KU LEUVEN

Branes in symplectic manifolds

Marco Zambon

joint work with Charlotte Kirchhoff-Lukat

Rui Fest

Coisotropic submanifolds

Let (M, ω_M) be a symplectic manifold.

Definition

A submanifold Y is coisotropic if

$$TY^{\omega_M} \subset TY.$$

Remark

We have

$$TY^{\omega_M} = \ker(\omega),$$

where $\omega := \iota^*\omega_M$.
This is an involutive distribution.
ω descends to a symplectic form on $Y/\sim$ (if smooth).

Examples

- M itself,
- codimension 1 submanifolds,
- Lagrangian submanifolds.

Branes

Let (M, ω_M) be a symplectic manifold.

Definition

A brane (Y, F) consists of

- Y a coisotropic submanifold
- $F \in \Omega^2_{cl}(Y)$

such that

1) $$\ker(F) = \ker(\omega) =: E$$

   where $\omega := \iota^* \omega_M$
2) on TY/E have: $I := \omega^{-1} \circ F$ satisfies

$$I^2 = -1.$$

Remark

If smooth, $Y/\sim$ is a complex manifold.
$F + i\omega$ induces a holomorphic symplectic form there (so $\dim(Y/\sim) \in 4\mathbb{Z}$).

Branes (cont.)

Remark

Branes arose in the study of the A-model [Kapustin-Orlov 2001]

Remark

Branes are natural in terms of the generalized geometry [Gualtieri 2003].

A generalized submanifold of M is (Y, L) where

- Y is a submanifold
- $L \subset (TM \oplus T^*M)|_Y$ is a transitive Dirac structure supported on Y. Equivalently, for a unique $F \in \Omega_{cl}(Y)$,

$$L = \{(X, \xi) : X \in TY, \xi|_{TY} = \iota_X F\}.$$

Let $J \colon TM \oplus T^*M \to TM \oplus T^*M$ be a generalized complex structure. A brane is a generalized submanifold (Y, L) such that

$$J(L) = L.$$

Notice: ω_M symplectic form $\rightsquigarrow$ generalized complex structure

$$\begin{pmatrix} 0 & -\omega_M^{-1} \\ \omega_M & 0 \end{pmatrix}.$$

Goal

Given a brane (Y, F), understand the deformation theory of branes, i.e. the space of nearby branes:

- To produce new examples
- Is this space smooth? Is the moduli space smooth?
- Are all nearby coisotropic submanifolds also branes?

Notation
Given a 2-form σ, we denote by the same symbol the bundle map

$$TM \to T^*M,\ v \mapsto \iota_v \sigma.$$

Lagrangian branes

Example

Let Y be a Lagrangian submanifold of (M, ω_M).
Then

$$(Y, 0)$$

is a brane, where $0 \in \Omega^2(Y)$.

Spacefilling branes: $Y = M$

Let (M, ω) be symplectic.

Remark

(M, F) is a brane

$\overset{Def}{\Leftrightarrow}$ F is symplectic and $I := \omega^{-1} \circ F$ satisfies $I^2 = -1$

$\Leftrightarrow$ $F + i\omega$ is holomorphic symplectic. (The imaginary part of fixed)

Remark

Space-filling brane structures $F \leftrightarrow$ Complex structures I s.t. $\omega(I\cdot, \cdot)$ is skew

$$F \mapsto I := \omega^{-1} \circ F.$$

Examples of spacefilling branes

Example

Let $M = \mathbb{C}^2$ (or $M = \mathbb{T}^4$), with complex coordinates $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Take

$$\omega := Im(dz_1 \wedge dz_2)$$
$$F := Re(dz_1 \wedge dz_2)$$

Example

The $K3$ manifold is

$$M := \{z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\} \subset \mathbb{CP}^3.$$

It is a compact, simply connected, 4-dimensional smooth manifold.
Every complex structure on M admits a holomorphic symplectic form $F + i\omega$.

Infinitesimal deformations of spacefilling branes

Let F be a space-filling brane on (M, ω).

Lemma

Let $\{F_t\} \subset \Omega^2(M)$ *a curve with* $F_0 = F$ *and* $(\omega^{-1} \circ F_t)^2 = -1$*. Then*

$$\dot{F} := \frac{d}{dt}|_0 F_t \in \Omega^{1,1}(M, \mathbb{R}).$$

I.e.: $\dot{F}(Iv, Iw) = \dot{F}(v, w)$.

Thus $\Omega^{1,1}(M, \mathbb{R})$ is the formal tangent space at F to

$$\{\widetilde{F} \in \Omega^2(M) : (\omega^{-1} \circ \widetilde{F})^2 = -1\}.$$

Corollary

$$\Omega^{1,1}_{cl}(M, \mathbb{R}) = T_F \left\{ \textit{Spacefilling branes on } (M, \omega) \right\}.$$

Future aim

Given a spacefilling brane F, find a good parametrization of nearby branes.

