Geometric constructions for Poisson, Dirac and generalized complex manifolds.

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Marco Zambon

Abstract

This Habilitationsschrift consists of seven papers in which I present constructions within the framework of Poisson geometry and related geometries. The constructions are concerned with the existence/uniqueness of coisotropic embeddings in Poisson manifolds, with the geometry of circle bundles prequantizing Dirac manifolds, and with the reduction of exact Courant algebroids, Dirac structures, generalized complex structures, and contact structures.

Content

Preface

The works collected in this Habilitationsschrift are closely related to Poisson geometry, hence we would like to start with few general remarks concerning Poisson geometry. Then we give few crucial definitions, and in the four following subsections we give a summary of the content of the single papers as well as some background and motivational material.

1 Few words on Poisson geometry

Poisson manifolds [26] are generalizations of symplectic manifolds that arise naturally: symplectic manifolds (in particular cotangent bundles) arise as the phase space of classical mechanics, and Poisson manifolds as the phase space of a mechanical system with symmetry. More precisely, a Poisson manifold can be described as a smooth manifold M endowed with a bivector field $\Pi \in \Gamma(\wedge^2 TM)$ satisfying $[\pi, \pi] = 0$ or, equivalently, as a manifold for which $C^\infty(M)$ is a Poisson algebra, i.e. there is a Lie bracket on the space of functions which acts as a derivation of the product: $\{f, g \cdot h\} = \{f, g\} \cdot h + g \cdot \{f, h\}$. The latter, more algebraic characterization of Poisson manifolds is responsible for the natural appearance of objects which are both geometric and algebraic, such as Lie algebroids and Lie groupoids, and of purely algebraic objects, such as the L_∞ structure associated to any coisotropic submanifold of M. Basic examples of Poisson manifolds are symplectic manifolds and duals of Lie algebras.

The techniques involved in Poisson geometry vary a lot, and include not only differential-geometric (say in the study of reduction problems) and algebraic ones, but also techniques arising from physics. For instance let us consider the problem of quantization of Poisson manifolds. Within the formal framework of deformation quantization this has been achieved by Kontsevich [17] in 1997, who deformed the Poisson algebra $(C^\infty(M), \cdot, \{\cdot, \cdot\})$ to a non-commutative associative algebra; within the framework of geometric quantization this has been achieved in special cases by Weinstein [29], using interesting objects canonically associated to Poisson manifolds called symplectic groupoids. The use of a topological field theory called Poisson-Sigma model, which can also be expressed in terms of maps between graded manifolds, allowed Cattaneo and Felder [6] to give an interpretation of Kontsevich's construction and at the same time to provide a construction for symplectic groupoids [7]

Poisson manifolds are close relatives of other interesting geometric structures, such as Jacobi manifolds [18] and Dirac manifolds [12] (which give two generalizations of the notion of Poisson manifold) and Hitchin's generalized complex manifolds [16](which always come equipped with a Poisson structure).

2 Basic definitions

2.1 Poisson manifolds

Recall that a manifold P is called *Poisson manifold* [26] if it is endowed with a bivector field $\Pi \in \Gamma(\wedge^2 TP)$ satisfying $[\Pi, \Pi] = 0$, where $[\bullet, \bullet]$ denotes the Schouten bracket on multivector fields. Let us denote by $\sharp \colon T^*P \to TP$ the map given by contraction with Π. The image of $\sharp$ is a singular integrable distribution on P, whose leaves are endowed with a symplectic structure that encodes the bivector field Π. Hence one can think of a Poisson manifold as a manifold with a singular foliation by symplectic leaves.

Alternatively P is a Poisson manifold if there is a Lie bracket $\{\bullet, \bullet\}$ on the space of functions satisfying the Leibniz identity[1] $\{f, g \cdot h\} = \{f, g\} \cdot h + g \cdot \{f, h\}$. The Poisson bracket $\{\bullet, \bullet\}$ and the bivector field Π determine each other by the formula $\{f, g\} = \Pi(df, dg)$.

Symplectic manifolds (P, Ω) are examples of Poisson manifolds: the map $TP \to T^*P$ given by contracting with Ω is an isomorphism, and (the negative of) its inverse is the sharp map of the Poisson bivector field associated to Π. Symplectic manifolds are exactly the Poisson manifolds whose symplectic foliation consists of just one leaf.

A second standard example is the dual of a Lie algebra $\mathfrak{g}$, as follows. A linear function v on $\mathfrak{g}^*$ can be regarded as an element of $\mathfrak{g}$; one defines the Poisson bracket on linear functions as $\{v_1, v_2\} := [v_1, v_2]$, and the bracket for arbitrary functions is determined by this in virtue of the Leibniz rule. Duals of Lie algebras are exactly the Poisson manifolds whose Poisson bivector fields are linear. The symplectic foliation of $\mathfrak{g}^*$ is given by the orbits of the coadjoint action; the origin is a symplectic leaf, and unless the $\mathfrak{g}$ is an abelian Lie algebra the symplectic foliation will be singular.

2.2 Dirac manifolds

A *Dirac structure* [12] on a manifold P is a subbundle L of $TP \oplus T^*P$ which is maximal isotropic w.r.t. the symmetric pairing $\langle X_1 \oplus \xi_1, X_2 \oplus \xi_2 \rangle_+ = \frac{1}{2}(i_{X_2}\xi_1 + i_{X_1}\xi_2)$ and whose sections are closed under the Courant bracket

$$[X_1 \oplus \xi_1, X_2 \oplus \xi_2] = \big([X_1, X_2] \ \oplus \ \mathcal{L}_{X_1}\xi_2 - i_{X_2} d\xi_1\big). \tag{1}$$

If ω is a 2-form on P then its graph $\{X \oplus \omega(X, \bullet) : X \in TP\}$ is a Dirac structure iff $d\omega = 0$. Given a bivector Λ on P, the graph $\{\Lambda(\bullet, \xi) \oplus \xi : \xi \in T^*P\}$ is a Dirac structure iff Λ is a Poisson bivector. A Dirac structure L on P gives rise to (and is encoded by) a singular foliation of P, whose leaves are endowed with closed 2-forms. Further the so-called admissible functions on a Dirac manifold (P, L), defined as

$$C^\infty_{adm}(P) = \{f \in C^\infty(P) : \text{ there exists a smooth vector field } X \text{ s.t. } (X, df) \subset L\}$$

form a Poisson algebra; if L is the graph of a Poisson bivector all functions are admissible.

2.3 Jacobi manifolds

We saw above that Dirac manifolds provide a generalization of Poisson manifolds in the following way: only a subset of the functions on a Dirac manifold is naturally a Poisson

[1] In this case one says that $(C^\infty(P), \{\bullet, \bullet\}, \cdot)$ forms a Poisson algebra.

algebra. Jacobi manifolds provide a different generalization of Poisson manifolds: all the functions are endowed with a Lie bracket, which however is not a Poisson bracket.

More precisely, a *Jacobi manifold* is a smooth manifold M with a bivector field Λ and a vector field E such that

$$[\Lambda, \Lambda] = 2E \wedge \Lambda, \quad [\Lambda, E] = 0, \tag{2}$$

where $[\cdot, \cdot]$ is the Schouten bracket.

$$\{f, g\} = \sharp\Lambda(df, dg) + fE(g) - gE(f)$$

endows $C^\infty(M)$ with a Lie bracket satisfying the following equation (instead of the Leibniz rule):

$$\{f_1 f_2, g\} = f_1\{f_2, g\} + f_2\{f_1, g\} - f_1 f_2\{1, g\}, \tag{3}$$

i.e. $\{\bullet, g\}$ is a first order differential operator.

If $E = 0$, (M, Λ) is a Poisson manifold, so Poisson manifold are special cases of Jacobi manifolds. However often it is useful to take a different point of view about the relation between these two geometric structures: exactly as Poisson manifolds arise allowing symplectic manifolds to be "degenerate", Jacobi manifolds arise allowing contact manifolds (the odd-dimensional analogue of symplectic manifolds) to be "degenerate". Recall that a *contact manifold* is a $2n+1$-dimensional manifold equipped with a 1-form θ such that $\theta \wedge (d\theta)^n$ is a volume form. A Jacobi manifold always comes endowed with a (singular) foliation: odd dimensional leaves are contact manifolds, and even dimensional ones are locally conformal symplectic manifolds, i.e. they are endowed with a non-degenerate 2-form Ω and a closed 1-form ω such that $d\Omega = \omega \wedge \Omega$.

2.4 Lie groupoids

A *Lie algebroid* [4] over a manifold P is a vector bundle $E \to P$ with a Lie bracket $[\bullet, \bullet]$ on its space of sections and a bundle map $\rho\colon E \to TP$ satisfying $[e_1, fe_2] = \rho(e_1)f \cdot e_2 + f[e_1, e_2]$; standard examples are tangent bundles and Lie algebras. Every Poisson manifold P induces the structure of a Lie algebroid on its cotangent bundle T^*P: the bracket is given by $[df, dg] = d\{f, g\}$ and the bundle map $T^*P \to TP$ by $-\sharp$.

In analogy to the fact that finite dimensional Lie algebras integrate to Lie groups (uniquely if required to be simply connected), Lie algebroids - when integrable - integrate to objects called *Lie groupoids*. Recall that a Lie groupoid over P is given by a manifold Γ endowed with surjective submersions $\mathbf{s}$,$\mathbf{t}$ (called source and target) to the base manifold P, a smooth associative multiplication defined on elements $g, h \in \Gamma$ satisfying $\mathbf{s}(g) = \mathbf{t}(h)$, an embedding of P into Γ as the spaces of "identities" and a smooth inversion map $\Gamma \to \Gamma$; see for example [22] for the precise definition. The total space of the Lie algebroid associated to the Lie groupoid Γ is $ker(\mathbf{t}_*|_P) \subset T\Gamma|_P$, with a bracket on sections defined using left invariant vector fields on Γ and $\mathbf{s}_*|_P$ as anchor. A Lie algebroid A is said to be integrable if there exists a Lie groupoid whose associated Lie algebroid is isomorphic to A; in this case there is a unique (up to isomorphism) integrating Lie groupoid with simply connected source fibers.

The cotangent bundle T^*P of a Poisson manifold P carries more data then just a Lie algebroid structure; when it is integrable, the corresponding Lie groupoid Γ is actually a *symplectic groupoid*, i.e. [22] there is a symplectic form Ω on Γ such that the graph of the multiplication is a lagrangian submanifold of $(\Gamma \times \Gamma \times \Gamma, \Omega \times \Omega \times (-\Omega))$. Ω is uniquely determined (up to symplectic groupoid automorphism) by the requirement that $\mathbf{t}\colon \Gamma \to P$ be a Poisson map. For example, if P carries the zero Poisson structure, then the symplectic groupoid is T^*P with the canonical symplectic structure and fiberwise addition as multiplication.

3 Coisotropic embeddings[10, 11]

A submanifold C of a Poisson manifold P is called *coisotropic* if $\sharp N^*C \subset TC$. Here N^*C (the conormal bundle of C) is defined as the annihilator of TC, and the singular distribution $\sharp N^*C$ on C is called *characteristic distribution*. Notice that if the Poisson structure of P comes from a symplectic form Ω then the subbundle $\sharp N^*C$ is just the symplectic orthogonal of TC, so we are reduced to the usual definition of coisotropic submanifolds in the symplectic case.

Coisotropic submanifolds appear naturally and have interesting properties. For instance, the graph of a Poisson map $\Phi\colon (P_1, \Pi_1) \to (P_2, \Pi_2)$ (i.e. of a map satisfying $\Phi_*(\Pi_1) = \Pi_2$) is coisotropic [27]. Further, the quotient of a coisotropic submanifold by its characteristic distribution (when smooth) is Poisson manifold, and the quotient map is a Poisson map. If the quotient is not smooth, one can still consider the set of basic functions on C, which forms a Poisson subalgebra of $C^\infty(P)$. Last, for every coisotropic submanifold C of P the conormal bundle N^*C is a Lie subalgebroid of T^*P, the Lie algebroid associated to P.

Given a coisotropic submanifold, one considers the problem of deformation quantizing its Poisson algebra of basic functions, i.e. one asks if it is possible to deform the commutative multiplication "in direction of the Poisson bracket" to obtain an associative product. A solution to this problem, when certain obstructions vanish, was given by Cattaneo and Felder in [8, 9]. Another interesting problem is the following: given a coisotropic submanifold C of P, we saw that N^*C is a subalgebroid of T^*P. It is natural to wonder what subgroupoid of the symplectic groupoid (Γ, Ω) of P it integrates too; it turns out [5] that N^*C integrates to a subgroupoid which is lagrangian w.r.t. Ω.

In the works [10] and [11] we start either with an arbitrary submanifold of a Poisson manifold or with an arbitrary Dirac manifold. We ask under what assumptions we can view them as coisotropic submanifolds of some Poisson manifold, for in that case we can associate to our initial objects the variety of geometric and algebraic structures that coisotropic submanifolds carry with them.

3.1 Coisotropic embeddings in Poisson manifolds [10]

The following two results in symplectic geometry are well known. First: a submanifold C of a symplectic manifold (M, Ω) is contained coisotropically in some symplectic submanifold of M iff the pullback of Ω to C has constant rank; see Marle's work [19]. Second: a manifold endowed with a closed 2-form ω can be embedded coisotropically into a symplectic manifold (M, Ω) so that $i^*\Omega = \omega$ (where i is the embedding) iff ω has constant rank; see Gotay's work [13].

In [10] we extend these results to the setting of Poisson geometry. To give a Poisson-analog of Marle's result we consider *pre-Poisson* submanifolds, i.e. submanifolds C for which

$TC + \sharp N^*C$ has constant rank. Natural classes of pre-Poisson submanifolds are given by affine subspaces $\mathfrak{h}^\circ + \lambda$ of $\mathfrak{g}^*$, where $\mathfrak{h}$ is a Lie subalgebra of a Lie algebra $\mathfrak{g}$ and λ any element of $\mathfrak{g}^*$, and of course by coisotropic submanifolds and by points. Pre-Poisson submanifolds satisfy some functorial properties: preimages of pre-Poisson submanifolds under Poisson submersions are again pre-Poisson. This can be used to show that on a Poisson-Lie group G the graph of L_h (the left translation by some fixed $h \in G$, which clearly is not a Poisson map) is a pre-Poisson submanifold, giving rise to a natural constant rank distribution D_h on G that leads to interesting constructions. For instance, if the Poisson structure on G comes from an r-matrix and the point h is chosen appropriately, G/D_h (when smooth) inherits a Poisson structure from G, and $[L_h] : G \to G/D_h$ is a Poisson map which is moreover equivariant w.r.t. the natural Poisson actions of G.

In the first part of [10] we consider the Poisson-analog of Marle's result, i.e. we ask the following question:

- Given an arbitrary submanifold C of a Poisson manifold (P, Π), under what conditions does there exist some submanifold $\tilde{P} \subset P$ such that

  a) $\tilde{P}$ has a Poisson structure induced from Π

  b) C is a coisotropic submanifold of $\tilde{P}$?

  When the submanifold $\tilde{P}$ exists, is it unique up to neighborhood equivalence, (i.e. up to a Poisson diffeomorphism on a tubular neighborhood which fixes C)?

In Thm. 3.3 we give a positive answer when C is a pre-Poisson submanifold: for any pre-Poisson submanifold C of a Poisson manifold P there is a submanifold $\tilde{P}$ which is cosymplectic (and hence has a canonically induced Poisson structure) such that C lies coisotropically in $\tilde{P}$. Further, generalizing Weinstein's proof for the uniqueness of the Poisson structure transverse to a symplectic leaf, we show in Thm. 4.4 that this cosymplectic submanifold is unique up to neighborhood equivalence. When the submanifold C is not pre-Poisson it might still admit an embedding as in the above question; we provide examples.

We then use the embedding theorem to show that, when certain obstructions vanish, the Poisson algebra of basic functions on a pre-Poisson manifold (i.e. the functions whose differentials annihilate $TC \cap \sharp N^*C$) admits a deformation quantization. Also, assuming that the symplectic groupoid $\Gamma_s(P)$ of P exists, we describe two subgroupoids (an isotropic and a presymplectic one) naturally associated to a pre-Poisson submanifold C of P.

The second part of [10] deals with a different embedding problem, where we start with an abstract manifold instead of a submanifold of some Poisson manifold. The question we ask is:

- Let (M, L) be a Dirac manifold. Is there an embedding $i\colon (M, L) \to (P, \Pi)$ into a Poisson manifold such that

  a) $i(M)$ is a coisotropic submanifold of P

  b) the Dirac structure L is induced by the Poisson structure Π?

  Is such embedding unique up to neighborhood equivalence?

In the symplectic setting it is a classical theorem of Gotay [13] that both existence and uniqueness hold.

The above existence question admits a positive answer iff the distribution $L \cap TM$ on the Dirac manifold M is regular (Thm. 8.1). In that case there is a canonical (up to a Poisson diffeomorphism fixing M) Poisson manifold P in which M embeds. One expects the Poisson manifold P to be unique, provided P has minimal dimension. We are not able to prove this global uniqueness; we can just show that the Poisson vector bundle $TP|_M$ is unique (an infinitesimal statement along M) and that around each point of M a small neighborhood of P is unique (a local statement).

Using the above embedding theorem can draw conclusions about deformation quantization (Thm. 8.5) and we notice that the foliated de Rham cohomology of M w.r.t. the foliation integration $L \cap TM$ admits the structure of an L_∞-algebra (canonically up to L_∞-isomorphism), generalizing a result of Oh and Park [23] in the presymplectic setting .

3.2 Pre-Poisson submanifolds [11]

The work [11] is mainly an exposition of the results of [10] that involve pre-Poisson submanifold, in which we provide new examples. We notice that pre-images of pre-Poisson submanifolds under Poisson submersions are again pre-Poisson, and we use this to show that any translate C of $\mathfrak{h}^\circ$, where $\mathfrak{h}$ is a Lie subalgebra of a Lie algebra $\mathfrak{g}$, is a pre-Poisson submanifold of $\mathfrak{g}^*$ (recall that $\mathfrak{g}^*$ is canonically a Poisson manifold, with linear Poisson structure). Then we provide explicit examples for such pre-Poisson submanifolds C and for the subgroupoids that are associated to them as in [10].

4 Prequantization of Dirac manifolds [31, 34]

In classical mechanics one considers a symplectic manifold P; the observables are the functions on P, which together with the pointwise multiplication and the Poisson bracket form a Poisson algebra. In the quantum description of the system one replaces P with a suitable Hilbert space $\mathcal{H}$, and the observables are self-adjoint operators on $\mathcal{H}$.

The term "quantization" denotes the passage from the classical to the quantum system. In Dirac's original formulation the correspondence at the level of observables should be a linear map from $C(P)$ (a suitable class of functions on P) to the self-adjoint operators on $\mathcal{H}$ with the following properties: it should map the Poisson bracket $\{\bullet, \bullet\}$ to $\frac{1}{i\hbar}[\bullet, \bullet]$ and it should map constant functions to (multiples of) the identity; furthermore one asks that the resulting representation of $C(P)$ on $\mathcal{H}$ be faithful and irreducible.

This can not be achieved in general, not even for $\mathbb{R}^{2n}$ endowed with its canonical symplectic form, by the Grönewold- van Hove theorem. Among the attempts to perform quantization of symplectic manifolds we mention deformation quantization, where among other things one relaxes the condition that the quantization correspondence be a Lie algebra morphism, and geometric quantization, where one ends up quantizing only a subalgebra of $C(P)$.

Geometric quantization was introduced by Kostant and Souriau in the 70's. The first step, known as *prequantization*, consists of endowing the symplectic manifold (P, ω) with a hermitian line bundle K with connection ∇ whose curvature is $2\pi i\omega$. This requires

$[\omega] \in H^2(P,\mathbb{R})$ to be an integer class. The Poisson algebra $C^\infty(P)$ then acts faithfully on the space $\Gamma(K)$ of sections of K, sending the function 1 to a multiple of the identity (more precisely, a function f acts via $\hat{f} = \nabla_{X_f} + 2\pi i f$). The representation space $\Gamma(K)$ however is unsuitable from the physical point of view because much too large. Imposing a polarization cuts down $\Gamma(K)$ to a smaller, more "physically appropriate" space, on which however only a subalgebra of $C^\infty(P)$ still acts.

We will work with the following equivalent, very geometric description of prequantization: let $\pi : Q \to P$ be the principal $U(1)$-bundle associated to K. Denote by σ the connection form on Q corresponding to ∇ (so $d\sigma = \pi^*\omega$, where $\pi : Q \to P$), and by E the infinitesimal generator of the $U(1)$ action on Q. We can identify the sections of K with functions $\bar{s} : Q \to \mathbb{C}$ which are $U(1)$-antiequivariant, and then the operator $\hat{f}$ on $\Gamma(K)$ corresponds to the action of the vector field $-X_f^H + \pi^* fE$, where the superscript ^{H} denotes the horizontal lift to Q of a vector field on P. σ is actually a contact form on Q, and $X_f^H - \pi^* fE$ is just the hamiltonian vector field of $\pi^* f$ with respect to this contact form.

For systems with constraints or systems with symmetry, the phase space P may be a presymplectic or Poisson manifold, both of which are instances of Dirac manifolds (and, as such, are endowed with a Poisson algebra of functions). It is natural to consider the quantization problem for Dirac manifolds; in the two papers below we start by considering the prequantization of Dirac manifolds.

4.1 Variations on prequantization [31]

We saw above that to a prequantizable symplectic manifold one can associate a circle bundle with a contact 1-form, whose hamiltonian vector fields provide a prequantization representation for the functions on the symplectic manifold.

In [31] we extend this from symplectic to Dirac manifolds. Recall that a Dirac structure on a manifold P is a subbundle L of $TP \oplus T^*P$ satisfying certain properties. In particular L, with the restriction of the Courant bracket (1) on sections of $TP \oplus T^*P$, is a Lie algebroid over P. There is a canonical class $[\Upsilon]$ in the second Lie algebroid cohomology of L. When $[\Upsilon]$ satisfies a certain prequantization condition, making certain choices one can associate to (P, L) a triple (Q, σ, β) consisting of a $U(1)$-bundle $\pi : Q \to P$ with connection form σ and a section β of L^*. Theorem 4.1 of [31] is an explicit construction of a subbundle $\bar{L} \subset (TQ \times \mathbb{R}) \oplus (T^*Q \times \mathbb{R})$ which endows Q with a kind of geometric structure called[2] Jacobi-Dirac structure [25].

Using the hamiltonian vector fields given by the Jacobi-Dirac structure on Q, in Prop. 5.1 of [31] we construct a prequantization representation for the so-called admissible functions on P, which form Poisson algebra, on a suitable subspace of functions on Q: an admissible function f on P acts by the derivation $\hat{f} = -X_{\pi^* f}$.

As in the symplectic case, there is a corresponding "line bundle picture" of the above prequantization: to each triple (Q, σ, β) as above one can associate an L-connection on K (the line bundle corresponding to Q) with curvature $2\pi i \Upsilon$. Then, by a formula analog to Kostant's, the L-connection gives a representation of the admissible functions on P on certain sections of K.

[2] Jacobi-Dirac structures include contact 1-forms and Jacobi structures.

When P is a symplectic manifold the choice of connection of the line bundle corresponds exactly to a choice of contact 1-form on the circle bundle. In the Dirac case we show that the choice of L-connection on K with curvature $2\pi i\Upsilon$ determines the Jacobi-Dirac structure on Q (Prop. 2.5 of [34]).

Unfortunately the prequantization representation we constructed from Dirac manifolds is not faithful in general (Section 5 of [31]). However the prequantization of a Dirac manifold (P, L) gives rise to an interesting geometric structure, namely the Jacobi-Dirac structure $\bar{L}$ on the circle bundle Q over P, which we investigate in the next subsection.

4.2 On the geometry of prequantization spaces [34]

In [34] we investigate the relation between the geometric objects that arise from the prequantization of Dirac manifolds as in [31]. We start by giving an intrinsic description the Jacobi-Dirac structures $\bar{L}$ with which we endow the circle bundles ("prequantization spaces") Q over prequantizable Dirac manifolds: in Theorem 2.11 we show that the Jacobi-Dirac structure associated to an L-connection D (with curvature $2\pi i\Upsilon$) is obtained "lifting" the Dirac subbundle L by means of the the L-connection D. This description also sheds light on the Lie algebroid structure attached to $\bar{L}$, which will be used in what follows.

The approach to the prequantization of Dirac manifolds outlined in subsection 4.1 above is not the only one. An alternative approach consists of building the presymplectic groupoid of P first and constructing a circle bundle over the groupoid [30], with the hope to quantize Poisson manifolds "all at once" as proposed by Weinstein [28]. Since the presymplectic groupoid $\Gamma_s(P)$ of P is the canonical global object associated to P, the prequantization circle bundle over $\Gamma_s(P)$ can be considered as an "alternative prequantization space" for P. Furthermore, since there is a submersive Dirac map $\Gamma_s(P) \to P$, the admissible functions on P can be viewed as a Poisson subalgebra of the functions on $\Gamma_s(P)$, which can be prequantized whenever $\Gamma_s(P)$ is a prequantizable presymplectic manifold. The resulting representation is faithful but the representation space is again unsuitable because much too large.

As mentioned above, in [34] we are not interested in representations but only in the geometry that arises from the prequantization spaces associated to a given a Dirac manifold (P, L). In particular we are interested in the relation between the two prequantization spaces above; it turns out that it is given by a reduction "à la Marsen-Weinstein" (but using precontact forms, i.e. 1-forms).

The Lie groupoid integrating the Lie algebroid $\bar{L}$ is a precontact groupoid, denoted by $\Gamma_c(Q)$; the prequantization of the presymplectic groupoid $\Gamma_s(P)$ will be denoted by $\tilde{\Gamma}_c(P)$, and is itself a groupoid over P. In Prop. 3.8 we show that the natural S^1 action on Q lifts to an action on $A(\Gamma_c(Q)) \cong \bar{L}$, and that its precontact reduction is $A(\tilde{\Gamma}_c(P))$, endowed with the Lie algebroid and precontact structures given by the Lie groupoid $\tilde{\Gamma}_c(P)$. Here $A(\bullet)$ denotes the Lie algebroid functor.

This is the infinitesimal version of the relation between the Lie groupoid associated to Q and the prequantization of $\Gamma_s(P)$, namely that the latter is (a discrete quotient of) an S^1 precontact reduction of the former. The proof uses infinite dimensional spaces of Lie algebroid-paths and the information on the Lie algebroid structure on $\bar{L}$ we gathered earlier. More precisely the result (Thm. 4.9 and 4.11) is that if (P, L) is an integrable prequantizable Dirac manifold and $(Q, \bar{L})$ one of its prequantizations (which we assume to be integrable) then

a) The source-simply connected (s.s.c) contact groupoid $\Gamma_c(P)$ of (P, L) is obtained from the s.s.c. contact groupoid $\Gamma_c(Q)$ of $(Q, \bar{L})$ by S^1 contact reduction.

b) The prequantization $\tilde{\Gamma}_c(P)$ of the s.s.c. symplectic groupoid $\Gamma_s(P)$ is a discrete quotient of $\Gamma_c(P)$.

5 Reduction of Dirac and generalized complex structures and stretching [33, 3]

We saw in Section 2 above that for any manifold P the vector bundle $TP \oplus T^*P$ is endowed with the Courant bracket $[\bullet, \bullet]$ on its space of sections, with a symmetric pairing $\langle \bullet, \bullet \rangle_+$, and a projection (called "anchor") π onto TP. We refer to this collection of data as the standard (or untwisted) Courant algebroid over P. An essential feature is that the group of automorphisms of the standard Courant algebroid consists not only of the diffeomorphisms of P (acting via the tangent and cotangent lifts), but also of the closed 2-forms B, acting via $(X, \xi) \mapsto (X, \xi + i_X B)$. Exactly as differential forms or complex structures are geometric structures on P which are defined in terms of its tangent or cotangent bundle, there are geometric structures on P which are defined in terms the standard Courant algebroid $TP \oplus T^*P$. We are interested in two of them: Dirac structures (defined in Section 2) and generalized complex structures.

A (untwisted) *generalized complex structure* on P is an endomorphism $\mathcal{J}$ of $TP \oplus T^*P$ which preserves $\langle \bullet, \bullet \rangle_+$, squares to $-Id$, and satisfies an integrability condition involving the Courant bracket. Analogously to the way Dirac structures include Poisson and symplectic structures, generalized complex structures allow to consider within the same framework both symplectic and complex structures. More precisely, a symplectic form ω on P is encoded by a generalized complex structure which, in matrix form, is represented by $\begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$. Similarly a complex structure J on P is encoded by $\begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}$.

Abstracting the properties of the standard Courant algebroid $TP \oplus T^*P$ leads to the notion of Courant algebroid. We shall be working with a particular class of Courant algebroids, called *exact Courant algebroids*; the definitions of Dirac structure and generalized complex structure carry on in a straight-forward way. We remark that a generalized complex structure induces canonically a Poisson structure on P.

Any exact Courant algebroid E over P is isomorphic to $TP \oplus T^*P$ with pairing $\langle \bullet, \bullet \rangle_+$ and anchor as above, and with bracket (the "H-twisted Courant bracket") given by adding the term $(0, i_Y i_X H)$ to formula (1), where H is a closed 3-form on P. The isomorphism class of E is characterized exactly by the cohomology class of H (called Ševera class). The isomorphism above is not canonical, for it depends on a choice of splitting of $\pi : E \to TP$. Hence, when we want to make sure that our geometric constructions depend only on the isomorphism class of the objects we start with, we work with an abstract exact Courant algebroid rather than choose a presentation in terms of an H-twisted Courant bracket as above.

We will now discuss two procedures that out of geometric structures on an exact Courant algebroid produce new ones: reduction (for both Dirac and generalized complex structures) and stretching (for Dirac structures).

5.1 Reduction of branes in generalized complex geometry [33]

Consider the following setup in ordinary geometry: a manifold M and a submanifold C endowed with some integrable distribution $\mathcal{F}$ so that $\underline{C} := C/\mathcal{F}$ be smooth. Then we have a projection $pr : C \to \underline{C}$ which induces a vector bundle morphism $pr_* : TC \to T\underline{C}$. If M is endowed with some geometric structure, such as a symplectic 2-form ω, one can ask when ω induces a symplectic form on $\underline{C}$.

This happens for example when C is a coisotropic submanifold; a particular case is when there is a Lie group G acting hamiltonianly on M with moment map $\nu : M \to \mathfrak{g}^*$ and C is the zero level set of ν (Marsden-Weinstein reduction [20]).

In [33] we consider the geometry that arises when one replaces the tangent bundle TM with an exact Courant algebroid E over M. Unlike the tangent bundle case, knowing $\underline{C}$ does not automatically determine the exact Courant algebroid over it. We have to replace the foliation $\mathcal{F}$ by more data, namely a suitable subbundle K of $E|_C$ (projecting to $\mathcal{F}$ under the anchor map $\pi : E \to TM$).

In Theorem 3.7 we determine conditions on K that allow to construct by a quotienting procedure a Courant algebroid $\underline{E}$ on $\underline{C}$; $\underline{E}$ as a vector bundle is obtained identifying suitably the fibers of the bundle $K^\perp/K \to C$. Our construction is quite general, in that when E is the standard Courant algebroid on M one might end up with a quotient $\underline{E}$ which is not isomorphic to the standard Courant algebroid of $\underline{C}$. Our construction follows closely the one of Bursztyn-Cavalcanti-Gualtieri [2], in which a suitable group action on E provides an identification between fibers of E at different points; in our case we don't assume any group action, and we make up for this asking that there exist enough " K-invariant sections of $K^\perp$ ". We also describe how a submanifold C with a foliation $\mathcal{F}$, once equipped with a suitable maximal isotropic subbundle L of $E|_C$, naturally has a reduced Courant algebroid over its leaf-space $\underline{C}$ (Prop. 3.14). Further, we describe in a simple way which splittings of E induce 3-forms on M (representing the Ševera class of E) which descend to 3-forms on $\underline{C}$ (representing the Ševera class of $\underline{E}$).

Once we know how to reduce an exact Courant algebroid, we can ask when Dirac structures and generalized complex structures descend to the quotient Courant algebroid. We give sufficient conditions, one of which is an invariance condition w.r.t. the subbundle K.

The heart of [33] is Section 6, where we identify the objects that *automatically* satisfy the assumptions needed to perform generalized complex reduction. When $(M, \mathcal{J})$ is a generalized complex manifold we consider pairs consisting of a submanifold C of M and suitable maximal isotropic subbundle L of $E|_C$ (we call them "weak branes"). Weak branes turn out to always be coisotropic submanifolds w.r.t. Poisson structure on M induced by $\mathcal{J}$. We show in Prop. 6.10 that weak branes admit a canonical quotient $\underline{C}$ (indeed, the quotient by its characteristic distribution) which is endowed with an exact Courant algebroid and a generalized complex structure; this construction is inspired by Thm. 2.1 of Vaisman's work [24] in the setting of the standard Courant algebroid. We also show (Prop. 6.16), using the coisotropic embedding theorem of [10], that pairs (C, L) which are not quite weak branes can be regarded as weak branes of some cosymplectic submanifold of M.

Particular cases of weak branes are generalized complex submanifolds (C, L) (also known as "branes"), first introduced by Gualtieri [15], which are defined asking that the maximal isotropic subbundle $L \subset E|_C$ be closed under the Courant bracket and preserved by $\mathcal{J}$. Using our reduction of Dirac structures we show in Thm. 6.4 that the quotients $\underline{C}$ of branes, which by the above are generalized complex manifolds, are also endowed with the structure of a space-filling brane (i.e. $\underline{C}$ together with the reduction of L forms a brane). This is interesting also because space filling branes induce an honest complex structure on the underlying manifold [14]; hence our reduction of branes could be used to construct new examples of complex manifolds.

5.2 Reduction of Dirac structures along isotropic subbundles [3]

Given a Dirac structure L in an exact Courant algebroid E over M, we introduce a way to "deform" L by means of an isotropic subbundle $K \subset E$ and obtain a maximally isotropic subbundle L^K. Explicitly, the *stretching* L^K is defined as $(L \cap K^{\perp}) + K$; L^K turns out the be the maximally isotropic subbundle "closest" to L among all those containing K. The way that we like to interpret this construction is as follows: if the assumptions of our theorems on reduction of Courant algebroids and Dirac structures (Thm. 3.7 and Prop. 4.1 of [33]) were satisfied, we would obtain a reduced Dirac structure $\underline{L}$ on a quotient $\underline{M}$ of M; then L^K would be the pullback of this Dirac structure to M.

The sections of L^K are usually not closed under the Courant bracket, hence L^K is usually not a Dirac structure. However in Thm. 3.2 of [3] we show that the sections $e \subset L^K$ which are K-invariant (in the sense that $[\Gamma(K), e] \subset \Gamma(K)$) are closed under the Courant bracket; this is consistent with the above interpretation in terms of reduced structures. Along the way we also show that K is a symmetry for L^K (in the sense that $[\Gamma(K), \Gamma(L^K)] \subset \Gamma(L^K)$) iff L^K is spanned at every point by K-invariant sections.

One of the original motivations for [3] was to generalize for any Dirac structure the Marsden-Ratiu reduction of Poisson manifolds [21]. In [3] we could describe the Marsden-Ratiu reduction in terms of pushforwards of stretched Dirac structures, but we needed to make assumptions which differ from those of [21]. We believe it is possible to improve the Marsden-Ratiu reduction theorem by applying in a more suitable way the stretching construction.

6 Contact reduction and groupoid actions [35]

Willett [32] and Albert [1] independently developed reduction procedures for contact manifolds: given a Hamiltonian action of a Lie group G on a contact manifold (M, θ_M) with moment map $J : M \to \mathfrak{g}^*$ (the dual of the Lie algebra of G), they showed that the quotient of suitable preimages of J by certain subgroups of G are again contact manifolds. However neither method is as natural as the classical Marsden-Weinstein reduction: the contact structure of Albert's reduction depends on the choice of the contact 1-form; Willett's requires additional conditions on the reduction points.

In [35] we interpret these reductions making use of the fact that unit spheres in $\mathfrak{g}^*$ are Jacobi manifolds (see Section 2 above), and that to a Jacobi manifold one can associate canonically a *contact groupoid*, i.e. a Lie groupoid with a compatible contact structure.

Using contact groupoids we are able to perform reduction. In the set-up described above this allows us to see (Thm. 5.4) that if G is compact then Willett's reduced spaces are prequantizations of our reduced spaces (which are symplectic manifolds). This explains Willett's conditions on the reduction points as an integrality condition.

Now we outline our reduction procedure via groupoids. In analogy to the well-known fact in symplectic geometry that the moment map allows one to reconstruct the corresponding Hamiltonian action, we show the following in Theorem 3.8: any complete Jacobi map J which is a surjective submersion from a contact manifold (M, θ_M) to a Jacobi manifold Γ_0 naturally induces an action on M of the contact groupoid Γ of Γ_0. Our main result on reduction is Theorem 4.1: if the contact groupoid Γ acts on (M, θ_M) and $x \in \Gamma_0$ satisfies mild assumptions then, denoting by $\Gamma_x \subset \Gamma$ a certain Lie group, the reduced space $M_x := J^{-1}(x)/\Gamma_x$ has an induced

1. contact structure, if x belongs to a contact leaf
2. conformal locally conformal symplectic structure, if x belongs to a locally conformal symplectic leaf.

This is the point-wise version of a result about global reduction: the quotient of a contact manifold by the action of a contact groupoid is naturally a Jacobi manifold, the leaves of which are the above reduced spaces M_x (therefore not necessarily contact). This shows that performing any natural reduction procedure on a contact manifold one should not expect to obtain contact manifolds in general. Notice that combining the two results above we are able to obtain contact manifolds by reduction starting with a simple piece of data, namely a complete Jacobi map, without even mentioning groupoids.

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Variations on Prequantization

Alan Weinstein and Marco Zambon

Abstract

We extend known prequantization procedures for Poisson and presymplectic manifolds by defining the prequantization of a Dirac manifold P as a principal $U(1)$-bundle Q with a compatible Dirac-Jacobi structure. We study the action of Poisson algebras of admissible functions on P on various spaces of locally (with respect to P) defined functions on Q, via hamiltonian vector fields. Finally, guided by examples arising in complex analysis and contact geometry, we propose an extension of the notion of prequantization in which the action of $U(1)$ on Q is permitted to have some fixed points.

Dedicated to the memory of Professor Shiing-Shen Chern

Contents

1 Introduction

Prequantization in symplectic geometry attaches to a symplectic manifold P a hermitian line bundle K (or the corresponding principal $U(1)$-bundle Q), with a connection whose curvature form is the symplectic structure. The Poisson Lie algebra $C^\infty(P)$ then acts faithfully on the space $\Gamma(K)$ of sections of K (or antiequivariant functions on Q). Imposing a polarization Π cuts down $\Gamma(K)$ to a smaller, more "physically appropriate" space $\Gamma_\Pi(K)$ on which a subalgebra of $C^\infty(P)$ may still act. By polarizing and looking at the "ladder" of sections of tensor powers $K^{\otimes n}$ (or functions on Q transforming according to all the negative tensor powers of the standard representation of $U(1)$), one gets an "asymptotic representation" of the full algebra $C^\infty(P)$. All of this often goes under the name of geometric quantization, with the last step closely related to deformation quantization.

For systems with constraints or systems with symmetry, the phase space P may be a presymplectic or Poisson manifold. Prequantization, and sometimes the full procedure of geometric quantization, has been carried out in these settings by several authors; their work is cited below.

The principal aim of this paper is to suggest two extensions of the prequantization construction which originally arose in an example coming from contact geometry. The first, which unifies the presymplectic and Poisson cases and thus permits the simultaneous application of constraints and symmetry, is to allow P to be a Dirac manifold. The second is to allow the $U(1)$ action on Q to have fixed points when P has a boundary, so that the antiequivariant functions become sections of a sheaf rather than a line bundle over P. In the course of the paper, we also make some new observations concerning the Poisson and presymplectic cases.

1.1 Symplectic prequantization

On a symplectic manifold (P, ω), one defines the hamiltonian vector field X_f of the function f by $\omega(X_f, \cdot) = df$, and one has the Lie algebra bracket $\{f, g\} = \omega(X_f, X_g)$ on $C^\infty(P)$. A closed 2-form ω is called integral if its de Rham cohomology class $[\omega] \in H^2(M, \mathbb{R})$ is integral, i.e. if it is in the image of the homomorphism $i_* : H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R})$ associated with the inclusion $i : \mathbb{Z} \to \mathbb{R}$ of coefficient groups.

When ω is integral, following Kostant [21], we prequantize (P, ω) by choosing a hermitian line bundle K bundle over P with first Chern class in $i_*^{-1}[\omega]$. Then there is a connection ∇ on K with curvature $2\pi i\omega$. Associating to each function f the operator $\hat{f}$ on $\Gamma(K)$ defined by[1] $\hat{f}(s) = -[\nabla_{X_f} s + 2\pi i f s]$, we obtain a faithful Lie algebra representation of $C^\infty(P)$ on $\Gamma(K)$.

The construction above is equivalent to the following, due to Souriau [27]: let Q be the principal $U(1)$-bundle associated to K. Denote by σ the connection form on Q cor-

[1] Our convention for the Poisson bracket differs by a sign from that of [15] and [21]; consequently our formula for $\hat{f}$ and Equation (1.1) below differ by a sign too. Our sign has the property that the map from functions to their hamiltonian vector fields is an antihomomorphism from Poisson brackets to Lie brackets.

responding to ∇ (so $d\sigma = \pi^*\omega$, where $\pi : Q \to P$), and by E the infinitesimal generator of the $U(1)$ action on Q. We can identify the sections of K with functions $\bar{s} : Q \to \mathbb{C}$ which are $U(1)$-antiequivariant (i.e. $\bar{s}(x \cdot t) = \bar{s}(x) \cdot t^{-1}$ for $x \in Q$, $t \in U(1)$, or equivalently $E(\bar{s}) = -2\pi i \bar{s}$), and then the operator $\hat{f}$ on $\Gamma(K)$ corresponds to the action of the vector field

$$-X_f^H + \pi^* f E, \tag{1.1}$$

where the superscript ^{H} denotes the horizontal lift to Q of a vector field on P. Notice that σ is a contact form on Q and that $X_f^H - \pi^* f E$ is just the hamiltonian vector field of $\pi^* f$ with respect to this contact form (viewed as a Jacobi structure; see Section 3).

1.2 Presymplectic prequantization

Prequantization of a presymplectic manifolds (P, ω) for which ω is integral and of constant rank[2] was introduced by Günther [15] (see also Gotay and Sniatycki [12] and Vaisman [33]). Günther represents the Lie algebra of functions constant along the leaves of $\ker \omega$ by assigning to each such function f the equivalence class of vector fields on Q given by formula (1.1), where X_f now stands for the equivalence class of vector fields satisfying $\omega(X_f, \cdot) = df$.

1.3 Poisson prequantization

Prequantization of Poisson manifolds (P, Λ) was first investigated algebraically by Huebschmann [16], in terms of line bundles by Vaisman [31], and then in terms of circle bundles by Chinea, Marrero, and de Leon [5]. When the Poisson cohomology class $[\Lambda] \in H^2_\Lambda(P)$ is the image of an integral de Rham class $[\Omega]$ under the map given by contraction with Λ, a $U(1)$-bundle Q with first Chern class in $i_*^{-1}[\Omega]$ may be given a Jacobi structure for which the map that assigns to $f \in C^\infty(P)$ the hamiltonian vector field (with respect to the Jacobi structure) of $-\pi^* f$ is a Lie algebra homomorphism. This gives a (not always faithful) representation of $C^\infty(P)$.

