# Homotopy moment maps

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# Introduction

A moment map a for the action of a Lie algebra $\mathfrak{g}$ on a symplectic manifold $(M, \omega)$ is given by a Lie algebra morphism

$$\mathfrak{g} \to \text{observables of } (M, \omega).$$

This suggests that the definition of moment map in the case of a closed $3$-form $c$ be a Lie-2 algebra morphism

$$\mathfrak{g} \to \text{observables of } (M, c).$$

We show that these moment maps are quite easy to construct in practice.

This can be generalized letting $c$ be an arbitrary closed $n$-form (or even a higher Dirac structure) and letting $\mathfrak{g}$ be an arbitrary $L_\infty$-algebra. A moment map is then a $L_\infty$-morphism

$$\mathfrak{g} \to \text{observables of } (M, c).$$

In general, there is no induced action of $\mathfrak{g}$ on $M$, but we expect an action on some object associated to $M$. When $c$ is a 3-form, for instance, we expect $\mathfrak{g}$ to act on the $c$-twisted Courant algebroid $(TM \oplus T^*M)_c$.

# Reminder: the symplectic case

Given $G \circlearrowleft (M, \omega)$ a Lie group action on a symplectic manifold:

## Definition

A moment map is

$$J : \mathfrak{g} \to C^\infty(M)$$

such that

(A) $v_M = X_{J^v}$ for all $v \in \mathfrak{g}$

(B) $J : (\mathfrak{g}, [\cdot, \cdot]) \to (C^\infty(M), \{\cdot, \cdot\})$ is a Lie algebra morphism.

## Remark

Equivalently, a moment map is $J$ such that $\omega + J$ is a closed degree $2$ element of

$$\left( (\Omega(M) \otimes S\mathfrak{g}^*)^G \ , \ d_G \right) ,$$

the Cartan model for equivariant cohomology.

## Definition

A Lie 2-algebra consists of

$$d := [\cdot]_1 : V_{-1} \to V_0$$

together with linear maps

$$\begin{aligned} [\cdot,\cdot]&\colon \wedge^2 V_0 \to V_0 \\ [\cdot,\cdot]&\colon V_0 \otimes V_{-1} \to V_{-1} \\ T := [\cdot,\cdot,\cdot]&\colon \wedge^3 V_0 \to V_{-1} \end{aligned}$$

such that

$$\begin{aligned} d[x,f] &= [x,df] \quad \text{for all } x \in V_0, f \in V_{-1}, \\ [x,[y,z]] + c.p. &= -d\left(T(x,y,z)\right) \quad \text{for all } x,y,z \in V_0 \end{aligned}$$

and for all $x, y, z, w \in V_0$:

$$\begin{aligned} &[x, T(y,z,w)] - [y, T(x,z,w)] + [z, T(x,y,w)] - [w, T(x,y,z)] = \\ &T([x,y],z,w) + T(y,[x,z],w) + T(y,z,[x,w]) \\ &-T(x,[y,z],w) - T(x,z,[y,w]) + T(x,y,[z,w]). \end{aligned}$$

# The case of closed 3-forms

Now let $c \in \Omega^3_{cl}(M)$ be non-degenerate.
Define

$$\Omega^1_{ham}(M) := \{\alpha \in \Omega^1(M) : \exists \text{ a vector field } X_\alpha \text{ with } \iota_{X_\alpha} c = -d\alpha\}$$
$$\{\alpha, \beta\} := c(X_\alpha, X_\beta, \cdot) \in \Omega^1_{ham}(M)$$

Notice: $\{\{\alpha, \beta\}, \gamma\} + c.p. = -d(c(X_\alpha, X_\beta, X_\gamma))$.

## Proposition (Baez,Rogers,Hoffnung 2008)

*The complex*

$$C^\infty(M) \xrightarrow{d} \Omega^1_{ham}(M)$$

*together with*

- $\{\alpha, \beta\}$
- $\{\alpha, \beta, \gamma\} := c(X_\alpha, X_\beta, X_\gamma)$

*is a Lie-2 algebra.*
*Notation:* $\mathcal{O}(M, c)$.

From now on, let $G \circlearrowleft (M, c)$.

## Definition

A homotopy moment map is a Lie 2-algebra morphism

$$(J, \mu)\colon \mathfrak{g} \rightsquigarrow \mathcal{O}(M, c)$$

such that $v_M = X_{J^v}$ for all $v \in \mathfrak{g}$.

