLA } (\mathfrak{X}^{\mathrm{multi}}(M)[1], d_\pi, [\cdot, \cdot]).
\end{aligned}
$$

The goal

Goal: Find an algebraic structure governing deformations of η.

Notice: {2-forms of constant rank k} is not an affine subspace of $\Omega^2(M)$.

Strategy:

Step A: identify

$$\{\text{constant rank 2-form near } \eta\} \;\cong\; \text{a vector space}$$

Step B: Find an L_∞-algebra structure such that

$$\{\text{pre-symplectic forms near } \eta\} \;\cong\; \text{solutions of the MC equation}$$

The symplectic case

Suppose that ω is symplectic.

2-form approach
If $\tilde{\omega} \in \Omega^2(M)$ is small, then

$$\begin{aligned} & \omega + \tilde{\omega} \text{ is symplectic} \\ \Leftrightarrow\ & d\tilde{\omega} = 0 \\ \Leftrightarrow\ & \tilde{\omega} \text{ satisfies the Maurer-Cartan equation of } (\Omega(M)[1], d). \end{aligned}$$

The symplectic case

Poisson approach

Let π be the Poisson structure corresponding to ω.

$$\wedge\pi^\sharp\colon (\Omega(M)[1], d, [\cdot,\cdot]_\pi) \to (\mathfrak{X}^{\mathrm{multi}}(M)[1], -d_\pi, [\cdot,\cdot])$$

is an isomorphism of DGLAs.
Here $[\cdot,\cdot]_\pi$ is the Koszul bracket (it extends $[df, dg]_\pi = d\{f,g\}$).

$$\begin{array}{ccc} \{\text{symplectic forms near } \omega\} & \xleftrightarrow{\text{inversion}} & \{\text{Poisson structures near } \pi\} \\ & & \updownarrow {\scriptstyle \bullet+\pi} \\ \{\text{small } \beta \text{ s.t. } d\beta + \frac{1}{2}[\beta,\beta]_\pi = 0\} & \xleftrightarrow{\wedge^2\pi^\sharp} & \{\text{small } \tilde{\pi} \text{ s.t. } -d_\pi\tilde{\pi} + \frac{1}{2}[\tilde{\pi},\tilde{\pi}] = 0\} \end{array}$$

We obtain a bijection between

- symplectic forms nearby ω
- small Maurer-Cartan elements of $(\Omega(M)[1], d, [\cdot,\cdot]_\pi)$

It is this approach that we extend to pre-symplectic forms.

Parametrizing constant rank 2-forms

Let $\eta \in \Omega^2(M)$ be of constant rank $\rightsquigarrow K := \ker(\eta)$ is a distribution.
Denote

$$\Omega^i_{hor}(M) := \{\alpha \in \Omega^i(M) : \alpha|_{\wedge^i K} = 0\}.$$

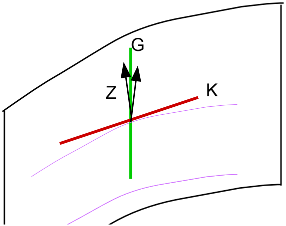
Lemma

Let $\{\eta_t\}$ be a path of constant rank 2-forms with $\eta_0 = \eta$. Then

$$\frac{d}{dt}|_0 \eta_t \in \Omega^2_{hor}(M).$$

Parametrizing constant rank 2-forms

Choose G such that $K \oplus G = TM$
$\rightsquigarrow \eta|_{G\times G}$ non-degenerate
$\rightsquigarrow Z \in \Gamma(\wedge^2 G)$.



Define $F\colon \Omega^2_{small}(M) \to \Omega^2(M)$ by

$$(F(\beta))^\sharp = \beta^\sharp(1 + Z^\sharp\beta^\sharp)^{-1}.$$

Proposition (A)

There is a bijection

$$\Omega^2_{hor,small}(M) \xrightarrow{\cong} \{\textit{Constant rank 2-forms with kernel transverse to } G\}$$
$$\beta \mapsto \eta + F(\beta)$$

Remark: $\Omega^2_{hor}(M) = \Gamma(\wedge^2 G^*) \oplus \Gamma(K^* \otimes G^*)$.