1.4 Dirac prequantization

We will unite the results in the previous two paragraphs by using Dirac manifolds. These were introduced by Courant [6] and include both Poisson and presymplectic manifolds as special cases. On the other hand, Jacobi manifolds had already been introduced by Kirillov [20] and Lichnerowicz [24], including Poisson, conformally symplectic, and contact manifolds as special cases. All of these generalizations of Poisson structures were encompassed in the definition by Wade [34] of Dirac-Jacobi[3] manifolds.

To prequantize a Dirac manifold P, we will impose an integrality condition on P which implies the existence of a $U(1)$-bundle $\pi : Q \to P$ with a connection which will be used to construct a Dirac-Jacobi structure on Q. Prequantization of (suitable) functions $g \in C^\infty(P)$

[2]Unlike many other authors (including some of those cited here), we will use the work "presymplectic" to describe any manifold endowed with a closed 2-form, even if the form does not have constant rank.

[3]Wade actually calls them $\mathcal{E}^1(M)$-Dirac manifolds; we will stick to the terminology "Dirac-Jacobi", as introduced in [13].

is achieved "Kostant-style" by associating to g the equivalence class of the hamiltonian vector fields of $-\pi^* g$ and by letting this equivalence class act on a suitable subset of the $U(1)$-antiequivariant functions on Q, or equivalently by letting $\pi^* g$ act by the bracket of functions on Q. The same prequantization representation can be realized as an action on sections of a hermitian line bundle over P with an L-connection, where L is the Lie algebroid given by the Dirac manifold P.

We also look at the following very natural example, discovered by Claude LeBrun. Given a contact manifold M with contact distribution $C \subset TM$, the nonzero part of its annihilator C° is a symplectic submanifold of T^*M. When the contact structure is cooriented, we may choose the positive half C°_+ of this submanifold. By adjoining to C°_+ the "the section at infinity of T^*M" we obtain a manifold with boundary, on which the symplectic structure on C°_+ extends to give a Poisson structure. We call this a "LeBrun-Poisson manifold". If now we additionally adjoin the zero section of T^*M we obtain a Dirac manifold P.

First we will describe the prequantization $U(1)$-bundle of P, then we will modify it by collapsing to points the fibers over one of the two boundary components and by applying a conformal change. At the end, restricting this construction to the LeBrun-Poisson manifold (which sits as an open set inside P), we will obtain a contact manifold in which M sits as a contact submanifold.

1.5 Organization of the paper

In Sections 2 and 3 we collect known facts about Dirac and Dirac-Jacobi manifolds. In Section 4 we state our prequantization condition and describe the Dirac-Jacobi structure on the prequantization space of a Dirac manifold. In Section 5 we study the corresponding prequantization representation, and in Section 6 we derive the same representation by considering hermitian line bundles endowed with L-connections. In Section 7 we study the prequantization of LeBrun's examples, and in Section 8 we allow prequantization $U(1)$-bundles to have fixed points, and we endow them with contact structures. We conclude with some remarks in Section 9.

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2 Dirac manifolds

We start by recalling some facts from [6].

Definition 2.1 ([6], Def 1.1.1)**.** A **Dirac structure** on a vector space V is a maximal isotropic subspace $L \subset V \oplus V^*$ with respect to the symmetric pairing

$$\langle X_1 \oplus \xi_1, X_2 \oplus \xi_2 \rangle_+ = \frac{1}{2}(i_{X_2}\xi_1 + i_{X_1}\xi_2). \tag{2.1}$$

L necessarily has the same dimension as V, and denoting by ρ_V and ρ_{V^*} the projections of $V \oplus V^*$ onto V and V^* respectively, we have

$$\rho_V(L) = (L \cap V^*)^\circ \text{ and } \rho_{V^*}(L) = (L \cap V)^\circ \tag{2.2}$$

where the symbol $^\circ$ denotes the annihilator. It follows that L induces (and is equivalent to) a skew bilinear form on $\rho_V(L)$ or a bivector on $V/L \cap V$ ([6], Prop. 1.1.4). If (V, L) is a Dirac vector space and $i : W \to V$ a linear map, then one obtains a pullback Dirac structure on W by $\{Y \oplus i^*\xi : iY \oplus \xi \in L\}$; one calls a map between Dirac vector spaces "backward Dirac map" if it pulls back the Dirac structure of the target vector space to the one on the source vector space [3]. Similarly, given a linear map $p : V \to Z$, one obtains a pushforward Dirac structure on Z by $\{pX \oplus \xi : X \oplus p^*\xi \in L\}$, and one thus has a notion of "forward Dirac map" as well.

On a manifold M, a maximal isotropic subbundle $L \subset TM \oplus T^*M$ is called an **almost Dirac structure** on M. The appropriate integrability condition was discovered by Courant ([6], Def. 2.3.1):

Definition 2.2. A **Dirac structure** on M is an almost Dirac structure L on M whose space of sections is closed under the Courant bracket on sections of $TM \oplus T^*M$, which is defined by

$$[X_1 \oplus \xi_1, X_2 \oplus \xi_2] = \big([X_1, X_2] \;\oplus\; \mathcal{L}_{X_1}\xi_2 - \mathcal{L}_{X_2}\xi_1 + \frac{1}{2}d(i_{X_2}\xi_1 - i_{X_1}\xi_2)\big). \tag{2.3}$$

When an almost Dirac structure L is integrable, $(L, \rho_{TM}|_L, [\cdot,\cdot])$ is a Lie algebroid[4] ([6], Thm. 2.3.4). The singular distribution $\rho_{TM}(L)$ is then integrable in the sense of Stefan and Sussmann [28] and gives rise to a singular foliation of M. The Dirac structure induces a closed 2-form (presymplectic form) on each leaf of this foliation ([6], Thm. 2.3.6). The distribution $L \cap V$, called the **characteristic distribution**, is singular in a different way. Its annihilator $\rho_{T^*M}(L)$ is closed in the cotangent bundle, but the distribution itself is not closed unless it has constant rank. It is not always integrable, either. (See Example 2.1 and the beginning of Section 7.)

Next we define hamiltonian vector fields and put a Lie algebra structure on a subspace of $C^\infty(M)$.

Definition 2.3. A function f on a Dirac manifold (M, L) is **admissible** if there exists a smooth vector field X_f such that $X_f \oplus df$ is a section of L. A vector field X_f as above is called a hamiltonian vector field of f. The set of admissible functions forms a subspace $C^\infty_{adm}(M)$ of $C^\infty(M)$.

If f is admissible then $df|_{L\cap TM} = 0$. The converse holds where the characteristic distribution $L \cap TM$ has constant rank, but not in general. In other words, df can be contained in $\rho_{T^*M}(L)$ without being the image of a smooth section of L; see Example 2.1. Since any two hamiltonian vector fields of an admissible function f differ by a characteristic vector field, which annihilates any other admissible function, we can make the following definition.

[4]Recall that a Lie algebroid is a vector bundle A over a manifold M together with a Lie bracket $[\cdot,\cdot]$ on its space of sections and a bundle map $\rho : A \to TM$ (the "anchor") such that the Leibniz rule $[s_1, fs_2] = \rho s_1(f) \cdot s_2 + f \cdot [s_1, s_2]$ is satisfied for all sections s_1, s_2 of A and functions f on M.

Definition 2.4. The bracket on $C^\infty_{adm}(M)$ is given by $\{f,g\} = X_g \cdot f$.

This bracket differs by a sign from the one in the original paper of Courant [6], but it allows us to recover the usual conventions for presymplectic and Poisson manifolds, as shown below. The main feature of this bracket is the following (see [6], Prop. 2.5.3):

Proposition 2.1. *Let (M, L) be a Dirac manifold. If X_f and X_g are any hamiltonian vector fields for the admissible functions f and g, then $-[X_f, X_g]$ is a hamiltonian vector field for $\{f,g\}$, which is therefore admissible as well. The integrability of L implies that the bracket satisfies the Jacobi identity, so $(C^\infty_{adm}(M), \{\cdot,\cdot\})$ is a Lie algebra.*

We remark that the above can be partially extended to the space $C^\infty_{bas}(M)$ of **basic** functions, i.e. of functions ϕ satisfying $d\phi|_{L\cap TM} = 0$, which contains the admissible functions. (This two spaces of functions coincide when $L \cap TM$ is regular). Indeed, if h is admissible and ϕ is basic, then $\{\phi, h\} := X_h \cdot \phi$ is well defined and basic, since the flow of a hamiltonian vector field X_h induces vector bundle automorphisms of $TM \oplus T^*M$ that preserve $L\cap TM$ (see Section 2.4 in [6]). If f is an admissible function, then the Jacobiator of $f, h,$ and ϕ vanishes (adapt the proof of Prop. 2.5.3 in [6]).

We recall how manifolds endowed with 2-forms or bivectors fit into the framework of Dirac geometry. Let ω be a 2-form on M, $\tilde{\omega} : TM \to T^*M$ the bundle map $X \mapsto \omega(X, \cdot)$. Its graph $L = \{X \oplus \tilde{\omega}(X) : X \in TM\}$ is an almost Dirac structure; it is integrable iff ω is closed. If ω is symplectic, i.e. nondegenerate, then every function f is admissible and has a unique hamiltonian vector field satisfying $\tilde{\omega}(X_f) = df$; the bracket is given by $\{f,g\} = \omega(X_f, X_g)$.

Example 2.1. Let ω be the presymplectic form $x_1^2 dx_1 \wedge dx_2$ on $M = \mathbb{R}^2$, and let L be its graph. The characteristic distribution $L \cap TM$ has rank zero everywhere except along $\{x_1 = 0\}$, where it has rank two, and it is clearly not integrable (compare the discussion following Definition 2.2). The differential of $f = x_1^2$ takes all its values in the range of ρ_{T^*M}, but f is not admissible. This illustrates the remark following Definition 2.3, i.e. it provides an example of a function which is basic but not admissible.

Let Λ be a bivector field on M, $\tilde{\Lambda} : T^*M \to TM$ the corresponding bundle map $\xi \mapsto \Lambda(\cdot, \xi)$. (Note that the argument ξ is in the second position.) Its graph $L = \{\tilde{\Lambda}(\xi) \oplus \xi : \xi \in T^*M\}$ is an almost Dirac structure which is integrable iff Λ is a Poisson bivector (i.e. the Schouten bracket $[\Lambda, \Lambda]_S$ is zero). Every function f is admissible with a unique hamiltonian vector field $X_f = \{\cdot, f\}$, and the bracket of functions is $\{f,g\} = \Lambda(df, dg)$.

3 Dirac-Jacobi manifolds

Dirac-Jacobi structures were introduced by Wade [34] (under a different name) and include Jacobi (in particular, contact) and Dirac structures as special cases. Like Dirac structures, they are defined as maximal isotropic subbundles of a certain vector bundle.

Definition 3.1. A Dirac-Jacobi structure on a vector space V is a subspace $\bar{L} \subset (V \times \mathbb{R}) \oplus (V^* \times \mathbb{R})$ which is maximal isotropic under the symmetric pairing

$$\big\langle (X_1, f_1) \oplus (\xi_1, g_1)\ ,\ (X_2, f_2) \oplus (\xi_2, g_2) \big\rangle_+ = \frac{1}{2}(i_{X_2}\xi_1 + i_{X_1}\xi_2 + g_1 f_2 + g_2 f_1). \tag{3.1}$$

A Dirac-Jacobi structure on V necessarily satisfies $\dim \bar{L} = \dim V + 1$. Furthermore, Equations (2.2) hold for Dirac-Jacobi structures too:

$$\rho_V(\bar{L}) = (\bar{L} \cap V^*)^\circ \text{ and } \rho_{V^*}(\bar{L}) = (\bar{L} \cap V)^\circ. \tag{3.2}$$

As in the Dirac case, one has notions of pushforward and pullback structures and as well as forward and backward maps. For example, given a Dirac-Jacobi structure $\bar{L}$ on V and a linear map $p : V \to Z$, one obtains a pushforward Dirac-Jacobi structure on Z by $\{(pX, f) \oplus (\xi, g) : (X, f) \oplus (p^*\xi, g) \in \bar{L}\}$.

On a manifold M, a maximal isotropic subbundle $\bar{L} \subset \mathcal{E}^1(M) := (TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R})$ is called an **almost Dirac-Jacobi structure** on M.

Definition 3.2 ([34], Def. 3.2)**.** A **Dirac-Jacobi structure** on a manifold M is an almost Dirac-Jacobi structure $\bar{L}$ on M whose space of sections is closed under the extended Courant bracket on sections of $\mathcal{E}^1(M)$, which is defined by

$$\begin{aligned}
[(X_1, f_1) \oplus (\xi_1, g_1)\ ,\ (X_2, f_2) \oplus (\xi_2, g_2)] = \big(&[X_1, X_2], X_1 \cdot f_2 - X_2 \cdot f_1\big) \\
&\oplus \big(\mathcal{L}_{X_1}\xi_2 - \mathcal{L}_{X_2}\xi_1 + \frac{1}{2} d(i_{X_2}\xi_1 - i_{X_1}\xi_2) \\
&+ f_1\xi_2 - f_2\xi_1 + \frac{1}{2}(g_2 df_1 - g_1 df_2 - f_1 dg_2 + f_2 dg_1), \\
&X_1 \cdot g_2 - X_2 \cdot g_1 + \frac{1}{2}(i_{X_2}\xi_1 - i_{X_1}\xi_2 - f_2 g_1 + f_1 g_2)\big).
\end{aligned} \tag{3.3}$$

By a straightforward computation (see also Section 4 of [13]) this bracket can be derived from the Courant bracket (2.3), as follows. Denote by U the embedding $\Gamma(\mathcal{E}^1(M)) \to \Gamma(T(M \times \mathbb{R}) \oplus T^*(M \times \mathbb{R}))$ given by

$$(X, f) \oplus (\xi, g) \mapsto (X + f\frac{\partial}{\partial t}) \oplus e^t(\xi + g dt),$$

where t is the coordinate on the $\mathbb{R}$ factor of the manifold $M \times \mathbb{R}$. Then U is a bracket-preserving map from $\Gamma(\mathcal{E}^1(M))$ with the extended bracket (3.3), to $\Gamma(T(M\times\mathbb{R})\oplus T^*(M\times\mathbb{R}))$ with the Courant bracket (2.3) of the manifold $M \times \mathbb{R}$.

Furthermore in Section 5 of [17] it is shown that any Dirac-Jacobi manifold $(M, \bar{L})$ gives rise to a Dirac structure on $M \times \mathbb{R}$ given by

$$\tilde{\bar{L}}_{(x,t)} = \{(X + f\frac{\partial}{\partial t})\ \oplus\ e^t(\xi + g dt) : (X, f) \oplus (\xi, g) \in \bar{L}_x\},$$

where t is the coordinate on $\mathbb{R}$. This procedure extends the well known symplectization of contact manifolds and Poissonization of Jacobi manifolds, and may be called "Diracization".

If $\bar{L}$ is a Dirac-Jacobi structure, $(\bar{L}, \rho_{TM}, [\cdot, \cdot])$ is a Lie algebroid ([34], Thm. 3.4), and each leaf of the induced foliation on M has the structure of a precontact manifold (i.e. simply a 1-form) or of a locally conformal presymplectic manifold (i.e. a 2-form Ω and a closed 1-form ω satisfying $d\Omega = \omega \wedge \Omega$). See Section 4.1 for a description of the induced foliation. As in the Dirac case, one can define hamiltonian vector fields and endow a subset of $C^\infty(M)$ with a Lie algebra structure.

Definition 3.3 ([34], Def. 5.1)**.** A function f on a Dirac-Jacobi manifold $(M, \bar{L})$ is **admissible** if there exists a smooth vector field X_f and a smooth function φ_f such that $(X_f, \varphi_f) \oplus (df, f)$ is a section of $\bar{L}$. Pairs (X_f, φ_f) as above are unique up to smooth sections of $\bar{L} \cap (TM \times \mathbb{R})$, and X_f is called a hamiltonian vector field of f. The set of admissible functions is denoted by $C^\infty_{adm}(M)$.

Definition 3.4. The bracket on $C^\infty_{adm}(M)$ is given by $\{f, g\} = X_g \cdot f + f\varphi_g$

This bracket, which differs by a sign from that in [34], enjoys the same properties stated in Proposition 2.1 for Dirac manifolds (see [34], Prop. 5.2 and Lemma 5.3).

Proposition 3.1. *Let $(M, \bar{L})$ be a Dirac-Jacobi manifold. If f and g are admissible functions, then*

$$\begin{aligned} &[(X_f, \varphi_f) \oplus (df, f)\ ,\ (X_g, \varphi_g) \oplus (dg, g)] = \\ &([X_f, X_g], X_f \cdot \varphi_g - X_g \cdot \varphi_f) \oplus (-d\{f, g\}, -\{f, g\}), \end{aligned} \tag{3.4}$$

hence $\{f, g\}$ is again admissible. The integrability of $\bar{L}$ implies that the admissible functions form a Lie algebra.

We call a function ψ on M **basic** if $X \cdot \psi + \psi f = 0$ for all elements $(X, f) \in \bar{L} \cap (TM \times \mathbb{R})$. This is equivalent to requiring $(d\psi, \psi) \in \rho_{T^*M \times \mathbb{R}}(\bar{L})$ at each point of M, so the basic functions contain the admissible ones. As in the case of Dirac structures, we have the following properties:

Lemma 3.1. *If ψ is a basic and h an admissible function, then the bracket $\{\psi, h\} := X_h \cdot \psi + \psi h$ is well-defined and again basic.*

Proof. It is clear that the bracket is well defined. To show that $X_h \cdot \psi + \psi h$ is again basic we reduce the problem to the Dirac case. Let $(X, f) \in \bar{L}_x \cap (TM \times \mathbb{R})$ Fix a choice of (X_h, φ_h) for the admissible function h. The vector field $X_h + \varphi_h \frac{\partial}{\partial t}$ on the Diracization $(M \times \mathbb{R}, \tilde{\bar{L}})$ (which is just a Hamiltonian vector field of $e^t h$) has a flow $\tilde{\phi}_\epsilon$, which projects to the flow ϕ_ϵ of X_h under $pr_1 : M \times \mathbb{R} \to M$. For each ϵ the flow $\tilde{\phi}_\epsilon$ induces a vector bundle automorphism Φ_ϵ of $\mathcal{E}^1(M)$, covering the diffeomorphism ϕ_ϵ of M, as follows:

$$(X, f) \oplus (\xi, g) \in \mathcal{E}^1_x(M) \mapsto (\tilde{\phi}_\epsilon)_*(X \oplus f\frac{\partial}{\partial t})_{(x,0)} \oplus (\tilde{\phi}_\epsilon^{-1})^*(\xi + gdt)_{(x,0)} \cdot e^{-pr_2(\tilde{\phi}_\epsilon(x,0))},$$

where we identify $T_{\tilde{\phi}_\epsilon(x,0)}(M \times \mathbb{R}) \oplus T^*_{\tilde{\phi}_\epsilon(x,0)}(M \times \mathbb{R})$ with $\mathcal{E}^1_{\phi_\epsilon(x)}(M)$ to make sense of the second term. Since the vector bundle maps induced by the flow $\tilde{\phi}_\epsilon$ preserve the Dirac structure $\tilde{\bar{L}}$ (see Section 2.4 in [6]), using the definition of the Diracization $\tilde{\bar{L}}$ one sees that Φ_ϵ preserves $\bar{L}$, and therefore also $\bar{L} \cap (TM \times \mathbb{R})$. Notice that we can pull back sections of $\mathcal{E}^1(M)$ by setting $(\Phi_\epsilon^*((X, f) \oplus (\xi, g)))_x := \Phi_\epsilon^{-1}((X, f) \oplus (\xi, g))_{\phi_\epsilon(x)}$. A computation shows that

$$(0, 0) \oplus (d(X_h \cdot \psi + \varphi_h\psi)\ ,\ X_h \cdot \psi + \varphi_h\psi) = \frac{\partial}{\partial\epsilon}\Big|_0 \Phi_\epsilon^*((0, 0) \oplus (d\psi, \psi)),$$

so that

$$\begin{aligned} &\langle (0, 0) \oplus (d(X_h \cdot \psi + \varphi_h\psi), X_h \cdot \psi + \varphi_h\psi)\ ,\ (X, f) \oplus (0, 0)\rangle_+ = \\ &\frac{\partial}{\partial\epsilon}\Big|_0 \big[\langle (0, 0) \oplus (d\psi, \psi)_{\phi_\epsilon(x)}\ ,\ \Phi_\epsilon((X, f) \oplus (0, 0))_x\rangle_+ e^{pr_2(\tilde{\phi}_\epsilon(x,0))}\big] = 0, \end{aligned}$$

as was to be shown. $\square$

Furthermore, the Jacobiator of admissible functions f, h and a basic function ψ is zero. One can indeed check that Wade's proof of the Jacobi identity for admissible functions ([34] Prop. 5.2) applies in this case too. Alternatively, this follows from the analogous statement for the Diracization $M \times \mathbb{R}$, since the map

$$C^{\infty}_{adm}(M) \to C^{\infty}_{adm}(M \times \mathbb{R}) \ , \ g \mapsto e^t g \tag{3.5}$$

is a well-defined Lie algebra homomorphism[5] and maps basic functions to basic functions.

Now we display some examples of Dirac-Jacobi manifolds.

There is a one-to-one correspondence between Dirac structures on M and Dirac-Jacobi structures on M contained in $TM \oplus (T^*M \times \mathbb{R})$: to each Dirac structure L one associates the Dirac-Jacobi structure $\{(X,0) \oplus (\xi, g) : X \oplus \xi \in L, g \in \mathbb{R}\}$ ([34], Remark 3.1).

A Jacobi structure on a manifold M is given by a bivector field Λ and a vector field E satisfying the Schouten bracket conditions $[E, \Lambda]_S = 0$ and $[\Lambda, \Lambda]_S = 2E \wedge \Lambda$. When $E = 0$, the Jacobi structure is a Poisson structure. Any skew-symmetric vector bundle morphism $T^*M \times \mathbb{R} \to TM \times \mathbb{R}$ is of the form $\left(\begin{smallmatrix} \tilde{\Lambda} & -E \\ E & 0 \end{smallmatrix}\right)$ for a bivector field Λ and a vector field E, where as in Section 2 we have $\tilde{\Lambda}\xi = \Lambda(\cdot, \xi)$. Graph $\left(\begin{smallmatrix} \tilde{\Lambda} & -E \\ E & 0 \end{smallmatrix}\right) \subset \mathcal{E}^1(M)$ is a Dirac-Jacobi structure iff (M, Λ, E) is a Jacobi manifold ([34], Sect 4.1). In this case all functions are admissible, the unique hamiltonian vector field of f is[6] $X_f = \tilde{\Lambda} df - fE$, $\varphi_f = E \cdot f$ and the bracket is given by $\{f, g\} = \Lambda(df, dg) + fE \cdot g - gE \cdot f$.

Similarly (see [34], Sect. 4.3), any skew-symmetric vector bundle morphism $TM \times \mathbb{R} \to T^*M \times \mathbb{R}$ is of the form $\left(\begin{smallmatrix} \tilde{\Omega} & \sigma \\ -\sigma & 0 \end{smallmatrix}\right)$ for a 2-form Ω and a 1-form σ, and graph $\left(\begin{smallmatrix} \tilde{\Omega} & \sigma \\ -\sigma & 0 \end{smallmatrix}\right) \subset \mathcal{E}^1(M)$ is a Dirac-Jacobi structure iff $\Omega = d\sigma$.

Any contact form σ on a manifold M defines a Jacobi structure (Λ, E) (where E is the Reeb vector field of σ and $\tilde{\Lambda}\tilde{d\sigma}|_{\ker \sigma} = \mathrm{Id}$; see for example [18], Sect. 2.2), and graph $\left(\begin{smallmatrix} \tilde{d\sigma} & \sigma \\ -\sigma & 0 \end{smallmatrix}\right)$ is equal to graph $\left(\begin{smallmatrix} \tilde{\Lambda} & -E \\ E & 0 \end{smallmatrix}\right)$. Further, by considering suitably defined graphs, one sees that locally conformal presymplectic structures and homogeneous Poisson manifolds (given by a Poisson bivector Λ and a vector field Z satisfying $\mathcal{L}_Z \Lambda = -\Lambda$) are examples of Dirac-Jacobi structures ([34], Sect. 4).

4 The prequantization spaces

In this section we determine the prequantization condition for a Dirac manifold (P, L), and we describe its "prequantization space" (i.e. the geometric object that allows us to find a representation of $C^{\infty}_{adm}(P)$).

We recall the prequantization of a Poisson manifold (P, Λ) by a $U(1)$-bundle as described in [5]. The bundle map $\tilde{\Lambda} : T^*P \to TP$ extends to a cochain map from forms to multivector fields, which descends to a map from de Rham cohomology $H^{\bullet}_{dR}(P, \mathbb{R})$ to Poisson cohomology $H^{\bullet}_{\Lambda}(P)$ (the latter having the set of p-vector fields as p-cochains). The

[5] For the well-definedness notice that, if $(X_g, \varphi_g) \oplus (dg, g) \in \Gamma(\bar{L})$, then $(X_g + \varphi_g \frac{\partial}{\partial t}) \oplus d(e^t g) \in \Gamma(\tilde{\bar{L}})$. Notice that in particular $X_g + \varphi_g \frac{\partial}{\partial t}$ is a hamiltonian vector field for $e^t g$. Using this, the equation $e^t\{f, g\}_M = \{e^t f, e^t g\}_{M \times \mathbb{R}}$ follows at once from the definitions of the respective brackets of functions.

[6] Again, this is opposite to the usual sign convention.

prequantization condition, first formulated in this form in [31], is that $[\Lambda] \in H^2_\Lambda(P)$ be the image under $\tilde{\Lambda}$ of an integral de Rham class, or equivalently that

$$\tilde{\Lambda}\Omega = \Lambda + \mathcal{L}_A \Lambda \tag{4.1}$$

for some integral closed 2-form Ω and vector field A on P. Assuming this prequantization condition to be satisfied, let $\pi : Q \to P$ be a $U(1)$-bundle with first Chern class $[\Omega]$, σ a connection on Q with curvature Ω (i.e. $d\sigma = \pi^*\Omega$), and E the generator of the $U(1)$-action (so that $\sigma(E) = 1$ and $\pi_* E = 0$). Then (see Thm. 3.1 in [5])

$$(\Lambda^H + E \wedge A^H, E) \tag{4.2}$$

is a Jacobi structure on Q which pushes down to $(\Lambda, 0)$ on P via π_*. (The superscript ^{H} denotes horizontal lift, with respect to the connection σ, of multivector fields on P.) We say that π is a Jacobi map.

It follows from the Jacobi map property of π that assigning to a function f on P the hamiltonian vector field of $-\pi^* f$, which is $-(\Lambda^H + E \wedge A^H)(\pi^* df) + (\pi^* f)E$, defines a Lie algebra homomorphism from $C^\infty(P)$ to the operators on $C^\infty(Q)$.

Now we carry out an analogous construction on a Dirac manifold (P, L). Recall that L is a Lie algebroid with the restricted Courant bracket and anchor $\rho_{TP} : L \to TP$ (which is just the projection onto the tangent component). This anchor gives a Lie algebra homomorphism from $\Gamma(L)$ to $\Gamma(TP)$ with the Lie bracket of vector fields. The pullback by the anchor therefore induces a map $\rho^*_{TP} : \Omega^\bullet_{dR}(P, \mathbb{R}) \to \Omega^\bullet_L(P)$, descending to a map from de Rham cohomology to the Lie algebroid cohomology $H^2_L(P)$. (We recall from [8] that $\Omega^\bullet_L(P)$ is the graded differential algebra of sections of the exterior algebra of L^*.) There is a distinguished class in $H^2_L(P)$: on $TP \oplus T^*P$, in addition to the natural symmetric pairing (2.1), there is also an anti-symmetric one given by

$$\langle X_1 \oplus \xi_1, X_2 \oplus \xi_2 \rangle_- = \frac{1}{2}(i_{X_2}\xi_1 - i_{X_1}\xi_2). \tag{4.3}$$

Its restriction Υ to L satisfies $d_L \Upsilon = 0$. Our prequantization condition is

$$[\Upsilon] \in \rho^*_{TP}(i_*(H^2(P, \mathbb{Z}))) \tag{4.4}$$

or equivalently

$$\rho^*_{TP}\Omega = \Upsilon + d_L \beta, \tag{4.5}$$

where Ω is a closed integral 2-form and β a 1-cochain for the Lie algebroid L, i.e. a section of L^*.

Remark 4.1. If L is the graph of a presymplectic form ω then $\Upsilon = \rho^*_{TP}(\omega)$. If L is graph$(\tilde{\Lambda})$ for a Poisson bivector Λ and Ω is a 2-form, then $\rho^*_{TP}[\Omega] = [\Upsilon]$ if and only if $\tilde{\Lambda}[\Omega] = [\Lambda]$.[7] $[\Omega]$ to $[\Upsilon]$. This shows that (4.4) generalizes the prequantization conditions for presymplectic and Poisson structures mentioned in the introduction and in formula (4.1).

[7]This is consistent with the fact that, if ω is symplectic, then graph$(\tilde{\omega})$ = graph$(\tilde{\Lambda})$, where the bivector Λ is defined so that the vector bundle maps $\tilde{\omega}$ and $\tilde{\Lambda}$ are inverses of each other (so if $\omega = dx \wedge dy$ on $\mathbb{R}^2$, then $\Lambda = \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$).

Remark 4.2. The prequantization condition above can not even be formulated for twisted Dirac structures. We recall the definition of these structures [25]. If ϕ is a closed 3-form on a manifold P, adding the term $\phi(X_1, X_2, \cdot)$ to the Courant bracket (i.e. to the right hand side of Equation (2.3)) determines a new bracket $[\cdot,\cdot]^{\phi}$ so that $TP \oplus T^*P$, together with this bracket, the original anchor ρ_{TP} and the symmetric pairing $\langle\cdot,\cdot\rangle_+$, form a Courant algebroid. A ϕ-twisted Dirac structure L is then a maximal isotropic subbundle which is closed under $[\cdot,\cdot]^{\phi}$; it is automatically a Lie algebroid (whose Lie algebroid differential we denote by d_L^{ϕ}). The orbits of the Lie algebroid carry 2-forms Ω_L given as in the remark following Definition 2.1, satisfying $d\Omega_L = j^*\phi$ where j is the inclusion of a leaf in P and d is the de Rham differential on the leaves. Since

$$d_L^{\phi}\Upsilon = d_L^{\phi}\rho_{TP}^*\Omega_L = \rho_{TP}^* d\Omega_L = \rho_{TP}^* j^*\phi,$$

we conclude that Υ is usually not d_L^{ϕ}-closed, so we cannot expect Ω to be closed in (4.5), and hence we cannot require that it be integral. The correct notion of prequantization should probably involve a gerbe.

Now, assuming the prequantization condition (4.4) and proceeding as in the Poisson case, let $\pi : Q \to P$ be a $U(1)$-bundle with connection form σ having curvature Ω; denote by E the infinitesimal generator of the $U(1)$-action.

Theorem 4.1. *The subbundle $\bar{L}$ of $\mathcal{E}^1(Q)$ given by the direct sum of*

$$\{(X^H + \langle X \oplus \xi, \beta\rangle E, 0) \oplus (\pi^*\xi, 0) : X \oplus \xi \in L\}$$

and the line bundles generated by $(-E, 0) \oplus (0, 1)$ and $(-A^H, 1) \oplus (\sigma - \pi^*\alpha, 0)$ is a Dirac-Jacobi structure on Q. Here, $A \oplus \alpha$ is an isotropic section of $TP \oplus T^*P$ satisfying $\beta = 2\langle A \oplus \alpha, \cdot\rangle_+|_L$. Such a section always exists, and the subbundle above is independent of the choice of $A \oplus \alpha$.

Proof. Let C be a maximal isotropic (with respect to $\langle\cdot,\cdot\rangle_+$) complement of L in $TP \oplus T^*P$. Such a complement always exists, since the space of complements at each point is contractible (an affine space modeled on a space of skew-symmetric forms). Now extend β to a functional $\tilde{\beta}$ on $TP \oplus T^*P$ by setting $\tilde{\beta}|_C = 0$. There exists a unique section $A \oplus \alpha$ of $TP \oplus T^*P$ satisfying $\beta = 2\langle A \oplus \alpha, \cdot\rangle_+$ since the symmetric pairing is non-degenerate. Since $\langle A \oplus \alpha, \cdot\rangle_+|_C = 0$ and C is maximal isotropic we conclude that $A \oplus \alpha$ belongs to C and is hence isotropic itself. This shows the existence of $A \oplus \alpha$ as above.

Now clearly $A \oplus \alpha + Y \oplus \eta$ satisfies the property stated in the theorem iff $Y \oplus \eta \subset L$, and in this case it is isotropic (i.e. $\langle A + Y, \alpha + \eta\rangle = 0$) iff $Y \oplus \eta \subset \ker \beta$. So a section $A \oplus \alpha$ as in the theorem is unique up to sections $Y \oplus \eta$ of $\ker \beta$. By inspection one sees that replacing $A \oplus \alpha$ by $A \oplus \alpha + Y \oplus \eta$ in the formula for $\bar{L}$ defines the same subbundle.

That $\bar{L}$ is isotropic with respect to the symmetric pairing on $\mathcal{E}^1(Q)$ follows from the fact that L is isotropic, together with the properties of $A \oplus \alpha$. $\bar{L}$ is clearly a subbundle of dimension $\dim P + 2$, so it is an almost Dirac-Jacobi structure.

To show that $\bar{L}$ is integrable, we use the fact that $\bar{L}$ is integrable if and only if $\langle [e_1, e_2], e_3\rangle_+ = 0$ for all sections e_i of $\bar{L}$ and that $\langle [\cdot,\cdot],\cdot\rangle_+$ is a totally skew-symmetric tensor if restricted to sections of $\bar{L}$, i.e. an element of $\Gamma(\wedge^3 \bar{L}^*)$ ([17], Prop. 2.2). Each section of $\bar{L}$ can be written as a $C^\infty(Q)$-linear combination of the following three types of sections of $\bar{L}$: $a =: (X^H + \langle X\oplus\xi, \beta\rangle E, 0)\oplus(\pi^*\xi, 0)$ where $X\oplus\xi \in \Gamma(L)$, $b := (-E, 0)\oplus(0, 1)$ and $c := (-A^H, 1) \oplus (\sigma - \pi^*\alpha, 0)$. We will use subscripts to label more than one section of a given type. It is immediate that brackets of the form $[a, b]$,$[b_1, b_2]$, and $[c_1, c_2]$ all vanish, and a computation shows that $\langle[a_1, a_2], a_3\rangle_+ = 0$ since $L \subset TP \oplus T^*P$ is a Dirac structure. Finally $\langle[a_1, a_2], c\rangle_+ = 0$ using $d\sigma = \pi^*\Omega$ and the prequantization condition (4.5), which when applied to sections $X_1 \oplus \xi_1$ and $X_2 \oplus \xi_2$ of L reads

$$\Omega(X_1, X_2) = \langle\xi_1, X_2\rangle + X_1\langle\beta, X_2 \oplus \xi_2\rangle - X_2\langle\beta, X_1 \oplus \xi_1\rangle - \langle\beta\ ,\ \left[X_1 \oplus \xi_1, X_2 \oplus \xi_2\right]\rangle.$$

By skew-symmetry, the vanishing of these expressions is enough to prove the integrability of $\bar{L}$. □

Remark 4.3. When (P, L) is a Poisson manifold, $\bar{L}$ is exactly the graph of the Jacobi structure (4.2), i.e. it generalizes the construction of [5]. If (P, L) is given by a presymplectic form Ω, then $\bar{L}$ is the graph of $(d\sigma, \sigma)$.

Remark 4.4. The construction of Theorem 4.1 also works for complex Dirac structures (i.e., integrable maximal isotropic complex subbundles of the complexified bundle $T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M$). It can be adapted to the setting of generalized complex structures [14] (complex Dirac structures which are transverse to their complex conjugate) and generalized contact structures [18] (complex Dirac-Jacobi structures which are transverse to their complex conjugate) as follows. If (P, L) is a generalized complex manifold, assume all of the previous notation and the following prequantization condition:

$$\rho^*_{TP}\Omega = i\Upsilon + d_L\beta, \tag{4.6}$$

where Ω is (the complexification of) a closed integer 2-form and β a 1-cochain for the Lie algebroid L. Then the direct sum of

$$\{(X^H + \langle X \oplus \xi, \beta\rangle E, 0) \oplus (\pi^*\xi, 0) : X \oplus \xi \in L\}$$

and the complex line bundles generated by $(-iE, 0) \oplus (0, 1)$ and $(-A^H, i) \oplus (\sigma - \pi^*\alpha, 0)$ is a generalized contact structure on Q, where $A \oplus \alpha$ is the unique section of the conjugate of L satisfying $\beta = 2\langle A \oplus \alpha, \cdot\rangle_+|_L$.

4.1 Leaves of the Dirac-Jacobi structure

Given any Dirac-Jacobi manifold $(M, \bar{L})$, each leaf of the foliation integrating the distribution $\rho_{TM}(\bar{L})$ carries one of two kinds of geometric structures [17], as we describe now. $\rho_1 : \bar{L} \to \mathbb{R}, (X, f) \oplus (\xi, g) \mapsto f$ determines an algebroid 1-cocycle, and a leaf $\bar{F}$ of the foliation will be of one kind or the other depending on whether $\ker\rho_1$ is contained in the kernel of the anchor ρ_{TM} or not. (This property is satisfied either at all points of $\bar{F}$ or at none). As with Dirac structures, the Dirac-Jacobi structure $\bar{L}$ determines a field of skew-symmetric bilinear forms $\Psi_{\bar{F}}$ on the image of $\rho_{TM} \times \rho_1$.

If $\ker\rho_1 \not\subset \ker\rho_{TM}$ on $\bar{F}$ then $\rho_{TM} \times \rho_1$ is surjective, hence $\Psi_{\bar{F}}$ determines a 2-form and a 1-form on $\bar{F}$. The former is the differential of the latter, so the leaf $\bar{F}$ is simply endowed with a 1-form , i.e. it is a precontact leaf. If $\ker\rho_1 \subset \ker\rho_{TM}$ on $\bar{F}$ then the image of $\rho_{TQ} \times \rho_1$ projects isomorphically onto $T\bar{F}$, which therefore carries a 2-form $\Omega_{\bar{F}}$. It turns out that $\omega_{\bar{F}}(Y) := -\rho_1(e)$, for any $e \in \bar{L}$ with $\rho_{TM}(e) = Y$, is a well-defined 1-form on $\bar{F}$, and that $(\bar{F}, \Omega_{\bar{F}}, \omega_{\bar{F}})$ is a locally conformal presymplectic manifold, i.e. $\omega_{\bar{F}}$ is closed and $d\Omega_{\bar{F}} = \Omega_{\bar{F}} \wedge \omega_{\bar{F}}$.

On our prequantization $(Q, \bar{L})$ the leaf $\bar{F}$ through $q \in Q$ will carry one or the other geometric structure depending on whether A is tangent to F, where F denotes the presymplectic leaf of (P, L) passing through $\pi(q)$. Indeed one can check that at q we have $\ker \rho_1 \not\subset \ker \rho_{TQ} \Leftrightarrow A \in T_{\pi(q)}F$. When $\ker \rho_1 \not\subset \ker \rho_{TQ}$ on a leaf $\bar{F}$ we hence deduce that $\bar{F}$, which is equal to $\pi^{-1}(F)$, is a precontact manifold, and a computation shows that the 1-form is given by the restriction of

$$\sigma + \pi^*(\xi_A - \alpha)$$

where ξ_A is any covector satisfying $A \oplus \xi_A \in L$.

A leaf $\bar{F}$ on which $\ker \rho_1 \subset \ker \rho_{TQ}$ is locally conformal presymplectic, and its image under π is an integral submanifold of the integrable distribution $\rho_{TP}(L) \oplus \mathbb{R}A$ (hence a one parameter family of presymplectic leaves). A computation shows that the locally conformal presymplectic structure is given by

$$(\omega_{\bar{F}}, \Omega_{\bar{F}}) = \Big(\pi^*\tilde{\gamma}\ ,\ (\sigma - \pi^*\alpha) \wedge \pi^*\tilde{\gamma} + \pi^*\tilde{\Omega}_L\Big).$$

Here $\tilde{\gamma}$ is the 1-form on $\pi(\bar{F})$ with kernel $\rho_{TP}(L)$ and evaluating to one on A, while $\tilde{\Omega}_L$ is the two form on $\pi(\bar{F})$ which coincides with Ω_L (the presymplectic form on the leaves of (P, L)) on $\rho_{TP}(L)$ and annihilates A.

4.2 Dependence of the Dirac-Jacobi structure on choices

Let (P, L) be a prequantizable Dirac manifold , i.e. one for which there exist a closed integral 2-form Ω and a section of β of L^* such that

$$\rho_{TP}^*\Omega = \Upsilon + d_L\beta. \tag{4.7}$$

The Dirac-Jacobi manifold $(Q, \bar{L})$ as defined in Theorem 4.1 depends on three data: the choice (up to isomorphism) of the $U(1)$-bundle Q, the choice of connection σ on Q whose curvature has cohomology class $i_*c_1(Q)$, and the choice of β, subject to the condition that Equation (4.7) be satisfied. We will explain here how the Dirac-Jacobi structure $\bar{L}(Q, \sigma, \beta)$ depends on these choices.

First, notice that the value of Ω outside of $\rho_{TP}(L)$ does not play a role in (4.7). In fact, different choices of σ agreeing over $\rho_{TP}(L)$ give rise to the same Dirac-Jacobi structure. This is consistent with the following lemma, which is the result of a straightforward computation:

Lemma 4.1. *For any 1-form γ on P the Dirac-Jacobi structures $\bar{L}(Q, \sigma, \beta)$ and $\bar{L}(Q, \sigma + \pi^*\gamma, \beta + \rho_{TP}^*\gamma)$ are equal.*

Two Dirac-Jacobi structures on a given $U(1)$-bundle Q over P give isomorphic quantizations if they are related by an element of the gauge group $C^\infty(P, U(1))$ acting on Q. Noting that the Lie algebroid differential d_L descends to a map $C^\infty(P, U(1)) \to \Omega^1_L(P)$ we denote by $H^1_L(P, U(1))$ the quotient of the closed elements of $\Omega^1_L(P)$ by the space $d_L(C^\infty(P, U(1))$ of $U(1)$-exact forms.