Infinitesimal deformations of spacefilling branes (cont.)

Let F be a space-filling brane on (M, ω).

Remark

Let $f \in C^\infty(M)$

$\rightsquigarrow$ Hamiltonian vector field X_f (w.r.t. ω).

$\rightsquigarrow$ Flow Φ_t, such that $\Phi_t^* F$ is again a brane.

We have

$$\frac{d}{dt}|_0 \Phi_t^* F = \mathcal{L}_{X_f} F = dI^* df.$$

Corollary

Consider the cochain complex

$$C^\infty(M) \xrightarrow{dI^* d} \Omega^{(1,1)}(M, \mathbb{R}) \xrightarrow{d} \Omega^3(M, \mathbb{R}) \xrightarrow{d} \dots$$

It models infinitesimal deformations of the spacefilling brane F in (M, ω):

$$H^1 = \frac{T_F \{\textit{Spacefilling branes}\}}{T_F \{\textit{Orbit of hamiltonian diffeomorphisms}\}}.$$

Moduli spaces in dimension 4

Theorem (Kirchhoff-Lukat, Z.)

For any symplectic form on $M = K3$ manifold (resp. $M = \mathbb{T}^4$) admitting a spacefilling brane,

$$\{\textit{Spacefilling branes}\}/Sympl(M,\omega)_*$$

is smooth, non-compact, of dimension 20 (resp. 4).

Here $Sympl(M,\omega)_*$ denotes the symplectomorphisms inducing $Id_{H^\bullet(M,\mathbb{R})}$.

Remark

The argument relies on the Local Torelli Theorem on complex structures.

An example of codimension 1 brane

Non-Example

$S^5 \subset (\mathbb{R}^6, \omega_{can})$ does not admit a brane structure.
Reason: $S^5/\sim = \mathbb{CP}^2$ doesn't admit a holomorphic sympl form: $H^2(\mathbb{CP}^2) = \mathbb{R}$.

Main example (Spacefilling $\times$ Lagrangian)

Let (N, ω_N) be symplectic. Then

$$Y := N \times S^1 \times \{0\}$$

is coisotropic in the symplectic manifold $(N \times S^1 \times \mathbb{R}\ ,\ \omega_N \times \omega_{T^*S^1})$.

Let F_N be a spacefilling brane[1].
Then

$$(Y, F)$$

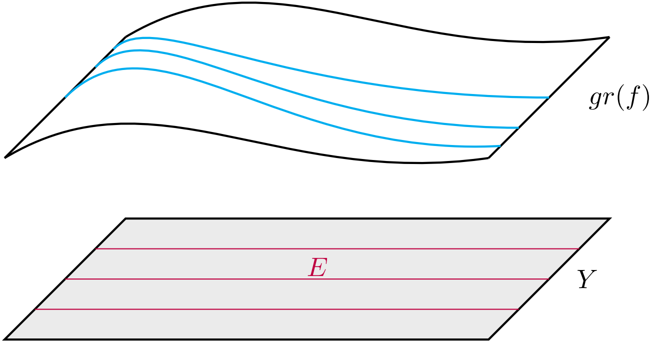
is a brane, where $F :=$ trivial extension of F_N.

[1] I.e. $F_N \in \Omega^2(N)$ symplectic s.t. $I := \omega_N^{-1} \circ F_N$ satisfies $I^2 = -1$.

Branes nearby $Y = N \times S^1$

Question:

Let $f \in C^\infty(Y)$. When does $gr(f) := \{(y, f(y))\}$ admit a brane structure?



Branes nearby $Y = N \times S^1$

Question:

Let $f \in C^\infty(Y)$. When does $gr(f) := \{(y, f(y))\}$ admit a brane structure?

Denote

$$\Phi_f := \text{time } 1 \text{ flow of the time-dependent vector field } \{X^{\omega_N}_{f|_{N\times\{q\}}}\}_{q\in[0,1]}.$$

Proposition

There is a bijection between

a) *brane structures on $gr(f)$,*
b) *spacefilling branes $\widetilde{F}_N$ on N preserved by Φ_f.*

Using $gr(f) \cong Y$, the bijection reads

$$\widetilde{F} \in \Omega^2(Y) \mapsto \widetilde{F}|_{N\times\{0\}}$$

$$\widetilde{F}_N \in \Omega^2(N) \mapsto \text{the unique extension invariant along } \ker(i^*_{gr(f)}\omega_M)$$

Remark

Hence $gr(f)$ has “less” brane structures than Y.

Proof of $a) \Rightarrow b)$

Proposition

There is a bijection between

a) *brane structures on* $gr(f)$,
b) *spacefilling branes* $\widetilde{F}_N$ *on* N *preserved by* Φ_f.