$L_\infty[1]$-algebras

Definition

A $L_\infty[1]$-algebra consists of a graded vector space $V = \oplus_{i\in\mathbb{Z}} V_i$ and

$$[\cdot, \cdots, \cdot]_n : \otimes^n V \longrightarrow V \qquad (n \geq 1)$$

graded symmetric, of degree 1, satisfying “higher Jacobi identities”:

- $d^2 = 0$, where $d := [\cdot]_1$
- $d[a,b]_2 = [da,b]_2 \pm [a,db]_2$
- $[[a,b]_2,c]_2 \pm c.p. = \pm d[a,b,c]_3 \pm ([da,b,c]_3 \pm c.p.)$
- ...

Definition

A Maurer-Cartan element of a $L_\infty[1]$-algebra V is an element $Q \in V_0$ satisfying

$$\sum_{n=1}^{\infty} \frac{1}{n!}[Q,\ldots,Q]_n = 0.$$

$L_\infty[1]$-algebras

Examples

i) If V is concentrated in degree 0, i.e. $V = V_0$, then V is a Lie algebra.

ii) If only $d := [\cdot]_1$ is non-zero, then V is a chain complex.
The Maurer-Cartan equation reads

$$dQ = 0.$$

iii) If only $d := [\cdot]_1$ and $[\cdot,\cdot]_2$ are non-zero, then W is a differential graded Lie algebra (DGLA).
The Maurer-Cartan equation reads

$$dQ + \frac{1}{2}[Q, Q]_2 = 0.$$

The $L_\infty[1]$-algebra associated to a bivector field

Let Z be a bivector field on M.

Proposition (B)

- $\Omega(M)[2]$ *has a* $L_\infty[1]$*-algebra structure, whose only non-trivial multibrackets are:*
  1. *the de Rham differential* d
  2. *the Koszul bracket* $[\cdot,\cdot]_Z$
  3. $$(\alpha,\beta,\gamma)\mapsto\left(\alpha^\sharp\wedge\beta^\sharp\wedge\gamma^\sharp\right)(\frac{1}{2}[Z,Z])$$
- *a 2-form* β *is a small Maurer-Cartan element of* $\Omega(M)[2]$ $\Leftrightarrow$ *the* 2*-form* $F(\beta)$ *is closed.*

Main theorem: the $L_\infty[1]$-algebra associated to η

Let $\eta \in \Omega^2(M)$ be a pre-symplectic form $\rightsquigarrow K := \ker(\eta)$ is a distribution.
Choose G such that $K \oplus G = TM$ $\rightsquigarrow Z \in \Gamma(\wedge^2 G)$ bivector field.

Remark:

i) Z is not Poisson in general (it is iff G is involutive)

ii) $[Z, Z]$ has no component in $\wedge^3 G$, because Z descends to a Poisson structure on M/K (locally).

Theorem

- $\Omega_{hor}(M)[2]$ *is a* $L_\infty[1]$*-subalgebra of* $\Omega(M)[2]$ *(with the* $L_\infty[1]$*-algebra structure associated to the bivector field* Z*)*
- *The map* $\beta \mapsto \eta + F(\beta)$ *gives a bijection*

$$\{\text{small Maurer-Cartan elements of } \Omega_{hor}(M)[2]\} \longrightarrow \{\text{pre-symplectic forms with kernel transverse to } G\}$$

- First item: follows from Remark
- Second item: follows from Propositions A and B

Dirac structures

The vector bundle $TM \oplus T^*M$ is an instance of Courant algebroid, with the non-degenerate pairing

$$\langle (X,\alpha),(Y,\beta)\rangle := \alpha(Y)+\beta(X)$$

and the Dorfman bracket

$$[\![(X,\alpha),(Y,\beta)]\!] = ([X,Y], \mathcal{L}_X\beta - \iota_Y d\alpha).$$

Definition

A Dirac structure is a Lagrangian subbundle $L \subset TM \oplus T^*M$ such that $\Gamma(L)$ is involutive.