Now we show:

Proposition 4.1. *The set of isomorphism classes of Dirac-Jacobi manifolds prequantizing (P, L) maps surjectively to the space $(\rho_{TP}^* \circ i_*)^{-1}[\Upsilon]$ of topological types of compatible $U(1)$-bundles; the prequantizations of a given topological type are a principal homogeneous space for $H^1_L(P, U(1))$.*

Proof. Make a choice of prequantizing triple (Q, σ, β). With Q and σ fixed, we are allowed to change β by a d_L-closed section of L^*. If we fix only Q, we are allowed to change σ in such a way that the resulting curvature represents the cohomology class $i_* c_1(Q)$, so we can change σ by $\pi^*\gamma$ where γ is a 1-form on P. Now $\bar{L}(Q, \sigma + \pi^*\gamma, \tilde{\beta}) = \bar{L}(Q, \sigma, \tilde{\beta} - \rho_{TP}^*\gamma)$ by Lemma 4.1, so we obtain one of the Dirac-Jacobi structures already obtained above. Now, if we replace β by $\beta + d_L\phi$ for $\phi \in C^\infty(P, U(1))$, we obtain an isomorphic Dirac-Jacobi structure: in fact $\bar{L}(Q, \sigma, \beta)$ is equal to $\bar{L}(Q, \sigma + \pi^* d\phi, \beta + d_L\phi)$ by Lemma 4.1, which is isomorphic to $\bar{L}(Q, \sigma, \beta + d_L\phi)$ because the gauge transformation given by ϕ takes the connection σ to $\sigma + \pi^* d\phi$. So we see that the difference between two prequantizing Dirac-Jacobi structures on the fixed $U(1)$-bundle Q corresponds to an element of $H^1_L(P, U(1))$. □

In Dirac geometry, a B-field transformation (see for example [25]) is an automorphism of the Courant algebroid $TM \oplus T^*M$ arising from a closed 2-form B and taking each Dirac structure into another one with an isomorphic Lie algebroid. There is a similar construction for Dirac-Jacobi structures. Given any 1-form γ on any manifold M, the vector bundle endomorphism of $\mathcal{E}^1(M) = (TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R})$ that acts on $(X, f) \oplus (\xi, g)$ by adding $(0, 0) \oplus \begin{pmatrix} \tilde{d\gamma} & \gamma \\ -\gamma & 0 \end{pmatrix} (X, f)$ preserves the extended Courant bracket and the symmetric pairing. Thus, it maps each Dirac-Jacobi structure to another one. We call this operation an extended B-field transformation.

Lemma 4.2. *Let γ be a closed 1-form on P. Then $\bar{L}(Q, \sigma + \pi^*\gamma, \beta)$ is obtained from $\bar{L}(Q, \sigma, \beta)$ by the extended B-field transformation associated to γ.*

In the statements that follow, until the end of this subsection, we assume that the distribution $\rho_{TP}(L)$ has constant rank, and we denote by $\mathcal{F}$ the regular distribution integrating it.

Corollary 4.1. *Assume that $\rho_{TP}(L)$ has constant rank. Then the isomorphism classes of prequantizing Dirac-Jacobi structures on the fixed $U(1)$-bundle Q, up to extended B-field transformations, form a principal homogeneous space for*

$$H^1_L(P, U(1))/H^1_{\rho_{TP}(L)}(P, U(1)),$$

where $H^\bullet_{\rho_{TP}(L)}(P)$ denotes the foliated (i.e. tangential de Rham) cohomology of $\rho_{TP}(L)$.

Proof. We saw in the proof of Prop. 4.1 that, if (P, L) is prequantizable, the prequantizing Dirac-Jacobi structures on a fixed $U(1)$-bundle Q are given by $\bar{L}(Q, \sigma, \beta + \beta')$ where Q, σ, β are fixed and β' ranges over all d_L-closed sections of L^*. Consider $\rho_{TP}^*\gamma$ for a closed 1-form γ. Then $\bar{L}(Q, \sigma, \beta + \rho_{TP}^*\gamma) = L(Q, \sigma - \pi^*\gamma, \beta)$ by Lemma 4.1, and this is related to $\bar{L}(Q, \sigma, \beta)$ by an extended B-field transformation because of Lemma 4.2. To finish the argument, divide by the $U(1)$-exact forms. □

We will now give a characterization of the β's appearing in a prequantization triple.

Lemma 4.3. *Let (P, L) be a Dirac manifold for which $\rho_{TP}(L)$ is a regular foliation. Given a section β' of L^*, write $\beta' = \langle A' \oplus \alpha', \cdot \rangle|_L$. Then $d_L\beta' = \rho_{TP}^*\Omega'$ for some 2-form along $\mathcal{F}$ iff the vector field A' preserves the foliation $\mathcal{F}$. In this case, $\Omega' = d\alpha' - \mathcal{L}_{A'}\Omega_L$ where Ω_L is the presymplectic form on the leaves of $\mathcal{F}$ induced by L.*

Proof. For all sections $X_i \oplus \xi_i$ of L we have

$$d_L\beta'(X_1 \oplus \xi_1, X_2 \oplus \xi_2) = d\alpha'(X_1, X_2) + (\mathcal{L}_{A'}\xi_2)X_1 - (\mathcal{L}_{A'}\xi_1)X_2 + A' \cdot \langle \xi_1, X_2 \rangle.$$

Clearly $d_L\beta'$ is of the form $\rho_{TP}^*\Omega'$ iff $L \cap T^*P \subset \ker d_L\beta'$ (and in this case Ω' is clearly unique). Using the constant rank assumption to extend appropriately elements of $L \cap T^*P$ to some neighborhood in P, one sees that this is equivalent to $(\mathcal{L}_{A'}\xi)X = 0$ for all sections ξ of $L \cap T^*P = (\rho_{TP}(L))^\circ$ and vectors X in $\rho_{TP}(L)$, i.e. to A' preserving the foliation.

The formula for Ω' follows from a computation manipulating the above expression for $d_L\beta'$ by means of the Leibniz rule for Lie derivatives. $\square$

We saw in the proof of Prop. 4.1 that, if (P, L) is prequantizable, the prequantizing Dirac-Jacobi structures on a fixed $U(1)$-bundle Q are given by $\bar{L}(Q, \sigma, \beta + \beta')$ where Q, σ, β are fixed and β' ranges over all d_L-closed sections of L^*. Since $\Upsilon = \rho_{TP}^*\Omega_L$, it follows from (4.7) that $d_L\beta$ is the pullback by ρ_{TP} of some 2-form along $\mathcal{F}$. So, by the above lemma, $\beta = \langle A \oplus \alpha, \cdot \rangle|_L$ for some vector field A preserving the regular foliation $\mathcal{F}$. Also, $\beta' = \langle A' \oplus \alpha', \cdot \rangle|_L$ where A' is a vector field preserving $\mathcal{F}$ and $d\alpha' - \mathcal{L}_{A'}\Omega_L = 0$, and conversely every d_L-closed β' arises this way (but choices of $A' \oplus \alpha'$ differing by sections of L will give rise to the same β').

Example 4.1. Let F be an integrable distribution on a manifold P (tangent to a regular foliation $\mathcal{F}$), and $L = F \oplus F^\circ$ the corresponding Dirac structure. By Lemma 4.3 (or by a direct computation) one sees that the d_L-closed sections β of L^* are sums of sections of TP/F preserving the foliation and closed 1-forms along F. By Prop. 4.1, the set of isomorphism classes of prequantizing Dirac-Jacobi structures maps surjectively to the set $\ker(\rho_{TP}^* \circ i_*)$ of topological types; the inverse image of a given type is a principal homogeneous space for

$$\{\text{Sections of } TP/F \text{ preserving the foliation}\} \times H_F^1(P, U(1)),$$

where the Lie algebroid cohomology $H_F^\bullet(P)$ is the tangential de Rham cohomology of $\mathcal{F}$ (and $\ker(\rho_{TP}^* \circ i_*)$ denotes the kernel in degree two).

5 The prequantization representation

In this section, assuming the prequantization condition (4.5) for the Dirac manifold (P, L) and denoting by $(Q, \bar{L})$ its prequantization as in Theorem 4.1, we construct a representation of the Lie algebra $C^\infty_{adm}(P)$. We will do so by first mapping this space of functions to a set of "equivalence classes of vector fields" on Q and then by letting these act on $C^\infty_{bas}(Q, \mathbb{C})_{P-loc}$, a sheaf over P. Here $C^\infty_{bas}(Q, \mathbb{C})$ denotes the complex basic[8] functions on $(Q, \bar{L})$, as defined in Section 3, which in the case at hand are exactly the functions whose

[8]We use basic instead of admissible functions in order to obtain the same representation as in Section 6.

differentials annihilate $\bar{L} \cap TQ$. The subscript "$_{P-loc}$" indicates that we consider functions which are defined on subsets $\pi^{-1}(U)$ of Q, where U ranges over the open subsets[9] of P. We will decompose this representation and make some comments on the faithfulness of the resulting subrepresentations.

Let $\tilde{L} = \{(X, 0) \oplus (\xi, g) : X \oplus \xi \in L, g \in \mathbb{R}\}$ be the Dirac-Jacobi structure associated to the Dirac structure L on P. It is immediate that $\tilde{L}$ is the push-forward of $\bar{L}$ via $\pi : Q \to P$, i.e. $\tilde{L} = \{(\pi_* Y, f) \oplus (\xi, g) : (Y, f) \oplus (\pi^* \xi, g) \in \bar{L}\}$. From this it follows that if functions f, g on P are admissible then their pullbacks $\pi^* f, \pi^* g$ are also admissible[10] and

$$\{\pi^* f, \pi^* g\} = \pi^* \{f, g\}. \tag{5.1}$$

Proposition 5.1. *The map*

$$\begin{aligned} (C^\infty_{adm}(P), \{\cdot, \cdot\}) &\to Der(C^\infty_{bas}(Q, \mathbb{C})_{P-loc}) \\ g \qquad &\mapsto \{\pi^* g, \cdot\} \end{aligned} \tag{5.2}$$

determines a representation of $C^\infty_{bas}(Q, \mathbb{C})_{P-loc}$.

Proof. Recall that the expression $\{\pi^* g, \phi\}$ for $\phi \in C^\infty_{bas}(Q, \mathbb{C})_{P-loc}$ was defined in Section 3 as $-X_{\pi^* g}(\phi) - \phi \cdot 0 = -X_{\pi^* g}(\phi)$, for any choice $X_{\pi^* g}$ of hamiltonian vector field for $\pi^* g$. The proposition follows from the versions of the following statements for basic functions (see Lemma 3.1 and the remark following it). First: the map (5.2) is well-defined since the set of admissible functions on the Dirac-Jacobi manifold Q is closed under the bracket $\{\cdot, \cdot\}$. Second: it is a Lie algebra homomorphism because of Equation (5.1) and because the bracket of admissible functions on Q satisfies the Jacobi identity. Alternatively, for the second statement we can make use of the relation $[-X_{\pi^* f}, -X_{\pi^* g}] = -X_{\{\pi^* f, \pi^* g\}}$ (see Proposition 3.1). □

Since the Dirac-Jacobi structure on Q is invariant under the $U(1)$ action, the infinitesimal generator E is a derivation of the bracket. We can decompose $C^\infty_{bas}(Q, \mathbb{C})_{P-loc}$ into the eigenspaces H^n_{bas} of E corresponding to the eigenvalues $2\pi i n$, where n must be an integer, and similarly for H^n_{adm}. The derivation property implies that $\{H^n_{adm}, H^{n'}_{bas}\} \subseteq H^{n+n'}_{bas}$. The Lie algebra of admissible functions on P may be identified with the real-valued global functions in H^0_{adm}, which acts on each H^n_{bas} by the bracket, i.e. by the representation (5.2). In particular, the action on H^{-1}_{bas} is the usual prequantization action. The classical limit is obtained by letting $n \to -\infty$. Clearly all of the above applies if we restrict the representation (5.2) to $C^\infty_{adm}(Q, \mathbb{C})_{P-loc}$, i.e. if we replace "$H^n_{bas}$" by "$H^n_{adm}$" above.

Now we will comment on the faithfulness of the above representations. The map that assigns to an admissible function g on P the equivalence class of hamiltonian vector fields of $-\pi^* g$ depends on the choices of Ω and β in Equation (4.5) as well as on the prequantizing $U(1)$ bundle Q and connection σ. In general, there is no choice for which it is injective,

[9] We use the space of *P-local* instead of *global* basic functions because the latter could be too small for certain injectivity statements. See Proposition 5.2 below and the remarks following it, as well as Section 9.

[10] To show the smoothness of the hamiltonian vector fields of $\pi^* f$ and $\pi^* g$, we actually have to use the particular form of $\bar{L}$.

as the following example shows. It follows that the prequantization representation on H^n_{bas} or H^n_{adm} (given by restricting suitably the representation (5.2)) is generally not faithful for any n.

Example 5.1. Consider the Poisson manifold $(S^2 \times \mathbb{R}^+, \Lambda = t\Lambda_{S^2})$ where t is the coordinate on $\mathbb{R}^+$ and Λ_{S^2} is the product of the Poisson structure on S^2 corresponding to the standard symplectic form ω_{S^2} and the zero Poisson structure on $\mathbb{R}^+$. (This is isomorphic to the Lie-Poisson structure on $\mathfrak{su}(2)^* - \{0\}$.) We first claim that for all choices of Ω and A in (4.1) (which, as pointed out in Remark 4.1, is equivalent to (4.5)), the $\frac{\partial}{\partial t}$-component of the vector field A has the form $(ct^2 - t)\frac{\partial}{\partial t}$ for some real constant c.

Indeed, notice that $\Lambda + [-t\frac{\partial}{\partial t}, \Lambda] = 0$, so

$$\tilde{\Lambda} cp^*\omega_{S^2} = ct^2\Lambda_{S^2} = \Lambda + [A, \Lambda] \tag{5.3}$$

where $A = ct^2\frac{\partial}{\partial t} - t\frac{\partial}{\partial t}$. Now any vector field B satisfying $[B, \Lambda] = 0$ must map symplectic leaves to symplectic leaves, and since all leaves have different areas, B must have no $\frac{\partial}{\partial t}$-component. Hence any vector field satisfying Equation (5.3) has the same $\frac{\partial}{\partial t}$-component as A above. Now any closed 2-form Ω on $S^2 \times \mathbb{R}^+$ is of the form $cp^*\omega_{S^2} + d\beta$ for some 1-form β, where $p : S^2 \times \mathbb{R}^+ \to S^2$. Since $\tilde{\Lambda} d\beta = -[\tilde{\Lambda}\beta, \Lambda]$ and $-\tilde{\Lambda}\beta$ has no $\frac{\partial}{\partial t}$ component, our first claim is proved.

Now, for any choice of Q and σ, let g be a function on $S^2 \times \mathbb{R}^+$ such that $X_{\pi^* g} = X^H_g + (\langle dg, A\rangle - g)E$ vanishes. This means that g is a function of t only, satisfying $(ct^2 - t)g' = g$. For any real number c, there exist non-trivial functions satisfying these conditions, for example $g = \frac{ct-1}{t}$, therefore for all choices the homomorphism $g \mapsto -X_{\pi^* g}$ is not injective.

This example also shows that one can not simply omit the vector field A from the definition of prequantizability, since no choice of c makes A vanish here.

Even though the prequantization representation for functions acting on H^n_{adm} and H^n_{bas} is usually not faithful for any integer n, we still have the following result, which shows that hamiltonian vector fields do act faithfully.

Proposition 5.2. *For each integer $n \neq 0$, the map that assigns to an equivalence class of hamiltonian vector fields $X_{\pi^* g}$ the corresponding operator on H^n_{adm} or H^n_{bas} is injective.*

Proof. Since $H^n_{adm} \subset H^n_{bas}$, it is enough to consider the H^n_{adm} case. Since the hamiltonian vector field of any function on Q is determined up to smooth sections of the singular distribution $F := \bar{L} \cap TQ = \{X^H + \langle \alpha, X\rangle E : X \in L \cap TP\}$, we have to show that, if a $U(1)$-invariant vector field Y on Q annihilates all functions in H^n_{adm}, then Y must be a section of F.

We start by characterizing the functions in H^n_{adm} on neighborhoods where a constant rank assumption holds:

Lemma 5.1. *Let U be an open set in P on which the rank of $L \cap TP$ is constant and $\bar{U} = \pi^{-1}(U)$. Then a function ϕ on $\bar{U}$ is admissible iff ϕ is constant along the leaves of F. Further $\cap_{\phi \in H^n_{adm}} \ker d\phi = F$.*

Proof. We have

$$\phi \text{ admissible } \Leftrightarrow (d\phi, \phi) \subset \rho_{T^*Q\times\mathbb{R}}(\bar{L}) \Leftrightarrow d\phi \subset \rho_{T^*Q}(\bar{L}), \tag{5.4}$$

where the first equivalence follows from the formula for $\bar{L}$, the remark following Definition 2.3 and the fact that $\dim(L \cap TP)$ is constant. For any Dirac-Jacobi structure one has $\rho_{T^*Q}(\bar{L}) = (\bar{L} \cap TQ)^\circ$, so the first statement follows.

Now consider the regular foliation of $\bar{U}$ with leaves equal to $U(1) \cdot \mathcal{F}$, where $\mathcal{F}$ ranges over the leaves of $F|_{\bar{U}}$[11]. Fix $p \in \bar{U}$ and choose a submanifold S through p which is transverse to the foliation $U(1) \cdot \mathcal{F}$. Given any covector $\xi \in T_p^*S$ we can find a function ϕ on S with differential ξ at p, and we extend ϕ to $\bar{U}$ so that it is constant on the leaves of F and equivariant with respect to the n-th power of the standard $U(1)$ action on $\mathbb{C}$. Then ϕ will lie in H^n_{adm} and $d_p\phi$ will be equal to ξ on T_pS, equal to $2\pi in$ on E_p, and will vanish on F_p. Since we can construct such a function $\phi \in H^n_{adm}$ for any choice of ξ, it is clear that a vector at p annihilated by all functions in H^n_{adm} must lie in F_p, so $\cap_{\phi \in H^n_{adm}} \ker d\phi \subset F$. The other inclusion is clear. $\square$

Now we make use of the fact that for any open subset V of P there exists a nonempty open subset $U \subset V$ on which $\dim(L \cap TP)$ is constant[12], and prove Proposition 5.2.

End of proof of Proposition 5.2. Suppose now the $U(1)$-invariant vector field Y on Q annihilates all functions in H^n_{adm} but is not a section of F. Then $Y \notin F$ at all points of some open set $\bar{U}$. By the remark above, we can assume that on $\bar{U}$ $\dim(L \cap TP)^H = \dim F$ is constant. By Lemma 5.1 on $\bar{U}$ the vector field Y must be contained in F, a contradiction. $\square$

If we modified the representation (5.2) to act on *global* admissible or basic functions, the injectivity statement of Proposition 5.2 could fail, as the following example shows.

Example 5.2. Let P be $(\mathbb{T}^2 \times \mathbb{R}, d\varepsilon)$, where $\varepsilon = x_3(dx_1 + x_3 dx_2)$ with (x_1, x_2) and x_3 standard coordinates on the torus and $\mathbb{R}$ respectively. This is a regular presymplectic manifold, so by Lemma 5.1 all basic functions on any prequantization Q are admissible. P is clearly prequantizable, and we can choose $\Omega = 0$ and $\beta = -\rho^*_{TP}\varepsilon$ in the prequantization condition (4.5). Therefore Q is the trivial $U(1)$ bundle over P, with trivial connection $\sigma = d\theta$ (where θ is the standard fiber coordinate). The distribution F on Q, as defined at the beginning of the proof of Proposition 5.2, is one dimensional, spanned by $2x_3\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} - x_3^2\frac{\partial}{\partial \theta}$. The coefficients $2x_3$, -1, and $-x_3^2$ are linearly independent over $\mathbb{Z}$ unless x_3 is a quadratic algebraic integer, so the closures of the leaves of F will be of the form $\mathbb{T}^2 \times \{x_3\} \times U(1)$ for a dense set of x_3's. Therefore $C^\infty_{adm}(Q, \mathbb{C}) = C^\infty_{bas}(Q, \mathbb{C})$ consists exactly of complex functions depending only on x_3. For similar reasons, the admissible functions on P are exactly those depending only on x_3. But the vector field X_{π^*g} on Q associated to such a function g has no $\frac{\partial}{\partial x_3}$ component, so it acts trivially on $C^\infty_{adm}(Q, \mathbb{C})$.

Next we illustrate how the choices involved in the prequantization representation affect injectivity.

Example 5.3. Let $P = S^2 \times \mathbb{R} \times S^1$, with coordinate t on the $\mathbb{R}$-factor and s on the S^1-factor. Endow P with the Poisson structure Λ which is the product of the zero Poisson

[11]The distribution $F = \bar{L} \cap TQ$ is clearly involutive; see Definition 3.2.

[12]Indeed, if q is a point of V where $\dim(L \cap TP)$ is minimal among all points of V, in a small neighborhood of q $\dim(L \cap TP)$ can not decrease, nor it can increase because $L \cap TP$ is an intersection of subbundles.

structure on $\mathbb{R} \times S^1$ and the inverse of an integral symplectic form ω_{S^2} on S^2. This Poisson manifold is prequantizable; in Equation (4.1) we can choose $\Omega = p^*\omega_{S^2}$ (where $p : P \to S^2$) and as A any vector field that preserves the Poisson structure. Each $g \in C^\infty(P)$ is prequantized by the action of the negative of 1its hamiltonian vector field $X_{\pi^*g} = (\tilde{\Lambda}dg)^H + (A(g) - g)E$. Therefore the kernel of the prequantization representation is given by functions of t and s satisfying $A(g) = g$. It is clear that if A is tangent to the symplectic leaves the representation will be faithful. If A is not tangent to the symplectic leaves, then $A(g) = g$ is an honest first order differential equation. However, even in this case the representation might be faithful: it is faithful if we choose $A = \frac{\partial}{\partial t}$, but not if $A = \frac{\partial}{\partial s}$.

Remark 5.1. Let (P, Λ) be a Poisson manifold such that its symplectic foliation $\mathcal{F}$ has constant rank, and assume that (P, Λ) is prequantizable (i.e. (4.1), or equivalently (4.5), is satisfied). It follows from the discussion following Lemma 4.3 that, after we fix a prequantizing $U(1)$-bundle Q, the prequantizing Dirac-Jacobi structures on Q are given by $\bar{L}(Q, \sigma, A)$ where σ is fixed and A is unique up to vector fields A' preserving $\mathcal{F}$ such that $\mathcal{L}_{A'}\Omega_L = 0$, i.e. up to vector fields whose flows are symplectomorphisms between the symplectic leaves. If the topology and geometry of the symplectic leaves of P "varies" sufficiently from one leaf to another (as in Example 5.1 above), then the projection of the A's as above to $TP/T\mathcal{F}$ will all coincide. Therefore the kernels of the prequantization representations (5.2), which associate to $g \in C^\infty(P)$ the negative of the hamiltonian vector field $X_{\pi^*g} = (\tilde{\Lambda}dg)^H + (A(g) - g)E$, will coincide for all representations arising from prequantizing Dirac-Jacobi structures over Q.

We end this section with two remarks linked to Kostant's work [22].

Remark 5.2. Kostant ([22], Theorem 0.1) has observed that the prequantization of a symplectic manifold can be realized by the Poisson bracket of a symplectic manifold two dimensions higher, i.e. that prequantization is "classical mechanics two dimensions higher". In the general context of Dirac manifolds we have seen in (5.2) that prequantization is given by a Jacobi bracket[13]; we will now show that Kostant's remark applies in this context too.

Let (P, L) be a prequantizable Dirac manifold, $(Q, \bar{L})$ its prequantization and $(Q \times \mathbb{R}, \bar{\bar{L}})$ the "Diracization" of $(Q, \bar{L})$. To simplify the notation, we will denote pullbacks of functions (to Q or $Q \times \mathbb{R}$) under the obvious projections by the same symbol. Using the homomorphism (3.5) we can re-write the representation (5.2) of $C^\infty_{adm}(P)$ on $C^\infty_{adm}(Q, \mathbb{C})_{P-loc}$ (or $C^\infty_{bas}(Q, \mathbb{C})_{P-loc}$) as

$$g \mapsto e^{-t}\{e^t g, e^t \cdot\}_{Q \times \mathbb{R}} = \{e^t g, \cdot\}_{Q \times \mathbb{R}},$$

i.e. g acts by the Poisson bracket on $Q \times \mathbb{R}$.

Remark 5.3. Kostant [22] also shows that a prequantizable symplectic manifold (P, Ω) can be recovered by reduction from the symplectization $(Q \times \mathbb{R}, d(e^t\sigma))$ of its prequantization (Q, σ). More precisely, the inverse of the natural $U(1)$ action on $Q \times \mathbb{R}$ is hamiltonian with momentum map e^t, and symplectic reduction at $t = 0$ delivers (P, Ω). We will show now how to extend this construction[14] to prequantizable Dirac manifolds.

[13]The bracket on functions on the prequantization $(Q, \bar{L})$ of a Dirac manifold makes $C^\infty_{adm}(Q)$ into a Jacobi algebra. See Section 5 of [34], which applies because the constant functions are admissible for the Dirac-Jacobi structure $\bar{L}$.

[14]Kostant calls the procedure of taking the symplectization of the prequantization "symplectic induction"; the term seems to be used here in a different sense from that in [19].

Let (P, L), $(Q, \bar{L})$ and $(Q \times \mathbb{R}, \tilde{\bar{L}})$ be as in Remark 5.2. Since $-E \oplus de^t \in \tilde{\bar{L}}$ we see that e^t is a "momentum map" for the inverse $U(1)$ action on $Q \times \mathbb{R}$, and by Dirac reduction [2] at the regular value 1 we obtain L: indeed, the pullback of $\tilde{\bar{L}}$ to $Q \times \{0\}$ is easily seen to be $\{(X^H + (\langle X \oplus \xi, \beta\rangle - g)E) \oplus \pi^*\xi : X \oplus \xi \in L\}$, and its pushforward via $\pi : Q \to P$ is exactly L.

6 The line bundle approach

In this section we will prequantize a Dirac manifold P by letting its admissible functions act on sections of a hermitian line bundle K over P. This approach was first taken by Kostant for symplectic manifolds and was extended by Huebschmann [16] and Vaisman [31] to Poisson manifolds. The construction of this section generalizes Vaisman's and turns out to be equivalent to the one we described in Sections 4 and 5.

Definition 6.1. [11] Let $(A, [\cdot,\cdot], \rho)$ be a Lie algebroid over the manifold M and K a real vector bundle over M. An A-connection on the vector bundle $K \to M$ is a map $D : \Gamma(A) \times \Gamma(K) \to \Gamma(K)$ which is $C^\infty(M)$-linear in the $\Gamma(A)$ component and satisfies

$$D_e(h \cdot s) = h \cdot D_e s + \rho e(h) \cdot s,$$

for all $e \in \Gamma(A)$, $s \in \Gamma(K)$ and $h \in C^\infty(M)$. The curvature of the A-connection is the map $\Lambda^2 A^* \to End(K)$ given by

$$R_D(e_1, e_2)s = D_{e_1} D_{e_2} s - D_{e_2} D_{e_1} s - D_{[e_1,e_2]} s.$$

If K is a complex vector bundle, we define an A-connection on K as above, but with $C^\infty(M)$ extended to the complex-valued smooth functions.

Remark 6.1. When $A = TM$ the definitions above specialize to the usual notions of covariant derivative and curvature. Moreover, given an ordinary connection ∇ on K, we can pull it back to a A-connection by setting $D_e = \nabla_{\rho e}$.

With this definition we can easily adapt Vaisman's construction [31] [32], extending it from the case where $L = T^*P$ is the Lie algebroid of a Poisson manifold to the case where L is a Dirac structure. We will act on locally defined, basic sections.

Lemma 6.1. *Let (P, L) be a Dirac manifold and K a hermitian line bundle over P endowed with an L-connection D. Then $R_D = 2\pi i \Upsilon$, where $\Upsilon = \langle\cdot,\cdot\rangle_-|_L$, iff the correspondence*

$$\hat{g}s = -(D_{X_g \oplus dg} s + 2\pi i g s)$$

defines a Lie algebra representation of $C^\infty_{adm}(P)$ on $\{s \in \Gamma(K)_{loc} : D_{Y \oplus 0} s = 0$ for $Y \in L \cap TP\}$, where X_g is any choice of hamiltonian vector field for g.

Proof. If $\hat{g}$ and s are as above, then clearly $\hat{g}s$ is a well-defined section of K. We will now show that $\hat{g}s \in \{s \in \Gamma(K)_{loc} : D_{Y \oplus 0} s = 0 \text{ for } Y \in L \cap TP\}$, so that the above "representation" is well-defined. The case where $Y \in L \cap T_pP$ can be locally extended to a smooth section of $L \cap TP$ is easy, whereas the techniques (see Section 2.5 of [11]) needed for general case are much more involved.

The section $X_g \oplus dg$ of L induces a flow ϕ_t on P (which is just the flow of the vector field X_g) and a one-parameter family of bundle automorphisms Φ_t on $TP \oplus T^*P$ which (see Section 2.4 in [6]) preserves L, and which takes L-paths to L-paths[15]. Further, Φ_t acts on the sections s of the line bundle K too, as follows: $(\Phi_t^* s)_p$ is the parallel translation of $s_{\phi_t(p)}$ along the L-path $\Phi_\bullet(X_g \oplus dg)_p = (X_g \oplus dg)_{\phi_\bullet(p)}$. Now $(D_{(X_g \oplus dg)} s)_p = \frac{\partial}{\partial t}|_0 (\Phi_t^* s)_p$, and $(D_{(Y\oplus 0)} D_{X_g \oplus dg} s)_p = \frac{\partial}{\partial t}|_0 (D_{(Y \oplus 0)} \Phi_t^* s)_p$. For every t, since ϕ_t preserves $L \cap TP$, we have

$$0 = (D_{(\phi_{t_*} Y \oplus 0)} s)_{\phi_t(p)} = \frac{\partial}{\partial \epsilon}\Big|_0 \mathbin{\backslash\!\backslash}_0^\epsilon s_{\phi_t(\gamma(\epsilon))} \tag{6.1}$$

where Γ is an L-path starting at $(Y \oplus 0) \in L_p$, γ is its base path, and $\mathbin{\backslash\!\backslash}_0^\epsilon$ is parallel translation along the L-path $\Phi_t(\Gamma(\bullet))$. (This notation denotes the path $\epsilon \mapsto \Phi_t(\Gamma(\epsilon))$.) Now we parallel translate the element (6.1) of $K_{\phi_t(p)}$ to p using the L-path $\Phi_\bullet(X_g \oplus dg)_p$, and compare the result with

$$(D_{(Y \oplus 0)} \Phi_t^* s)_p = \frac{\partial}{\partial \epsilon}\Big|_0 \mathbin{\backslash\!\backslash}_0^\epsilon \mathbin{\backslash\!\backslash}_0^t s_{\phi_t(\gamma(\epsilon))}, \tag{6.2}$$

where the parallel translation is taken first along $\Phi_\bullet(X_g \oplus dg)_{\gamma(\epsilon)}$ and then along $\Gamma(\bullet)$.

The difference between (6.2) and the parallel translation to p of (6.1) lies only in the order in which the parallel translations are taken. Now applying $\frac{\partial}{\partial t}|_0$ to this difference (and recalling that $\Phi_t(X_g \oplus dg)_p = (X_g \oplus dg)_{\phi_t(p)}$) we obtain the evaluation at p of

$$D_{\Phi_t \Gamma(\epsilon)} D_{(X_g \oplus dg)} s - D_{(X_g \oplus dg)} D_{\Phi_t \Gamma(\epsilon)} s,$$

which by the definition of curvature is just

$$(D_{[\Phi_t \Gamma(\epsilon), X_g \oplus dg]} s)_p + \Upsilon(Y \oplus 0, (X_g \oplus dg)_p) s.$$

The second term vanishes because $Y \in L \cap T_pP$, and using the fact that Φ_t is the flow generated by X_g one sees that the Courant bracket in the first term is also zero. Altogether we have proven that $(D_{(Y \oplus 0)} D_{X_g \oplus dg} s)_p$ vanishes, and from this is follows easily that the "representation" in the statement of the lemma is well defined.

Since

$$[\hat{f}, \hat{g}] = D_{X_f \oplus df} D_{X_g \oplus dg} - D_{X_g \oplus dg} D_{X_f \oplus df} + 2\pi i (X_f(g) - X_g(f)),$$

using $-[X_f \oplus df, X_g \oplus dg] = X_{\{f,g\}} \oplus d\{f,g\}$ ([6], Prop. 2.5.3) we see that the condition on R_D holds iff $[\hat{f}, \hat{g}] = \widehat{\{f,g\}}$. $\square$

Now assume that the prequantization condition (4.5) is satisfied, i.e. that there exists a closed integral 2-form Ω and a Lie algebroid 1-cochain β for such that

$$\rho_{TP}^* \Omega = \Upsilon + d_L \beta.$$

Then we can construct an L-connection D satisfying the property of the previous lemma:

[15]For any algebroid A over P an A-path is a defined as a path $\Gamma(t)$ in A such that the anchor maps $\Gamma(t)$ to the velocity of the base path $\pi(\Gamma(t))$.

Lemma 6.2. *Let $(A, [\cdot,\cdot], \rho)$ be a Lie algebroid over the manifold M, Ω a closed integral 2-form on M, and ∇ a connection (in the usual sense) on a hermitian line bundle K with curvature $R_\nabla = 2\pi i\Omega$. If $\rho^*\Omega = \Upsilon + d_L\beta$ for a 2-cocycle Υ and a 1-cochain β on A, then the A-connection D defined by*

$$D_e = \nabla_{\rho e} - 2\pi i\langle e, \beta\rangle$$

has curvature $R_D = 2\pi i\Upsilon$.

Proof. An easy computation shows

$$R_D(e_1, e_2) = R_\nabla(\rho e_1, \rho e_2) + 2\pi i(-\rho e_1\langle e_2, \beta\rangle + \rho e_2\langle e_1, \beta\rangle + \langle [e_1, e_2], \beta\rangle),$$

which using $\rho^*\Omega = \Upsilon + d_L\beta$ reduces to $2\pi i\Upsilon(e_1, e_2)$. □

Altogether we obtain that

$$\hat{g} = -[\nabla_{X_g} - 2\pi i(\langle X_g \oplus dg, \beta\rangle - g)]$$

determines a representation of $C^\infty_{adm}(P)$ on $\{s \in \Gamma(K)_{loc} : \nabla_Y s - 2\pi i\langle Y \oplus 0, \beta\rangle s = 0 \text{ for } Y \in L \cap TP\}$. Notice that, when P is symplectic, we recover Kostant's prequantization mentioned in the introduction. Now let $Q \to P$ be the $U(1)$-bundle corresponding to K, with the connection form σ corresponding to ∇. If $\bar{s}$ is the $U(1)$-antiequivariant complex valued function on Q corresponding to the section s of K, then $X^H(\bar{s})$ corresponds to $\nabla_X s$ and $E(\bar{s})$ to $-2\pi i s$. Here $X \in TP$, $X^H \in \ker\sigma$ its horizontal lift to Q, and E is the infinitesimal generator of the $U(1)$ action on Q (so $\sigma(E) = 1$). Translating the above representation to the $U(1)$-bundle picture, we see that $\hat{g} = -[X_g^H + (\langle X_g \oplus dg, \beta\rangle - g)E]$ defines a representation of $C^\infty_{adm}(P)$ on

$$\{\bar{s} \in C^\infty(Q, \mathbb{C})_{P-loc} : \bar{s} \text{ is } U(1)\text{-antiequivariant and}$$
$$(Y^H + \langle Y \oplus 0, \beta\rangle E)\bar{s} = 0 \text{ for } Y \in L \cap TP\},$$

which is nothing else than H^{-1}_{bas} as defined in Section 5. Since $X_g^H + (\langle X_g \oplus dg, \beta\rangle - g)E$ is the hamiltonian vector field of π^*g (with respect to the Dirac-Jacobi structure $\bar{L}$ on Q as in Theorem 4.1), we see that this is exactly our prequantization representation given by Equation (5.2) restricted to H^{-1}_{bas}.

6.1 Dependence of the prequantization on choices: the line bundle point of view

In Subsection 4.2 we gave a classification the Dirac-Jacobi structures induced on the prequantization of a given Dirac manifold, and hence also a classification of the corresponding prequantization representations. Now we will see that the line bundle point of view allows for an equivalent but clearer classification.

Recall that, given a Dirac manifold satisfying the prequantization condition (4.5), we associated to it a hermitian line bundle K and a representation as in Lemma 6.1, where the L-connection D is given as in Lemma 6.2

Proposition 6.1. *Fix a line bundle K over P with $(\rho_{TP}^* \circ i_*)c_1(K) = [\Upsilon]$. Then all the hermitian L-connections of K with curvature Υ are given by the L-connections constructed in Lemma 6.2. Therefore there is a surjective map from the set of isomorphism classes of prequantization representations of (P, L) to the space $(\rho_{TP}^* \circ i_*)^{-1}[\Upsilon]$ of topological types; the set with a given type is a principal homogeneous space for $H^1_L(P, U(1))$.*

Proof. Exactly as in the case of ordinary connections one shows that the difference of two hermitian L-connections on K is a section of L^*, whose d_L-derivative is the difference of the curvatures. Fix a choice of L-connection D as in Lemma 6.2, say given by $D_{(X\oplus\xi)} = \nabla_X - 2\pi i\langle X \oplus \xi, \beta\rangle$. Another L-connection D' with curvature Υ is given by $D'_{(X\oplus\xi)} = \nabla_X - 2\pi i\langle X \oplus \xi, \beta + \beta'\rangle$ for some d_L-closed section β' of L^*, hence it arises as in Lemma 6.2. This shows the first claim of the proposition. Since, as we have just seen, the L-connections with given curvature differ by d_L-closed sections of L^* and since $U(1)$-exact sections of L^* give rise to gauge equivalences of hermitian line bundles with connections, the second claim follows as well. □

Using Lemma 4.1 it is easy to see that choices of (σ, β) giving rise to the same L-connection (as in Lemma 6.2) also give rise to the same Dirac-Jacobi structure $\bar{L}$, in accord with the results of Section 4.2. Given this, it is natural to try to express the Dirac-Jacobi structure $\bar{L}$ intrinsically in terms of the L-connection to which it corresponds; this is subject of work in progress.

7 Prequantization of Poisson and Dirac structures associated to contact manifolds

We have already mentioned in Remark 5.3 the symplectization construction, which associates to a manifold M with contact form σ the manifold $M \times \mathbb{R}$ with symplectic form $d(e^t\sigma)$. The construction may also be expressed purely in terms of the cooriented contact distribution C annihilated by σ. In fact, given any contact distribution, its nonzero annihilator C° is a (locally closed) symplectic submanifold of T^*M. When C is cooriented, we can select the positive component C°_+. Either of these symplectic manifolds is sometimes known as the symplectization of (M, C). It is a bundle over M for which a trivialization (which exists in the cooriented case) corresponds to the choice of a contact form σ and gives a symplectomorphism between this "intrinsic" symplectization and $(M \times \mathbb{R}, d(e^t\sigma))$. The contact structure on M may be recovered from its symplectization along with the conformally symplectic $\mathbb{R}$ action generated by $\partial/\partial t$.

One may partially compactify C°_+ (we stick to the cooriented case for simplicity) at either end to get a manifold with boundary diffeomorphic to M. The first, and simplest way, is simply to take its closure $C^\circ_{0,+}$ in the cotangent bundle by adjoining the zero section. The result is a presymplectic manifold with boundary, diffeomorphic to $M \times [0, \infty)$ with the exact 2-form $d(s\sigma) = ds \wedge \sigma + sd\sigma$, where s is the exponential of the coordinate t in $\mathbb{R}$. For positive s, this is symplectic; the characteristic distribution of $C^\circ_{0,+}$ lives along the boundary $M \times \{0\}$, where it may be identified with the contact distribution C. This is highly nonintegrable even though $d(s\sigma)$ is closed, so we have another example of the phenomenon alluded to in the discussion after Definition 2.2.

We also note that the basic functions on $C^\circ_{0,+}$ are just those which are constant on $M \times \{0\}$. One can prove that all of these functions are admissible as well, even though the characteristic distribution is singular. It would be interesting to characterize the Dirac structures for which these two classes of functions coincide.

To compactify the other end of C°_+, we begin by identifying C°_+ with the positive part of its dual $(TM/C)_+$, using the "inversion" map j which takes $\phi \in C^\circ_+$ to the unique element $X \in (TM/C)_+$ for which $\phi(X) = 1$. We then form the union $C^\circ_{+,\infty}$ of C°_+ with the zero section in TM/C and give it the topology and differentiable structure induced via j from the closure of $(TM/C)_+$. It was discovered by LeBrun [23] that the Poisson structure on C°_+ corresponding to its symplectic structure extends smoothly to $C^\circ_{+,\infty}$. We call $C^\circ_{+,\infty}$ with this Poisson structure the **LeBrun-Poisson manifold** corresponding to the contact manifold (M, C).

To analyze the LeBrun-Poisson structure more closely, we introduce the inverted coordinate $r = 1/s$, which takes values in $[0, \infty)$ on $C^\circ_{+,\infty}$. In suitable local coordinates on M, the contact form σ may be written as $du + \sum p_i dq^i$. On the symplectization, we have the form $d(r^{-1}(du + \sum p_i dq^i))$. The corresponding Poisson structure turns out to be

$$\Lambda = r\left[\left(r\frac{\partial}{\partial r} + \sum p_i \frac{\partial}{\partial p_i}\right) \wedge \frac{\partial}{\partial u} + \sum \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i}\right].$$

From this formula we see not only that Λ is smooth at $r = 0$ but also that its linearization

$$r \sum \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i}$$

at the origin (which is a "typical" point, since M looks the same everywhere) encodes the contact subspace in terms of the symplectic leaves in the tangent Poisson structure.

We may take the union of the two compactifications above to get a manifold $C^\circ_{0,+,\infty}$ diffeomorphic to M times a closed interval. It is presymplectic at the 0 end and Poisson at the ∞ end, so it can be treated globally only as a Dirac manifold. In what follows, we will simply denote this Dirac manifold as (P, L).

To prequantize (P, L), we first notice that its Dirac structure is "exact" in the sense that the cohomology class $[\Upsilon]$ occurring in the condition (4.4) is zero. In fact, on the presymplectic end, L is isomorphic to TP, and Υ is identified with the form $d(s\sigma)$, so we can take the cochain β to be the section of L^* which is identified with $-s\sigma$. To pass to the other end, we compute the projection of this section of L^* into TP and find that it is just the Euler vector field $A = s\frac{\partial}{\partial s}$. In terms of the inverse coordinate r, $A = -r\frac{\partial}{\partial r}$. (The reader may check that the Poisson differential of this vector field is $-\Lambda$, either by direct computation or using the degree 1 homogeneity of Λ with respect to r.) On the Poisson end, L^* is isomorphic to TP, so $-r\frac{\partial}{\partial r}$ defines a smooth continuation of β to all of P.

Continuing with the prequantization, we can take the 2-form Ω to be zero and the $U(1)$-bundle Q to be the product $P \times U(1)$ with the trivial connection $d\theta$, where θ is the (2π-periodic) coordinate on $U(1)$. On the presymplectic end, the Dirac-Jacobi structure is defined by the 1-form $\sigma = s\sigma + \theta$, which is a contact form when $s \neq 0$.

On the Poisson end, we get the Jacobi structure $(\Lambda^H + E \wedge A^H, E)$ which in coordinates becomes

$$\left(r\left[\left(r\frac{\partial}{\partial r} + \sum p_i \frac{\partial}{\partial p_i}\right) \wedge \frac{\partial}{\partial u} + \sum \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} - \frac{\partial}{\partial \theta} \wedge \frac{\partial}{\partial r}\right], \frac{\partial}{\partial \theta}\right). \tag{7.1}$$

8 Prequantization by circle actions with fixed points

Inspired by a construction of Engliš [10] in the complex setting, we modify the prequantization in the previous section by "pinching" the boundary component $M \times U(1)$ at the Poisson end and replacing it by a copy of M. To do this, we identify $U(1)$ with the unit circle in the plane $\mathbb{R}^2$ with coordinates (x, y). In addition, we make a choice of contact form on M so that P is identified with $M \times [0, \infty]$, with the coordinate r on the second factor. Next we choose a smooth nonnegative real valued function $f : [0, \infty] \to \mathbb{R}$ such that, for some $\epsilon > 0$, $f(r) = r$ on $[0, \epsilon]$ and $f(r)$ is constant on $[2\epsilon, \infty]$. Let Q' be the submanifold of $P \times \mathbb{R}^2$ defined by the equation $x^2 + y^2 = f(r)$.