Proof: Under $gr(f) \cong Y$, we have

$$\ker(i^*_{gr(f)}\omega_M) \quad \leftrightarrow \quad \mathbb{R}(\partial_q - X_f^{\omega_N}).$$

Let $\widetilde{F} \in \Omega^2(Y)$ with this kernel.
Then $\widetilde{F}$ is closed iff

- $(\iota_{N \times \{q\}})^* \widetilde{F}$ is closed for all $q \in S^1$
- The flow of $\partial_q - X_f^{\omega_N}$ preserves this family.

In this case, $(\iota_{N \times \{0\}})^* \widetilde{F}$ is closed and preserved by Φ_f.

Stability of branes

Let (Y, F) be a brane in (M, ω_M).

Question:
Is the forgetful map

$$\{\text{Branes}\} \to \{\text{Coisotropic submanifolds}\}$$
$$(\widetilde{Y}, \widetilde{F}) \mapsto \widetilde{Y}$$

surjective near Y?

Answer: No

Recall the Main Example:
spacefilling brane F_N on the sympl. manifold $(N, \omega_N) \rightsquigarrow$

$$(\underbrace{N \times S^1 \times \{0\}}_{Y}, F)$$

brane in $(N \times S^1 \times \mathbb{R}, \omega_N \times \omega_{T^*S^1})$.

A) Answer by example

Proposition

Let N be the K3 manifold, and ω_N a symplectic form admitting a space-filling brane F_N.
Let $f \in C^\infty(Y)$. Then

$$gr(f) \text{ admits a brane structure} \quad \Leftrightarrow \quad \Phi_f = Id_N.$$

Proof of "$\Rightarrow$":
There is a brane structure $\widetilde{F}$ on $gr(f)$
$\Rightarrow \Phi_f$ preserves $\widetilde{F}_N$
$\Rightarrow \Phi_f$ preserves $\widetilde{I} := \omega_N^{-1} \circ \widetilde{F}_N$.
Use: the only automorphism of $(N, \widetilde{I})$ inducing $Id_{H^2(N,\mathbb{R})}$ is Id_N.

Corollary

There exists a compact brane Y such that:
for every $\epsilon > 0$ there is a coisotropic submanifold which

- *is ϵ-close to Y in the C^2-sense*
- *does not admit a brane structure.*

B) "Answer" by comparing first order deformations

Remark

Let Y be a codimension 1 submanifold of (M, ω_M).
Denote $E = \ker(\omega)$ for $\omega := \iota^* \omega_M$. Choose G s.t. $G \oplus E = TY$.
Gotay's theorem $\rightsquigarrow$

$$M \cong E^* \text{ as symplectic manifolds.}$$

So {deformations of Y} $= \Gamma(E^*)$.

Lemma

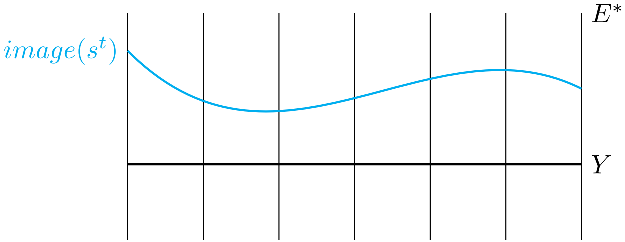
Let (Y, F) be a codimension 1 brane. Assume G involutive.
The first order deformations of Y as a brane are

$$(r, B) \in \Gamma(E^*) \oplus \Omega^2(Y) \qquad \text{such that}$$

- *closeness:* $dB = 0$
- *condition on $B|_{E \otimes G}$: for all $v \in E$, on TY/E we have*

  $$[\iota_v B] = -[\iota_v d\bar{r}] \circ I,$$

  where $\bar{r} \in \Omega^1(Y)$ is the extension of r annihilating G,
- *condition on $B|_{\wedge^2 G}$: $B|_{\wedge^2 G}$ is of type $(1, 1)$.*



Proof:

Take a curve

$$(s^t, F^t) \in \Gamma(E^*) \oplus \Omega^2(Y)$$

such that $(image(s^t), (\pi|_{s^t})^* F^t)$ is a curve of branes through (Y, F). Then

$$(r, B) := \frac{d}{dt}|_0 (s^t, F^t)$$

satisfy the above equations, since for $\omega^t := (s^t)^* \omega$ we have

- $dF^t = 0$
- $\ker(\omega^t) = \ker(F^t)$
- $(\omega^t)^{-1} \circ F^t$ squares to -1.

Proposition

For a brane $Y = N \times S^1$ *as above, consider*

$$\Upsilon \colon \{\textit{First order def. of } Y \textit{ as a brane}\} \to \Gamma(E^*),$$
$$(r, B) \mapsto r.$$

Its image consists of $r \in \Gamma(E^*)$ *s.t.*

$$d_N I^* d_N \left(\int_{S^1} r \right) = 0.$$

Here $\int_{S^1} r \in C^\infty(N)$ is obtained integrating r along the fibers of $Y \to N$.

Corollary

Υ *is not surjective in general.*

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Thank you for your attention