Examples:

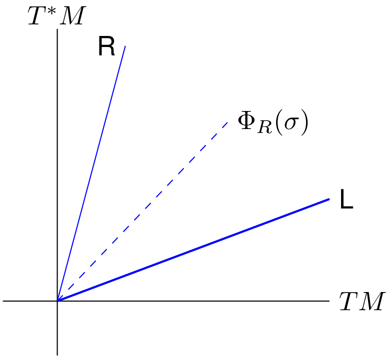
Given $\omega \in \Omega^2(M)$: its graph is Dirac $\Leftrightarrow$ ω is closed.

Given $\pi \in \mathfrak{X}^2(M)$: its graph is Dirac $\Leftrightarrow$ π is Poisson.

Dirac structures

Suppose L, R are transverse Lagrangian subbundles. There is a bijection

$$\Phi_R\colon \Gamma(\wedge^2 L^*) \cong \{\text{Lagrangian subbundles transverse to } R\}$$
$$\sigma \mapsto \text{ (graph of the map } L \xrightarrow{\sigma^\sharp} L^* \cong R).$$



When is $\Phi_R(\sigma)$ Dirac?

Dirac structures

Proposition (Deformations of Dirac structures, [LWX]...)

Let L be a Dirac structure and R a complementary Lagrangian subbundle.

- $\Gamma(\wedge L^*)[2]$ *has an* $L_\infty[1]$*-algebra structure, whose only non-trivial multibrackets are:*
  1. d_L, *the differential associated to the Lie algebroid* L,
  2. *the extention of* $[\cdot,\cdot]_{L^*} := \mathrm{pr}_R[\![\cdot,\cdot]\!]$, *the bracket of the almost Lie algebroid* $R \cong L^*$,
  3. $$\alpha, \beta, \gamma \mapsto (\alpha^\sharp \wedge \beta^\sharp \wedge \gamma^\sharp)\psi$$

     where $\psi \in \Gamma(\wedge^3 L)$ *is* $\langle \mathrm{pr}_L([\![\cdot,\cdot]\!]), \cdot\rangle|_{\wedge^3 L^*}$.
- *An element* $\sigma \in \Gamma(\wedge^2 L^*)[2]$ *is a Maurer-Cartan element* $\Leftrightarrow$ *the graph* $\{(X - \iota_X \sigma) : X \in L\}$ *is a Dirac structure.*

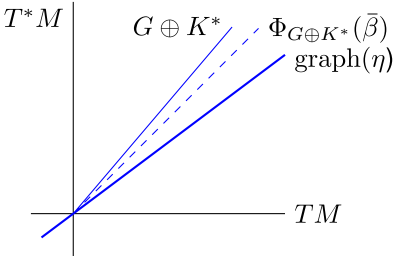
Deforming η as a Dirac structure

Let η be a pre-symplectic form $\rightsquigarrow K = \ker(\eta)$.
Choose a complement G $\rightsquigarrow Z \in \Gamma(\wedge^2 G)$.

Idea: View η as a Dirac structure, and deform it using $G \oplus K^*$ as a complement.

The Lagrangian subbundles near $\mathrm{graph}(\eta)$ are of the form $\Phi_{G\oplus K^*}(\bar{\beta})$, where

$$\bar{\beta} \in \Gamma(\wedge^2((\mathrm{graph}(\eta))^*) \;\cong\; \Omega^2(M) \ni \beta.$$

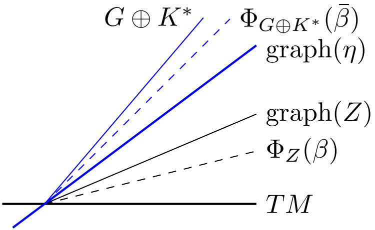


Lemma

$$\mathrm{rank}(\Phi_{G\oplus K^*}(\bar{\beta}) \cap TM) = \mathrm{rank}(K) \quad \Leftrightarrow \quad \beta \in \Omega^2_{hor}(M).$$