Radial projection in the (x, y) plane determines a map $F : Q \to Q'$ which is smooth, and in fact a diffeomorphism, where $r > 0$. The boundary $M \times U(1)$ of Q is projected smoothly to $M \times (0, 0)$ in Q', but F itself is not smooth along the boundary. We may still use F to transport the Jacobi structure on Q to the part of Q' where $r > 0$. For small r, we have $x = \sqrt{r}\cos\theta$ and $y = \sqrt{r}\sin\theta$, so $r = x^2 + y^2$, $r\frac{\partial}{\partial r} = \frac{1}{2}(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})$, and $\frac{\partial}{\partial\theta} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$. Using these substitutions to write the Jacobi structure (7.1) with polar coordinates (r, θ) replaced by rectangular coordinates (x, y), we see immediately that the structure extends smoothly to a Jacobi structure on the Poisson end of Q' and to a Dirac-Jacobi structure on all of Q', and that the projection $Q' \to P$, like $Q \to P$ pushes the Dirac-Jacobi structure on Q' to the Dirac structure on P. (Thus, the projection is a "forward Dirac-Jacobi map"; see the beginning of Section 3.) The essential new feature of Q' is that the vector field $E' = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ of the Jacobi structure on Q' vanishes along the locus $x = y = 0$ where the projection is singular.

The vanishing of E' at some points means that the Jacobi structure on Q' does not arise from a contact form, even on the Poisson end, where $r < \infty$. However, it turns out that we can turn it into a contact structure by making a conformal change, i.e. by multiplying the bivector by $1/f$ and replacing E' by $E'/f + X_{1/f}$. The resulting Jacobi structure still extends smoothly over Q', and now comes from a contact structure over the Poisson end; the price we pay is that the projection to P is now a conformal Jacobi map rather than a Jacobi map.

Remark 8.1. Looking back at the construction above, we see that we have embedded any given contact manifold M as a codimension 2 submanifold in another contact manifold. Our construction depended only on the choice of a contact form. On the other hand, Eliashberg and Polterovich [9] construct a similar embedding in a canonical way, without the choice of a contact form. It is not hard to show that the choice of a contact form defines a canonical isomorphism between our contact manifold and theirs.

Example 8.1. Let M be the unit sphere in $\mathbb{C}^n$, with the contact structure induced from the Cauchy-Riemann structure on the boundary of the disc D^{2n}. It turns out that a neighborhood U of M in the disc can be mapped diffeomorphically to a neighborhood V of M at the Poisson end in its LeBrun-Poisson manifold P so that the symplectic structure on the interior of V pulls back to the symplectic structure on U coming from the Kähler structure on the open disc, viewed as complex hyperbolic space. If we now pinch the end of the prequantization Q, as above, the part of the contact manifold Q' lying over V can be glued to the usual prequantization of the open disc so as to obtain a compact contact manifold Q'' projecting by a "conformal Jacobi map" to the closed disc. The fibres of the map are the orbits of a $U(1)$-action which is principal over the open disc. In fact, Q'' is just the unit sphere in $\mathbb{C}^{n+1}$ with *its* usual contact structure. All this is the symplectic analogue of the complex construction by Engliš [10], who enlarges a bounded pseudoconvex domain D in $\mathbb{C}^n$ to one in $\mathbb{C}^{n+1}$ with a $U(1)$ action on its boundary which degenerates just over the boundary of D.

The "moral" of the story in this section is that, in prequantizing a Poisson manifold P whose Poisson structure degenerates along a submanifold, one might want to allow the prequantization bundle to be a Jacobi manifold Q whose vector field E generates a $U(1)$ action having fixed points and for which the quotient projection $Q \to P$ is a Jacobi map.

9 Final remarks and questions

We conclude with some suggestions for further research along the lines initiated in this paper.

9.1 Cohomological prequantization

Cohomological methods have already been used in geometric quantization of symplectic manifolds: rather than the space of global polarized sections, which may be too small or may have other undesirable properties, one looks at the higher cohomology of the sheaf of local polarized sections. (An early reference on this approach is [26].) When we deal with Dirac (e.g. presymplectic) manifolds, it may already be interesting to introduce cohomology at the prequantization stage. There are two ways in which this might be done.

The first approach, paralleling that which is done with polarizations, is to replace the Lie algebra of global admissible functions on a Dirac manifold P by the cohomology of the sheaf of Lie algebras of local admissible functions. Similarly, one would replace the sheaf of P-local functions on Q by its cohomology. The first sheaf cohomology should then act on the second.

The other approach, used by Cattaneo and Felder [4] for the deformation quantization of coisotropic submanifolds of Poisson manifolds, would apply to Dirac manifolds P whose characteristic distribution is regular. Here, one introduces the "longitudinal de Rham complex" of differential forms along the leaves of the characteristic foliation on P. The zeroth cohomology of this foliation is just the admissible functions, so it is natural to consider the full cohomology, or even the complex itself. It turns out that, if one chooses a transverse distribution to the characteristic distribution, the transverse Poisson structure induces the structure of an L_∞ algebra on the longitudinal de Rham complex. Carrying out a similar construction on a prequantization Q should result in an L_∞ representation of this algebra.

9.2 Noncommutative prequantization

If the characteristic distribution of a Dirac structure P is regular, we may consider the groupoid algebra associated to the characteristic foliation as a substitute for the admissible functions. By adding some extra structure, as in [1][29][36], we can make this groupoid algebra into a noncommutative Poisson algebra. This means that the Poisson bracket is not a

Lie algebra structure, but rather a class with degree 2 and square 0 in the Hochschild cohomology of the groupoid algebra. It should be interesting to define a notion of representation for an algebra with such a cohomology class, and to construct such representations from prequantization spaces. Such a construction should be related to the algebraic quantization of Dirac manifolds introduced in [30].

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On the geometry of prequantization spaces

Marco Zambon and Chenchang Zhu

Abstract

Given a Poisson (or more generally Dirac) manifold P, there are two approaches to its geometric quantization: one involves a circle bundle Q over P endowed with a Jacobi (or Jacobi-Dirac) structure; the other one involves a circle bundle with a (pre)contact groupoid structure over the (pre)symplectic groupoid of P. We study the relation between these two prequantization spaces. We show that the circle bundle over the (pre)symplectic groupoid of P is obtained from the Lie groupoid of Q via an S^1 reduction that preserves both the Lie groupoid and the geometric structures.

Contents

1 Introduction

The geometric quantization of symplectic manifolds is a classical problem that has been much studied over years. The first step is to find a prequantization. A symplectic manifold (P, ω) is prequantizable iff $[\omega]$ is an integer cohomology class. Finding a prequantization means finding a faithful representation of the Lie algebra of functions on (P, ω) (endowed with the Poisson bracket) mapping the function 1 to a multiple of the identity. Such a representation space consists usually of sections of a line bundle over P [14], or equivalently of S^1-antiequivariant complex functions on the total space Q of the corresponding circle bundle [18].

For more general kinds of geometric structure on P, such as Poisson or even more generally Dirac [5] structures, there are two approaches to extend the geometric quantization of symplectic manifolds, at least as far as prequantization is concerned:

- To build a circle bundle Q over P compatible with the Possion (resp. Dirac) structure on P (see Souriau [18] for the symplectic case, [12][20][4] for the Poisson case, and [25] for the Dirac case)

- To build the symplectic (resp. presymplectic) groupoid of P first and construct a circle bundle over the groupoid [24], with the hope to quantize Poisson manifolds "all at once" as proposed by Weinstein [23].

We call Q as above a "prequantization space" for P because, when P is prequantizable, out of the hamiltonian vector fields on Q one can construct a representation of the admissible functions on P, which form a Poisson algebra, on the space of S^1 anti-equivariant functions on Q (see Prop. 5.1 of [25]). Usually however this representation is not faithful.

Since the (pre)symplectic groupoid $\Gamma_s(P)$ of P is the canonical global object associated to P, the prequantization circle bundle over $\Gamma_s(P)$ can be considered an "alternative prequantization space" for P. Furthermore, since there is a submersive Poisson (Dirac) map $\Gamma_s(P) \to P$, the admissible functions on P can be viewed as a Poisson subalgebra of the functions on $\Gamma_s(P)$, which can be prequantized whenever $\Gamma_s(P)$ is a prequantizable (pre)symplectic manifold. The resulting representation is faithful but the representation space is unsuitable because much too large.

In this paper we will not be interested in representations but only in the geometry that arises from the prequantization spaces associated to a given a Dirac manifold (P, L). Indeed our main aim is to study the relation between the two prequantization spaces above, which we will explain in Thm. 4.2, Thm. 4.9 and Thm. 4.11.

We start searching for a more transparent description of the geometric structures on the circle bundles Q, which are Jacobi-Dirac structures [25] $\bar{L}$. This will be done in Section 2, both in terms of subbundles and in terms of brackets of functions, paying particular attention to the Lie algebroid structure that $\bar{L}$ carries.

Secondly, in Section 3, we relate the Lie algebroid $\bar{L}$ associated to Q to the Lie algebroid of the prequantization of $\Gamma_s(P)$. We do this using S^1 precontact reduction, paralleling one of the motivating examples of symplectic reduction: $T^*M//_0G = T^*(M/G)$. This gives us evidence at the infinitesimal level for the relation between the Lie groupoid associated to Q and the prequantization of $\Gamma_s(P)$. The latter relation between Lie groupoids will be described in Section 4, again as an S^1 precontact reduction. We provide a direct proof in the Poisson case. In the general Dirac case, the proof is done by integrating the results of Section 3 to the level of Lie groupoids with the help of Lie algebroid path spaces. As a byproduct, we obtain the prequantization condition for $\Gamma_s(P)$ in terms of period groups on P. Then we show that this condition is automatically satisfied when the Dirac manifold P

admits a prequantization circle bundle Q over it. This generalizes some of the results in [8] and [2].

This paper ends with three appendices. Appendix A provides a useful tool to perform computations on precontact groupoids, and Appendix B describes explicitly the Lie groupoid of a locally conformal symplectic manifold. In Appendix C we apply a construction of Vorobjev to the setting of Section 2.

Notation: Throughout the paper, unless otherwise specified, (P, L) will always denote a Dirac manifold, $\pi : Q \to P$ will be a circle bundle and $\bar{L}$ will be a Jacobi-Dirac structure on Q. By Γ_s and Γ_c we will denote presymplectic and precontact groupoids respectively, and we adopt the convention that the source map induces the (Dirac and Jacobi-Dirac respectively) structures on the bases of the groupoids. By "precontact structure" on a manifold we will just mean a 1-form on the manifold.

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2 Constructing the prequantization of P

The aim of this section is to describe in an intrinsic way the geometric structures (Jacobi-Dirac structures $\bar{L}$) on the circle bundles Q induced by prequantizable Dirac manifolds (P, L), paying particular attention to the associated Lie algebroid structures. In Subsection 2.1 we will recall the non-intrinsic construction of $\bar{L}$ given in [25]. In Subsection 2.2 we will describe $\bar{L}$ intrinsicly in terms of subbundles and in Subsection 2.3 by specifying the bracket on functions that it induces.

We first recall few definitions from [25].

Definition 2.1. A *Dirac structure* on a manifold P is a subbundle of $TP \oplus T^*P$ which is maximal isotropic w.r.t. the symmetric pairing $\langle X_1 \oplus \xi_1, X_2 \oplus \xi_2 \rangle_+ = \frac{1}{2}(i_{X_2}\xi_1 + i_{X_1}\xi_2)$ and whose sections are closed under the Courant bracket

$$[X_1 \oplus \xi_1, X_2 \oplus \xi_2]_{Cou} = \big([X_1, X_2] \;\oplus\; \mathcal{L}_{X_1}\xi_2 - \mathcal{L}_{X_2}\xi_1 + \frac{1}{2}d(i_{X_2}\xi_1 - i_{X_1}\xi_2)\big).$$

If ω is a 2-form on P then its graph $\{X \oplus \omega(X, \bullet) : X \in TP\}$ is a Dirac structure iff $d\omega = 0$. Given a bivector Λ on P, the graph $\{\Lambda(\bullet, \xi) \oplus \xi : \xi \in T^*P\}$ is a Dirac structure iff Λ is a Poisson bivector. A Dirac structure L on P gives rise to (and is encoded by) a singular foliation of P, whose leaves are endowed with presymplectic forms.

A function f on a Dirac manifold (Q, L) is *admissible* if there exists a smooth vector field X_f such that $X_f \oplus df$ is a section of L. A vector field X_f as above is called a *hamiltonian vector field* of f. The set of admissible functions, with the bracket $\{f, g\} = X_g \cdot f$, forms a Lie (indeed a Poisson) algebra. Given a map $\pi : Q \to P$ and a Dirac structure L on Q, for every $q \in Q$ one can define the subspace $(\pi_\star L)_{\pi(q)} := \{\pi_* X \oplus \mu : X \oplus \pi^*\mu \in L_q\}\}$ of $T_{\pi(q)}P \oplus T^*_{\pi(q)}P$. Whenever $\pi_\star L$ is a well-defined and smooth subbundle of $TP \oplus T^*P$ it is automatically a Dirac structure on P. In this case $\pi : (Q, L) \to (P, \pi_\star L)$ is said to be a *forward Dirac map*. Similarly, if P is endowed with some Dirac structure L, $(\pi^\star L)(q) := \{Y \oplus \pi^*\xi : \pi_* Y \oplus \xi \in L_{\pi(q)}\}$ (when a smooth subbundle) defines a Dirac structure on Q, and $\pi : (Q, \pi^\star L) \to (P, L)$ is said to be a *backward Dirac map*.

Definition 2.2. A *Jacobi-Dirac structure* on Q is defined as a subbundle of $\mathcal{E}^1(Q) := (TQ \times \mathbb{R}) \oplus (T^*Q \times \mathbb{R})$ which is maximal isotropic w.r.t. the symmetric pairing

$$\langle (X_1, f_1) \oplus (\xi_1, g_1)\ ,\ (X_2, f_2) \oplus (\xi_2, g_2)\rangle_+ = \frac{1}{2}(i_{X_2}\xi_1 + i_{X_1}\xi_2 + g_2 f_1 + g_1 f_2)$$

and whose space of sections is closed under the extended Courant bracket on $\mathcal{E}^1(Q)$ given by

$$\begin{aligned}
[(X_1, f_1) \oplus (\xi_1, g_1)\ ,\ (X_2, f_2) \oplus (\xi_2, g_2)]_{\mathcal{E}^1(Q)} &= \big([X_1, X_2], X_1 \cdot f_2 - X_2 \cdot f_1\big) \\
&\oplus \big(\mathcal{L}_{X_1}\xi_2 - \mathcal{L}_{X_2}\xi_1 + \frac{1}{2} d(i_{X_2}\xi_1 - i_{X_1}\xi_2) \\
&+ f_1\xi_2 - f_2\xi_1 + \frac{1}{2}(g_2 df_1 - g_1 df_2 - f_1 dg_2 + f_2 dg_1), \\
&X_1 \cdot g_2 - X_2 \cdot g_1 + \frac{1}{2}(i_{X_2}\xi_1 - i_{X_1}\xi_2 - f_2 g_1 + f_1 g_2)\big).
\end{aligned} \tag{1}$$

We mention two examples. Given any 1-form (precontact structure) σ on Q, Graph $\left(\begin{smallmatrix} d\sigma & \sigma \\ -\sigma & 0 \end{smallmatrix}\right) \subset \mathcal{E}^1(Q)$ is a Jacobi-Dirac structure. Given a bivector field Λ and a vector field E on Q and with the notation $\tilde{\Lambda}\xi := \Lambda(\bullet, \xi)$, Graph $\left(\begin{smallmatrix} \tilde{\Lambda} & -E \\ E & 0 \end{smallmatrix}\right) \subset \mathcal{E}^1(Q)$ is a Jacobi-Dirac structure iff (Λ, E) is a Jacobi structure, i.e. by definition if it satisfies the Schouten bracket conditions $[E, \Lambda] = 0$ and $[\Lambda, \Lambda] = 2E \wedge \Lambda$. Further to a Dirac structure $L \subset TQ \oplus T^*Q$ there is an associated Jacobi-Dirac structure

$$L^c := \{(X, 0) \oplus (\xi, g) : (X, \xi) \in L, g \in \mathbb{R}\} \subset \mathcal{E}^1(Q).$$

A function f on a Jacobi-Dirac manifold $(Q, \bar{L})$ is *admissible* if there exists a smooth vector field X_f and a smooth function φ_f such that $(X_f, \varphi_f) \oplus (df, f)$ is a section of $\bar{L}$, and X_f is called a *hamiltonian vector field* of f. The set of admissible functions, denoted by $C^\infty_{adm}(Q)$, together with the bracket $\{f, g\} = X_g \cdot f + f\varphi_g$ forms a Lie algebra. There is a notion of forward and backward Jacobi-Dirac maps analogous to the one for Dirac structures.

Definition 2.3. A *Lie algebroid* over a manifold P is a vector bundle A over P together with a Lie bracket $[\cdot, \cdot]$ on its space of sections and a bundle map $\rho : A \to TP$ (the *anchor*) such that the Leibniz rule $[s_1, f s_2] = \rho s_1(f) \cdot s_2 + f \cdot [s_1, s_2]$ is satisfied for all sections s_1, s_2 of A and functions f on P.

One can think of Lie algebroids as generalizations of tangent bundles. To every Lie algebroid A one associates cochains (the sections of the exterior algebra of A^*) and a certain differential d_A; the associated *Lie algebroid cohomology* $H^\bullet_A(P)$ can be thought of as a generalization of deRham cohomology. One also defines an A-*connection* on a vector bundle $K \to P$ as map $\Gamma(A) \times \Gamma(K) \to \Gamma(K)$ satisfying the usual properties of a contravariant connection.

A Dirac structure $L \subset TP \oplus T^*P$ is automatically a Lie algebroid over P, with bracket on sections of L given by the Courant bracket and anchor the projection $\rho_{TP} : L \to TP$. Similarly, a Jacobi-Dirac structure $\bar{L} \subset \mathcal{E}^1(Q)$, with the extended Courant bracket and projection onto the first factor as anchor, is a Lie algebroid.

2.1 A non-instrinsic description of $\bar{L}$

We now recall the prequantization construction of [25], which associates to a Dirac manifold a circle bundle Q with a Jacobi-Dirac structure.

Let (P, L) be a Dirac structure. We saw above that L is a Lie algebroid with the restricted Courant bracket and anchor $\rho_{TP} : L \to TP$ (which is just the projection onto the tangent component). This anchor gives a Lie algebra homomorphism from $\Gamma(L)$ to $\Gamma(TP)$ endowed with the Lie bracket of vector fields. The pullback by the anchor therefore induces a map $\rho_{TP}^* : \Omega_{dR}^\bullet(P, \mathbb{R}) \to \Omega_L^\bullet(P)$, the sections of the exterior algebra of L^*, which descends to a map from de Rham cohomology to the Lie algebroid cohomology $H_L^\bullet(P)$ of L. There is a distinguished class in $H_L^2(P)$: on $TP \oplus T^*P$ there is an anti-symmetric pairing given by

$$\langle X_1 \oplus \xi_1, X_2 \oplus \xi_2 \rangle_- = \frac{1}{2}(i_{X_2}\xi_1 - i_{X_1}\xi_2). \tag{2}$$

Its restriction Υ to L satisfies $d_L \Upsilon = 0$. The *prequantization condition* (which for Poisson manifolds was first formulated by Vaisman) is

$$[\Upsilon] = \rho_{TP}^*[\Omega] \tag{3}$$

for some integer deRham 2-class $[\Omega]$. (3) can be equivalently phrased as

$$\rho_{TP}^* \Omega = \Upsilon + d_L \beta, \tag{4}$$

where Ω is a closed integral 2-form and β a 1-cochain for the Lie algebroid L, i.e. a section of L^*. Let $\pi : Q \to P$ be an S^1-bundle with connection form σ having curvature Ω; denote by E the infinitesimal generator of the S^1-action. In Theorem 4.1 of [25] Q was endowed with the following geometric structure, described in terms of the triple (Q, σ, β):

Theorem 2.4. *The subbundle $\bar{L}$ of $\mathcal{E}^1(Q)$ given by the direct sum of*

$$\{(X^H + \langle X \oplus \xi, \beta \rangle E, 0) \oplus (\pi^*\xi, 0) : X \oplus \xi \in L\}$$

and the line bundles generated by $(-E, 0) \oplus (0, 1)$ and $(-A^H, 1) \oplus (\sigma - \pi^*\alpha, 0)$ is a Jacobi-Dirac structure on Q. Here, $A \oplus \alpha$ is an isotropic section of $TP \oplus T^*P$ satisfying $\beta = 2\langle A \oplus \alpha, \cdot \rangle_+|_L$. Such a section always exists, and the subbundle above is independent of the choice of $A \oplus \alpha$.

We call $(Q, \bar{L})$ a "prequantization space" for (P, L) because the assignment $g \mapsto \{\pi^*g, \bullet\} = -X_{\pi^*g}$ is a representation of $C^\infty_{adm}(P)$ on the space of S^1 anti-equivariant functions on Q [25].

Triples (Q,σ,β) as above define a hermitian L-connection with curvature $2\pi i\Upsilon$ on the line bundle K corresponding to Q, via the formula

$$D_{\bullet} = \nabla_{\rho_{TP}\bullet} - 2\pi i\langle \bullet, \beta\rangle \tag{5}$$

where ∇ is the covariant connection corresponding to σ (Lemma 6.2 in [25]). We have

Proposition 2.5. *For a prequantizable Dirac manifold (P,L), the Jacobi-Dirac structure $\bar{L}$ constructed in Thm. 2.4 on Q is determined by a choice of hermitian L-connection on K with curvature $2\pi i\Upsilon$.*

Proof. We described above how the triples (Q,σ,β) used to construct $\bar{L}$ give rise to hermitian L-connections with curvature $2\pi i\Upsilon$. Conversely, all hermitian L-connections with curvature $2\pi i\Upsilon$ arise from triples (Q,σ,β) as above (Proposition 6.1 in [25]). A short computation shows that the triples that define the same L-connection as (Q,σ,β) are exactly those of the form $(Q,\sigma+\pi^*\gamma,\beta+\rho_{TP}^*\gamma)$ for some 1-form γ on P, and that these triples all define the same Jacobi Dirac structure $\bar{L}$ (Lemma 4.1 in [25]; see also the last comment in Sect. 6.1 there). □

In the next two subsections we will construct $\bar{L}$ directly from the L-connection. We end this subsection by commenting on how the various Jacobi-Dirac structure $\bar{L}$ defined above are related.

Remark 2.6. Two L-connections on K are *gauge equivalent* if the differ by $d_L\phi$ for some function $\phi: P \to S^1$. Gauge-equivalent L-connections D on K with curvature $2\pi i\Upsilon$ give rise to isomorphic Jacobi-Dirac structures: denoting by Φ the bundle automorphism of Q given by $q \mapsto q\cdot\pi^*\phi$, using the proof of Proposition 4.1 in [25] one can show that if $D_2 = D_1 - 2\pi i d_L\phi$ then $(\Phi_*, Id)\oplus((\Phi^{-1})^*, Id)$ is an isomorphism from the Jacobi-Dirac structure induced by D_1 to the one induced by D_2. (Alternatively one can check directly that for the bracket of functions, which by Remark 2.17 determine the Jacobi-Dirac structures, $\Phi^*\{\cdot,\cdot\}_{D_2} = \{\Phi^*\cdot,\Phi^*\cdot\}_{D_1}$. The gauge-equivalence classes of L-connections with curvature $2\pi i\Upsilon$ are a principal homogeneous space for $H^1_L(P,U(1))$ (see the proof of Prop. 6.1 in [25]).

Remark 2.7. It's easy to see that the prequantization space Q of a prequantizable Dirac manifold (P,L) can be endowed with various non-isomorphic Jacobi-Dirac structures $\bar{L}$. Even more is true: $(Q,\bar{L}_1)$ and $(Q,\bar{L}_2)$ will usually not even be Morita equivalent, for any reasonable notion of Morita equivalence of Jacobi-Dirac manifold (or of their respective precontact groupoids). Indeed for $P=\mathbb{R}$ with the zero Poisson structure, choosing $(Q,\sigma,\beta) = (S^1\times\mathbb{R}, d\theta, x\partial_x)$ as in Example 4.13 one obtains a Jacobi structure on Q with three leaves, whereas choosing $(S^1\times\mathbb{R}, d\theta, 0)$ one obtains a Jacobi structure with uncountably many leaves (namely all $S^1\times\{q\}$). On the other hand, one of the general properties of Morita equivalence is to induce a bijection on the space of leaves.

2.2 An intrinsic characterization of $\bar{L}$

In this subsection we fix an L-connection D on the line bundle $K\to P$ with curvature $2\pi i\Upsilon$ and construct the Lie algebroid $\bar{L}$ from L and D directly. (In Prop. 3.4 we will perform the inverse construction, i.e. we will recover L from $\bar{L}$). An alternative approach that works in particular cases is presented in Appendix C.

We begin with a useful lemma concerning flat Lie algebroid connections (compare also to Lemma 6.1 in [25]).

Lemma 2.8. *Let E be any Lie algebroid over a manifold M, K a line bundle over M, and D a Hermitian E-connection on K. Consider the central extension $E \oplus_\eta \mathbb{R}$, where $2\pi i\eta$ equals the curvature of D; then $\tilde{D}_{(Y,g)} = D_Y + 2\pi i g$ defines an $E \oplus_\eta \mathbb{R}$-connection on K which is moreover flat.*

Proof. One checks easily that $\tilde{D}$ is indeed a Lie algebroid connection. Recall that the bracket on $E \oplus_\eta \mathbb{R}$ is defined as $[(a_1, f_1), (a_2, f_2)]_{E\oplus_\eta\mathbb{R}} = ([e_1, e_2]_E, \rho(a_1)f_2 - \rho(a_2)f_1 + \eta(a_1, a_2))$, where ρ is the anchor, and that the curvature of $\tilde{D}$ is

$$R_{\tilde{D}}(e_1, e_2)s = \tilde{D}_{e_1}\tilde{D}_{e_2}s - \tilde{D}_{e_2}\tilde{D}_{e_1}s - \tilde{D}_{[e_1,e_2]}s$$

for elements e_i of $E \oplus_\eta \mathbb{R}$ and s of K. The flatness of $\tilde{D}$ follows by a straightforward calculation. $\square$

We will use of this construction, which is just a way to make explicit the structure of a transformation algebroid (see Remark 2.10 below).

Lemma 2.9. *Let A be any Lie algebroid over a manifold P, $\pi_Q : Q \to P$ a principle $SO(n)$-bundle, $\pi_K : K \to P$ the vector bundle associated to the standard representation of $SO(n)$ on $\mathbb{R}^n$, and $\tilde{D}$ a flat A-connection on K preserving its fiber-wise metric. The A-connection induces a bundle map $h_Q : \pi_Q^*A \to TQ$ (the "horizontal lift") that can be used to extend, by the Leibniz rule, the obvious bracket on $SO(n)$-invariant sections of π_Q^*A to all sections of π_Q^*A. The vector bundle π_Q^*A, with this bracket and h_Q as an anchor, is a Lie algebroid over Q.*

Proof. We first recall some facts from Section 2.5 in [11]. The A-connection $\tilde{D}$ on the vector bundle K defines a map (the "horizontal lift") $h_K : \pi_K^*A \to TK$ covering the anchor $A \to TP$ by taking parallel translations of elements of K along A-paths. See Section 4.2 for the definition of A-paths. Explicitly, fix an A-path $a(t)$ with base path $\gamma(t)$, a point $x \in \pi_K^{-1}(\gamma(0))$ and let $\tilde{\gamma}(t)$ the unique path in K (over $\gamma(t)$) starting at x with $\tilde{D}_{a(t)}\tilde{\gamma}(t) = 0$. We can always write $\tilde{D} = \nabla_{\rho\bullet} - \tilde{\beta}$ where ∇ is a metric TP-connection on A and $\tilde{\beta} \in \Gamma(A^*) \otimes \mathfrak{so}(K)$; then $\nabla_{\rho a(t)}\tilde{\gamma}(t) = \langle\tilde{\beta}, a(t)\rangle\tilde{\gamma}(t)$. Since the left hand side is the projection of the velocity of $\tilde{\gamma}(t)$ along the Ehresmann distribution H corresponding to ∇, we obtain $\frac{d}{dt}\tilde{\gamma}(t) = (\frac{d}{dt}\gamma(t))^H + \langle\tilde{\beta}, a(t)\rangle\tilde{\gamma}(t)$, so that

$$h_K(a(0), x) := \frac{d}{dt}|_{t=0}\tilde{\gamma}(t) = \rho(a(0))^H + \langle\tilde{\beta}, a(0)\rangle x. \tag{6}$$

Of course h_K does not depend on ∇ or $\tilde{\beta}$ directly, but just on $\tilde{D}$. By our assumptions h_K is induced by a "horizontal lift" for the principle bundle Q, i.e. by a $SO(n)$-equivariant map $h_Q : \pi_Q^*A \to TQ$ covering the anchor of A. Since our A-connection $\tilde{D}$ is flat, the map that associates to a section s of A the vector field $h_Q(\pi_Q^*s)$ on Q is a Lie algebra homomorphism.

On sections $\pi_Q^*s_1$, $\pi_Q^*s_2$ of π_Q^*A which are pullbacks of sections of A we define the bracket to be $\pi_Q^*[s_1, s_2]$, and we extend it to all sections of π_Q^*A by using h_Q as an anchor and forcing the Leibniz rule. We have to show that the resulting bracket satisfies the Jacobi identity. Given sections s_i of A and a function f on Q one can show that the Jacobiator $[[\pi_Q^* s_1, f \cdot \pi_Q^* s_2], \pi_Q^* s_3] + c.p. = 0$ by using the facts that the bracket on sections of A satisfies the Jacobi identity and that the correspondence $\pi_Q^* s_i \mapsto h_Q(\pi_Q^* s_i)$ is a Lie algebra homomorphism. Similarly, the Jacobiator of arbitrary sections of Q is also zero due to fact that h_Q actually induces a homomorphism on *all* sections of $\pi_Q^* A$. □

Remark 2.10. Using h_K instead of h_Q in the construction of the previous lemma leads to a Lie algebroid structure on $\pi_K^* A \to K$. As Kirill Mackenzie pointed out to us, $\pi_K^* A$ is just the transformation algebroid arising from the Lie algebroid action of A on K given by the flat connection $\tilde{D}$. Similarly, the Lie algebroid structure on $\pi_Q^* A$ we constructed in the lemma is the transformation algebroid structure coming from h_Q, which is viewed here as a Lie algebroid action of A on Q.

Now we come back to our original setting, where we consider the Lie algebroid L over P and a hermitian L-connection D on the line bundle K over P. Consider L^c, the Jacobi-Dirac structure on P naturally associated to L. There is a canonical isomorphism $L^c \to L \oplus_\Upsilon \mathbb{R}$, $(X,0) \oplus (\xi, g) \mapsto (X, \xi, g)$ of Lie algebroids over P [8]. Lemma 2.8 provides us with a flat $L \oplus_\Upsilon \mathbb{R}$-connection $\tilde{D}$ on K , and by Lemma 2.9 the pullback of $L \oplus_\Upsilon \mathbb{R}$ to Q (the circle bundle associated to K) is endowed with a Lie algebroid structure. Using equation (6) one sees that its anchor $h_Q : \pi_Q^*(L \oplus_\Upsilon \mathbb{R}) \to TQ$, at any point of Q, is given by

$$h_Q(X, \xi, g) = X^H + (\langle X \oplus \xi, \beta \rangle - g)E \tag{7}$$

(here we make immaterial choices to write D as in equation (5) and denote by ^{H} the horizontal lift w.r.t. $\ker \sigma$). This formula for the anchor suggests how to identify $\pi_Q^*(L \oplus_\Upsilon \mathbb{R})$ with a subbundle of $\mathcal{E}^1(Q)$: we will show that the natural injection

$$I : \pi_Q^*(L \oplus_\Upsilon \mathbb{R}) \to \bar{L} \subset \mathcal{E}^1(Q), \quad I(X, \xi, g) = (h_Q(X, \xi, g), 0) \oplus (\pi^* \xi, g)$$

is a Lie algebroid morphism, whose image is a codimension one subalgebroid of $\bar{L}$ which we denote by $\bar{L}_0$. We regard $\bar{L}_0$ as a "lift" of L (or rather L^c) obtained using the hermitian L-connection D. Now we can describe the Jacobi-Dirac structure $\bar{L}$ prequantizing L in invariant terms and characterize partially (see also Remark 2.14) its Lie algebroid structure:

Theorem 2.11. *Assume that the Dirac manifold (P, L) satisfies the prequantization condition* (3)*. Fix the line bundle K over P associated with $[\Omega]$ and a Hermitian L-connection D on K with curvature $2\pi i \Upsilon$. Denote as above by $\bar{L}_0$ the lift of L^c by the connection D. Then $\bar{L}$, the subbundle defined in Thm. 2.4, is characterized as the unique Jacobi-Dirac structure on Q which contains $\bar{L}_0$ and which is different from $(\pi^\star L)^c$ (where $\pi^\star L$ denotes the pullback Dirac structure of L). Further $\bar{L}_0$ is canonically isomorphic to $\pi_Q^*(L \oplus_\Upsilon \mathbb{R})$ as a Lie algebroid.*

Proof. We first show that $I : \pi_Q^*(L \oplus_\Upsilon \mathbb{R}) \to \bar{L}$ is indeed a Lie algebroid morphism. We compute for S^1 invariant sections

$$\begin{aligned}
&[I(X_1, \xi_1, 0), I(X_2, \xi_2, 0)]_{\mathcal{E}^1(Q)} \\
&\quad = I([(X_1, \xi_1), (X_2, \xi_2)]_{Cou}, 0) + \langle (X_1, \xi_1), (X_2, \xi_2) \rangle_- ((-E, 0) \oplus (0, 1)) \qquad (8) \\
&\quad = I([(X_1, \xi_1, 0), (X_2, \xi_2, 0)]_{\pi_Q^*(L \oplus_\Upsilon \mathbb{R})})
\end{aligned}$$

and $[I(X,\xi,0), I(0,0,1)]_{\mathcal{E}^1(Q)} = 0$; then one checks that I respects the anchor maps of $\pi_Q^*(L \oplus_\Upsilon \mathbb{R})$ and $\bar{L}$.

To prove the above characterization of $\bar{L}$ we show that there are exactly two maximally isotropic subbundles of $\mathcal{E}^1(Q)$ containing $\bar{L}_0$. Indeed, denoting by $(\bar{L}_0)^\perp$ the orthogonal of $\bar{L}_0$ w.r.t. the pairing $\langle \bullet, \bullet \rangle_+$, the quotient $(\bar{L}_0)^\perp / \bar{L}_0$ is a rank 2 vector bundle over Q which inherits a non-degenerate symmetric pairing on its fibers. Every fiber of such bundle is isomorphic to $\mathbb{R}^2$ with pairing $\langle (a,b),(a',b') \rangle = \frac{1}{2}(ab' + ba')$, which clearly contains exactly two isotropic subspaces of rank one (namely $\mathbb{R}(1,0)$ and $\mathbb{R}(0,1)$). So there are at most two maximally isotropic subbundles of $\mathcal{E}^1(Q)$ containing $\bar{L}_0$; indeed there are exactly two: $\bar{L}$ and $\bar{L}_0 \oplus \mathbb{R}((0,0)\oplus(0,1))$. The latter is $\pi^\star L = \{Y \oplus \pi^*\xi : \pi_*(Y) \oplus \xi \in L\}$ viewed as a Jacobi-Dirac structure on Q, hence we are done. □

Remark 2.12. Using the canonical identifications of Lie algebroids $L \oplus_\Upsilon \mathbb{R} \cong L^c$ and $\pi_Q^*(L \oplus_\Upsilon \mathbb{R}) \cong \bar{L}_0$ the natural Lie algebroid morphism $\pi_Q^*(L \oplus_\Upsilon \mathbb{R}) \to L \oplus_\Upsilon \mathbb{R}$ is

$$\Phi : \bar{L}_0 \to L^c, (X,0) \oplus (\pi^*\xi, g) \mapsto (\pi_* X, 0) \oplus (\xi, g). \tag{9}$$

Remark 2.13. The construction of Thm. 2.11 gives a quick way to see that the subbundle $\bar{L}$ of $\mathcal{E}^1(Q)$, as defined in Thm. 2.4, is indeed closed under the extended Courant bracket: $\bar{L}_0$ is closed since we realized it as a Lie algebroid, and the sum with the span of the section $(-A^H, 1) \oplus (\sigma - \pi^*\alpha, 0)$ is closed under the bracket because $\langle [s_1, s_2]_{\mathcal{E}^1(Q)}, s_3 \rangle_+$ (for s_i sections of $\mathcal{E}^1(Q)$) is a totally skew-symmetric tensor [13].

Remark 2.14. The characterization of $\bar{L}_0$ as the transformation algebroid of some action of $L \oplus_\Upsilon \mathbb{R} \cong L^c$ on Q (Thm. 2.11) shows that if the Lie algebroid L^c is integrable then $\bar{L}_0$ is integrated by the corresponding transformation groupoid. Unfortunately using Thm. 2.11 we are not able to make the same conclusion for $\bar{L}$. Looking at the brackets on $\bar{L}$ is not very illuminating: it is determined by (8) and

$$\begin{aligned} [I(X,\xi,0), (-A^H, 1) \oplus (\sigma - \pi^*\alpha, 0)]_{\mathcal{E}^1(Q)} &= I(-[(X,\xi),(A,\alpha)]_{Cou}, 0) \\ &+ I(0, \Omega(X) - \xi + \frac{1}{2} d\langle X \oplus \xi, \beta \rangle, 0) - \langle A, \xi \rangle \left((-E,0) \oplus (0,1)\right). \end{aligned} \tag{10}$$

The remaining brackets between sections of the form $I(X,\xi,0)$, $I(0,0,1)$ and $(-A^H,1) \oplus (\sigma - \pi^*\alpha, 0)$ vanish, and by the Leibniz rule these brackets determine the bracket for arbitrary sections of $\bar{L}$.

Remark 2.15. Different choices of L-connection on the line bundle K with curvature $2\pi i \Upsilon$ usually lead to Lie algebroids $\bar{L}$ with different foliations (see Remark 2.7), which therefore can not be isomorphic. However the subalgebroids $\bar{L}_0$ are always isomorphic. Indeed any two connections with the same curvature are of the form D and $D' = D + 2\pi i \gamma$, where γ is a closed section of L^* (see Prop. 6.1 in [25]). A computation using $d_L \gamma = 0$ shows that $(X,\xi) \oplus g \mapsto (X,\xi) \oplus (g - \langle (X,\xi), \gamma \rangle)$ is a Lie algebroid automorphism of $L \oplus_\Upsilon \mathbb{R}$. Further this automorphism intertwines the Lie algebroid actions (7) of $L \oplus_\Upsilon \mathbb{R}$ on Q given by the "horizontal lifts" of the flat connections $\tilde{D}$ and $\tilde{D}'$. Hence the transformation algebroids of the two actions are isomorphic, as is clear from the description of Lemma 2.9.

We exemplify the fact that actions coming from different flat connections are intertwined by a Lie algebroid automorphism (something that can not occur if the anchor of the Lie algebroid is injective) in the simple case when the Dirac structure on P comes from a close 2-form ω: the Lie algebroid action of $TP \oplus_\omega \mathbb{R}$ on Q via a connection ∇ (with curvature $2\pi i\omega$) is intertwined to the obvious action of the Atiyah algebroid TQ/S^1 on Q (essentially given by the identity map) via $TP \oplus_\omega \mathbb{R} \cong TQ/S^1$ is $(X,g) \mapsto X^H - \pi^* gE$, where σ is the connection on the circle bundle Q corresponding to ∇.

2.3 Describing $\bar{L}$ via the bracket on functions

In this subsection we will describe the geometric structure $\bar{L}$ on the circle bundle Q in terms of the bracket on the admissible functions on Q; by Remark 2.17 below the bracket on functions uniquely determines $\bar{L}$.

We adopt the following notation. F_S denotes the function on Q associated to a section S of the line bundle K: F_S is just the restriction to the bundle of unit vectors Q of the fiberwise linear function on K given by $\langle \cdot, S\rangle$, where $\langle \cdot,\cdot\rangle$ is the S^1-invariant *real* inner product on K corresponding to the chosen Hermitian form on K. Alternatively F_S can be described as the real part of the S^1-antiequivariant function on Q that naturally corresponds to the section S. By iS we denote the image of the section S by the action of $i \in S^1$ (i.e. S rotated by 90°), and f and g are functions on P.

Proposition 2.16. *Assume that the Dirac manifold (P,L) satisfies the prequantization condition (3). Fix the line bundle K over P associated with $[\Omega]$ and a Hermitian L-connection D on K with curvature $2\pi i\Upsilon$. Denote by $\tilde{D}$ the flat connection induced as in Lemma 2.8 and by $h_Q : \pi_Q^*(L \oplus_\Upsilon \mathbb{R}) \to TQ$ the horizontal lift associated to $\tilde{D}$ given by (7).*

Suppose a Jacobi-Dirac structure $\hat{L}$ on Q has the following two properties: first, nearby any $q \in Q$ such that $TP \cap L$ is regular near $\pi(q)$, the admissible functions for $\hat{L}$ are exactly those that are constant along the leaves of $\{h_Q(X,0,0) : X \in TP \cap L\}$. Second, the bracket on locally defined admissible functions is given by

- $\{\pi^* f, \pi^* g\}_Q = \pi^*\{f,g\}_P$
- $\{\pi^* f, F_S\}_Q = F_{-\tilde{D}_{X_f,df,f}S}$
- $\{\pi^* f, 1\}_Q = 0$
- $\{F_S, 1\}_Q = -2\pi F_{iS}$.

Then $\hat{L}$ must be the Jacobi-Dirac structure $\bar{L}$ given in Thm. 2.4.

Conversely, the Jacobi-Dirac structure $\bar{L}$ given in Thm. 2.4 has the two properties above.

Proof. We start by showing that the Jacobi-Dirac structure $\bar{L}$ constructed in Thm. 2.4 satisfies the above two properties. On the set of points where the "characteristic distribution" $C := \bar{L} \cap (TQ \times \mathbb{R}) \oplus (0,0)$ of any Jacobi-Dirac structure has constant rank the admissible functions are exactly the functions f such that (df, f) annihilate C. In our case $C = \{X^H + \langle \alpha, X\rangle E : X \in L \cap TP\} = \{h_Q(X,0,0) : X \in TP \cap L\}$ is actually contained in TQ, so the admissible functions are those constant on the leaves of C as claimed.

Now we check that the four formulae for the bracket hold. The first equation follows from the fact that the pushforward of $\bar{L}$ is the Jacobi-Dirac structure associated to L (see Section 5 in [25]).