## Remark

“Lie 2-algebra morphism” means:

$$J\colon \mathfrak{g} \to \Omega^1_{ham}(M)$$
$$\mu\colon \wedge^2 \mathfrak{g} \to C^\infty(M)$$

such that

$$J^{[v,w]} - \underbrace{\{J^v, J^w\}}_{c(v_M, w_M, \cdot\,)} = d\, \mu^{v\wedge w}$$
$$-\underbrace{\{J^v, J^w, J^z\}}_{c(v_M, w_M, z_M)} = \mu^{v\wedge[w,z]} - \mu^{w\wedge[v,z]} + \mu^{z\wedge[v,w]}$$

# Constructing moment maps

## Theorem

*Suppose we have a linear*

$$J\colon \mathfrak{g} \to \Omega^1_{ham}(M)$$

*with*

(A) $v_M = X_{J^v}$ *for all* $v \in \mathfrak{g}$

(B) $J$ *is* $G$*-equivariant*

(C) $\iota_{v_M} J^v = 0$ *for all* $v \in \mathfrak{g}$.

*Then* $(J, \mu)\colon \mathfrak{g} \rightsquigarrow \mathcal{O}(M, c)$ *is a homotopy moment map, where*

$$\mu^{v \wedge w} := \iota_{v_M} J^w.$$

## Remark

The assumptions of the theorem are equivalent to:
$c + J$ is a closed degree $3$ element of $((\Omega(M) \otimes S\mathfrak{g}^*)^G\ ,\ d_G)$.

## Remark

- $H^1(\mathfrak{g}) = \{0\}$ (for ex. take $\mathfrak{g}$ semisimple) or $H^2(M) = 0$
  $\Rightarrow\ \exists\, J$ satifsying (A).
- $G$ compact
  $\Rightarrow\ \exists\, J$ satisfying (B).
- (A),(B), every $v_M$ vanishes at some point of $M$ (for ex. take $\chi(M) \neq 0$)
  $\Rightarrow$ any $J$ satisfies (C).

## Example

Let $b \in \Omega^2(M)$ be $G$-invariant, and consider

$$G \circlearrowleft (M, c := db).$$

Then $J^v := \iota_{v_M} b$ satisfies the assumption of the theorem.
Hence we obtain a homotopy moment map.

# Reduction à la Marsden-Weinstein

## Lemma

*Let* $G \circlearrowleft (M, c)$ *with homotopy moment map* $(J, \mu)$*. Assume that*

$$M_0 := \{x \in M : J^v(x) = 0 \; \forall \, v \in \mathfrak{g}\}$$

*is a smooth submanifold and* $M_0/G$ *is smooth.*
*Then* $c$ *descends to*

$$\underline{c} \in \Omega^3_{cl}(M_0/G).$$

## Example

Let $G \circlearrowleft N$ freely and properly. Then

$$G \circlearrowleft (M := \wedge^2 T^* N, db^N_{can}),$$

and $J^v := \iota_{v_M} b^N_{can}$ gives rise to a homotopy moment map.
We have

$$M_0/G \cong \wedge^2 T^*(N/G) \quad \text{and} \quad \underline{c} = db^{N/G}_{can}.$$

# Relation to Courant algebroids

## Lemma (Bursztyn-Cavalcanti-Gualtieri)

*$c + J$ is a closed degree $3$ element of $((\Omega(M) \otimes S\mathfrak{g}^*)^G \ , \ d_G)$*

$$\Updownarrow$$

$$\mathfrak{g} \to \Gamma(TM \oplus T^*M), \ \ v \mapsto v_M - J^v$$

*is a trivially extended action on the Courant algebroid $(TM \oplus T^*M)_c$ with isotropic image, integrating to the (co)tangent lift.*

Expected:

the above extended action is the composition of

$$\underbrace{\mathfrak{g} \rightsquigarrow \mathcal{O}(M, c)}_{\text{hom. moment map}}$$

with

$$\underbrace{\mathcal{O}(M, c) \rightsquigarrow \text{Lie 2-algebra associated to } (TM \oplus T^*M)_c}_{\text{Weinstein-Roytenberg 1999, Rogers 2009}}.$$

All of this should work for closed forms of any degree.

# Thank you!

# Bibliography I


H. Bursztyn, G. R. Cavalcanti, and M. Gualtieri.
Reduction of Courant algebroids and generalized complex structures.
*Adv. Math.*, 211(2):726–765, 2007.

J. F. Cariñena, M. Crampin, and L. A. Ibort.
On the multisymplectic formalism for first order field theories.
*Differential Geom. Appl.*, 1(4):345–374, 1991.

Y. Fregier and C. L. Rogers.
Equivariant cohomology and formal geometry views on n-plectic moment maps.
in progress.

T. B. Madsen and A. Swann.
Multi-moment maps.
12 2010, Arxiv 1012.2048.

C. L. Rogers.
L-infinity algebras from multisymplectic geometry.
05 2010, Arxiv 1005.2230.