Deforming η as a Dirac structure

Apply the Courant algebroid automorphism $\mathfrak{t}_{-\eta}\colon (X,\xi) \mapsto (X, \xi - \iota_X \eta)$:

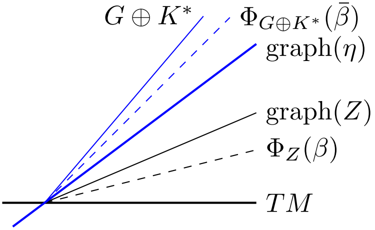


Consequence 1:

$$\Phi_Z(\beta) = \mathrm{graph}(F(\beta))$$
$$\Rightarrow \ \Phi_{G\oplus K^*}(\bar{\beta}) = \mathrm{graph}(\eta + F(\beta))$$
$\Rightarrow$ Proposition A.

Deforming η as a Dirac structure

Apply the Courant algebroid automorphism $\mathfrak{t}_{-\eta}\colon (X,\xi) \mapsto (X, \xi - \iota_X \eta)$.



Consequence 2:

Deforming $\mathrm{graph}(\eta)$ using the complement $G \oplus K^* \leftrightarrow$
Deforming TM using the complement $\mathrm{graph}(Z)$

Apply the proposition on deformations of Dirac structures to the latter
$\Rightarrow$ Proposition B.

Infinitesimal deformations

Let $\beta(t)$ be a curve of Maurer-Cartan elements in $\Omega^2_{hor}(M)$, with $\beta(0) = 0$.
Since

$$0 = d(\beta(t)) + \frac{1}{2}[\beta(t), \beta(t)]_2 + \frac{1}{3!}[\beta(t), \beta(t), \beta(t)]_3$$

we have

$$d(\beta'(0)) = 0$$

$$[\beta'(0), \beta'(0)]_2 = d(\beta''(0)).$$

Definition

An infinitesimal deformation of η is $B \in \Omega^2_{hor}(M)$ such that

$$\underbrace{dB = 0}_{\text{linearized Maurer-Cartan equation}}$$

Corollary (Kuranishi criterium)

An infinitesimal deformation B can be extended to a curve of deformations

$$\Rightarrow \quad [B, B]_2 \in d\Omega^2_{hor}(M).$$

Obstructed infinitesimal deformations

Proposition

The infinitesimal deformations of η are generally obstructed, i.e. they might not be extended to a smooth curve of pre-symplectic forms.

Example

Let $(M, \eta) = (\mathbb{T}^4, dx_3 \wedge dx_4)$.

Then $B = \cos(x_3) dx_1 \wedge dx_3 + \cos(x_4) dx_2 \wedge dx_4$ is an infinitesimal deformation, but

$$[B, B]_Z \notin d\Omega^2_{hor}(M).$$

(Here $Z = \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4}$.)

Remarks on the $L_\infty[1]$-algebra $\Omega_{hor}(M)[2]$

Let be $\Omega_{hor}(M)[2]$ the $L_\infty[1]$-algebra associated to a pre-symplectic form η.

- It is independent of the choice of complement G, up to $L_\infty[1]$-isomorphism. [GUALTIERI, MATVIICHUK, SCOTT]
- Two small Maurer-Cartan elements are gauge equivalent $\Leftrightarrow$ the corresponding pre-symplectic forms are related by an isotopy of M.
- The map of Maurer-Cartan elements

  $$\text{(pre-symplectic form } \eta'\text{)} \mapsto \ker(\eta')$$

  lifts to a strict $L_\infty[1]$-morphism

  $\Omega_{hor}(M)[2] \to (L_\infty[1]$-algebra governing deformations of the foliation $K)$.

Future work:

- Check whether these two deformation problems are equivalent:
  - pre-symplectic forms
  - coisotropic submanifolds and ambient symplectic forms
- Apply these ideas to deformations of regular Poisson structures.

Thank you for your attention!