For the second equation we make use of the formulae

$$E(F_S) = -2\pi F_{iS} \qquad \text{and } X^H(F_S) = F_{\nabla_X S},$$

where we make some choice to express D as in equation (5) and X^H denotes to horizontal lift of $X \in TP$ using the connection on Q corresponding to the covariant derivative ∇ on K. Using these formulae we see

$$\begin{aligned}
\{\pi^* f, F_S\}_Q &= -\langle dF_S, X_f^H + \langle (X_f, df), \beta\rangle E - fE\rangle \\
&= F_{-\nabla_{X_f} S + 2\pi i(\langle (X_f, df), \beta\rangle - f} \\
&= F_{-\tilde{D}_{X_f, df, f} S}.
\end{aligned}$$

For the last two equations just notice that, since $(-E,0) \oplus (0,1)$ is a section of $\bar{L}$, the bracket of any admissible function with the constant function 1 amounts to applying $-E$ to that function.

Now we show that if a Jacobi-Dirac structure $\hat{L}$ satisfies the two properties in the statement of the proposition, then it must be $\bar{L}$. By Remark 2.17, the bracket of $dimQ - rkC + 1$ independent functions at regular points of $C := \hat{L} \cap (TQ \times \mathbb{R}) \oplus (0,0)$ determines $\hat{L}$, so we have to show that our two properties carry the information of the bracket of $dimQ - rkC + 1$ independent functions at regular points of C .

It will be enough to consider the open dense subset of the regular points of C where $C = \{h_Q(X,0,0) : X \in TP \cap L\}$ (This subset is dense because it includes the points q such that C is regular near q and $TP \cap L$ is regular near $\pi(q)$). Since there C is actually contained in TQ it is clear that 1 and $\pi^* f$ are admissible functions, for f any admissible function on P (this means that f is constant along the leaves of $L \cap TP$; there are $dimP - rkC$ such f which are linearly independent at $\pi(q)$). Further we can construct an admissible function F_S as follows: take a submanifold Y near $\pi(q)$ which is transverse to the foliation given by $L \cap TP$, and define the section $S|_Y$ so that it has norm one (i.e. its image lies in $Q \subset K$). Then extend S to a neighborhood of $\pi(q)$ by starting at a point y of Y and "following" the leaf of C through $S(y)$ (notice that C is a flat partial connection on $Q \to P$ covering the distribution $L \cap TP$ on P). Since C is S^1 invariant, the resulting function F_S is clearly constant along the leaves of C, hence admissible. Altogether we obtain $dimQ - rkC + 1$ admissible functions in a neighborhood of q for which we know the brackets, so we are done. □

Remark 2.17. On any Jacobi-Dirac manifold $(Q, \hat{L})$ the bracket on the sheaf of admissible functions $(C^\infty_{adm}(Q), \{\cdot,\cdot\})$ determines the subbundle $\hat{L}$ of $\mathcal{E}^1(Q)$. (This might seem a bit surprising at first, since the set of admissible functions is usually much smaller than $C^\infty(Q)$).

The set of points where $C := \hat{L} \cap (TQ \times \mathbb{R}) \oplus (0,0)$ (an analog of a "characteristic distribution") has locally constant rank is an open dense subset of Q, since C is an intersection of subbundles. Hence by continuity it is enough to reconstruct the subbundle $\bar{L}$ on each point q of this open dense set.

Since we assume that C has constant rank near q, given $C^\infty_{adm}(Q)$ in a neighborhood of q we can reconstruct C as the distribution annihilated by (df, f) where f ranges over $C^\infty_{adm}(Q)$. We can clearly find $dimQ - rkC + 1$ admissible functions f_i such that $\{(df_i, f_i)\}$

forms a basis of $\rho_{T^*Q\times\mathbb{R}}(\hat{L}) = C^\circ$ near q. The fact that each f_i is an admissible function means that there exist (X_i, ϕ_i) such that $(X_i, \phi_i) \oplus (df_i, f_i)$ is a smooth section of $\hat{L}$. Now knowing the bracket of any f_j with the other f_i's, i.e. the pairing of (X_j, ϕ_j) with all elements of $\rho_{T^*Q\times\mathbb{R}}(\hat{L})$, does not quite determine (X_j, ϕ_j). However it determines (X_j, ϕ_j) up to sections of C, hence the direct sum of the span of all $(X_i, \phi_i) \oplus (df_i, f_i)$ and of C is a well defined subbundle of $\mathcal{E}^1(Q)$. Moreover it has the same dimension as $\hat{L}$ and it is spanned by sections of $\hat{L}$, so it is $\hat{L}$.

3 Prequantization and reduction of Jacobi-Dirac structures

In the last section we considered a prequantizable Dirac manifold (P, L) and endowed Q (the total space of the circle bundle over P) with distinguished Jacobi-Dirac structures $\bar{L}$.

We are interested in the relation between the *Lie algebroid* structures on $\bar{L}$ and L^c (the Jacobi-Dirac structure canonically associated to L), because they will give an indication of the relation between the Lie groupoids integrating them. The map Φ of (9) is a natural surjective morphism of Lie algebroids from the codimension one subalgebroid $\bar{L}_0$ of $\bar{L}$ to L^c, so one may hope to extend Φ to a Lie algebroid morphism defined on $\bar{L}$. However in general there cannot be any Lie algebroid morphism from $\bar{L}$ to L^c or L with base map π: recall that a morphism of Lie algebroids maps each orbit of the source Lie algebroid into an orbit of the target Lie algebroid. If the map $\pi : Q \to P$ induced a morphism of Lie algebroids, then the orbits[1] of $\bar{L}$ would be mapped into the orbits of L^c (which coincide with those of L). However this happens exactly when (one and hence all choices of) the vector field A appearing in Thm. 2.4 is tangent to the foliation of L (see Section 4.1 of [25]). In the case of Example 4.13, i.e. $Q = S^1 \times \mathbb{R}$ and $P = \mathbb{R}$, the orbits of $T^*Q \times \mathbb{R}$ are exactly three (namely $S^1 \times \mathbb{R}_+, S^1 \times \{0\}$ and $S^1 \times \mathbb{R}_-$), and π does not map them into the orbits of T^*P, which are just points.

In this section we will take advantage of the fact that $\bar{L}$, in addition to the Lie algebroid structure, also carries a geometric structure, namely a precontact structure $\theta_{\bar{L}} \in \Omega^1(\bar{L})$ defined as follows:

$$\theta_{\bar{L}} := pr^*(\theta_c + dt), \tag{11}$$

where θ_c is the canonical 1-form on the cotangent bundle T^*Q, t is the coordinate on $\mathbb{R}$, and pr is the projection of $\bar{L} \subset \mathcal{E}^1(Q)$ onto $T^*Q \times \mathbb{R}$. We will use the the 1-form $\theta_{\bar{L}}$ to recover the Lie algebroid L^c from $\bar{L}$ via a precontact reduction procedure, which we will globalize to the corresponding Lie groupoids in the next Section.

3.1 Reduction of Jacobi-Dirac structures as precontact reduction

We recall a familiar fact: in symplectic geometry, we have the well-known motivating example of symplectic reduction $T^*M//_0G = T^*(M/G)$. In [9], it is extended to contact geometry by replacing T^*M by the cosphere bundle of M. Here we prove a similar result by replacing T^*M by $T^*M \times \mathbb{R}$–another natural contact manifold associated to any manifold M. Later on we will use this to reduce a G-invariant Jacobi-Dirac structure on M to a

[1]The orbits of a Lie algebroid are the leaves integrating the (singular) distribution given by the image of the anchor map.

Jacobi-Dirac structure on M/G.

Let a Lie group G act on a contact manifold (C, θ) preserving the contact form θ.

Then, a moment map is a map J from the manifold M to $\mathfrak{g}^*$ (the dual of the Lie algebra) such that for all v in the Lie algebra $\mathfrak{g}$:

$$\langle J, v \rangle = \theta_M(v_M), \tag{12}$$

where v_M is the infinitesimal generator of the action on M given by v. The moment map J is automatically equivariant with respect to the coadjoint action of G on $\mathfrak{g}^*$ given by $\xi \cdot g = L_g^* R_{g^{-1}}^* \xi$. A group action as above together with its moment map is called *Hamiltonian*. Notice that any group action preserving the contact form is Hamiltonian. In the above setting there are two ways to perform contact reduction, developed by Albert [1] and Willett [26] respectively, which agree when one performs reduction at $0 \in \mathfrak{g}^*$:

$$C//_0 G := J^{-1}(0)/G$$

is again a smooth contact manifold with induced contact form $\bar{\theta}$ such that $\pi^*(\bar{\theta}) = \theta|_{J^{-1}(0)}$.

Lemma 3.1. *Let the group G act on manifold M freely and properly. Then G has an induced action on the contact manifold $(C := T^*M \times \mathbb{R}, \theta := \theta_c + dt)$ where θ_c is the canonical 1-form on T^*M and t is the coordinate on $\mathbb{R}$. Then this action is Hamiltonian and the contact reduction at 0 is*

$$T^*M \times \mathbb{R}//_0 G = T^*(M/G) \times \mathbb{R}.$$

Proof. The induced G action on $T^*M \times \mathbb{R}$ is by $g \cdot (\xi, t) = ((g^{-1})^* \xi, t)$, and it preserves the 1-form $\theta_c + dt$. The projection of this action on M is the G action on M so it is also free and proper. Then the moment map J is determined by

$$\langle J(\xi, t), v \rangle = (\theta_c + dt)_{(\xi,t)}(v_C) = \theta_c(v_C) = \langle \xi, v_M \rangle,$$

where v_C (resp. v_M) denotes the vector filed corresponding to the infinitesimal action of G on the manifold C (resp. M). Since all infinitesimal generators v_C are nowhere proportional to the Reeb vector field $\frac{\partial}{\partial t}$, by Remark 3.2 in [26] all points of $T^*M \times \mathbb{R}$ are regular points of J. So $J^{-1}(0) = \{(\xi, t) : \langle \xi, v_M \rangle = 0 \ \forall v \in \mathfrak{g}\} = \{(\pi^*\mu, t) : \mu \in T^*(M/G)\}$ (with $\pi : M \to (M/G)$) is a smooth manifold. Therefore it is not hard to see that there is a well-defined

$$\Phi : J^{-1}(0)/G \to T^*(M/G) \times \mathbb{R}, \quad ([\xi], t) \mapsto (\mu, t),$$

where μ is uniquely determined by $\pi^*\mu = \xi$ and we used the notation $[\cdot]$ to denote the quotient of points (and later tangent vectors) of $J^{-1}(0)$ by the G action. It is not hard to see that Φ is an isomorphism since the two sides have the same dimension and Φ is obviously surjective. The contact form on $T^*(M/G) \times \mathbb{R}$ corresponding to the reduced contact form $\bar{\theta}$ via the isomorphism Φ is the canonical one: for a tangent vector $([v], \lambda\frac{\partial}{\partial t}) \in T_{[\xi],t}(J^{-1}(0)/G)$,

$$\bar{\theta}_{[\xi],t}([v], \lambda\frac{\partial}{\partial t}) = \theta_{\xi,t}(v, \lambda\frac{\partial}{\partial t}) = \xi(p_* v) + \lambda = \mu(\bar{p}_* \Phi_*[v]) + \lambda,$$

where $p : T^*M \to M$ and $\bar{p} : T^*(M/G) \to M/G$. Here we used $\bar{p}_* \Phi_*[v] = \pi_* p_* v$, which follows from the fact that Φ is a vector bundle map, and we abuse notation by denoting with the same symbol a restriction of Φ. $\square$

This result extends to the precontact situation: instead of the contact manifold $T^*M\times\mathbb{R}$ we consider a Jacobi-Dirac subbundle $\bar{L}\subset\mathcal{E}^1(M)$, which together with the 1-form $\theta_{\bar{L}}\in\Omega^1(\bar{L})$ defined in (11) is a precontact manifold.

Proposition 3.2. *When $(Q,\bar{L})$ is a Jacobi-Dirac manifold, $\bar{L}$ is a precontact manifold as described above. If the group G acts freely and properly on Q preserving the Jacobi-Dirac structure, the action lifts to a free proper Hamiltonian action on $\bar{L}$ with moment map J,*

$$\langle J((X,f)\oplus(\xi,g)),v\rangle=\theta_{\bar{L}}|_{(X,f)\oplus(\xi,g)}(v_{\bar{L}})=\xi(v_Q).$$

Write $\mathfrak{g}_Q$ as a short form for $\{v_Q : v\in\mathfrak{g}\}\subset TQ$, and let $\pi_\star\bar{L}\subset\mathcal{E}^1(P)$ be the pushforward of $\bar{L}$ via $\pi:Q\to P:=Q/G$. Then

1. *$J^{-1}(0)$ is a subalgebroid of $\bar{L}$ iff $\bar{L}\cap(\mathfrak{g}_Q,0)\oplus(0,0)$ has constant rank, and in that case $\bar{L}//_0G:=J^{-1}(0)/G$ has an induced Lie algebroid structure;*

2. *$J^{-1}(0)/G\cong\pi_\star\bar{L}$ both as Lie algebroids and precontact manifolds, iff $\bar{L}\cap(\mathfrak{g}_Q,0)\oplus(0,0)=\{0\}$. Here the precontact forms are the reduced 1-form on $J^{-1}(0)/G$ and the one defined as in (11) on $\pi_\star\bar{L}$ respectively.*

Proof. The G action on Q lifts to $\bar{L}$ by $g\cdot(X,f)\oplus(\xi,g)=(g_*X,f)\oplus((g^{-1})^*\xi,g)$, and the resulting moment map J is clearly as claimed in the statement.

To prove (1) we start with some linear algebra and fix $x\in Q$. We have a map $\pi_*: T_xQ\to T_{\pi(x)}(Q/G)$, hence we can push forward $\bar{L}_x$ to

$$(\pi_\star\bar{L})_{\pi(x)}=\{(\pi_*X,f)\oplus(\mu,g):(X,f)\oplus(\pi^*\mu,g)\in\bar{L}_x\}$$

to obtain a linear Jacobi-Dirac subspace of $\mathcal{E}^1(Q/G)_{\pi(x)}$. Since $\bar{L}$ is G invariant, doing this at every $x\in Q$ we obtain a well defined subbundle of $\mathcal{E}^1(Q/G)$, which however might fail to be smooth[2]. We have a surjective map

$$\begin{aligned}&\Phi: J^{-1}(0)=\{(X,f)\oplus(\xi,g)\in\bar{L}:\xi=\pi^*\mu\text{ for some }\mu\in T_{\pi(x)}(Q/G)\}\to\pi_\star\bar{L}\\&(X,f)\oplus(\xi,g)\mapsto(\pi_*X,f)\oplus(\mu,g)\end{aligned}\tag{13}$$

whose kernel is exactly $J^{-1}(0)\cap(\mathfrak{g}_Q,0)\oplus(0,0)$ (Notice that the map is well defined for π is a submersion). So $J^{-1}(0)$ has constant rank iff $J^{-1}(0)\cap(\mathfrak{g}_Q,0)\oplus(0,0)=\bar{L}\cap(\mathfrak{g}_Q,0)\oplus(0,0)$ does. In this case it is easy to see that $J^{-1}(0)$ is closed under the Courant bracket: the Courant bracket of two sections of $J^{-1}(0)$ lie in $\bar{L}$ (because $\bar{L}$ is closed under the bracket), therefore one just has to show that its cotangent component is annihilated by $\mathfrak{g}_Q$. By a straight-forward computation this is true for G-invariant sections, and by the Leibniz rule it follows for all sections of $J^{-1}(0)$, i.e. $J^{-1}(0)$ is a subalgebroid. Clearly $J^{-1}(0)/G$ becomes a Lie algebroid with the bracket induced from the one on $J^{-1}(0)$ and anchor $([X],f)\oplus([\xi],g)\mapsto\pi_*X$ (where $[\cdot]$ denotes the equivalence relation given by the G action).

To prove (2) consider the map Φ above. It induces an isomorphism of vector bundles over P between $J^{-1}(0)/G$ and $\pi_\star\bar{L}$ iff it is fiberwise injective, i.e. iff $\bar{L}\cap(\mathfrak{g}_Q,0)\oplus(0,0)=\{0\}$. Since $J^{-1}(0)/G$ (being a precontact reduction) is a smooth manifold and $J^{-1}(0)/G\cong\pi_\star\bar{L}$ is point-wise a subbundle of $\mathcal{E}^1(P)$, it follows that $\pi_\star\bar{L}$ is a smooth vector bundle over P.

[2]For example it is not smooth when $G=\mathbb{R}$, $Q=\mathbb{R}^2$, $v_Q=\frac{\partial}{\partial x}$ and $\bar{L}$ is the graph of the 1-form $\frac{y^2}{2}dx$.

We are left with showing that Φ induces an isomorphism of Lie algebroids and precontact manifolds. Using the fact that operations appearing in the definition of Courant bracket such as taking Lie derivatives commute with taking quotient of G (for example $\pi^*(L_{\pi_* X}\mu) = L_X \pi^*\mu$) we deduce that $\Phi : J^{-1}(0) \to \pi_\star \bar{L}$ is a surjective morphism of Lie algebroids, hence the induced map $\Phi : J^{-1}(0)/G \to \pi_\star \bar{L}$ an isomorphism of Lie algebroids.

The isomorphism of precontact manifolds follows from an entirely similar argument as in Lemma 3.1. We consider a tangent vector $([w], \kappa\frac{\partial}{\partial s}) \oplus ([v], \lambda\frac{\partial}{\partial t}) \in T_{([X],f)\oplus([\xi],g)}(J^{-1}(0)/G)$, then $\Phi(([X], f) \oplus ([\xi], g)) = (\pi_* X, f) \oplus (\mu, g)$, where $\pi^*\mu = \xi$. So the induced 1-form $\bar{\theta}$ on $J^{-1}(0)/G$ satisfies,

$$\bar{\theta}_{[X],f,[\xi],g}([w], \kappa\frac{\partial}{\partial s}) \oplus ([v], \lambda\frac{\partial}{\partial t}) = \theta_{X,f,\xi,g}(w, \kappa\frac{\partial}{\partial t}) \oplus (v, \lambda\frac{\partial}{\partial t}) = \xi(p_* v) + \lambda = \mu(\bar{p}_*\Phi_*[v]) + \lambda,$$

where $p : \bar{L} \to Q$ and $\bar{p} : \pi_\star \bar{L} \to P$ are projections. Therefore $\bar{\theta} = \Phi^*\theta_{\pi_\star \bar{L}}$ with $\theta_{\pi_\star \bar{L}}$ the canonical 1-form as in (11). □

Remark 3.3. A special case of Prop. 3.2 is the usual reduction of basic 1-forms: if the Jacobi-Dirac structure $\bar{L}$ of Prop. 3.2 comes from 1-form σ on Q such that $\mathfrak{g}_Q \subset ker\sigma$, then the pushforward $\pi_\star \bar{L}$ is given by the unique 1-form σ_{red} on $P = Q/G$ satisfying $\pi^*\sigma_{red} = \sigma$.

3.2 Reduction of prequantizing Jacobi-Dirac structures

Now we adapt the general theory of reduction of Jacobi-Dirac manifolds discussed in the previous subsection to our situation, namely we consider a prequantization Q of Dirac manifold (P, L). Then Q is Jacobi-Dirac with a free and proper S^1 action which preserves the Jacobi-Dirac structure $\bar{L}$. Let $L^c = \{(X, 0) \oplus (\xi, g) : (X, \xi) \in L, g \in \mathbb{R}\}$ denote the Jacobi-Dirac structure associated to the Dirac manifold (P, L). Then L^c naturally has a precontact form as described in (11). The algebroids $\bar{L}$, L^c and L fit into the following diagram (where we denote dimensions and ranks by superscripts):

$$\begin{tikzcd}
\bar{L}^{n+2} \arrow[d] & (L^c)^{n+1} \arrow[r] \arrow[d] & L^n \arrow[dl] \\
Q^{n+1} \arrow[r, "\pi"] & P^n &
\end{tikzcd}$$

The left two Lie algebroids in the diagram are related by the reduction described in the next proposition:

Proposition 3.4. *When $(Q, \bar{L})$ is a prequantization of Dirac manifold (P, L) we have $J^{-1}(0) = \bar{L}_0$ (recall that $\bar{L}_0$ was defined at the end of Section 2.2) and the isomorphisms of precontact manifolds and Lie algebroids,*

$$\bar{L}//_0 S^1 \cong L^c.$$

Proof. The equality is clear from the characterization of $J^{-1}(0)$ in eq. (13) and from the definition of $\bar{L}_0$. For the isomorphism notice that $L^c = \pi_\star \bar{L}$ (this is equivalent to saying that π is a forward Jacobi-Dirac map) and apply Prop. 3.2 (which holds because the assumption $\bar{L} \cap (\mathfrak{g}_q, 0) \oplus (0, 0) = \{0\}$ is satisfied, as is clear from the definition of $\bar{L}$ in Theorem 2.4). □

In the rest of this subsection we want to see what Lemma 3.4 says about the objects that integrate the Lie algebroids $\bar{L}$ and L^c. We first recall few definitions.

Definition 3.5. A *Lie groupoid* over a manifold P is a manifold Γ endowed with surjective submersions $\mathbf{s}$,$\mathbf{t}$ (called source and target) to the base manifold P, a smooth associative multiplication m defined on elements $g, h \in \Gamma$ satisfying $\mathbf{s}(g) = \mathbf{t}(h)$, an embedding of P into Γ as the spaces of "identities" and a smooth inversion map $\Gamma \to \Gamma$ satisfying certain compatibility conditions (see for example [16])

Every Lie algebroid Γ has an associated Lie algebroid, whose total space is $ker(\mathbf{s}_*|_P) \subset T\Gamma|_P$, with a bracket on sections defined using right invariant vector fields on Γ and $\mathbf{t}_*|_P$ as anchor. A Lie algebroid A is said to be integrable if there exists a Lie groupoid whose associated Lie algebroid is isomorphic to A; in this case there is a unique (up to isomorphism) source simply connected (s.s.c.) Lie groupoid integrating A.

The following two definition are adapted from [3] and [17] respectively to match up the conventions of [8] and [27].

Definition 3.6. A *presymplectic groupoid* is a Lie groupoid Γ over a manifold P, with $\dim \Gamma = 2 \dim P$, equipped with a closed 2-form Ω_Γ satisfying

$$m^*\Omega_\Gamma = pr_1^*\Omega_\Gamma + pr_2^*\Omega_\Gamma$$

and the non-degeneracy condition

$$\ker \mathbf{t}_* \cap \ker \mathbf{s}_* \cap \ker \Omega_\Gamma = \{0\}.$$

By [3] the Dirac structure on Γ given by the graph of Ω pushes down via $\mathbf{s}$ to a Dirac structure L on the base P which, as a Lie algebroid, is isomorphic to the Lie algebroid of Γ. Conversely, if (P, L) is any Dirac manifold, then L (if integrable) integrates to a s.s.c. presymplectic groupoid as above. The latter is unique (up to presymplectic groupoid automorphism), and will be denoted by $\Gamma_s(P)$ in this paper.

Hence presymplectic groupoids are the objects integrating Dirac structures. The objects integrating Jacobi-Dirac structures are the following:

Definition 3.7. A *precontact groupoid* is a Lie groupoid Γ over a manifold Q, $\dim \Gamma = 2 \dim Q + 1$, equipped with a 1-form θ_Γ and a function f_Γ satisfying $f_\Gamma(gh) = f_\Gamma(g) f_\Gamma(h)$ and

$$m^*\theta_\Gamma = pr_1^*\theta_\Gamma pr_2^* f_\Gamma + pr_2^*\theta_\Gamma$$

and the non-degeneracy condition

$$\ker \mathbf{t}_* \cap \ker \mathbf{s}_* \cap \ker \theta_\Gamma \cap \ker d\theta_\Gamma = \{0\}.$$

The 1-form θ_Γ, viewed as a Jacobi-Dirac structure on Γ, pushes forward via the source map to a Jacobi-Dirac structure on M which is isomorphic to the Lie algebroid of Γ. (The formula for a canonical isomorphism is given in Appendix A). Conversely, if $(Q, \bar{L})$ is any Jacobi-Dirac manifold, then $\bar{L}$ (if integrable) integrates to a s.s.c. unique precontact groupoid as above, which will be denoted by $\Gamma_s(P)$ in this paper. Notice that a Dirac manifold (P, L), in addition to the presymplectic groupoid $\Gamma_s(P)$ associated as above, also has has an associated precontact groupoid $\Gamma_c(P)$ integrating the Jacobi-Dirac structure L^c corresponding to L.

When the presymplectic groupoid $\Gamma_s(P)$ is prequantizable its prequantization circle bundle can be view as an "alternative prequantization space" for (P, L), because $\Gamma_s(P)$ is the global object that corresponds to the Dirac manifold (P, L). We will see in items (4) and (5) of Thm. 4.11 that the prequantizability and integrability of (P, L) implies that $\Gamma_s(P)$ is prequantizable, and that the prequantization bundle $\tilde{\Gamma}_c(P)$ is a groupoid integrating L^c, so $A(\tilde{\Gamma}_c(P)) \cong L^c$ where "A" denote the functor that takes the Lie algebroid of a Lie groupoid. (In the Poisson case this follows from [8] and [2]).

There is a canonical Lie algebroid isomorphism between $\ker \mathbf{s}_*|_P \subset T\tilde{\Gamma}_c(P)|_P$ and L^c, given by Lemma A.1. It matches the restriction to $\ker \mathbf{s}_*|_P$ of the 1-form on $\tilde{\Gamma}_c(P)$ and the precontact form θ_{L^C} on L^c (see eq. (11)) at points of P (notice that at points of the zero section P the precontact form on L^c is just pr^*dt, i.e. the projection onto the last component). Similarly the canonical isomorphism between $\ker \mathbf{s}_*|_Q$ (where here $\mathbf{s}$ denotes the source map of $\Gamma_c(Q)$) and $\bar{L}$ matches the restriction of the 1-form on $\Gamma_c(Q)$ and $\theta_{\bar{L}}$. Hence the reduction of Prop. 3.4 matches the 1-forms on the groupoids $\Gamma_c(Q)$ and $\tilde{\Gamma}_c(P)$ at points of the identity sections.

As we will see in the next section, there is an S^1 action on the precontact groupoid $(\Gamma_c(Q), \theta_\Gamma, f_\Gamma)$ of $(Q, \bar{L})$ which is canonically induced by the S^1 action on Q and which hence makes the source map equivariant and which respects the 1-form and multiplicative function on the groupoid. The equivariance makes sure that taking derivatives along the identity one gets an S^1 action on $\ker \mathbf{s}_*|_Q$ by vector bundle isomorphism. Further, under the canonical isomorphism (see Lemma A.1) $\ker \mathbf{s}_*|_Q \cong \bar{L}$, the S^1 action is the natural one described at the beginning of the proof of Prop. 3.2, because the S^1 action on $\Gamma_c(Q)$ respects $\mathbf{t}$,r_Γ and θ_Γ. We conclude that the S^1 action we considered in this subsection is the infinitesimal version of the S^1 action on $(\Gamma_c(Q), \theta_\Gamma)$. We summarize:

Proposition 3.8. *The natural S^1 action on Q lifts to an action on $A(\Gamma_c(Q)) \cong \bar{L}$, whose precontact reduction is $L^c \cong A(\tilde{\Gamma}_c(P))$, endowed with the Lie algebroid and precontact structures given by the Lie groupoid $\tilde{\Gamma}_c(P)$.*

In the next section we will show that the precontact reduction of $\Gamma_c(Q)$ is isomorphic, both as precontact manifold and a groupoid, to the s.s.c. precontact groupoid of P, and that $\tilde{\Gamma}_c(Q)$ is a discrete quotient of it. This means that precontact reduction commutes with the Lie algebroid functor:

$$A(\Gamma_c(Q)//_0 S^1) = A(\Gamma_c(Q))//_0 S^1.$$

Further we also have a correspondence at the intermediate step of the reduction, namely for the zero level sets of the moment maps (see item (3) of Thm. 4.9).

4 Prequantization and reduction of precontact groupoids

In this section we analyze the relation between the groupoids associated to (P, L) and $(Q, \bar{L})$, leading to an "integrated" version of Proposition 3.4 (i.e. to reduction of groupoids).

In Subsection 4.1 we will perform the reduction using finite dimensional arguments, restricting ourselves for simplicity to the case when P is a Poisson manifold. If on one hand our finite dimensional proof might appeal more to geometric intuition, it will not allow to conclude whether the reduced groupoids we obtain are source simply connected. In Subsection 4.2, for the general case when P is a Dirac manifold, we will obtain a complete description of the reduction using path spaces. We will conclude with two examples.

4.1 The Poisson case

In this subsection we show our results for Poisson manifold without using the infinite dimensional path spaces.

We start displaying a simple example, which was also a motivating example in [6].

Example 4.1. Let (P,ω) be a simply connected integral symplectic manifold, and (Q,θ) a prequantization. We have the following diagram of groupoids:

$$
\begin{tikzcd}
(Q\times Q\times\mathbb{R}, -e^{-s}\theta_1+\theta_2, e^{-s}) \arrow[d, shift left] \arrow[d, shift right] & (Q\times_{S^1} Q, [-\theta_1+\theta_2]) \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & (P\times P, -\omega_1+\omega_2) \arrow[dl, shift left] \arrow[dl, shift right] \\
Q \arrow[r] & P &
\end{tikzcd}
$$

The first groupoid is a (usually not s.s.c.) contact groupoid of (Q,θ), with coordinate s on the $\mathbb{R}$ factor. The second is a contact groupoid of (P,ω) which is a prequantization of the third groupoid (the s.s.c. symplectic groupoid of (P,ω)). The S^1 action on Q induces a circle action on its contact groupoid with moment map given by $\langle J,1\rangle = -e^{-s}+1$, so that its zero level set is obtained setting $s=0$, and dividing by the circle action we obtain exactly the second groupoid above, i.e. the prequantization of the s.s.c. groupoid of (P,ω).

Let P be a Poisson manifold, consider the Dirac structure L given by the graph of the Poisson bivector, and assume that (P,L) is prequantizable and that it is integrable, in which case it integrates to a s.s.c symplectic[3] groupoid $\Gamma_s(P)$. The prequantizability of (P,L) implies that the period group of any source fiber of $\Gamma_s(P)$ is contained in $\mathbb{Z}$ (see Section 3.3 of [2], or Theorem 4.2 below for a straightforward generalization). This last condition is equivalent to saying that the symplectic groupoid $\Gamma_s(P)$ is prequantizable in the sense of [6] (see Prop. 2 in [2] or Thm. 3 in [8]). Its unique prequantization will be denoted by $\tilde{\Gamma}_c(P)$ and turns out to be a (usually not s.s.c.) contact[4] groupoid of P, i.e. it integrates the Lie algebroid L^c. Fix a prequantization $(Q,\bar{L})$ and assume that the Lie algebroid $\bar{L}$ is integrable; denote by $\Gamma_c(Q)$ the integrating s.s.c. contact groupoid. Now, "integrating" the reduction statements of the last section, we will clarify the relation between $\Gamma_c(Q)$ (the global object attached to the prequantization bundle Q) and the prequantization of $\Gamma_s(P)$ (which can be thought of as a different way to prequantize (P,L)).

The (smooth) groupoids we consider fit into the following diagram; we omitted $\tilde{\Gamma}_c(P)$, which is just a discrete quotient of the s.s.c. contact groupoid $\Gamma_c(P)$. This diagram corresponds to the diagram of Lie algebroids in Subsection 3.2, and again we denote dimensions by superscripts.

[3]This means that the 2-form on the presymplectic groupoid intergrating L is non-degenerate.

[4]This means that the 1-form on the precontact groupoid satisfies $\theta_\Gamma \wedge (d\theta_\Gamma)^{dim(P)} \neq 0$.

$$\begin{tikzcd}
\Gamma_c(Q)^{2n+3} \arrow[d, shift left] \arrow[d, shift right] & \Gamma_c(P)^{2n+1} \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \Gamma_s(P)^{2n} \arrow[dl, shift left] \arrow[dl, shift right] \\
Q^{n+1} \arrow[r, "\pi"] & P^n &
\end{tikzcd}$$

Theorem 4.2. *Let (P, L) be an integrable prequantizable Poisson manifold, and $(Q^{n+1}, \bar{L})$ one of its prequantizations as in Subsection 2.1, which we assume to be integrable. Then:*

a) The s.s.c contact groupoid $\Gamma_c(P)$ of (P, L) is obtained from the s.s.c. contact groupoid $\Gamma_c(Q)$ of $(Q, \bar{L})$ by S^1 contact reduction.

b) The prequantization of the s.s.c. symplectic groupoid $\Gamma_s(P)$ is a discrete quotient of $\Gamma_c(P)$.

Proof. S^1 acts on Q, and it acts also on $TQ \oplus T^*Q$ by the tangent and cotangent lifts. The S^1 action preserves the subbundle given by the Jacobi-Dirac structure $\bar{L}$, hence we obtain an S^1 action on the Lie algebroid $\bar{L} \to Q$. The source simply connected (s.s.c.) contact groupoid $(\Gamma_c(Q), \theta_\Gamma, f_\Gamma)$ of $(Q, \bar{L})$ is constructed canonically from the Lie algebroid $\bar{L}$ via the path-space construction [7], so it inherits an S^1 action that preserves its geometric and groupoid structures. In particular the source and target maps are S^1 equivariant, and similarly the multiplication map $\Gamma_c(Q)_{\mathbf{s}} \times_{\mathbf{t}} \Gamma_c(Q) \to \Gamma_c(Q)$. Also, the S^1 action preserves the contact form, so there is a moment map $J_\Gamma : \Gamma_c(Q) \to \mathbb{R}$ by $J_\Gamma(g) = \theta_\Gamma(v_\Gamma(g))$ where v_Γ denotes the infinitesimal generator of the S^1 action. We divide the proof in three steps.

Step 1: $J_\Gamma^{-1}(0)$ is a s.s.c. Lie subgroupoid of $\Gamma_c(Q)$.
We start by showing that $J_\Gamma = 1 - f_\Gamma$; this explicit[5] formula will turn out to be necessary in Step 2.

To do this we will use several properties of contact groupoids, for which to refer to Remark 2.2 in [27]. The identity $J_\Gamma + f_\Gamma = 1$ is clear along the identity section Q, since f_Γ is a multiplicative function and v_Γ is tangent to Q which is a Legendrian submanifold of $(\Gamma_c(Q), \theta_\Gamma)$. So to show that the statement holds at any point of $\Gamma_c(Q)$ it is enough to show that $\langle d(f_\Gamma + J_\Gamma), X_{f_\Gamma \mathbf{t}^* u}\rangle = 0$ for functions $u \in C^\infty(Q)$, since hamiltonian vector fields $X_{f_\Gamma \mathbf{t}^* u}$ span $\ker \mathbf{s}_*$. The statement follows by two computations: first

$$\begin{aligned}
&\langle df_\Gamma, X_{f_\Gamma \mathbf{t}^* u}\rangle = \langle df_\Gamma, f_\Gamma \mathbf{t}^* u E_\Gamma + \Lambda_\Gamma d(f_\Gamma \mathbf{t}^* u)\rangle \\
=&f_\Gamma \cdot \langle df_\Gamma, \Lambda_\Gamma d(\mathbf{t}^* u)\rangle = -f_\Gamma \cdot d(\mathbf{t}^* u) X_{f_\Gamma} = f_\Gamma \cdot E(u),
\end{aligned} \tag{14}$$

where we used twice $E_\Gamma(f_\Gamma) = 0$ and the fact that $\mathbf{t}$ is a $-f_\Gamma$-Jacobi map. Second,

$$\langle d(\theta_\Gamma(v_\Gamma)), X_{f_\Gamma \mathbf{t}^* u}\rangle = -d\theta_\Gamma(v_\Gamma, X_{f_\Gamma \mathbf{t}^* u}) = \langle -d(f_\Gamma \mathbf{t}^* u), (v_\Gamma - \theta_\Gamma(v_\Gamma) E_\Gamma)\rangle = -f_\Gamma \cdot E(u),$$

[5] The claim of Step 1 follows even without knowing the explicit formula for J_Γ. Indeed one can show that $J_\Gamma^{-1}(0)$ is a subgroupoid by means of the identity $J_\Gamma(gh) = f(h) J_\Gamma(g) + J_\Gamma(h)$, which is derived using the multiplicativity of θ_Γ and the fact that v_Γ is a multiplicative vector field (i.e. $v_\Gamma(g) \cdot v_\Gamma(h) = v_\Gamma(gh)$; this is just the infinitesimal version of the statement that the multiplication map is S^1 equivariant). Since $J_\Gamma^{-1}(0)$ is a smooth wide subgroupoid it is transverse to the $\mathbf{s}$ fibers nearby the identity, therefore its source and target maps are submersions and hence it is actually a Lie subgroupoid.

where we use the fact that $\mathcal{L}_{v_\Gamma}\theta_\Gamma = 0$ in the first equality, the formula $d\theta_\Gamma(X_\phi, w) = -\langle d\phi, w^H\rangle$ valid for any function ϕ on a contact groupoid (where w^H is the projection of the tangent vector w to $\ker\theta_\Gamma$ along the Reeb vector field E_Γ) in the second one, and in the last equality that $E_\Gamma(f_\Gamma)$,$v_\Gamma(f_\Gamma)$,$\mathbf{t}_*E_\Gamma$ all vanish and that the S^1 actions on $\Gamma_c(Q)$ and Q are intertwined by the target map $\mathbf{t}$.

Since f_Γ is multiplicative, it is clear that $J_\Gamma^{-1}(0) = f_\Gamma^{-1}(1)$ is a subgroupoid.

Further $J_\Gamma^{-1}(0)$ is a smooth submanifold of $\Gamma_c(Q)$: by Prop. 3.1.4 in [26] $g \in \Gamma_c(Q)$ is a singular point of J_Γ iff $v_\Gamma(g)$ is a non-zero multiple of $E_\Gamma(g)$. Since $\theta_\Gamma(E_\Gamma) = 1$ this is never the case if $g \in J_\Gamma^{-1}(0)$, so 0 is a regular value of J_Γ.

To show that $J_\Gamma^{-1}(0)$ is a Lie subgroupoid we still need to show that its source and target maps are submersions onto Q. We do so by showing explicitly that $(\ker \mathbf{s}_* \cap \ker df_\Gamma)$ (which along Q will be the Lie algebroid of $J_\Gamma^{-1}(0)$) has rank one less than $\ker \mathbf{t}_*$; this is clear since by the first equation of Step 1 it is just $\{X_{f_\Gamma \mathbf{t}^*\pi^* v} : v \in C^\infty(P)\}$.

For the proof of the source simply connectedness of the subgroupoid $J_1^{-1}(0)$ we refer to Thm. 4.9.

Step 2: The contact reduction $J_\Gamma^{-1}(0)/S^1$ is the s.s.c. contact groupoid $\Gamma_c(P)$ of P.

$J_\Gamma^{-1}(0)/S^1$ is smooth because the S^1 action is free and proper, and by contact reduction it is a contact manifold, so we just have to show that the Lie groupoid structure descends and is a compatible one.

The S^1 equivariance of the source and target maps of $\Gamma_c(Q)$ ensure that source and target descend to maps $J_\Gamma^{-1}(0)/S^1 \to P(= Q/S^1)$. Since the multiplication on $\Gamma_c(Q)$ is S^1 equivariant, the multiplication on $J_\Gamma^{-1}(0)$ induces a multiplication on $J_\Gamma^{-1}(0)/S^1$. It is routine to check this makes $J_\Gamma^{-1}(0)/S^1$ into a groupoid over P. Further, since the source map intertwines the S^1 action on $J^{-1}(0)$ and the free S^1 action on the base Q, the source fibers of $J_\Gamma^{-1}(0)/S^1$ will be diffeomorphic to the corresponding source fibers of $J_\Gamma^{-1}(0)$, hence we obtain a s.s.c. Lie groupoid. Since $J_\Gamma^{-1}(0) \to J_\Gamma^{-1}(0)/S^1$ is a surjective submersion, the f_Γ-twisted multiplicativity of θ_Γ implies that the induced 1-form $\hat{\theta}_\Gamma$ is multiplicative, i.e. $(J_\Gamma^{-1}(0)/S^1, \hat{\theta}_\Gamma, \hat{f}_\Gamma)$ is a contact groupoid.

In order to prove that the above contact groupoid corresponds to the original Poisson structure Λ_P on P, we have to show that the source map $\hat{\mathbf{s}} : J_\Gamma^{-1}(0)/S^1 \to P$ is a Jacobi map (i.e. a forward Jacobi-Dirac map). Consider the diagram

$$\begin{array}{ccc} J_\Gamma^{-1}(0) & \xrightarrow{\pi_{J_\Gamma}} & J_\Gamma^{-1}(0)/S^1 \\ \mathbf{s}\Big\downarrow & & \hat{\mathbf{s}}\Big\downarrow \\ Q & \xrightarrow{\pi} & P. \end{array}$$

We adopt the following short-form notation: for a 1-form α, L_α will denote the Jacobi-Dirac structure associated to α [22]. Then for the pullback Jacobi-Dirac structure we have $i^*L_{\theta_\Gamma} = L_{i^*\theta_\Gamma}$, where i is the inclusion of $J_\Gamma^{-1}(0)$ into $\Gamma_c(Q)$, and the reduced 1-form is recovered as ${\pi_{J_\Gamma}}_* i^* L_{\theta_\Gamma} = L_{\hat{\theta}_\Gamma}$. So by the functoriality of the pushforward, it is enough to show that $\pi_*\mathbf{s}_*L_{i^*\theta_\Gamma}$, which by definition is

$$\{((\pi\circ\mathbf{s})_*Y, f) \oplus (\xi, g) : (Y, f) \oplus ((\pi\circ\mathbf{s})^*\xi, g) \in L_{i^*\theta_\Gamma}\}, \tag{15}$$

equals the Jacobi-Dirac structure given by Λ_P. First we determine which tangent vectors Y to $J_\Gamma^{-1}(0)$ and $f \in \mathbb{R}$ have the property that $i^*(d\theta_\Gamma(Y) + f\theta_\Gamma)$ annihilates $\ker(\pi\circ\mathbf{s})_*$, which using equation (14) is equal to $\{X_{f_\Gamma \mathbf{t}^*\pi^*v} : v \in C^\infty(P)\} \oplus \mathbb{R}v_\Gamma$. A computation similar to those carried out in Step 1 and using the explicit formula $J = 1 - f_\Gamma$ shows that this is the case when $f = 0$ and $\pi_*\mathbf{t}_*Y = 0$, which by a computation similar to (14) amounts to $Y \in \{X_{\mathbf{s}^*\pi^*v} : v \in C^\infty(P)\} \oplus \mathbb{R}v_\Gamma$. These will be exactly the "Y" and "f" appearing in (15); a short computation using the facts that the source map of $\Gamma_c(Q)$ and π are Jacobi maps shows that (15) equals $\{(-\Lambda_P\xi, 0) \oplus (\xi, g) : \xi \in T^*P, g \in \mathbb{R}\}$, as was to be shown.

Step 3: $((J_\Gamma^{-1}(0)/S^1)/\mathbb{Z}, \hat{\theta}_\Gamma)$ *is the prequantization of the s.s.c. symplectic groupoid* $\Gamma_s(P)$ *of* P*. Here* $\mathbb{Z}$ *acts as a subgroup of* $\mathbb{R}$ *by the flow of the Reeb vector field* $\hat{E}_\Gamma$*.*
Consider the action on $J_\Gamma^{-1}(0)/S^1$ by its Reeb vector field $\hat{E}_\Gamma$, which by the contact reduction procedure is the projection of the Reeb vector field E_Γ of $\Gamma_c(Q)$ under $J_\Gamma^{-1}(0) \to J_\Gamma^{-1}(0)/S^1$.

The $\mathbf{t}$-image of a v_Γ orbit is an orbit of the S^1 action on Q, since the target map is S^1 equivariant. Hence each v_Γ orbit meets each $\mathbf{t}$-fiber at most once. Further each E_Γ-orbit is contained in a single $\mathbf{t}$-fiber (since $\mathbf{t}_*E_\Gamma = 0$), so an E_Γ orbit meets any orbit of the S^1 action on $\Gamma_c(Q)$ at most once. Therefore the period of an E_Γ orbit and of the corresponding $\bar{E}_\Gamma$ orbit are equal, and the first period is always an integer number (because $\mathbf{s}_*E_\Gamma = E_Q$, the generator of the circle action on Q).

Now the we know that the periods of $\bar{E}_\Gamma$ are integers, we can just apply Theorems 2 and 3 of [8] to prove our claim. □

4.2 Path space constructions and the general Dirac case

In this subsection we generalize Thm. 4.2 allowing P to be a general Dirac manifold, using the explicit description of Lie groupoids as quotients of path spaces as a powerful tool. The generalization will be presented in Thm. 4.9 and Thm. 4.11.

Definition 4.3. Let $\pi : A \to M$ be a Lie algebroid with anchor ρ. The *A-path space* $P_a(A)$ consists of all paths $a : [0,1] \to A$ satisfying $\frac{d}{dt}(\pi \circ a)(t) = \rho(a(t))$.

There is an equivalence relation in P_aA, called *A-homotopy* [7].

Definition 4.4. Let $a(t,s)$ be a family of A-paths which is C^2 in s. Assume that the base paths $\gamma(t,s) := \pi \circ a(t,s)$ have fixed end points. For a connection ∇ on A, consider the equation

$$\partial_t b - \partial_s a = T_\nabla(a,b), \quad b(0,s) = 0. \tag{16}$$

Here T_∇ is the torsion of the connection defined by $T_\nabla(\alpha,\beta) = \nabla_{\rho(\beta)}\alpha - \nabla_{\rho(\alpha)}\beta + [\alpha,\beta]$. Two paths $a_0 = a(0,\cdot)$ and $a_1 = a(1,\cdot)$ are homotopic if the solution $b(t,s)$ satisfies $b(1,s) = 0$.

More geometrically, for every Lie algebroid A, (notice that tangent bundles are Lie algebroids), we associate A a simplicial set $S(A) = [...S_2(A) \Rrightarrow S_1(A) \rightrightarrows S_0(A)]$ with,

$$S_i(A) = \hom_{algd}(T\Delta^i, A) := \{\text{Lie algebroid morphisms } T\Delta^i \xrightarrow{f} A\}, \tag{17}$$

and face and degeneracy maps $d_i^n : S_n(A) \to S_{n-1}(A)$ and $s_i^n : S_n(A) \to S_{n+1}(A)$ induced from the natual face and degeneracy maps $\Delta^n \to \Delta^{n-1}$ and $\Delta^n \to \Delta^{n+1}$. Here Δ^i is the i-dimensional standard simplex viewed as a smooth Riemannian manifold with boundary, hence it is isomorphic to the i-dimensional closed ball. Then as explained in [28, Section 2],

- it is easy to check that $S_0 = M$;
- S_1 is exactly the A-path space P_aA;
- bigons in S_2 are exactly the A-homotopies in P_aA since a bigon $f : T(d_2^2)^{-1}(Ts_0^1(T\Delta^0)) \to A$ can be written as $a(t,s)dt + b(t,s)ds$ over the base map $\gamma(t,s)$ after a suitable choice of parametrization[6] of the disk $(d_2^2)^{-1}(s_0^1(\Delta^0))$. Then we naturally have $b(0,s) = f(0,s)(\frac{\partial}{\partial s}) = 0$ and $b(1,s) = f(1,s)(\frac{\partial}{\partial s}) = 0$. Moreover the morphism is a Lie algebroid morphism if and only if $a(t,s)$ and $b(t,s)$ satisfy equation (16) which defines the A-homotopy.

The s.s.c. groupoid of any integrable Lie algebroid A can be constructed as the quotient of the A-path space by a foliation $\mathcal{F}$, whose leaves consists of the A-paths that are A-homotopic to each other [7]. In particular the precontact groupoid $(\Gamma_c(Q), \theta, f)$ of a Jacobi-Dirac manifold Q can be constructed via the A-path space $P_a(\bar{L})$, with θ and f coming from a corresponding 1-form and function on the path space. We refer to [8] [6] [17] and summarize the results in Thm. 4.5 below. The advantage of this method is that it can be used to generalize Theorem 4.2 to the setting of Dirac manifolds (see Theorems 4.9 and 4.11) and that it can be applied to a general group G action as in [10].

Theorem 4.5. *The s.s.c. precontact groupoid $(\Gamma_c(Q), \theta_\Gamma, f_\Gamma)$ of an integrable Jacobi-Dirac manifold $(Q, \bar{L})$ is the quotient space of the A-path space $P_a(\bar{L})$ by A-homotopies, and θ_Γ and f_Γ come from a 1-form $\tilde{\theta}$ and a function $\tilde{f}$ on $P_a(\bar{L})$. At the point $a = (a_4, a_3, a_1, a_0) \in P_a(\bar{L})$, where (a_4, a_3, a_1, a_0) are components in $TQ \oplus \mathbb{R} \oplus T^*Q \oplus \mathbb{R}$, $\tilde{\theta}$ and $\tilde{f}$ are*

$$\begin{aligned} \tilde{\theta}_a(X) =& -\int_0^1 \left\langle e(t)X(t), d\left(\int_0^1 a_0(t)dt\right)\right\rangle dt + \int_0^1 \langle e(t)X(t), pr^*\theta_c\rangle\, dt, \\ \tilde{f}(a) =& e(1), \quad \text{with } e(t) := e^{\int_0^t -a_3} \end{aligned} \tag{18}$$

where X is a tangent vector to $P_a(\bar{L})$, hence a path itself (parameterized by t), and $pr^*\theta_c$ is the pull-back via $pr : \bar{L} \to T^*Q$ of the canonical 1-form on T^*Q.

Proof. The equation for $\tilde{f}$ is taken from Prop. 3.5(i) of [8]. It is shown there that $\tilde{f}$ descends to the function f_Γ on $\Gamma_c(Q)$. To get the formula for $\tilde{\theta}$, we recall from Section 3.4 of [8] that the following map ϕ is an isomorphism preserving A-homotopies:

$$\phi : P_a(\bar{L}) \times \mathbb{R} \to P_a(\bar{L} \times_\psi \mathbb{R}),$$

mapping (a, s) with base path γ_1 to $\tilde{a} := e^{\gamma_0(t)}a$ with base path (γ_1, γ_0), where $\gamma_0 := s - \int_0^t a_3$. Here ψ is the 1-cocycle on $\bar{L}$ given by $(X, f) \oplus (\xi, g) \mapsto f$; $\bar{L} \times_\psi \mathbb{R}$ is the Lie algebroid on $Q \times \mathbb{R}$ obtained from the Lie algebroid $\bar{L}$ and the 1-cocycle ψ, and it is isomorphic to the Lie algebroid given by the Dirac structure on $Q \times \mathbb{R}$ obtained from the "Diracization" of $(Q, \bar{L})$ (see Section 2.3 in [17]).

[6] We need the one with $\gamma(0,s) = x$ and $\gamma(1,s) = y$ for all $s \in [0,1]$.

The correspondence on the level of tangent spaces given by $T\phi$ maps $(\delta\gamma_1, \delta s, \delta a)$ to $(\delta\gamma_1, \delta\gamma_0, \delta\tilde{a})$ and satisfies

$$
\begin{aligned}
\delta\gamma_0 &= \delta s - \int_0^t a_3, \\
\delta\tilde{a}_1 &= e^{\gamma_0}(\delta a_1 + (\delta s - \int_0^t \delta a_3)a_1), \\
\delta\tilde{a}_0 &= e^{\gamma_0}(\delta a_0 + (\delta s - \int_0^t \delta a_3)a_0).
\end{aligned}
$$

We identify $\bar{L} \times_\psi \mathbb{R}$ with the Dirac structure on $Q \times \mathbb{R}$ given by the Diracization of $(Q, \bar{L})$. Then on the whole space $P(\bar{L} \times_\psi \mathbb{R})$ of paths in $\bar{L} \times_\psi \mathbb{R}$ there is a symplectic form ω coming from integrating the pull-back of the canonical symplectic form on $T^*(Q \times \mathbb{R})$ (see Section 5 in [3]). This form restricted to the A-path space $P_a(\bar{L} \times_\psi \mathbb{R})$ is homogeneous w.r.t. the $\mathbb{R}$ component, i.e. $\varphi_s\omega = e^s\omega$, where φ_s is the flow of $\frac{\partial}{\partial s}$ with s the coordinate of $\mathbb{R}$. This is because φ_s acts on vector fields $\delta\tilde{a}_1$ and $\delta\tilde{a}_0$ by rescaling by an e^s factor as the formula of $T\phi$ and γ_0 show. This homogeneity survives the quotient to groupoids as shown in [8]. Therefore θ_Γ comes from the 1-form $\tilde{\theta}$ whose associated homogeneous symplectic form is ω, i.e. $\tilde{\theta} = -i_0^* i(\frac{\partial}{\partial s})\omega$. With a straightforward calculation and the formula of $T\phi$, we have the formula for $\tilde{\theta}$ in (18). $\square$

Remark 4.6. The formula for $\tilde{\theta}$ is a generalization of Theorem 4.2 in [6] in the case $\bar{L}$ that comes from a Dirac structure. To get the formula of the 1-form there up to sign[7], one just has to put $e(t) = 1$ which corresponds to the case that $a_3 = 0$.

In Lemma 2.9 we constructed a Lie algebroid structure on π^*A, the pull back via $\pi : Q \to P$ of any Lie algebroid A on P, provided there is a flat A-connection $\tilde{D}$ on the vector bundle K corresponding to the principal bundle Q. (π^*A turned out to be the transformation algebroid w.r.t. the action by the flat connection). Now we show some functorial property of algebroid paths in π^*A. Later in this section we will apply them to $A = L^c$, for π^*L^c is identified with a Lie subalgebroid of $\bar{L}$ (Thm. 2.11), whose integrating groupoid we can describe in term of A-paths (Thm. 4.5).

Lemma 4.7. *An A-path a in A can be lifted to an A-path in π^*A. The same is true for A-homotopies. In other words, in the following diagram (for $n = 1, 2$),*

$$
\begin{tikzcd}
T\Delta^n \arrow[d] \arrow[dr] \arrow[drr, "f"] & & \\
\Delta^n \arrow[dr] \arrow[drr, "f_0"] & \pi^*A \arrow[r] \arrow[d] & A \arrow[d] \\
& Q \arrow[r, "\pi"] & P
\end{tikzcd}
$$

any Lie algebroid morphism $f : T\Delta^n \to A$ lifts to a Lie algebroid morphism from $T\Delta^n$ to π^*A.

[7] In [6] 1-forms on contact groupoids are so that the target map is a Jacobi map, whereas here we adopt the convention (as in [27]) that the source map be Jacobi.

Proof. Let γ be the base path of an A-path a, and let $\tilde{\gamma}$ be the parallel translation along a of some $\tilde{\gamma}(0) \in \pi^{-1}(\gamma(0))$ as in the proof of Lemma 2.9. Denoting by π^*a the lift of a to π^*A with base path $\tilde{\gamma}$, we have $\rho(\pi^*a) = h_Q(a(\gamma(t)), \tilde{\gamma}(t)) = d/dt(\tilde{\gamma})$, with ρ the anchor of π^*A (see equation (6)). That is, π^*a is an A-path in π^*A over $\tilde{\gamma}$. The lifting of a is not unique. In fact it is determined by the choice of a point in $\pi^{-1}(\gamma(0))$ as initial value.

Now we prove the same statement for A-homotopies. Suppose $a(\epsilon, t)$ is an A-homotopy over $\gamma(\epsilon, t)$, i.e. there exist A-paths (w.r.t. parameter ϵ) $b(\epsilon, t)$ also over γ satisfying

$$\partial_t b - \partial_\epsilon a = \nabla_{\rho(b)} a - \nabla_{\rho(a)} b + [a, b], \tag{19}$$

and the boundary condition $b(\epsilon, 0) = b(\epsilon, 1) = 0$, for any choice of connection ∇ on TP. As above, we can lift γ to $\tilde{\gamma}(\epsilon, t)$. In fact, once we choose $\tilde{\gamma}(0,0)$, we can use $\tilde{\gamma}(0,0)$ to obtain the lift $\tilde{\gamma}(\epsilon, 0)$ and then $\tilde{\gamma}(\epsilon, t)$. (The lift does not depend on whether we lift $\gamma(\epsilon, 0)$ or $\gamma(0, t)$ first, because the connection $\tilde{D}$ is flat). Then π^*a and π^*b are A-paths over $\tilde{\gamma}$ w.r.t. parameters t and ϵ respectively. Moreover, we choose a connection $\tilde{\nabla}$ on Q induced from the connection ∇ on P such that $\tilde{\nabla}_{X^H} Y^H = (\nabla_X Y)^H$, $\tilde{\nabla}_{X^H} E = 0$, $\tilde{\nabla}_E Y^H = 0$ and $\tilde{\nabla}_E E = 0$, where the superscript H denotes the horizontal lift with respect to some connection we fix on the circle bundle $\pi : Q \to P$. (Since $E(\pi^* f) = 0$ and $X^H(\pi^* f) = X(f)$ these requirements are consistent. In fact, the connection $\tilde{\nabla}$ on $TQ = \pi^*TP \oplus \mathbb{R}E$ is just the sum of the pullback connection on π^*TP and of the trivial connection). Now we will prove that π^*a and π^*b satisfy (19) w.r.t. $\tilde{\nabla}$. Notice that $\langle \pi^*\eta, \tilde{\nabla}_E X\rangle = 0$ for all vector fields X, so we have

$$\tilde{\nabla}_E \pi^*\eta = 0, \quad \tilde{\nabla}_{(\frac{\partial}{\partial\epsilon}\gamma)^H} \pi^*\eta = \pi^*(\nabla_{\frac{\partial}{\partial\epsilon}\gamma}\eta).$$

Therefore $\tilde{\nabla}_{\frac{\partial}{\partial\epsilon}\tilde{\gamma}} \pi^*\eta = \pi^*(\nabla_{\frac{\partial}{\partial\epsilon}\gamma}\eta)$. So $\partial_\epsilon \pi^* a = \pi^*(\partial_\epsilon a)$. The same is true for π^*b. Moreover, since $\rho(\pi^*a) = (\rho(a))^H + \langle\tilde{\beta}, a\rangle E$ (upon writing $\tilde{D}$ as in equation (5) and denoting by ^{H} the horizontal lift w.r.t. $\ker\sigma$), similarly we have $\tilde{\nabla}_{\rho(\pi^*a)} \pi^*b = \pi^*(\nabla_{\rho(a)} b)$ as well as the analog term obtained switching a and b. By the definition of Lie bracket on π^*A, we also have $[\pi^*a, \pi^*b] = \pi^*([a,b])$. Therefore a, b satisfying (19) implies that the same equation holds for π^*a and π^*b. The boundary condition $\pi^*b(\epsilon, 0) = \pi^*b(\epsilon, 1) = 0$ is obvious. Hence, π^*a is an A-homotopy in π^*A. □

Remark 4.8. We claim that all the A-paths and A-homotopies in π^*A are of the form π^*a. Indeed consider a π^*A path $\hat{a}$ over a base path $\hat{\gamma}$, i.e. $\rho(\hat{a}(t)) = \frac{d}{dt}\hat{\gamma}(t)$. Let $\gamma := \pi \circ \hat{\gamma}$ and let $a(t)$ be equal to $\hat{a}(t)$, seen as an element of $A_{\gamma(t)}$. The commutativity of

$$\begin{array}{ccc} \pi^*A & \xrightarrow{h_Q = \rho} & TQ \\ \downarrow & & \pi_* \downarrow \\ A & \xrightarrow{\rho_A} & TP \end{array}$$

implies that a is an A-path over γ. Further, the horizontal lift of a starting at $\hat{\gamma}(0)$ satisfies by definition $\frac{d}{dt}\tilde{\gamma}(t) = h_Q(a(\gamma(t)), \tilde{\gamma}(t))$, so it coincides with $\hat{\gamma}$. The same holds for A-homotopies.

The next theorem generalizes Thm. 4.2a).

Theorem 4.9. *Let (P, L) be an integrable prequantizable Dirac manifold and $(Q, \bar{L})$ one of its prequantization. We use the notation $[\cdot]_A$ to denote A-homotopy classes in the Lie algebroid A. Then we have the following results:*

1. *there is an S^1 action on the precontact groupoid $\Gamma_c(Q)$ with moment map $J_\Gamma = 1 - f_\Gamma$;*

2. *$J_\Gamma^{-1}(0)$ is a source connected and simply connected subgroupoid of $\Gamma_c(Q)$ and is isomorphic to the action groupoid $\Gamma_c(P) \ltimes Q \rightrightarrows Q$.*

3. *In terms of path spaces,*

$$J_\Gamma^{-1}(0) = \{[\pi^* a]_{\bar{L}}\} = \{[\pi^* a]_{\bar{L}_0}\},$$

where a is an A-path in L^c and $\pi^* a$ is defined as in Lemma 4.7 (we identify $\pi^* L^c$ with $\bar{L}_0 \subset \bar{L}$ as in Thm. 2.11). Hence the Lie algebroid of $J_\Gamma^{-1}(0)$ is $\bar{L}_0 = J^{-1}(0)$ (see Prop. 3.4).

4. *the precontact reduction $\Gamma_c(Q)//_0 S^1$ is isomorphic to the s.s.c. contact groupoid $\Gamma_c(P)$ via the inverse of the following map*

$$p : [a]_{L^c} \mapsto [\pi^* a]_{\bar{L}, S^1},$$

where $[\cdot]_{\bar{L}, S^1}$ denotes S^1 equivalence classes of $[\cdot]_{\bar{L}}$.

Remark 4.10. The isomorphism p gives the same contact groupoid structure on $\Gamma_c(Q)//_0 S^1$ as in Theorem 4.2 in the case when P is Poisson.

Proof. 1) The definition of the S^1 action is the same as in Theorem 4.2. J_Γ is defined by $J_\Gamma(g) = \theta_\Gamma(v_\Gamma(g))$, where v_Γ is induced by the S^1 action on Q hence on $\bar{L}$. More explicitly, $T(P_a(\bar{L}))$ is a subspace of the space of paths in $T\bar{L}$. If we take a connection ∇ on Q, then $T\bar{L}$ decomposes as $TQ \oplus \bar{L}$. At $(a_4, a_3, a_1, a_0) \in P_a(\bar{L})$ the infinitesimal S^1 action $\tilde{v}$ on the path space is $\tilde{v} = (E(\gamma(t)), *, *, *, 0)$. So

$$J_\Gamma([a]) = \tilde{\theta}_a(\tilde{v}) = \int_0^1 (\langle a_1(t), E\rangle e^{-\int_0^t \langle a_1, E\rangle dt}) dt = -\int_0^1 d(e^{-\int_0^t \langle a_1, E\rangle dt}) = 1 - f_\Gamma.$$

2) By 1) $J_\Gamma^{-1}(0) = f_\Gamma^{-1}(1)$. Since f_Γ is multiplicative, it is clear that $f_\Gamma^{-1}(1)$ is a subgroupoid. Moreover using Thm. 4.5 we see that $f_\Gamma^{-1}(1)$ is made up by paths $a = (a_4, a_3, a_1, a_0)$ such that

$$\int_0^1 \langle a_1(t), E\rangle dt = 0. \tag{20}$$

Notice that this are not exactly the same as A-paths in $\bar{L}_0$, which are the A-paths such that $\langle a_1(t), E\rangle \equiv 0$ for all $t \in [0, 1]$ (see Thm. 2.11).

Now we show that $J_\Gamma^{-1}(0)$ is source connected. Take $g \in \mathbf{s}^{-1}(x)$, and choose an A-path $a(t)$ representing g over a base path $\gamma(t) : I \to Q$. We will connect g to x within $J_\Gamma^{-1}(0) \cap \mathbf{s}^{-1}(x)$ in two steps: first we deform g to some other point h which can be represented by an A-path in $\bar{L}_0$; then we "linearly shrink" h to x.

Suppose the vector bundle $\bar{L}$ is trivial on a neighborhood U of the image of γ in Q. Choose a frame $Y_0, \dots, Y_{\dim Q}$ for $\bar{L}|_U$, with the property that $Y_0 = (-A^H, 1) \oplus (\sigma - \pi^*\alpha, 0)$ (with σ, A and α as in Thm. 2.4) and that all other Y_i satisfy $\langle a_1, E\rangle = 0$. In this frame, $a(t) = \sum_{i=0}^{\dim Q} p_i(t) Y_i|_{\gamma(t)}$ for some time-dependent coefficients $p_i(t)$. Define the following section of $\bar{L}|_U$: $Y_{t,\epsilon} = (1-\epsilon)p_0(t)Y_0 + \sum_{i=1}^{\dim Q} p_i(t) Y_i$. Define a deformation $\gamma(\epsilon, t)$ of $\gamma(t)$ by

$$\frac{d}{dt}\gamma(\epsilon, t) = \rho(Y_{t,\epsilon}), \quad \gamma(\epsilon, 0) = x,$$

where ρ is the anchor of $\bar{L}$ (one might have to extend U to make $\gamma(\epsilon, t) \in U$ for $t \in [0,1]$). Let $a(\epsilon, t) := Y_{t,\epsilon}|_{\gamma(\epsilon,t)}$. For each ϵ it is an A-path by construction, and $a(0,t) = a(t)$. Using $g \in J_\Gamma^{-1}(0)$ (so that $\int_I p_0(t)dt = 0$) we have

$$\int_0^1 \langle a_1(\epsilon, t), E\rangle dt = \int_0^1 \langle (1-\epsilon)p_0(t)Y_0 + \sum_{i=1}^{\dim Q} p_i(t)Y_i, (E, 0, 0, 0)\rangle_- dt = (1-\epsilon)\int_I p_0(t)dt = 0,$$

so $[a(\epsilon, \cdot)]$ lies in $J_\Gamma^{-1}(0)$. Notice that $a(1,t)$ satisfies $\langle a_1(1,t), E\rangle \equiv 0$ for all t; hence an A-path in $\bar{L}_0$. We denote $h := [a(1,t)]$ and define a continuous map $pr : P_a(\bar{L}|_U) \to P_a(\bar{L}_0|_U)$ by $a(t) \mapsto a(1,t)$.

Then we can shrink linearly $a(1,t)$ to the zero path, via $a^\delta(1,t) := \delta a(1, \delta t)$ which is an A-path over $\gamma(1, \delta t)$. Taking equivalence classes we obtain a path from h to x, which moreover lies in $J_\Gamma^{-1}(0)$ because $\langle a_1(1,t), E\rangle \equiv 0$.

Now we show that $J_\Gamma^{-1}(0)$ is source simply connected. If there is a loop $g(s) = [a(1,s,t)]$ in a source fibre of $J_\Gamma^{-1}(0)$, then $g(s)$ can shrink to $x := \mathbf{s}(g(s))$ inside the big (s.s.c.!) groupoid $\Gamma_c(Q)$ via $g(\epsilon, s) = [a(\epsilon, s, t)]$. We can assume $a(\epsilon, s, t) = sa(\epsilon, 1, st)$. This is easy to realize since we can simply take $a(\epsilon, s, t) = g(\epsilon, st)^{-1} d/dt(g(\epsilon, st))$. Then the $a(i, 1, \cdot)$'s are A-paths in $\bar{L}_0$ for $i = 0, 1$. This is because both $g(s)$ and x are paths in $J_\Gamma^{-1}(0)$ which implies $\int_0^1 sa(i, 1, st) = 0$ for all $s \in [0,1]$. Moreover the base paths $\gamma(\epsilon, s, t)$ form an embedded disk (one can assume that the deformation $g(\epsilon, s)$ has no self-intersections) in Q. So we can take a simply connected open set (for example a tubular neighborhood of this disk) $U \subset Q$ containing $\gamma(\epsilon, s, t)$. Then $L|_U$ is trivial. Therefore there is a continuous map pr such that $\bar{a}(\epsilon, 1, \cdot) = pr(a(\epsilon, 1, \cdot))$ is an A-path in $\bar{L}_0$ and $\bar{a}(1,1,\cdot) = a(1,1,\cdot)$. Then we can shrink $g(s) = \bar{g}(1,s)$ to $x = \bar{g}(0,s)$ via

$$\bar{g}(\epsilon, s) := [s\bar{a}(\epsilon, 1, st)],$$

which is inside of $J_\Gamma^{-1}(0)$ since $\langle \bar{a}_1(\epsilon, 1, t), E\rangle \equiv 0$.

3) To show that $J_\Gamma^{-1}(0) = \{[\pi^* a]_{\bar{L}}\}$, we just have to show that an A-path in $\bar{L}$ satisfying (20) is A-homotopic (equivalent) to an A-path lying contained in $\bar{L}_0$. Since $J_\Gamma^{-1}(0)$ has connected source fibres, given a point $g = [a]$ in $J_\Gamma^{-1}(0)$, there is a path $g(t)$ connecting g to $\mathbf{s}(g)$ lying in $J_\Gamma^{-1}(0)$. Differentiating $g(t)$ we get an A-path $b(t) = g(t)^{-1}\dot{g}(t)$ which is A-homotopic to a and $sb(st)$ represents the point $g(st) \in J^{-1}(0)$. Therefore $\int_0^1 \langle sb_1(st), E\rangle dt = 0$, for all $s \in [0,1]$. Hence $\langle b_1(t), E\rangle \equiv 0$ for all $t \in [0,1]$, i.e. b is a path in $\bar{L}_0$.

To further show that $J_\Gamma^{-1}(0) = \{[\pi^* a]_{\bar{L}_0}\}$, we only have to show that if two A-paths in $\bar{L}_0$ are A-homotopic in $\bar{L}$ then they are also A-homotopic in $\bar{L}_0$. Let $a(1,\cdot)$ and $a(0,\cdot)$ be two A-paths in $\bar{L}_0$, A-homotopic in $\bar{L}$ and representing an element $g \in J_\Gamma^{-1}(0)$. Integrate $sa(i,st)$ to get $g(i,t)$ for $i = 0, 1$. Namely we have $sa(i,st) = g(i,s)^{-1}\frac{d}{dt}|_{t=s} g(i,t)$. Then $g(i,t)$ are two paths connecting g and $x := \mathbf{s}(g)$ lying in the subgroupoid $J_\Gamma^{-1}(0)$ since $a(i,t)$ are paths in $\bar{L}_0$. Since the source fibre of $J_\Gamma^{-1}(0)$ is simply connected, there is a homotopy $g(\epsilon,t) \in J_\Gamma^{-1}(0)$ linking $g(0,t)$ and $g(1,t)$. So $sa(\epsilon,st) := g(\epsilon,s)^{-1}\frac{d}{dt}|_{t=s} g(\epsilon,t)$ is an A-path in the variable t representing the element $g(\epsilon,s) \in J_\Gamma^{-1}(0)$ for every fixed s. Hence $sa(\epsilon,st)$ satisfies (20) for every $s \in [0,1]$. Therefore $\langle a_1(\epsilon,t), E\rangle \equiv 0$. Then $a(\epsilon,t) \subset \bar{L}_0$ is an A-homotopy between $a(0,t)$ and $a(1,t)$.

Therefore $J_\Gamma^{-1}(0)$ is the s.s.c. Lie groupoid integrating $J^{-1}(0) = \bar{L}_0$.

4) First of all, given an A-path a of L^c over the base path γ and a point $\tilde{\gamma}(0)$ over $\gamma(0)$ in Q, we lift it to an A-path π^*a of $\bar{L}$ as described in Lemma 4.7. By the same lemma, we see that (L^c) A-homotopic A-paths in L^c lift to ($\bar{L}_0$) A-homotopic A-paths in $\pi^*L^c \cong \bar{L}_0 \subset \bar{L}$, so the map p is well defined Different choices of $\tilde{\gamma}(0)$ give exactly the S^1 orbit of (some choice of) $[\pi^*a]_{\bar{L}}$. Surjectivity of the map p follows from the statement about A-paths in Remark 4.8. Injectivity follows from the fact that $\{[\pi^*a]_{\bar{L}}\} = \{[\pi^*a]_{\bar{L}_0}\}$ in 3) and the statement about A-homotopies in Remark 4.8. $\square$

We saw in Subsection 3.2 that, given any integrable Dirac manifold (P,L), there are two groupoids attached to it. One is the presymplectic groupoid $\Gamma_s(P)$ integrating L; the other is the precontact groupoid $\Gamma_c(P)$ integrating L^c. In the non-integrable case, these two groupoids still exist as stacky groupoids carrying the same geometric structures (presymplectic and precontact) [19]. In this paper, to simplify the treatment, we view them as topological groupoids carrying the same name and when the topological groupoids are smooth manifolds they have additional presymplectic and precontact structures. Item (4) of the following theorem generalizes Thm. 4.2b). The other items generalize from the Poisson case to the Dirac case Theorem 2 and 3 in [8] and a result in [2].

Theorem 4.11. *For a Dirac manifold (P,L), there is a short exact sequence of topological groupoids*

$$1 \to \mathcal{G} \to \Gamma_c(P) \xrightarrow{\tau} \Gamma_s(P) \to 1,$$

where $\mathcal{G}$ is the quotient of the trivial groupoid $\mathbb{R} \times P$ by a group bundle $\mathcal{P}$ over P defined by

$$\mathcal{P}_x := \{\int_{[\gamma]} \omega_F : [\gamma] \in \pi_2(F,x) \text{ and } \gamma \text{ is the base of an } A\text{-homotopy between paths representing } 1_x \text{ in } L.\},$$

with F the presymplectic leaf passing through $x \in P$ and ω_F the presymplectic form on F. In the case that (P,L) is integrable as a Dirac manifold, then

1. *the presymplectic form Ω on $\Gamma_s(P)$ is related to the precontact form θ on $\Gamma_c(P)$ by*

$$\tau^* d\theta = \Omega,$$

and the infinitesimal action R of $\mathbb{R}$ on $\Gamma_c(P)$ via $\mathbb{R} \times P \to \mathcal{G}$ satisfies

$$\mathcal{L}_R \theta = 0, \quad i(R)\theta = 1.$$

2. *R is the left invariant vector field extending the section $(0,0)\oplus(0,-1)$ of $L^c \subset \mathcal{E}^1(P)$ as in Cor. A.2;*

3. *the group $\mathcal{P}_x$ is generated by the periods of R;*

4. *$\Gamma_s(P)$ is prequantizable iff $\mathcal{P} \subset P \times \mathbb{Z}$; in this case its prequantization is $\Gamma_c(P)/\mathbb{Z}$, where $\mathbb{Z}$ acts on $\Gamma_c(P)$ as a subgroup of $\mathbb{R}$.*

5. *If P is prequantizable as a Dirac manifold, then $\Gamma_s(P)$ is prequantizable.*

Proof. The proof of (1) and (4) is the same as Section 4 of [8]. One only has to replace the Poisson bivector π by Υ and the leaf-wise symplectic form of π by ω_F. (3) is clear since R generates the $\mathbb{R}$ action and $\mathcal{G} = \mathbb{R}/\mathcal{P}$.

For (2), we identify $(0,0)\oplus(0,-1)$ with a section of $\ker \mathbf{t}_*$ using Lemma A.1 and then extend it to a left invariant vector field on $J^{-1}(0)/S^1$. Using Cor. A.2 we see that the resulting vector field is killed by $\mathbf{s}_*$, $\mathbf{t}_*$ and $d\theta_\Gamma$ and that it pairs to 1 with θ_Γ, so by the "non-degeneracy" condition in Def. 3.7 it must be equal to R.

For (5), if P is prequantizable as a Dirac manifold, then $\Upsilon = \rho^*\Omega + d_L\beta$ for some integral form Ω on P and $\beta \in \Gamma(L^*)$. Suppose $f = ad\epsilon + bdt$ is a Lie algebroid homomorphism from the tangent bundle $T\square$ of a square $[0,1]\times[0,1]$ to L over the base map $\gamma : \square \to P$, i.e. $a(\epsilon,t)$ is an A-homotopy over γ via $b(\epsilon,t)$ as in (19). Denoting by ω_F the presymplectic form of the leaf F in which $\gamma(\square)$ lies, we have (see also Sect. 3.3 of [2]),

$$\int_\gamma \omega_F = \int_\square \omega_F(\frac{\partial\gamma}{\partial t}, \frac{\partial\gamma}{\partial\epsilon}) = \int_\square \langle ad\epsilon, bdt\rangle_- = \int_\square f^*\Upsilon$$
$$= \int_\square f^*(\rho^*\Omega + d_L\beta) = \int_\square f^*(\rho^*\Omega) = \int_\square \gamma^*\omega = \int_\gamma \omega \in \mathbb{Z}$$

where we used $\Upsilon = \rho^*\omega_F$ in the second equation and $f^*d_L\beta = d_{dR}(f^*\beta)$ in the fifth. $\square$

4.3 Two examples

We present two explicit examples for Thm. 4.2, 4.9 and 4.11.

The first one generalizes Example 4.1.

Example 4.12. Let (P,ω) be an integral symplectic manifold (non necessarily simply connected), and (Q,θ) a prequantization. The s.s.c. contact groupoid of (Q,θ) is $(\bar{Q} \times_{\pi_1(Q)} \bar{Q} \times \mathbb{R}, -e^{-s}\theta_1 + \theta_2, e^{-s})$ where $\bar{Q}$ denotes the universal cover of Q. As in Example 4.1 the moment map is given by $J_\Gamma = -e^{-s} + 1$ and the reduced manifold at zero is $((\bar{Q} \times_{\pi_1(Q)} \bar{Q})/S^1, [-\theta_1 + \theta_2])$, where $\pi_1(Q)$ acts diagonally and the diagonal S^1 action is realized by following the Reeb vector field on $\bar{Q}$.

Notice that the Reeb vector field of $(\bar{Q} \times_{\pi_1(Q)} \bar{Q})/S^1$ is the Reeb vector field of the second copy of $\bar{Q}$. Dividing $\bar{Q}$ by $\mathbb{Z} \subset$ (Flow of Reeb v.f.) is the same as dividing by the $\pi_1(\tilde{Q})$ action on $\bar{Q}$, where $\tilde{Q}$ is the pullback of $Q \to P$ via the universal covering $\tilde{P} \to P$. To see this use that $\pi_1(\tilde{Q})$ is generated by any of its Reeb orbits (look at the long exact sequence corresponding to $S^1 \to \tilde{Q} \to \tilde{P}$), and that the Reeb vector field of $\bar{Q}$ is obtained lifting the one on $\tilde{Q}$. Also notice that $\pi_1(\tilde{Q})$ embeds into $\pi_1(Q)$ (as the subgroup generated by the Reeb orbits of Q) and that the quotient by the embedded image is isomorphic to $\pi_1(P)$, by the long exact sequence for $S^1 \to Q \to P$. So the quotient of $(\bar{Q} \times_{\pi_1(Q)} \bar{Q})/S^1$

by the $\pi_1(\tilde{Q})$ action on the second factor is $(\tilde{Q} \times_{\pi_1(P)} \tilde{Q})/S^1$ where we used $\bar{Q}/\pi_1(\tilde{Q}) = \tilde{Q}$ on each factor. This groupoid, together with the induced 1-form $[-\theta_1 + \theta_2]$, is clearly the prequantization of the s.s.c. symplectic groupoid $(\tilde{P} \times_{\pi_1(P)} \tilde{P}, -\omega_1 + \omega_2)$ of (P, ω).

In the second example we consider a Lie algebra $\mathfrak{g}$. Its dual $\mathfrak{g}^*$ is endowed with a linear Poisson structure Λ, called Lie-Poisson structure, and the Euler vector field A satisfies $\Lambda = -d_\Lambda A$ where d_Λ is the Poisson cohomology differential. So the prequantization condition (3) for $(\mathfrak{g}^*, \Lambda)$ is satisfied, with $\Omega = 0$ and $\beta = A$. We display the contact groupoid integrating the induced prequantization $(Q, \bar{L})$ for the simple case that $\mathfrak{g}$ be one dimensional; then we show that (a discrete quotient of) the S^1 contact reduction of this groupoid is the prequantization of the symplectic groupoid of $\mathfrak{g}^*$.

Example 4.13. Let $\mathfrak{g} = \mathbb{R}$ be the one-dimensional Lie algebra. We claim that the prequantization $Q = S^1 \times \mathfrak{g}^*$ of $\mathfrak{g}^*$ as above has as a s.s.c. contact groupoid $\Gamma_c(Q)$ the quotient of

$$(\mathbb{R}^5, x d\epsilon - e^t d\theta_1 + d\theta_2, e^t) \tag{21}$$

by the diagonal $\mathbb{Z}$ action on the variables (θ_1, θ_2). Here the coordinates on the five factors of $\mathbb{R}^5$ are $(\theta_1, t, \epsilon, \theta_2, x)$. The groupoid structure is the product of the following three groupoids: $\mathbb{R} \times \mathbb{R} = \{(\theta_1, \theta_2)\}$ the pair groupoid; $\mathbb{R} \times \mathbb{R} = \{(t, x)\}$ the action groupoid given by the flow of the vector field $-x\partial_x$ on $\mathbb{R}$, i.e. $(t', e^{-t}x) \cdot (t, x) = (t' + t, x)$; and $\mathbb{R} = \{\epsilon\}$ the group.

To see this, first determine the prequantization of $(\mathfrak{g}^*, \Lambda)$: it is $Q = S^1 \times \mathbb{R}$ with Jacobi structure $(E \wedge x\partial_x, E)$, where $E = \partial_\theta$ is the infinitesimal generator of the circle action and $x\partial_x$ is just the Euler vector field on $\mathfrak{g}^*$ (see [4]). This Jacobi manifold has two open leaves, and we first focus on one of them, say $Q_+ = S^1 \times \mathbb{R}_+$. This is a locally conformal symplectic leaf, with structure $(d\theta \wedge \frac{dx}{x}, \frac{dx}{x})$.

We determine the s.s.c contact groupoid $\Gamma_c(Q_+)$ of $(Q_+, d\theta \wedge \frac{dx}{x}, \frac{dx}{x})$ applying Lemma B.1 (choosing $\tilde{g} = \log x$, so that $e^{-\tilde{g}}\tilde{\Omega} = d(x^{-1}d\theta)$ there). We obtain the quotient of

$$(\tilde{Q}_+ \times \mathbb{R} \times \tilde{Q}_+, x_2 d\epsilon - \frac{x_2}{x_1} d\theta_1 + d\theta_2, \frac{x_2}{x_1})$$

by the diagonal $\mathbb{Z}$ action on the variables (θ_1, θ_2). Here (θ_i, x_i) are the coordinates on the two copies of the universal cover $\tilde{Q}_+ \cong \mathbb{R} \times \mathbb{R}_+$ and ϵ is the coordinate on the $\mathbb{R}$ factor. The groupoid structure is given by the product of the pair groupoid over $\tilde{Q}_+$ and group $\mathbb{R}$. This contact groupoid, and the one belonging to $Q_- = S^1 \times \mathbb{R}_-$, will sit as open contact subgroupoids in the contact groupoid of Q, and the question is how to "complete" the disjoint union of $\Gamma_c(Q_+)$ and $\Gamma_c(Q_-)$ to obtain the contact groupoid of Q. A clue comes from the simplest case of groupoid with two open orbits and a closed one to separate them, namely the transformation groupoid of a vector field on $\mathbb{R}$ with exactly one zero. The transformation groupoid associated to $-x\partial_x$ is $\mathbb{R} \times \mathbb{R} = \{(t, x)\}$ with source given by x, target given by $e^{-t}x$ and multiplication $(t', e^{-t}x) \cdot (t, x) = (t' + t, x)$. Notice that, on each of the two open orbits $\mathbb{R}_+$ and $\mathbb{R}_-$ the groupoid is isomorphic to a pair groupoid by the correspondence $(t, x) \in \mathbb{R} \times \mathbb{R}_\pm \mapsto (e^{-t}x, x) \in \mathbb{R}_\pm \times \mathbb{R}_\pm$, with inverse $(x_1, x_2) \mapsto (\log(\frac{x_2}{x_1}), x_2)$.

Now we embed $\Gamma_c(Q_+)$ into the groupoid $\Gamma_c(Q)$ described in (21) by the mapping

$$(\theta_1, x_1, \epsilon, \theta_2, x_2) \mapsto \left(\theta_1, t = \log(\frac{x_2}{x_1}), \epsilon, \theta_2, x = x_2\right),$$

and similarly for $\Gamma_c(Q_-)$. The contact forms and function translate to those indicated in (21), which as a consequence also satisfy the multiplicativity condition. One checks directly that the one form is a contact form also on the complement $\{x = 0\}$ of the two open subgroupoids. Therefore the one described in (21) is a contact groupoid, and since we know that the source map is a Jacobi map on the open dense set sitting over Q_+ and Q_-, it is the contact groupoid of $(Q, E \wedge x\partial_x, E)$.

Now we consider the S^1 contact reduction of the above s.s.c. groupoid $\Gamma_c(Q)$. As shown in the proof of Theorem 4.2 the moment map is $J_\Gamma = 1 - f_\Gamma = 1 - e^t$, so its zero level set is $\{t = 0\}$. The definition of moment map and the fact that the infinitesimal generator v_Γ of the S^1 action projects to E both via source and via target imply that on $\{t = 0\}$ we have $v_\Gamma = (\partial_{\theta_1}, 0, 0, \partial_{\theta_2}, 0)$. So $J^{-1}(0)/S^1$ is $\mathbb{R}^3$ with coordinates $(\theta := \theta_2 - \theta_1, \epsilon, x)$, 1-form $d\theta + x d\epsilon$, source and target both given by x and groupoid multiplication given by addition in the θ and ϵ factors. Upon division of the θ factor by $\mathbb{Z}$ (notice that the Reeb vector field of $\Gamma_c(Q)$ is ∂_{θ_2}) this is clearly just the prequantization of $T^*\mathbb{R}$, endowed with the canonical symplectic form $dx \wedge d\epsilon$ and fiber addition as groupoid multiplication, i.e. the prequantization of the symplectic groupoid of the Poisson manifold $(\mathbb{R}, 0)$.

A Lie algebroids of precontact groupoids

Lemma A.1. *Let $(\Gamma, \theta_\Gamma, f_\Gamma)$ be a precontact groupoid (as in Definition 3.7) over the Jacobi-Dirac manifold $(Q, \bar{L})$, so that the source map be a Jacobi-Dirac map. Then a Lie algebroid isomorphism between* $\ker \mathbf{s}_*|_Q$ *and $\bar{L}$ is given by*

$$Y \mapsto (\mathbf{t}_* Y, -r_{\Gamma *} Y) \oplus (-d\theta_\Gamma(Y)|_{TQ}, \theta_\Gamma(Y)) \tag{22}$$

where $e^{-r_\Gamma} = f_\Gamma$. A Lie algebroid isomorphism between $\ker \mathbf{t}_*|_Q$ *and $\bar{L}$ (obtained composing the above with i_* for i the inversion) is*

$$Y \mapsto (\mathbf{s}_* Y, r_{\Gamma *} Y) \oplus (d\theta_\Gamma(Y)|_{TQ}, -\theta_\Gamma(Y)) \tag{23}$$

Proof. Consider the groupoid $\Gamma \times \mathbb{R}$ over $Q \times \mathbb{R}$ with target map $\tilde{\mathbf{t}}(g, t) = (\mathbf{t}(g), t - r_\Gamma(g))$ and the obvious source $\tilde{\mathbf{s}}$ and multiplication. $(\Gamma \times \mathbb{R}, d(e^t\theta_\Gamma))$ is then a presymplectic groupoid with the property that $\tilde{\mathbf{s}}$ is a forward Dirac map onto $(Q \times \mathbb{R}, \tilde{L})$, where

$$\tilde{L}_{(q,t)} = \{(X, f) \oplus e^t(\xi, g) : (X, f) \oplus (\xi, g) \in L_q\}$$

is the "Diracization" ([25][17]) of the Jacobi-Dirac structure $\bar{L}$ and t is the coordinate on $\mathbb{R}$. In the special case that $\bar{L}$ corresponds to a Jacobi structure this is just Prop. 2.7 of [8]; in the general case (but assuming different conventions for the multiplicativity of θ_Γ and for which of source and target is a Jacobi-Dirac map) this is Prop. 3.3 in [17]. We will prove only the first isomorphism above (the one for $\ker \mathbf{s}_*|_Q$); the other one follows by composing the first isomorphism with i_*. Now we consider the following diagram of spaces of sections (on the left column we have sections over Q, on the right column sections over $Q \times \mathbb{R}$):

$$\begin{array}{ccc}
\Gamma(\ker \mathbf{s}_*|_Q) & \xrightarrow{\Phi_\mathbf{s}} & \Gamma(\ker \tilde{\mathbf{s}}_*|_{Q\times\mathbb{R}}) \\
\downarrow & & \Phi \downarrow \\
\Gamma(\bar{L}) & \xrightarrow{\Phi_L} & \tilde{L}.
\end{array}$$

The first horizontal arrow $\Phi_{\mathbf{s}}$ is $Y \mapsto \tilde{Y}$, where the latter denotes the constant extension of Y along the $\mathbb{R}$ direction of the base $Q \times \mathbb{R}$. Notice that the projection $pr : \Gamma \times \mathbb{R} \to \Gamma$ is a groupoid morphism, so it induces a surjective Lie algebroid morphism $pr_* : \ker \tilde{\mathbf{s}}_*|_{Q\times\mathbb{R}} \to \ker \mathbf{s}_*|_Q$. Since sections $\tilde{Y}$ as above are projectable, by Prop. 4.3.8. in [15] we have $pr_*[\tilde{Y}_1, \tilde{Y}_2] = [Y_1, Y_2]$, and since pr_* is a fiberwise isomorphism we deduce that $\Phi_{\mathbf{s}}$ is a bracket-preserving map.

The vertical arrow Φ is induced from the following isomorphism of Lie algebroids (Cor. 4.8 iii of [3][8]) valid for any presymplectic manifold $(\tilde{\Gamma}, \Omega)$ over a Dirac manifold $(N, \tilde{L})$ for which the source map is Dirac:

$$\ker \tilde{\mathbf{s}}_*|_N \to \tilde{L}\ ,\ Z \mapsto (\tilde{\mathbf{t}}_* Z, -\Omega(Z)|_{TN}).$$

In our case, as mentioned above, the presymplectic form is $d(e^t\theta_\Gamma)$.

The second horizontal arrow Φ_L is the natural map

$$(X, f) \oplus (\xi, g) \in L_q \mapsto (X, f) \oplus e^t(\xi, g) \in \bar{L}_{(q,t)}$$

which preserves the Lie algebroid bracket (see the remarks after Definition 3.2 of [25]).

One can check that $(\Phi \circ \Phi_{\mathbf{s}})(Y) = (\tilde{\mathbf{t}}_* \tilde{Y}) \oplus (-d(e^t\theta_\Gamma)(\tilde{Y})|_{TQ\times\mathbb{R}})$ lies in the image of the injective map Φ_L. The resulting map from $\Gamma(\ker \mathbf{s}_*)$ to $\Gamma(\bar{L})$ is given by (22) and the arguments above show that this map preserves brackets. Further it is clear that this map of sections is induced by a vector bundle morphism given by the same formula, which clearly preserves not only the bracket of sections but also the anchor, so that the map $\ker \mathbf{s}_*|_Q \to \bar{L}$ given by (22) is a Lie algebroid morphism.

To show that it is an isomorphism one can argue noticing that $\ker \mathbf{s}_*$ and $\bar{L}$ have the same dimension and show that the vector bundle map is injective, by using the "non-degeneracy condition" in Def. 3.7 and the fact that the source and target fibers of $\Gamma \times \mathbb{R}$ are presymplectic orthogonal to each other. □

The vector bundle morphisms in the above lemma give a characterization of vectors tangent to the $\mathbf{s}$ or $\mathbf{t}$ fibers of a precontact groupoid as follows. Consider for instance a vector λ in $\bar{L}_x$, where $\bar{L}$ is the Jacobi-Dirac structure on the base Q. This vector corresponds to some $Y_x \in \ker \mathbf{t}_*$ by the isomorphism (23), and by left translation we obtain a vector field Y tangent to $\mathbf{t}^{-1}(x)$. Of course, every vector tangent to $\mathbf{t}^{-1}(x)$ arises in this way for a unique λ. The vector field Y satisfies the following equations at every point g of $\mathbf{t}^{-1}(x)$, which follow by simple computation from the multiplicativity of θ_Γ: $\theta_\Gamma(Y_g) = \theta_\Gamma(Y_x)$, $d\theta_\Gamma(Y_g, Z) = d\theta_\Gamma(Y_x, \mathbf{s}_* Z) - r_{\Gamma *} Y_x \cdot \theta_\Gamma(Z)$ for all $Z \in T_g\Gamma$, $r_{\Gamma *} Y_g = r_{\Gamma *} Y_x$ and $\mathbf{s}_* Y_g = \mathbf{s}_* Y_x$. Notice that the right hand sides of this properties can be expressed in terms of the four components of $\lambda \in \mathcal{E}^1(Q)$, and that by the "non-degeneracy" of θ_Γ these properties are enough to uniquely determine Y_g. We sum up this discussion into the following corollary, which can be used as a tool in computations on precontact groupoids in the same way that hamiltonian vector fields are used on contact or symplectic groupoids (such as the proof of Thm. 4.2):

Corollary A.2. *Let $(\Gamma, \theta_\Gamma, f_\Gamma)$ be a precontact groupoid (as in Definition 3.7) and denote by $\bar{L}$ the Jacobi-Dirac structure on the base Q so that source map is Jacobi-Dirac. Then there*

[8] In [3] the authors adopted the convention that the target map be a Dirac map. Here we use their result applied to the pre-symplectic form $-\Omega$.

is bijection between sections of $\bar{L}$ and vector fields on Γ which are tangent to the $\mathbf{t}$*-fibers and are left invariant. To a section $(X, f) \oplus (\xi, g)$ of $\bar{L} \subset \mathcal{E}^1(Q)$ corresponds the unique vector field Y tangent to the* $\mathbf{t}$*-fibers which satisfies*

- $\theta_\Gamma(Y) = -g$
- $d\theta_\Gamma(Y) = \mathbf{s}^*\xi - f\theta_\Gamma$
- $\mathbf{s}_* Y = X.$

Y furthermore satisfies $r_{\Gamma *}Y = f$.

B Groupoids of locally conformal symplectic manifolds

A locally conformal symplectic (l.c.s.) manifold is a manifold (Q, Ω, ω) where Ω is a non-degenerate 2-form and ω is a closed 1-form satisfying $d\Omega = \omega \wedge \Omega$. Any Jacobi manifold is foliated by contact and l.c.s. leaves (see for example [27]); in particular a l.c.s. manifold is a Jacobi manifold, and hence, when it is integrable, it has an associated s.s.c. contact groupoid. In this appendix we will construct explicitly this groupoid; we make use of it in Example 4.13.

Lemma B.1. *Let (Q, Ω, ω) a locally conformal symplectic manifold. Consider the pullback structure on the universal cover $(\tilde{Q}, \tilde{\Omega}, \tilde{\omega})$, and write $\tilde{\omega} = d\tilde{g}$. Then Q is integrable as a Jacobi manifold iff the symplectic form $e^{-\tilde{g}}\tilde{\Omega}$ is a multiple of an integer form. In that case, choosing $\tilde{g}$ so that $e^{-\tilde{g}}\tilde{\Omega}$ is integer, the s.s.c. contact groupoid of (Q, Ω, ω) is the quotient of*

$$\left(\tilde{R} \times_{\mathbb{R}} \tilde{R}, e^{\tilde{\mathbf{s}}^*\tilde{g}}(-\tilde{\sigma}_1 + \tilde{\sigma}_2), \frac{e^{\tilde{\mathbf{s}}^*\tilde{g}}}{e^{\tilde{\mathbf{t}}^*\tilde{g}}}\right), \tag{24}$$

a groupoid over $\tilde{Q}$, by a natural $\pi_1(Q)$ action. Here $(\tilde{R}, \tilde{\sigma})$ is the universal cover (with the pullback 1-form) of a prequantization (R, σ) of $(\tilde{Q}, e^{-\tilde{g}}\tilde{\Omega})$, and the group $\mathbb{R}$ acts by the diagonal lift of the S^1 action on R.

Proof. Using for example the Lie algebroid integrability criteria of [7], one sees that (Q, Ω, ω) is integrable as a Jacobi manifold iff $(\tilde{Q}, \tilde{\Omega}, \tilde{\omega})$ is. Lemma 1.5 in Appendix I of [27] states that, given a contact groupoid, multiplying the contact form by $\mathbf{s}^*u$ and the multiplicative function by $\frac{\mathbf{s}^*u}{\mathbf{t}^*u}$ gives another contact groupoid, for any non-vanishing function u on the base. Such an operation corresponds to twisting the groupoid, viewed just as a Jacobi manifold, by the function $\mathbf{s}^*u^{-1}$, hence the Jacobi structure induced on the base by the requirement that the source be a Jacobi map is the twist of the original one by u^{-1}. So $(\tilde{Q}, \tilde{\Omega}, \tilde{\omega})$ is integrable iff the symplectic manifold $(\tilde{Q}, e^{-\tilde{g}}\tilde{\Omega})$ is Jacobi integrable, and by Section 7 of [8] this happens exactly when the class of $e^{-\tilde{g}}\tilde{\Omega}$ is a multiple of an integer one.

Choose $\tilde{g}$ so that this class is actually integer. A contact groupoid of $(\tilde{Q}, e^{-\tilde{g}}\tilde{\Omega})$ is clearly $(R \times_{S^1} R, [-\sigma_1 + \sigma_2], 1)$, where the S^1 action on $R \times R$ is diagonal and "[ ]" denotes the form descending from $R \times R$. This groupoid is not s.s.c.; the s.s.c. one is $\tilde{R} \times_{\mathbb{R}} \tilde{R}$, where the $\mathbb{R}$ action on $\tilde{R}$ is the lift of the S^1 action on R. The source simply connectedness follows since

$\mathbb{R}$ acts transitively (even though not necessarily freely) on each fiber of the map $\tilde{R} \to \tilde{Q}$, and this in turns holds because any S^1 orbit in R generates $\pi_1(R)$ and because the fundamental group of a space always acts (by lifting loops) transitively on the fibers of its universal cover.

By the above cited Lemma from [27] we conclude that (24) is the s.s.c. contact groupoid of $(\tilde{Q}, \tilde{\Omega}, \tilde{\omega})$. The fundamental group of Q acts on $\tilde{Q}$ respecting its geometric structure, so it acts on its Lie algebroid $T^*\tilde{Q} \times \mathbb{R}$. Since the path-space construction of the s.s.c. groupoid is canonical (see Subsection 4.2), $\pi_1(Q)$ acts on the s.s.c. groupoid (24) preserving the groupoid and geometric structure. Hence the quotient is a s.s.c. contact groupoid over (Q, Ω, ω), and its source map is a Jacobi map, so it is the s.s.c. contact groupoid of (Q, Ω, ω). $\square$

C On a construction of Vorobjev

In Section 2 we derived the geometric structure on the circle bundles Q from a prequantizable Dirac manifold (P, L) and a suitable choice of connection D. In this appendix we describe an alternative attempt; even though we can make our construction work only if we start with a symplectic manifold, we believe the construction is interesting on its own right.

First we recall Vorobjev's construction in Section 4 of [21], which the author there uses to study the linearization problem of Poisson manifolds near a symplectic leaf. Consider a transitive algebroid A over a base P with anchor ρ; the kernel $\ker \rho$ is a bundle of Lie algebras. Choose a splitting $\gamma : TP \to A$ of the anchor. Its curvature R_γ is a 2-form on P with values in $\Gamma(\ker \rho)$ (given by $R_\gamma(v, w) = [\gamma v, \gamma w]_A - \gamma[v, w]$). The splitting γ also induces a (TP-)covariant derivative ∇ on $\ker \rho$ by $\nabla_v s = [\gamma v, s]_A$. Now, if P is endowed with a symplectic form ω, a neighborhood of the zero section in $(\ker \rho)^*$ inherits a Poisson structure $\Lambda_{vert} + \Lambda_{hor}$ as follows (Theorem 4.1 in [21]): denoting by F_s the fiberwise linear function on $(\ker \rho)^*$ obtained by contraction with the section s of $\ker \rho$, the Poisson bivector has a vertical component determined by $\Lambda_{vert}(dF_{s_1}, dF_{s_2}) = F_{[s_1, s_2]}$. It also has a component Λ_{hor} which is tangent to the Ehresmann connection Hor given by the dual connection[9] to ∇ on the bundle $(\ker \rho)^*$; Λ_{hor} at $e \in (\ker \rho)^*$ is obtained by restricting the non-degenerate form $\omega - \langle R_\gamma, e\rangle$ to Hor_e and inverting it. (Here we are identifying Hor_e and the corresponding tangent space to P.)

To apply Vorobjev's construction in our setting, let (P, ω) be a prequantizatible symplectic manifold and (K, ∇_K) its prequantization line bundle with Hermitian connection of curvature $2\pi i\omega$. By Lemma 2.8 we obtain a flat $TP \oplus_\omega \mathbb{R}$-connection $\tilde{D}_{(X,f)} = \nabla_X + 2\pi i f$ on K. Now we make use of the following well know fact about extensions, which can be proven by direct computation:

Lemma C.1. *Let A be a Lie algebroid over M, V a vector bundle over M, and $\tilde{D}$ a flat A-connection on V. Then $A \oplus V$ becomes a Lie algebroid with the anchor of A as anchor and bracket*

$$[(Y_1, S_1), (Y_2, S_2)] = ([Y_1, Y_2]_A, \tilde{D}_{Y_1} S_2 - \tilde{D}_{Y_2} S_1).$$

Therefore $A := TP \oplus_\omega \mathbb{R} \oplus K$ is a transitive Lie algebroid over P, with isotropy bundle $\ker \rho = \mathbb{R} \oplus K$ and bracket $[(f_1, S_1), (f_2, S_2)] = [(0, 2\pi i(f_1 S_2 - f_2 S_1)]$ there. Now choosing the canonical splitting γ of the anchor $TM \oplus_\omega \mathbb{R} \oplus K \to TM$ we see that its curvature

[9] In [21] the author phrases this condition as $\mathcal{L}_{hor(X)} F_s = F_{\nabla_X s}$.

is $R_\gamma(X_1, X_2) = (0, \omega(X_1, X_2), 0)$. The horizontal distribution on the dual of the isotropy bundle is the product of the trivial one on $\mathbb{R}$ and of the one corresponding to ∇_K on K (upon identification of K and K^* by the metric). By the above, there is a Poisson structure on $\mathbb{R} \oplus K$, at least near the zero section: the Poisson bivector at (t, q) has a horizontal component given by lifting the inverse of $(1-t)\omega$ and a vertical component which turns out to be $2\pi(iq\partial_q) \wedge \partial t$, where "$iq\partial_q$" denotes the vector field tangent to the circle bundles in K obtained by turning by 90° the Euler vector field $q\partial_q$. A symplectic leaf is clearly given by $\{t < 1\} \times Q$ (where $Q = \{|q| = 1\}$). On this leaf the symplectic structure is seen to be given by $(1-t)\omega + \theta \wedge dt = d((1-t)\theta)$, where θ is the connection 1-form on Q corresponding to the connection ∇_K on K (which by definition satisfies $d\theta = \pi^*\omega$). This means that the leaf is just the symplectification $(\mathbb{R}_+ \times Q, d(r\theta))$ of (Q, θ) (here $r = 1 - t$), which is a "prequantization space" for our symplectic manifold (P, ω). Unfortunately we are not able to modify Vorobjev's construction appropriately when P is a Poisson or Dirac manifold.

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Coisotropic embeddings in Poisson manifolds

Alberto S. Cattaneo and Marco Zambon

Abstract

We consider existence and uniqueness of two kinds of coisotropic embeddings and deduce the existence of deformation quantizations of certain Poisson algebras of basic functions. First we show that any submanifold of a Poisson manifold satisfying a certain constant rank condition, already considered by Calvo and Falceto [4], sits coisotropically inside some larger cosymplectic submanifold, which is naturally endowed with a Poisson structure. Then we give conditions under which a Dirac manifold can be embedded coisotropically in a Poisson manifold, extending a classical theorem of Gotay.

Contents

1 Introduction

The following two results in symplectic geometry are well known. First: a submanifold C of a symplectic manifold (M, Ω) is contained coisotropically in some symplectic submanifold of M iff the pullback of Ω to C has constant rank; see Marle's work [17]. Second: a manifold endowed with a closed 2-form ω can be embedded coisotropically into a symplectic manifold

(M, Ω) so that $i^*\Omega = \omega$ (where i is the embedding) iff ω has constant rank; see Gotay's work [15].

In this paper we extend these results to the setting of Poisson geometry. Recall that P is a Poisson manifold if it is endowed with a bivector field $\Pi \in \Gamma(\wedge^2 TP)$ satisfying the Schouten-bracket condition $[\Pi, \Pi] = 0$. A submanifold C of (P, Π) is coisotropic if $\sharp N^*C \subset TC$, where the conormal bundle N^*C is defined as the annihilator of TC in $TP|_C$ and $\sharp\colon T^*P \to TP$ is the contraction with the bivector Π. Coisotropic submanifolds appear naturally; for instance the graph of any Poisson map is coisotropic, and for any Lie subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ the annihilator $\mathfrak{h}^\circ$ is a coisotropic submanifold of the Poisson manifold $\mathfrak{g}^*$. Further coisotropic submanifolds C are interesting for a variety of reasons, one being that the distribution $\sharp N^*C$ is a (usually singular) integrable distribution whose leaf space, if smooth, is a Poisson manifold.

To give a Poisson-analogue of Marle's result we consider *pre-Poisson* submanifolds, i.e. submanifolds C for which $TC + \sharp N^*C$ has constant rank (or equivalently $pr_{NC} \circ \sharp\colon N^*C \to TP|_C \to NC := TP|_C/TC$ has constant rank). Natural classes of pre-Poisson submanifolds are given by affine subspaces $\mathfrak{h}^\circ + \lambda$ of $\mathfrak{g}^*$, where $\mathfrak{h}$ is a Lie subalgebra of the Lie algebra $\mathfrak{g}^*$ and λ any element of $\mathfrak{g}^*$, and of course by coisotropic submanifolds and by points. More details are given in [12], where it is also shown that pre-Poisson submanifolds satisfy some functorial properties. This can be used to show that on a Poisson-Lie group G the graph of L_h (the left translation by some fixed $h \in G$, which clearly is not a Poisson map) is a pre-Poisson submanifold, giving rise to a natural constant rank distribution D_h on G that leads to interesting constructions. For instance, if the Poisson structure on G comes from an r-matrix and the point h is chosen appropriately, G/D_h (when smooth) inherits a Poisson structure from G, and $[L_h] : G \to G/D_h$ is a Poisson map which is moreover equivariant w.r.t. the natural Poisson actions of G.

In the following table we characterize submanifolds of a symplectic or Poisson manifold in terms of the bundle map $\rho := pr_{NC} \circ \sharp\colon N^*C \to NC$:

	P symplectic	P Poisson
$Im(\rho) = 0$	C coisotropic	C coisotropic
$Im(\rho) = NC$	C symplectic	C cosymplectic
$Rk(\rho)$ =const	C presymplectic	C pre-Poisson

In the first part of this paper (sections 3- 6) we consider the Poisson-analog of Marle's result, i.e. we ask the following question:

> Given an arbitrary submanifold C of a Poisson manifold (P, Π), under what conditions does there exist some submanifold $\tilde{P} \subset P$ such that
>
> a) $\tilde{P}$ has a Poisson structure induced from Π
>
> b) C is a coisotropic submanifold of $\tilde{P}$?
>
> When the submanifold $\tilde{P}$ exists, is it unique up to neighborhood equivalence (i.e. up to a Poisson diffeomorphism on a tubular neighborhood which fixes C)?

We show in section 3 that for any pre-Poisson submanifold C of a Poisson manifold P there is a submanifold $\tilde{P}$ which is cosymplectic (and hence has a canonically induced Poisson structure) such that C lies coisotropically in $\tilde{P}$. Further (section 4) this cosymplectic submanifold is unique up to neighborhood equivalence; to the best of our knowledge, this uniqueness result is new even in the symplectic setting. In section 5 we give sufficient conditions and necessary conditions for the existence of a submanifold $\tilde{P}$ as in the above question and we provide examples. Then in section 6 we deduce statements about the algebra $C^{\infty}_{bas}(C)$ of functions on C which are basic (invariant), meaning that their differentials annihilate the distribution $\sharp N^*C \cap TC$, and about and its deformation quantization. We show that if C is a pre-Poisson submanifold so that the first and second Lie algebroid cohomology of $N^*C \cap \sharp^{-1}TC$ vanish, then the Poisson algebra of basic functions on C admits a deformation quantization. Finally in section 7, assuming that the symplectic groupoid $\Gamma_s(P)$ of P exists, we describe two subgroupoids (an isotropic and a presymplectic one) naturally associated to a pre-Poisson submanifold C of P.

The second part of this paper (sections 8 and 9) deals with a different embedding problem, where we start with an abstract manifold instead of a submanifold of some Poisson manifold. This is the Poisson-analogue of Gotay's result. The question we ask is:

> Let (M, L) be a Dirac manifold. Is there an embedding $i\colon (M, L) \to (P, \Pi)$ into a Poisson manifold such that
>
> a) $i(M)$ is a coisotropic submanifold of P
>
> b) the Dirac structure L is induced by the Poisson structure Π?
>
> Is such an embedding unique up to neighborhood equivalence?

In the symplectic setting both existence and uniqueness hold [15]. One motivation for this question is the deformation quantization of the Poisson algebra of so-called admissible functions on (M, L), for a coisotropic embedding as above allows one to reduce the problem to [10], i.e. to the deformation quantization of the basic functions on a coisotropic submanifold of a Poisson manifold.

It turns out (section 8) that the above question admits a positive answer iff the distribution $L \cap TM$ on the Dirac manifold M is regular. In that case one expects the Poisson manifold $\tilde{P}$ to be unique (up to a Poisson diffeomorphism fixing M), provided $\tilde{P}$ has minimal dimension. We are not able to prove this global uniqueness; we can just show in section 9 that the Poisson vector bundle $T\tilde{P}|_M$ is unique (an infinitesimal statement along M) and that around each point of M a small neighborhood of $\tilde{P}$ is unique (a local statement). We remark that A. Wade [20] has been considering a similar question too. Our result about deformation quantization is the following (Thm. 8.5): let (M, L) be a Dirac manifold such that $L \cap TM$ has constant rank, and denote by $\mathcal{F}$ the regular foliation integrating $L \cap TM$. If the first and second foliated de Rham cohomologies of the foliation $\mathcal{F}$ vanish then the Poisson algebra of admissible functions on (M, L) has a deformation quantization. In Prop. 8.6 we also notice that the foliated de Rham cohomology $\Omega^{\bullet}_{\mathcal{F}}(M)$ admits the structure of an L_∞-algebra (canonically up to L_∞-isomorphism), generalizing a result of Oh and Park in the presymplectic setting (Thm. 9.4 of [18]).

We end this introduction describing one of our motivations for the first question above, namely an application of the Poisson sigma model to quantization problems. The Poisson sigma model is a topological field theory, whose fields are bundle maps from $T\Sigma$ (for Σ a surface) to the cotangent bundle T^*P of a Poisson manifold (P,Π). It was used by Felder and the first author [8] to derive and interpret Kontsevich's formality theorem and his star product on the Poisson manifold P. The Poisson sigma model with boundary conditions on a coisotropic submanifold C, when suitable assumptions on C are satisfied and P is assumed to be an open subset of $\mathbb{R}^n$, provides [9] a deformation quantization of the Poisson algebra of basic (invariant) functions $C^\infty_{bas}(C)$ on C. This result was globalized using a supergeometric version of Kontsevich's formality theorem [10]: when the first and second cohomology of the Lie algebroid N^*C vanish, for C a coisotropic submanifold of any Poisson manifold P, the Poisson algebra $C^\infty_{bas}(C)$ admits a deformation quantization. Notice that the quotient of C by the distribution $\sharp N^*C$ is usually not a smooth manifold. Hence $C^\infty_{bas}(C)$ is usually not the algebra of functions on any Poisson manifold, and one cannot apply Kontsevich's theorem [16] on deformation quantization of Poisson manifolds directly.

Calvo and Falceto observed that the most general boundary conditions for the Poisson sigma model are given by pre-Poisson submanifolds of (P,Π) (which they referred to as "strongly regular submanifolds"). They show [5] that when P is an open subset of $\mathbb{R}^n$ the problem of deformation quantizing the Poisson algebra of basic functions on C can be reduced to the results of [9]. The computations in [5] are carried out choosing local coordinates on P adapted to C. The strong regularity condition allows one to choose local constraints for C such that the number of first class constraints (X^μs whose Poisson bracket with all other constrains vanish on C) and second class constraints (the remaining constraints X^A, which automatically satisfy $det\{X^A,X^B\}\neq 0$ on C) be constant along C. Setting the second class constraints X^A to zero locally gives a submanifold with an induced Poisson structure, and the fact that only first class constraints are left means that C lies in it as a coisotropic submanifold. Our first question above can be seen as a globalization of Calvo and Falceto's results.

Conventions: We use the term "presymplectic manifold" to denote a manifold endowed with a closed 2-form of *constant rank*, i.e. such that its kernel have constant rank. However we stick to the denominations "presymplectic groupoid" coined in [2] and "presymplectic leaves" (of a Dirac manifold) despite the fact that the 2-forms on these objects do not have constant rank, for these denominations seem to be established in the literature.

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2 Basic definitions

We will use some notions from Dirac linear algebra [13] [3]. A Dirac structure on a vector space P is a subspace $L \subset P \oplus P^*$ which is maximal isotropic w.r.t. the natural symmetric inner product on $P \oplus P^*$ (i.e. L is isotropic and has same dimension as P). A Dirac structure L specifies a subspace $\mathcal{O}$, defined as the image of L under the projection $P \oplus P^* \to P$, and a skew-symmetric bilinear form ω on $\mathcal{O}$, given by $\omega(X_1, X_2) = \langle \xi_1, X_2 \rangle$ where ξ_1 is any element of P^* such that $(X_1, \xi_1) \in L$. The kernel of ω (which in terms of L is given as $L \cap P$) is called *characteristic subspace*. Conversely, any choice of bilinear form defined on a subspace of P determines a Dirac structure on P. Given this equivalence, we will sometimes work with the bilinear form ω on $\mathcal{O}$ instead of working with L.

We consider now Poisson vector spaces (P, Π) (i.e. $\Pi \in \wedge^2 P$; we denote by $\sharp \colon P^* \to P$ the map induced by contraction with Π). The Poisson structure on P is encoded by the Dirac structure $L_P = \{(\sharp\xi, \xi) \colon \xi \in P^*\}$. The image of L_P under the projection onto the first factor is $\mathcal{O} = \sharp P^*$, and the bilinear form ω is non-degenerate.

Remark 2.1. We recall that any subspace W of a Dirac vector space (P, L) has an induced Dirac structure L_W; the bilinear form characterizing L_W is just the pullback of ω (hence it is defined on $W \cap \mathcal{O}$). When (P, Π) is actually a Poisson vector space, one shows[1] that the symplectic orthogonal of $W \cap \mathcal{O}$ in $(\mathcal{O}, \omega)$ is $\sharp W^\circ$. Hence $\sharp W^\circ \cap W$ is the kernel of the restriction of ω to $W \cap \mathcal{O}$, i.e. it is the characteristic subspace of the Dirac structure L_W, and we will refer to it as the *characteristic subspace of W*. Notice that pulling back Dirac structure is functorial [3] (i.e. if W is contained in some other subspace W' of P, pulling back L first to W' and then to W gives the Dirac structure L_W), hence L_W, along with the corresponding bilinear form and characteristic subspace, is *intrinsic* to W.

Let W be a subspace of the Poisson vector space (P, Π). W is called *coisotropic* if $\sharp W^\circ \subset W$, which by the above means that $W \cap \mathcal{O}$ is coisotropic in $(\mathcal{O}, \omega)$.

W is called *Poisson-Dirac* subspace [14] when $\sharp W^\circ \cap W = \{0\}$; equivalent conditions are that $W \cap \mathcal{O}$ be a symplectic subspace of $(\mathcal{O}, \omega)$ or that the pullback Dirac structure L_P correspond to a Poisson bivector on W.

W is called *cosymplectic* subspace if $\sharp W^\circ \oplus W = P$, or equivalently if the pushforward of Π via the projection $P \to P/W$ is an invertible bivector. Notice that if W is cosymplectic then it has a canonical complement $\sharp W^\circ$ which is a symplectic subspace of $(\mathcal{O}, \omega)$. Clearly a cosymplectic subspace is automatically a Poisson-Dirac subspace, and the Poisson bivector on W can be expressed in a particularly simple way [14]: its sharp map $\sharp_W \colon W^* \to W$ is given by $\sharp_W \tilde{\xi} = \sharp\xi$, where $\xi \in P^*$ is the extension of $\tilde{\xi}$ which annihilates $\sharp W^\circ$.

Now we pass to the global definitions. A Dirac structure on P is a maximal isotropic subbundle $L \subset TP \oplus T^*P$ which is integrable, in the sense that its sections are closed under the so-called Courant bracket (see [13]). The image of L under the projection onto the first

[1] Indeed, writing vectors in $\mathcal{O}$ as $\sharp\xi$ for some $\xi \in P^*$, one sees that the symplectic orthogonal of $W \cap \mathcal{O}$ is $\sharp(W \cap \mathcal{O})^\circ$. This space coincides with $\sharp W^\circ$ because their respective annihilators $\sharp^{-1}(W \cap \mathcal{O})$ and $\sharp^{-1} W$ coincide.

factor is an integrable singular distribution, whose leaves (which are called presymplectic leaves) are endowed with closed 2-forms. A Poisson structure on P is a bivector Π such that $[\Pi,\Pi]=0$. Coisotropic and cosymplectic submanifolds of a Poisson manifold are defined exactly as in the linear case; a Poisson-Dirac submanifold additionally requires that the bivector induced on the submanifold by the point-wise condition be smooth [14]. Cosymplectic submanifolds are automatically Poisson-Dirac submanifolds[2]. The Poisson bracket on a cosymplectic submanifold $\tilde{P}$ of (P,Π) is computed as follows: $\{\tilde{f}_2,\tilde{f}_2\}_{\tilde{P}}$ is the restriction to $\tilde{P}$ of $\{f_1,f_2\}$, where the f_i are extensions of $\tilde{f}_i$ to P such that $df_i|_{\sharp N^*\tilde{P}}=0$ (for at least one of the two functions).

We will also need a definition which does not have a linear algebra counterpart.

Definition 2.2. A submanifold C of a Poisson manifold (P,Π) is called *pre-Poisson* if the rank of $TC+\sharp N^*C$ is constant along C.

Remark 2.3. An alternative characterization of pre-Poisson submanifolds is the requirement that the rank of $\Pi|_{\wedge^2 N^*C}$ be constant. Indeed the kernel of the corresponding sharp map $N^*C\to(N^*C)^*$ is $N^*C\cap\sharp^{-1}TC$, which is the annihilator of $TC+\sharp N^*C$.

Calvo and Falceto already considered [4][5] such submanifolds and called them "strongly regular submanifolds". We prefer to call them "pre-Poisson" because when P is a symplectic manifold they reduce to presymplectic submanifolds[3]. See Section 5 for several examples.

3 Existence of coisotropic embeddings for pre-Poisson submanifolds

In this section we consider the problem of embedding a submanifold of a Poisson manifold coisotropically in a Poisson-Dirac submanifold, and show that this can be always done for pre-Poisson submanifolds.

We start with some linear algebra. Given a subspace C of (P,Π), we want to determine the subspaces $\tilde{P}$ such that $\tilde{P}\subset(P,\Pi)$ is Poisson-Dirac and $C\subset(\tilde{P},L_{\tilde{P}})$ is coisotropic. We want to use the characterization given in section 2 of Poisson-Dirac and coisotropic subspaces in terms of the corresponding bilinear forms, hence we need a statement about symplectic vector spaces.

Lemma 3.1. *Let $(\mathcal{O},\omega)$ be a symplectic vector space and D any subspace. Then the symplectic subspaces of $\mathcal{O}$ in which D sits coisotropically are exactly those of the form $R'\oplus D$, where R' is such that $R'\oplus(D+D^\omega)=\mathcal{O}$. Here D^ω denotes the symplectic orthogonal to D.*

[2]Indeed the bivector induced on a cosymplectic submanifold $\tilde{P}$ is always smooth: denote by L_P the Dirac structure corresponding to the Poisson structure on P and by $L_{\tilde{P}}$ its pullback to $\tilde{P}$. $L_{\tilde{P}}\cong L_P\cap(T\tilde{P}\oplus T^*P)/L_P\cap(\{0\}\oplus N^*\tilde{P})$ (see [13]), and both numerator and denominator have constant rank because $L_P\cap(\{0\}\oplus N^*\tilde{P})=ker\sharp|_{N^*\tilde{P}}=\{0\}$.

[3]Further reasons are the following: the subgroupoid associated to a pre-Poisson manifold, when it exists, is presymplectic (see Prop. 7.5). The Hamiltonian version of the Poisson Sigma Model with boundary conditions on P (at $t=0$) and on a submanifold C (at $t=1$) delivers a space of solutions which is presymplectic iff C is pre-Poisson.

Proof. First we show that a subspace $R' \oplus D$ as above is symplectic. Notice that $R' \cap (D + D^\omega) = \{0\}$ implies that $(R' \oplus D) \cap D^\omega$ is contained in (hence equal to) $D \cap D^\omega$. Hence

$$(R' \oplus D) \cap (R' \oplus D)^\omega = ((R' \oplus D) \cap D^\omega) \cap R'^\omega = (D \cap D^\omega) \cap R'^\omega,$$

and this is zero because its symplectic orthogonal is $D^\omega + D + R'$, which we assumed to be the whole of $\mathcal{O}$. Next we show that D is coisotropic in $R' \oplus D$: $D \cap D^\omega$ is surely contained in the symplectic orthogonal of D in $R' \oplus D$, and by dimension counting we see that it is the whole symplectic orthogonal.

Conversely let us consider a symplectic subspace of $\mathcal{O}$ in which D sits coisotropically; we write this subspace as $R' \oplus D$ for some R'. By the coisotropicity condition $(R' \oplus D) \cap D^\omega$, the symplectic orthogonal of D in $R' \oplus D$, is contained in D. This has two consequences: first, using the fact that $R' \oplus D$ is a symplectic subspace,

$$\{0\} = ((R' \oplus D) \cap D^\omega) \cap R'^\omega = (D \cap D^\omega) \cap R'^\omega,$$

which taking symplectic orthogonals gives $\mathcal{O} = R' + (D + D^\omega)$. Second, this last sum is direct because $\dim R' = \dim(D \cap D^\omega) = codim(D + D^\omega)$. □

Lemma 3.2. *Let (P, Π) be a Poisson vector space and C a subspace. The Poisson-Dirac subspaces of P in which C sits coisotropically are exactly those of the form $R \oplus C$, where R is such that*

$$R \oplus (C + \sharp C^\circ) \supset \mathcal{O}, \tag{1}$$

where $\mathcal{O} = \sharp P^*$. Among the Poisson-Dirac subspaces above the cosymplectic ones are exactly those of maximal dimension, i.e. those for which $R \oplus (C + \sharp C^\circ) = P$.

Proof. Notice that the symplectic subspaces determined in Lemma 3.1 can be described (without making a choice of complement to D) as those whose sum with $D + D^\omega$ is the whole of $\mathcal{O}$ and whose intersection with $D + D^\omega$ is D. Hence[4] the Poisson-Dirac subspaces $\tilde{P}$ of P that contain C as a coisotropic subspace are characterized by

$$(\tilde{P} \cap \mathcal{O}) + ((C \cap \mathcal{O}) + \sharp C^\circ) \equiv (\tilde{P} \cap \mathcal{O}) + \sharp C^\circ \stackrel{!}{=} \mathcal{O}. \tag{2}$$

$$(\tilde{P} \cap \mathcal{O}) \cap ((C \cap \mathcal{O}) + \sharp C^\circ) \equiv \tilde{P} \cap ((C \cap \mathcal{O}) + \sharp C^\circ) \stackrel{!}{=} C \cap \mathcal{O} \tag{3}$$

The equality (2) is really $(\tilde{P} \cap \mathcal{O}) + \sharp C^\circ \supset \mathcal{O}$ and is equivalent[5] to $\tilde{P} + \sharp C^\circ \supset \mathcal{O}$. For any choice of splitting $\tilde{P} = R \oplus C$ this just means $R + (C + \sharp C^\circ) \supset \mathcal{O}$.

The equality (3) is really $\tilde{P} \cap ((C \cap \mathcal{O}) + \sharp C^\circ) \subset C \cap \mathcal{O}$ and is equivalent[6] to $\tilde{P} \cap (C + \sharp C^\circ) \subset C$. For any choice of splitting $\tilde{P} = R \oplus C$ this inclusion means[7] to $R \cap (C + \sharp C^\circ) = \{0\}$. This proves the first part of the Lemma.

[4] Here we use the characterization of subspaces of (P, Π) in terms of their intersections with $\mathcal{O}$, see section 2.

[5] The direction "⇒" is clear. The other direction follows because if $v \in \mathcal{O}$ is written as the sum of an element $v_{\tilde{P}}$ of $\tilde{P}$ and an element $v_{\sharp C^\circ}$ of $\sharp C^\circ$, then $v_{\tilde{P}} = v - v_{\sharp C^\circ} \in \mathcal{O}$.

[6] The "⇐" direction follows because the r.h.s. implies that $\tilde{P} \cap ((C \cap \mathcal{O}) + \sharp C^\circ)$ is contained in C, and it is clearly contained in $\mathcal{O}$ too. For the direction "⇒", write an element v of $\tilde{P} \cap (C + \sharp C^\circ)$ as the sum of an element v_C of C and an element $v_{\sharp C^\circ}$ of $\sharp C^\circ$; because of $C \subset \tilde{P}$ we have $v_{\sharp C^\circ} = v - v_C \in \tilde{P}$, so using the l.h.s. (i.e. eq. (3)) we get $v_{\sharp C^\circ} \in C \cap \mathcal{O}$. Hence v, as the sum of two elements of C, lies in C.

[7] The direction "⇒" is clear. The other direction follows by taking a vector $v \in (R \oplus C) \cap (C + \sharp C^\circ)$ and writing it as $v_R \oplus v_C$. Then $v_R \in C + \sharp C^\circ$, so it follows by the r.h.s. that $v_R = 0$, hence $v = v_C \in C$.

Now let $\tilde{P} = R \oplus C$ satisfy eq. (1); in particular $\tilde{P}$ is Poisson-Dirac. By dimension counting $\tilde{P}$ is cosymplectic iff the restriction of $\sharp$ to $\tilde{P}^\circ$ is injective, i.e. iff $\tilde{P}^\circ \cap \mathcal{O}^\circ = \{0\}$ or $\tilde{P} + \mathcal{O} = P$. This is equivalent to $\tilde{P} + \sharp C^\circ = P$: the direction "$\Rightarrow$" follows using eq. (1), the reverse direction simply because $\sharp C^\circ \subset \mathcal{O}$. □

Now we pass from linear algebra to global geometry. Given a submanifold C of a Poisson manifold P, one might try to construct a Poisson-Dirac submanifold in which C embeds coisotropically applying the corresponding symplectic contruction "leaf by leaf" in a smooth way. In view of Lemma 3.1 it would be then natural to require that the characteristic "distribution" $TC \cap \sharp N^*C$ of C have constant rank. However this approach generally does not work because even when it has constant rank $TC \cap \sharp N^*C$ might not be smooth (see example 5.4). Lemma 3.2 suggests instead to require that C be pre-Poisson and extend C *not* only "along the symplectic leaves of P".

Theorem 3.3. *Let C be a pre-Poisson submanifold of a Poisson manifold (P, Π). Then there exists a cosymplectic submanifold $\tilde{P}$ containing C such that C is coisotropic in $\tilde{P}$.*

Proof. Because of the rank condition on C we can choose a smooth subbundle R of $TP|_C$ which is a complement to $TC + \sharp N^*C$. Then by Lemma 3.2 at every point p of C we have that $T_pC \oplus R_p$ is a cosymplectic subspace of T_pP in which T_pC sits coisotropically. "Thicken" C to a smooth submanifold $\tilde{P}$ of P satisfying $T\tilde{P}|_C = TC \oplus R$. If we can show that, in a neighborhood of C, $\tilde{P}$ is a cosymplectic submanifold, then we are done.

First we show that at points p near C the restriction of $\sharp$ to $N_p^*\tilde{P}$ is injective. By the proof of Lemma 3.2 we know that this is equivalent to $T_p\mathcal{O}_p + T_p\tilde{P} = T_pP$ (where $\mathcal{O}_p$ the symplectic leaf of P through p) and that it is true if p belongs to C. The case $p \notin C$ is reduced to this using Weinstein's local structure theorem [21] which states that, near any $q \in C$, P is isomorphic (as a Poisson manifold) to the product of the symplectic leaf $\mathcal{O}_q$ and a Poisson manifold whose bivector vanishes at q. Under this isomorphism $T_q\mathcal{O}_q$ can be identified with a subspace of $T_p\mathcal{O}_p$, hence from $T\mathcal{O}_q + T\tilde{P} = TP$ at q we deduce $T\mathcal{O}_p + T\tilde{P} = TP$ at p. So we showed that the restriction of $\sharp$ to $N^*\tilde{P}$ is injective, hence $\sharp N^*\tilde{P}$ is a smooth constant rank subbundle of TP. The rank of $T\tilde{P} \cap \sharp N^*\tilde{P}$, which is the intersection of two smooth subbundles, can locally only decrease, and since it is zero along C it is zero also in a neighborhood of C. By dimension counting we deduce $T\tilde{P} \oplus \sharp N^*\tilde{P} = TP$, i.e. $\tilde{P}$ is cosymplectic. □

Remark 3.4. The above proposition says that if C is pre-Poisson then we can choose a subbundle R over C with fibers as in eq. (1) and "extend" C in direction of R to obtain a Poisson-Dirac submanifold of P containing C coisotropically. If C is not a pre-Poisson submanifold of (P, Π), we might still be able to find a smooth bundle R over C consisting of subspaces as in eq. (1). However "extending" C in direction of this subbundle will usually not give a submanifold with a smooth Poisson-Dirac structure, see Example 5.7 below.

Corollary 3.5. *Let $C, \tilde{P}$ be as in Thm. 3.3. The map $T^*\tilde{P} \to T^*P$ given by the splitting $T\tilde{P} \oplus \sharp N^*\tilde{P} = TP$ is a Lie algebroid map. Further $TC + \sharp N^*C = TC \oplus \sharp N^*\tilde{P}$.*

Proof. Recall that a Lie-Dirac submanifold of a Poisson manifold P is one for which there exists a subbundle E containing $\sharp N^*M$ such that $E \oplus TM = TP$ and such that the induced map $T^*M \to T^*P$ be a Lie algebroid map. By Cor. 2.11 of [22] any cosymplectic submanifold $\tilde{P}$ is automatically Lie-Dirac with $E = \sharp N^*\tilde{P}$.

To prove $TC + \sharp N^*C = TC \oplus \sharp N^*\tilde{P}$ we notice that the inclusion "$\supset$" is obvious because $C \subset \tilde{P}$. The other inclusion follows by dimension counting or by the following argument: write any $\xi \in N^*C$ uniquely as $\xi_1 + \xi_2$ where ξ_1 annihilates $\sharp N^*\tilde{P}$ and ξ_2 annihilates $T\tilde{P}$. Then $\sharp\xi_1 = \tilde{\sharp}(\xi_1|_{T\tilde{P}}) \in TC$, where $\tilde{\sharp}$ denotes the sharp map of $\tilde{P}$, since C is coisotropic in $\tilde{P}$. Hence $\sharp\xi = \sharp\xi_1 + \sharp\xi_2 \in TC \oplus \sharp N^*\tilde{P}$, and "$\subset$" follows. Finally, we have a direct sum in $TC \oplus \sharp N^*\tilde{P}$ because $\sharp N^*\tilde{P} \cap T\tilde{P} = \{0\}$ and $C \subset \tilde{P}$. $\square$

Now we deduce consequences about Lie algebroids. See section 7 for the corresponding integrated statement.

Proposition 3.6. *Let C be a submanifold of a Poisson manifold (P, Π). Then $N^*C \cap \sharp^{-1}TC$ is a Lie subalgebroid of T^*P iff C is pre-Poisson. Further, for any cosymplectic submanifold $\tilde{P}$ in which C sits coisotropically, $N^*C \cap \sharp^{-1}TC$ is isomorphic as a Lie algebroid to the annihilator of C in $\tilde{P}$.*

Proof. At every point $N^*C \cap \sharp^{-1}TC$ is the annihilator of $TC + \sharp N^*C$, so it is a vector bundle iff C is pre-Poisson. So assume that C be pre-Poisson. We saw in Corollary 3.5 that for any cosymplectic submanifold $\tilde{P}$ constructed as in Thm. 3.3, the natural Lie algebroid embedding $T^*\tilde{P} \to T^*P$ is obtained by extending a covector in $T^*\tilde{P}$ so that it annihilates $\sharp N^*\tilde{P}$. By the same corollary $TC + \sharp N^*C = TC \oplus \sharp N^*\tilde{P}$. Hence $N^*_{\tilde{P}}C$, the conormal bundle of C in $\tilde{P}$, is mapped *isomorphically* onto $(TC \oplus \sharp N^*\tilde{P})^\circ = (TC + \sharp N^*C)^\circ = N^*C \cap \sharp^{-1}TC$. Since $N^*_{\tilde{P}}C$ is a Lie subalgebroid of $T^*\tilde{P}$ [7], we are done. $\square$

Remark 3.7. The fact that $N^*C \cap \sharp^{-1}TC$ is a Lie algebroid if C is pre-Poisson can also be deduced as follows. The Lie algebra $(\mathcal{F} \cap I)/I^2$ forms a Lie-Rinehart algebra over the commutative algebra $C^\infty(P)/I$, where I is the vanishing ideal of C and $\mathcal{F}$ its Poisson-normalizer in $C^\infty(P)$. Lemma 1 of [4] states that C being pre-Poisson is equivalent to $N^*C \cap \sharp^{-1}TC$ being spanned by differentials of functions in $\mathcal{F} \cap I$. From this one deduces easily that $(\mathcal{F} \cap I)/I^2$ is identified with the sections of $N^*C \cap \sharp^{-1}TC$, and since $C^\infty(P)/I$ are just the smooth functions on C we deduce that $N^*C \cap \sharp^{-1}TC$ is a Lie algebroid over C.

4 Uniqueness of coisotropic embeddings for pre-Poisson submanifolds

Given a submanifold C of a Poisson manifold (P, Π) in this section we investigate the uniqueness (up Poisson diffeomorphisms fixing C) of *cosymplectic* submanifolds in which C is embedded coisotropically.

This lemma tells us that we need consider only the case that C be pre-Poisson and the construction of Thm. 3.3:

Lemma 4.1. *A submanifold C of a Poisson manifold (P, Π) can be embedded coisotropically in a cosymplectic submanifold $\tilde{P}$ iff it is pre-Poisson. In this case all such $\tilde{P}$ are constructed (in a neighborhood of C) as in Thm. 3.3.*

Proof. In Thm. 3.3 we saw that given any pre-Poisson submanifold C, choosing a smooth subbundle R with $R \oplus (TC + \sharp N^*C) = TP|_C$ and "thickening" C in direction of R gives a submanifolds $\tilde{P}$ with the required properties.

Now let C be any submanifold embedded coisotropically in a cosymplectic submanifold $\tilde{P}$. By Lemma 3.2, for any complement R of TC in $T\tilde{P}|_C$ we have $R \oplus (TC + \sharp N^*C) = TP|_C$. This has two consequences: first the rank of $TC + \sharp N^*C$ must be constant, concluding the proof of the "iff" statement of the lemma. Second, it proves the final statement of the lemma. □

When C is a point $\{x\}$ then $\tilde{P}$ as above is a slice transverse to the symplectic leaf through x (see Ex. 5.1) and $\tilde{P}$ is unique up Poisson diffeomorphism by Weinstein's splitting theorem (Lemma 2.2 in [21]; see also Thm. 2.16 in [19]). A generalization of its proof gives

Proposition 4.2. *Let $\tilde{P}_0$ be a cosymplectic submanifold of a Poisson manifold P and $\pi\colon U \to \tilde{P}_0$ a projection of some tubular neighborhood of $\tilde{P}_0$ onto $\tilde{P}_0$. Let $\tilde{P}_t$, $t \in [0,1]$, be a smooth family of cosymplectic submanifolds such that all $\tilde{P}_t$ are images of sections of π. Then, for t close enough to zero, there are Poisson diffeomorphisms ϕ_t mapping open sets of $\tilde{P}_0$ to open sets of $\tilde{P}_t$.*

Remark 4.3. Since each $\tilde{P}_t$ is cosymplectic it has a canonical transverse direction given by $\sharp N^*\tilde{P}_t$. The family of diffeomorphisms ϕ_t can be constructed[8] so that the curve $t \mapsto \phi_t(y)$ (for $y \in \tilde{P}_0$) is tangent to $\sharp N^*\tilde{P}_t$ at time t.

Proof. We will use the following fact, whose straightforward proof we omit: let $\tilde{P}_t$, $t \in [0,1]$, be a smooth family of submanifold of a manifold U, and Y_t a time-dependent vector field on U. Then $Y + \frac{\partial}{\partial t}$ (considered as a vector field on $U \times [0,1]$) is tangent to the submanifold $\bigcup_{t\in[0,1]}(\tilde{P}_t, t)$ iff for each $\bar{t}$ and each integral curve γ of Y_t in U with $\gamma(\bar{t}) \in \tilde{P}_{\bar{t}}$ we have $\gamma(t) \in \tilde{P}_t$ (at all times where γ is defined).

Denote by s_t the section of π whose image is $\tilde{P}_t$. We will be interested in time-dependent vector fields Y_t on U such that for all $\bar{t}$ and $y \in \tilde{P}_{\bar{t}}$

$$Y_{\bar{t}}(y) = s_{\bar{t}*}(\pi_* Y_y) + \frac{d}{dt}|_{\bar{t}} s_t(\pi(y)). \tag{4}$$

We claim that, for such a vector field, $(Y + \frac{\partial}{\partial t})$ will be tangent to $\bigcup_{t\in[0,1]}(\tilde{P}_t, t)$. Indeed

$$\begin{aligned}(Y + \frac{\partial}{\partial t})(y, \bar{t}) &= Y_{\bar{t}}(y) + \frac{\partial}{\partial t} &(5)\\ &= s_{\bar{t}*}(\pi_* Y_y) + \frac{d}{dt}|_{\bar{t}} s_t(\pi(y)) + \frac{\partial}{\partial t}. &(6)\end{aligned}$$

Since $s_{\bar{t}*}(\pi_* Y_y)$ is tangent to $(\tilde{P}_{\bar{t}}, \bar{t})$, and $\frac{d}{dt}|_{\bar{t}} s_t(\pi(y)) + \frac{\partial}{\partial t}$ is the velocity at time $\bar{t}$ of the curve $(s_t(\pi(y)), t)$, the claimed tangency follows. Hence by the fact recalled in the first paragraph we deduce that the flow ϕ_t of Y_t takes points y of $\tilde{P}_0$ to $\tilde{P}_{\bar{t}}$ (if $\phi_t(y)$ is defined until time $\bar{t}$).

So we are done if we realize such Y_t as the hamiltonian vector fields of a smooth family of functions H_t on U. For each fixed $\bar{t}$, eq. (4) for $Y_{\bar{t}}$ is just a condition on the vertical[9]

[8] To achieve this just choose H_t in the proof so that it vanishes on $\tilde{P}_t$.

[9] Vertical w.r.t. the splitting $T_yP = T_y\tilde{P}_{\bar{t}} \oplus ker_y\pi_*$.

component of $Y_{\bar{t}}$ at points of $\tilde{P}_{\bar{t}}$, and the latter is determined exactly by the effect of $Y_{\bar{t}}$ on functions f vanishing on $\tilde{P}_{\bar{t}}$. We have

$$Y_{\bar{t}}(f) = X_{H_{\bar{t}}}(f) = -dH_{\bar{t}}(\sharp df),$$

and the restriction of $\sharp$ to $N^*\tilde{P}_{\bar{t}}$ is injective because $\tilde{P}_{\bar{t}}$ is cosymplectic. Together we obtain that specifying the vertical component of $X_{H_{\bar{t}}}$ at points of $\tilde{P}_{\bar{t}}$ is equivalent to specifying the derivative of $H_{\bar{t}}$ in direction of $\sharp N^*\tilde{P}_{\bar{t}}$, which is transverse to $\tilde{P}_{\bar{t}}$. We can clearly find a function $H_{\bar{t}}$ satisfying the required conditions on its derivative at $\tilde{P}_{\bar{t}}$. Choosing H_t smoothly for every t we conclude that the vector field X_{H_t} will satisfy eq. (4), hence its flow ϕ_t, which obviously consists of Poisson diffeomorphisms, will take $\tilde{P}_0$ (or rather any subset of it on which the flow is defined up to time $\bar{t}$) to $\tilde{P}_{\bar{t}}$. □

Now we are ready to prove the uniqueness of $\tilde{P}$:

Theorem 4.4. *Let C be a pre-Poisson submanifold (P, Π), and $\tilde{P}_0$, $\tilde{P}_1$ cosymplectic submanifolds that contain C as a coisotropic submanifold. Then, shrinking $\tilde{P}_0$ and $\tilde{P}_1$ to a smaller tubular neighborhood of C if necessary, there is a Poisson diffeomorphism Φ from $\tilde{P}_0$ to $\tilde{P}_1$ which is the identity on C.*

Proof. In a neighborhood U of $\tilde{P}_0$ take a projection $\pi\colon U \to \tilde{P}_0$; choose it so that at points of $C \subset \tilde{P}_0$ the fibers of π are tangent to $\sharp N^*\tilde{P}_0|_C$. For $i = 0, 1$ make some choices of maximal dimensional subbundles R_i satisfying eq. (1) to write $T\tilde{P}_i|_C = TC \oplus R_i$, and join R_0 to R_1 by a smooth curve of subbundles R_t satisfying eq. (1) (there is no topological obstruction to this because R_0 and R_1 are both complements to the same subbundle $TC + \sharp N^*C$). By Thm. 3.3 we obtain a curve of cosymplectic submanifolds $\tilde{P}_t$, which moreover by Cor. 3.5 at points of C are all transverse to $\sharp N^*\tilde{P}_0|_C$, i.e. to the fibers of π.

Hence we are in the situation of Prop.4.2, which allows us to construct a Poisson diffeomorphism from $\tilde{P}_0$ to $\tilde{P}_t$ for small t. Since $C \subset \tilde{P}_t$ for all t, in the proof of Prop.4.2 we have that the sections s_t are trivial on C, hence by eq. (4) the vertical part of X_{H_t} at points of $C \subset \tilde{P}_t$ is zero. Choosing H_t to vanish on $\tilde{P}_t$ we obtain, $X_{H_t} = 0$ at points of $C \subset \tilde{P}_t$. From this we deduce two things: in a tubular neighborhood of C the flow ϕ_t of X_{H_t} is defined for all $t \in [0, 1]$, and each ϕ_t keeps points of C fixed. Now just let $\Phi := \phi_1$. □

The derivative at points of C of the Poisson diffeomorphism Φ constructed in Thm. 4.4 gives an isomorphism of Poisson vector bundles $T\tilde{P}_0|_C \to T\tilde{P}_1|_C$ which is the identity on TC. The construction of Φ involves many choices; we wish now to give a *canonical* construction for such a vector bundle isomorphism.

Proposition 4.5. *Let C be a pre-Poisson submanifold (P, Π), and $\tilde{P}$, $\hat{P}$ cosymplectic submanifolds that contain C as a coisotropic submanifold. Then there is a canonical isomorphism of Poisson vector bundles $\varphi\colon T\tilde{P}|_C \to T\hat{P}|_C$ which is the identity on TC.*

Proof. We construct φ in two steps, and to simplify notation we will omit the restriction to C in expressions like $T\tilde{P}|_C$.

First we consider the vector bundle map

$$A\colon T\tilde{P} \to \sharp N^*\tilde{P}$$

determined by the requirement that $T\hat{P} = \{v + Av : v \in T\tilde{P}\}$. A is well-defined since $\sharp N^*\tilde{P}$ is a complement in TP both to $T\tilde{P}$ (because $\tilde{P}$ is cosymplectic) and to $T\hat{P}$ (because $T\hat{P} \cap (TC + \sharp N^*C) = TC$ by Lemma 3.2 and $TC + \sharp N^*C = TC \oplus \sharp N^*\tilde{P}$ by Lemma 3.5). Notice that, since C lies in both $\tilde{P}$ and $\hat{P}$, the restriction of A to TC is zero. The map $A + Id\colon T\tilde{P} \to T\hat{P}$ is an isomorphism of vector bundles. Further at each x it maps $T_x\tilde{P} \cap T_x\mathcal{O}$ isomorphically onto $T_x\hat{P} \cap T_x\mathcal{O}$ (which has the same dimension since both vector spaces contain $T_xC \cap T_x\mathcal{O}$ as a coisotropic subspace) because $\sharp N^*\tilde{P} \subset T\mathcal{O}$, however it does not match the symplectic forms there. We deform $A + Id$ by adding the following vector bundle map[10]($x \in C$):

$$B\colon T_x\tilde{P} \to T_xC, v \mapsto \frac{1}{2}\tilde{\sharp}(\Omega_x(Av, A\bullet)).$$

Here $\tilde{\sharp}$ is the sharp map of the cosymplectic submanifold $\tilde{P}$, Ω_x denotes the symplectic form at x of the symplectic leaf $\mathcal{O}$ through x, and $\Omega_x(Av, A\bullet)$ is an element of $T_x^*\tilde{P}$. To show that B is a smooth vector bundle map, it is enough to show that if X is a smooth section of (the restriction to C of) $\sharp N^*\tilde{P}$, then $\Omega(X, \bullet)|_{\sharp N^*\tilde{P}} : \sharp N^*\tilde{P} \to \mathbb{R}$ is smooth. But this follows from the fact that $\tilde{P}$ is cosymplectic: since $\sharp\colon N^*\tilde{P} \to \sharp N^*\tilde{P}$ is bijective, there is a smooth section ξ of $N^*\tilde{P}$ with $\sharp\xi = X$, and $\Omega(X, \bullet)|_{\sharp N^*\tilde{P}} = \xi|_{\sharp N^*\tilde{P}}$. Next we show that B is well-defined and that it actually maps into $TC \cap \sharp N^*C$: this is true because the section $\Omega(Av, A\bullet)$ of $T^*\tilde{P}$ annihilates TC (recall that $A|_{TC} = 0$) and because C is coisotropic in $\tilde{P}$. Further it is clear that the restriction of B to TC is zero.

At this point we are ready to define

$$\varphi\colon T\tilde{P} \to T\hat{P}, v \mapsto v + Av + Bv.$$

This is a well-defined (since $TC \subset T\hat{P}$), smooth map of vector bundles, and it is an isomorphism: if $v + Bv + Av = 0$ then $v + Bv = 0$ and $Av = 0$ (because $T\tilde{P}$ is transversal to $\sharp N^*P$); from $Av = 0$ we deduce $Bv = 0$ hence $v = 0$. At each $x \in C$ the map φ restricts to an isomorphism from $T_x\tilde{P} \cap T_x\mathcal{O}$ to $T_x\hat{P} \cap T_x\mathcal{O}$ (because the images if A and B lie in $T_x\mathcal{O}$); we show that this restriction is a linear symplectomorphism. If $v_1, v_1 \in T_x\tilde{P} \cap T_x\mathcal{O}$ we have $\Omega(\varphi v_1, \varphi v_2) = \Omega(v_1 + Bv_1, v_2 + Bv_2) + \Omega(Av_1, Av_2)$, for the cross terms vanish since A takes values in $\sharp N^*\tilde{P}$. Now $\Omega(Bv_1, \bullet)|_{T_x\tilde{P}\cap T_x\mathcal{O}} = -\frac{1}{2}\Omega(Av_1, A\bullet)|_{T_x\tilde{P}\cap T_x\mathcal{O}}$ using the fact that $\Omega(\sharp\xi, \bullet) = -\xi|_{T\mathcal{O}}$ for any covector ξ of P. Further $\Omega(Bv_1, Bv_2)$ vanishes because B takes values in $T_xC \cap \sharp N_x^*C$. So altogether we obtain $\Omega(\varphi v_1, \varphi v_2) = \Omega(v_1, v_2)$ as desired. □

Remark 4.6. The isomorphism φ constructed in Prop. 4.5 can be extended to a Poisson vector bundle automorphism of $TP|_C$ as follows: define

$$(\varphi, pr)\colon T\tilde{P} \oplus \sharp N^*\tilde{P} \to T\hat{P} \oplus \sharp N^*\hat{P}$$

where pr denotes the projection of $N^*\tilde{P}$ onto $N^*\hat{P}$ along TC (recall from Cor. 3.5 that $TC \oplus N^*\tilde{P} = TC \oplus N^*\hat{P}$). (φ, pr) restricts to a linear automorphism of $T\mathcal{O} = (T\tilde{P} \cap T\mathcal{O}) \oplus \sharp N^*\tilde{P}$ which preserves the symplectic form: the only non-trivial check is $\Omega(pr(v_1), pr(v_2)) = \Omega(v_1, v_2)$ for $v_i \in \sharp N^*\tilde{P}$, which follows because $pr(v_1) - v_1 \in TC \cap \sharp N^*C$.

[10]Here we mimic a construction in symplectic linear algebra where one deforms canonically a complement of a coisotropic subspace C to obtain an isotropic complement of C; see [6] for the case when C is Lagrangian.

5 Conditions and examples

Let C be as usual a submanifold of the Poisson manifold (P, Π); in Section 3 we considered the question of existence of a Poisson-Dirac submanifold $\tilde{P}$ of P in which C is contained coisotropically. In Thm. 3.3 we showed that a *sufficient* condition is that C be pre-Poisson, which by Prop. 3.6 is equivalent to saying that $N^*C \cap \sharp^{-1}TC$ be a Lie algebroid.

A *necessary* condition is that the (intrinsically defined) characteristic distribution $TC \cap \sharp N^*C$ of C be the distribution associated to a Lie algebroid over C; in particular its rank locally can only increase. This is a necessary condition since the concept of characteristic distribution is an intrinsic one (see Remark 2.1), and the characteristic distribution of a coisotropic submanifold of a Poisson manifold is the image of the anchor of its conormal bundle, which is a Lie algebroid.

The submanifolds C which are not covered by the above conditions are those for which $N^*C \cap \sharp^{-1}TC$ is not a Lie algebroid but its image $TC \cap \sharp N^*C$ under $\sharp$ is the image of the anchor of some Lie algebroid over C. Diagrammatically:

$$\{C \text{ s.t. } N^*C \cap \sharp^{-1}TC \text{ is a Lie algebroid, i.e. } C \text{ is pre-Poisson } \} \subset$$

$$\{C \text{ sitting coisotropically in some Poisson-Dirac submanifold } \tilde{P} \text{ of } P\} \subset$$

$$\{C \text{ s.t. } TC \cap \sharp N^*C \text{ is the distribution of some Lie algebroid over } C\}$$

The following are examples of pre-Poisson submanifolds.

Example 5.1. An obvious example is when C is a coisotropic submanifold of P, and in this case the construction of Thm. 3.3 delivers $\tilde{P} = P$ (or more precisely, a tubular neighborhood of C in P).

Another obvious example is when C is just a point x: then the construction of Thm. 3.3 delivers as $\tilde{P}$ any slice through x transversal to the symplectic leaf $\mathcal{O}_x$.

Now if $C_1 \subset P_1$ and $C_2 \subset P_2$ are pre-Poisson submanifolds of Poisson manifolds, the cartesian product $C_1 \times C_2 \subset P_1 \times P_2$ also is, and if the construction of Thm. 3.3 gives cosymplectic submanifolds $\tilde{P}_1 \subset P_1$ and $\tilde{P}_2 \subset P_2$, the same construction applied to $C_1 \times C_2$ (upon suitable choices of complementary subbundles) delivers the cosymplectic submanifold $\tilde{P}_1 \times \tilde{P}_2$ of $P_1 \times P_2$. In particular, if C_1 is coisotropic and C_2 just a point x, then $C_1 \times \{x\}$ is pre-Poisson.

The following are two examples of submanifolds C which surely can not be imbedded coisotropically in any Poisson-Dirac submanifold:

Example 5.2. The submanifold $C = \{(x_1, x_2, x_2^2, x_1^2)\}$ of the symplectic manifold $(P, \omega) = (\mathbb{R}^4, dx_1 \wedge dx_3 + dx_2 \wedge dx_4)$ has characteristic distribution of rank 2 on the points with $x_1 = x_2$ and rank zero on the rest of C. The rank of the characteristic distribution locally decreases, hence C does not satisfies the necessary condition above.

Remark 5.3. If C is a submanifold of a symplectic manifold (P, ω), then the necessary and the sufficient conditions coincide, both being equivalent to saying that the characteristic distribution of C (which can be described as $\ker(i_C^*\omega)$ for i_C the inclusion) have constant rank, i.e. that C be presymplectic.

Example 5.4. Consider the Poisson[11] manifold $(\mathbb{R}^6, x_1\partial_{x_2} \wedge \partial_{x_4} + (\partial_{x_3} + x_1\partial_{x_5}) \wedge \partial_{x_6})$. Let C be the three-dimensional subspace given by setting $x_4 = x_5 = x_6 = 0$. The characteristic subspaces are all one-dimensional, spanned by ∂_{x_3} at points of C where $x_1 = 0$ and by ∂_{x_2} on the rest of C. Hence the characteristic subspaces don't form a smooth distribution, and can not be the image of the anchor map of any Lie algebroid over C. Hence C does not satisfies the necessary condition above.

The *sufficient* condition above is not necessary (i.e. the first inclusion in the diagram above is strict), as either of the following simple examples shows.

Example 5.5. Take C to be the vertical line $\{x = y = 0\}$ in the Poisson manifold $(P, \Pi) = (\mathbb{R}^3, f(z)\partial_x \wedge \partial_y)$, where f is any function with at least one zero. Then C is a Poisson-Dirac submanifold (with zero induced Poisson structure), hence taking $\tilde{P} := C$ we obtain a Poisson-Dirac submanifold in which C embeds coisotropically. The sufficient conditions here is not satisfied, for the rank of $TC + \sharp N^*C$ at $(0,0,z)$ is 3 at points where f does not vanishes and 1 at points where f vanishes.

Example 5.6. Consider the Poisson manifold $(P, \Pi) = (\mathbb{R}^4, x^2\partial_x \wedge \partial_y + z\partial_z \wedge \partial_w)$ as in Example 6 of [14] and the submanifold $C = \{(z^2, 0, z, 0) : z \in \mathbb{R}\}$. The rank of $TC + \sharp N^*C$ is 3 away from the origin (because there C is an isotropic submanifold in an open symplectic leaf of P) and 1 at the origin (since Π vanishes there). The submanifold $\tilde{P} = \{(z^2, 0, z, w) : z, w \in \mathbb{R}\}$ is Poisson-Dirac and it clearly contains C as a coisotropic submanifold.

The *necessary* condition above is not a sufficient (i.e. the second inclusion in the diagram above is strict):

Example 5.7. In Example 3 in Section 8.2 of [14] the authors consider the manifold $P = \mathbb{C}^3$ with complex coordinates x, y, z and specify a Poisson structure on it by declaring the symplectic leaves to be the complex lines given by $dy = 0, dz - ydx = 0$, the symplectic forms being the restrictions of the canonical symplectic form on $\mathbb{C}^3$. They consider submanifold C the complex plane $\{z = 0\}$ and show that C is point-wise Poisson-Dirac (i.e. $TC \cap \sharp N^*C = \{0\}$ at every point), but that the induced bivector field is not smooth. Being point-wise Poisson-Dirac, C satisfies the necessary condition above. However there exists no Poisson-Dirac submanifold $\tilde{P}$ of P in which C embeds coisotropically. Indeed at points p of C where $y \neq 0$ we have $T_pC \oplus T_p\mathcal{O} = TP$ (where as usual $\mathcal{O}$ is a symplectic leaf of P through p), from which follows that $\sharp|_{N_p^*C}$ is injective and $T_pC \oplus \sharp N_p^*C = TP$. From Lemma 3.2 (notice that the subspace R there must have trivial intersection with $T_pC \oplus \sharp N_p^*C$, so R must be the zero subbundle over C) it follows that the only candidate for $\tilde{P}$ is C itself. However, as we have seen, the Poisson bivector induced on C is not smooth. (More generally, examples are provided by any submanifold C of a Poisson manifold P which is point-wise Poisson-Dirac but not Poisson-Dirac and for which there exists a point p at which $T_pC \oplus T_p\mathcal{O} = TP$.) Notice that this provides an example for the claim made in Remark 3.4, because the zero subbundle R over C satisfies the condition of Lemma 3.2 at every point of C and is obviously a smooth subbundle.

[11] This is really a Poisson structure because the bracket of any two coordinates is a Casimir functions (indeed either a constant or x_1), so that the Jacobiator of any three coordinate functions vanishes.

We refer the reader to Section 6 of [12] for more examples in which the Poisson manifold P is the dual of a Lie algebra and C an affine subspace.

6 Reduction of submanifolds and deformation quantization of pre-Poisson submanifolds

In this section we consider the set of basic functions on a submanifold of a Poisson manifold, and show that in certain cases it is a Poisson algebra and that it can be deformation quantized.

Given any submanifold C of a Poisson manifold (P, Π), it is natural to consider the characteristic "distribution" $\sharp N^*C \cap TC$, which by Remark 2.1 consists of the kernels of the restriction to C of the symplectic forms on the symplectic leaves of P. We used quotation marks because $\sharp N^*C \cap TC$ usually does not have a constant rank and may not be smooth. We will consider the set of basic functions on C, i.e.

$$C^\infty_{bas}(C) = \{f \in C^\infty(C) : df|_{\sharp N^*C \cap TC} = 0\}.$$

When the characteristic distribution is regular and smooth and the quotient $\underline{C}$ is a smooth manifold, then these are exactly the pullbacks of functions on $\underline{C}$.

If we endow C with the (possibly non-smooth[12]) point-wise Dirac structure i^*L_P, where $i\colon C \to P$ is the inclusion and L_P is the Dirac structure corresponding to Π, then $C^\infty_{bas}(C)$ is exactly the set of basic functions in the sense of Dirac geometry, i.e. the set of functions whose differentials annihilate at each point the characteristic subspaces $i^*L_P \cap TC$. Given basic functions f, g the expression

$$\{f, g\}_C(p) := Y(g),$$

where Y is any element[13] of T_pC such that $(Y, df_p) \in i^*L_P$, is well-defined. However it does not usually vary smoothly[14] with p, so we can *not* conclude that $C^\infty_{bas}(C)$ with this bracket is a Poisson algebra.

As pointed out in [4] $\mathcal{F}/(\mathcal{F} \cap \mathcal{I})$ inherits a Poisson bracket from the Poisson manifold P, where $\mathcal{I}$ denotes the set of functions on P that vanish on C and $\mathcal{F} := \{\hat{f} \in C^\infty(P) : \{\hat{f}, \mathcal{I}\} \subset \mathcal{I}\}$ (the so-called first class functions) its normalizer. $\mathcal{F}/(\mathcal{F} \cap \mathcal{I})$ is exactly the subset of functions f on C which admits an extension to some function $\hat{f}$ on P whose

[12] A sufficient condition for the induced Dirac structure to be smooth and integrable is that the rank of $\sharp N^*C$ be constant, because $L_P \cap (\{0\} \oplus N^*C) = ker(\sharp|_{N^*C})$.

[13] Such a Y exists because the annihilator of $\sharp N^*C \cap TC = i^*L_P \cap TC$ is the projection onto T^*C of i^*L_P.

[14] In the case of smooth Dirac structure the set of so-called admissible functions, endowed with this bracket, is a Poisson algebra [13]. In our case it is tempting to define the set of admissible functions as functions f on C for which there is a smooth vector field X such that $(X, df) \subset i^*L_P$, however these does not seem to be closed under $\{\bullet, \bullet\}_C$.

We can instead consider a larger set of functions. Denote by C_{reg} the open, dense subset of C where the rank of $\sharp N^*C$ is locally constant; on this set the point-wise Dirac structure i^*L_P is actually smooth and integrable [13]. The set of functions f for which there is a smooth vector field X such that $(X, df)|_{C_{reg}} \subset i^*L_P|_{C_{reg}}$ is a Poisson algebra. The reason is essentially that the set of smooth sections of $TC \oplus T^*C$ whose restriction to C_{reg} lie in $i^*L_P|_{C_{reg}}$ are closed under the Courant bracket. However the latter set of functions is usually not contained in $C^\infty_{bas}(C)$.

differential annihilates $\sharp N^*C$ (or equivalently $X_{\hat{f}}|_C \subset TC$). The bracket is computed as follows:

$$\{f,g\} = \{\hat{f},\hat{g}\}_P|_C = X_{\hat{f}}(g)|_C$$

for extensions as above. Notice that $\mathcal{F}/(\mathcal{F}\cap\mathcal{I}) \subset C^\infty_{bas}(C)$, and that the Poisson bracket $\{\bullet,\bullet\}$ on $\mathcal{F}/(\mathcal{F}\cap\mathcal{I})$ coincides with $\{\bullet,\bullet\}_C$ (if f,g belong to $\mathcal{F}/(\mathcal{F}\cap\mathcal{I})$ we can compute $\{f,g\}_C$ by choosing $Y = X_{\hat{f}}$ for some extension $\hat{f} \in \mathcal{F}$).

In some cases $(C^\infty_{bas}(C), \{\bullet,\bullet\}_C)$ actually is a Poisson algebra:

Proposition 6.1. *Let C be any submanifold of a Poisson manifold (P,Π). If there exists a Poisson-Dirac submanifold $\tilde{P}$ of P in which C is contained coisotropically, then the set of basic functions on C has an intrinsic Poisson algebra structure, and $(\mathcal{F}/(\mathcal{F}\cap\mathcal{I}), \{\bullet,\bullet\})$ is a Poisson subalgebra.*

Proof. We add a tilde in the notation introduced above when we view C as a submanifold of the Poisson manifold $\tilde{P}$ instead of P. So is $\tilde{\mathcal{I}}$ the vanishing ideal of C in $\tilde{P}$ and by $\tilde{\mathcal{F}}$ its normalizer. Since $\sharp N^*C \subset TC$ it follows that $\tilde{\mathcal{F}}/\tilde{\mathcal{I}}$ fills up the space of basic functions of $(C, \tilde{i}^*L_{\tilde{P}})$. By the above the bracket on $\tilde{\mathcal{F}}/\tilde{\mathcal{I}}$ and the bracket coming from the point-wise Dirac structure $\tilde{i}^*L_{\tilde{P}}$ agree; in particular the latter endows $C^\infty_{bas}(C)$ with a Poisson algebra structure. It is intrinsic to C in the following sense: if $\bar{P}$ is any other submanifold of (P,Π) containing C, $L_{\bar{P}}$ the point-wise Dirac structure on $\bar{P}$ induced by P and $\bar{i} : C \to \bar{P}$ the inclusion, then the Poisson bracket on $C^\infty_{bas}(C)$ induced by $\bar{P}$ via $\bar{i}^*L_{\bar{P}}$ agrees with the above, because $\bar{i}^*L_{\bar{P}} = \tilde{i}^*L_{\tilde{P}}$ by the functoriality of pullback. Hence it makes sense to denote this bracket by $\{\bullet,\bullet\}_C$. This allows us to show that $(C^\infty_{bas}(C), \{\bullet,\bullet\}_C)$ contains $\mathcal{F}/(\mathcal{F}\cap\mathcal{I})$ as a Poisson subalgebra, because as we saw above the bracket on $\mathcal{F}/(\mathcal{F}\cap\mathcal{I})$ coincides with the bracket of basic functions on (C, i^*L_P). $\square$

By Thm. 3.3 pre-Poisson submanifolds C satisfy the assumption of Prop. 6.1, hence they admit a Poisson algebra structure on their space of basic functions. This fact was already established in Theorem 3 of [4], where furthermore it is shown that $\mathcal{F}/(\mathcal{F}\cap\mathcal{I})$ is the whole space of basic functions. We need our Prop. 6.1 because it tells us that the Poisson algebra $(C^\infty_{bas}(C), \{\bullet,\bullet\}_C)$ is intrinsic to C; this allows us to finally state our result about deformation quantization.

Theorem 6.2. *Let C be a pre-Poisson submanifold, and assume that the first and second Lie algebroid cohomology of $N^*C \cap \sharp^{-1}TC$ vanish. Then $(C^\infty_{bas}(C), \{\bullet,\bullet\}_C)$, the Poisson algebra of basic functions on C, admits a deformation quantization.*

Proof. By Thm. 3.3 we can embed C coisotropically in some cosymplectic submanifold $\tilde{P}$. Further by Prop. 6.1 the Poisson bracket $\{\bullet,\bullet\}_C$ on $C^\infty_{bas}(C)$ is induced by the embedding of C in $\tilde{P}$. Now we invoke Corollary 3.3 of [10]: if the first and second Lie algebroid cohomology of the conormal bundle of a coisotropic submanifold vanish, then the Poisson algebra of basic functions on the coisotropic submanifold admits a deformation quantization. The conditions in Corollary 3.3 of [10] translate into the conditions stated in the proposition because the conormal bundle of C in $\tilde{P}$ is isomorphic to $N^*C \cap \sharp^{-1}TC$ as a Lie algebroid, see Prop. 3.6. $\square$

7 Subgroupoids associated to pre-Poisson submanifolds

Let C be a pre-Poisson submanifold of a Poisson manifold (P, Π). In Prop. 3.6 we showed that $N^*C \cap \sharp^{-1}TC$ is a Lie subalgebroid of T^*P. When $\sharp N^*C$ has constant rank there is another Lie subalgebroid associated[15] to C, namely $\sharp^{-1}TC = (\sharp N^*C)^\circ$. Now we assume that T^*P is an integrable Lie algebroid, i.e. that the source simply connected (s.s.c.) symplectic groupoid $(\Gamma_s(P), \Omega)$ of (P, Π) exists. In this section we study the (in general only immersed) subgroupoids of $\Gamma_s(P)$ integrating $N^*C \cap \sharp^{-1}TC$ and $\sharp^{-1}TC$. Here, for any Lie subalgebroid A of T^*P integrating to a s.s.c. Lie groupoid G, we take "subgroupoid" to mean the (usually just immersed) image of the (usually not injective) morphism $G \to \Gamma_s(P)$ induced from the inclusion $A \to T^*P$.

By Thm. 3.3 we can find a cosymplectic submanifold $\tilde{P}$ in which C lies coisotropically. We first make a few remarks on the subgroupoid corresponding to $\tilde{P}$.

Lemma 7.1. *The subgroupoid of $\Gamma_s(P)$ integrating $\sharp^{-1}T\tilde{P}$ is $\mathbf{s}^{-1}(\tilde{P}) \cap \mathbf{t}^{-1}(\tilde{P})$ and is a symplectic subgroupoid. Its source (target) map is a Poisson (anti-Poisson) map onto $\tilde{P}$, where the latter is endowed with the Poisson structure induced by (P, Π).*

Proof. According to Thm. 3.7 of [22] the subgroupoid[16] of $\Gamma_s(P)$ corresponding to $\tilde{P}$, i.e. the one integrating $(\sharp N^*\tilde{P})^\circ$, is a symplectic subgroupoid of $\Gamma_s(P)$. It is given by $\mathbf{s}^{-1}(\tilde{P}) \cap \mathbf{t}^{-1}(\tilde{P})$, because[17] $(\sharp N^*\tilde{P})^\circ = \sharp^{-1}T\tilde{P}$.

To show that the maps $\mathbf{s}^{-1}(\tilde{P}) \cap \mathbf{t}^{-1}(\tilde{P}) \to \tilde{P}$ given by the source and target maps of $\mathbf{s}^{-1}(\tilde{P}) \cap \mathbf{t}^{-1}(\tilde{P})$ are Poisson (anti-Poisson) maps proceed as follows. Take a function $\tilde{f}$ on $\tilde{P}$, and extend it to a functions f on P so that X_f is tangent to $\tilde{P}$ along $\tilde{P}$ (i.e. exactly as was done in section 2 to compute the Poisson bracket on $\tilde{P}$ in terms of the one on P). Since $s : \Gamma_s(P) \to P$ is a Poisson map and s-fibers are symplectic orthogonal to t-fiber we know that the vector field X_{s^*f} on $\Gamma_s(P)$ is tangent to $\mathbf{s}^{-1}(\tilde{P}) \cap \mathbf{t}^{-1}(\tilde{P})$. Hence, denoting by $\tilde{s}$ the source map of $\mathbf{s}^{-1}(\tilde{P}) \cap \mathbf{t}^{-1}(\tilde{P})$, we have

$$\tilde{s}^*\{\tilde{f}_1, \tilde{f}_2\} = \tilde{s}^*(\{f_1, f_2\}|_{\tilde{P}}) = \{s^*f_1, s^*f_2\}|_{\mathbf{s}^{-1}(\tilde{P})\cap\mathbf{t}^{-1}(\tilde{P})} = \{\tilde{s}^*f_1, \tilde{s}^*f_1\},$$

i.e. $\tilde{s}$ is a Poisson map. A similar reasoning holds for $\tilde{t}$. $\square$

Now we describe the subgroupoid integrating $N^*C \cap \sharp^{-1}TC$:

Proposition 7.2. *Let C be a pre-Poisson submanifold of (P, Π). Then the subgroupoid of $\Gamma_s(P)$ integrating $N^*C \cap \sharp^{-1}TC$ is an isotropic subgroupoid of $\Gamma_s(P)$.*

Proof. The canonical vector bundle isomorphism $i : T^*\tilde{P} \cong (\sharp N^*\tilde{P})^\circ$ is a Lie algebroid isomorphism, where $T^*\tilde{P}$ is endowed with the cotangent algebroid structure coming from the Poisson structure on $\tilde{P}$. Indeed both the anchor and the brackets of exact (hence by

[15]More generally for any Lie algebroid $A \to M$ with anchor ρ, if N is a submanifold of M such that $\rho^{-1}TN$ has constant rank then $\rho^{-1}TN \to N$ is a Lie subalgebroid of $A \to M$.

[16]In [22] this is claimed only when the subgroupoid integrating $(\sharp N^*\tilde{P})^\circ$ is an embedded subgroupoid, however the proof there is valid for immersed subgroupoids too.

[17]More generally we claim the following: if $\Gamma \rightrightarrows M$ is any Lie groupoid integrating the Lie algebroid $A \to M$ and $N \subset M$ a submanifold such that $\rho^{-1}TN \to N$ has constant rank, then the Lie subalgebroid $\rho^{-1}TN$ is integrated by the source-connected part of the subgroupoid $\mathbf{s}^{-1}(N) \cap \mathbf{t}^{-1}(N)$, and this intersection is clean.

the Leibniz rule of all) 1-forms on $\tilde{P}$ match, as follows from section 2. Integrating this algebroid isomorphism we obtain a Lie groupoid morphism from $\Gamma_s(\tilde{P})$, the s.s.c. Lie groupoid integrating $T^*\tilde{P}$, to $\Gamma_s(P)$, and the image of this morphism is $\mathbf{s}^{-1}(\tilde{P}) \cap \mathbf{t}^{-1}(\tilde{P})$. Since by Lemma 7.1 the symplectic form on $\mathbf{s}^{-1}(\tilde{P}) \cap \mathbf{t}^{-1}(\tilde{P})$ is multiplicative, symplectic and the source map is a Poisson map, pulling back the symplectic form on $\mathbf{s}^{-1}(\tilde{P}) \cap \mathbf{t}^{-1}(\tilde{P})$ endows $\Gamma_s(\tilde{P})$ with the structure of the s.s.c. symplectic groupoid of $\tilde{P}$. The subgroupoid of $\Gamma_s(\tilde{P})$ integrating $N^*_{\tilde{P}}C$, the annihilator of C in $\tilde{P}$, is Lagrangian ([7], Prop. 5.5). Hence $i(N^*_{\tilde{P}}C)$, which by Prop. 3.6 is equal to $N^*C \cap \sharp^{-1}TC$, integrates to a Lagrangian subgroupoid of $\mathbf{s}^{-1}(\tilde{P}) \cap \mathbf{t}^{-1}(\tilde{P})$, which therefore is an isotropic subgroupoid of $\Gamma_s(P)$. □

Now we consider $\sharp^{-1}TC$. For any submanifold N, $\sharp^{-1}TN$ has constant rank iff it is a Lie subalgebroid of T^*P, integrating to the subgroupoid $\mathbf{s}^{-1}(N) \cap \mathbf{t}^{-1}(N)$ of $\Gamma_s(P)$. So the constant rank condition on $\sharp^{-1}TN$ corresponds to a smoothness condition on $\mathbf{s}^{-1}(N) \cap \mathbf{t}^{-1}(N)$.

Remark 7.3. 1) If $\sharp^{-1}TN$ has constant rank it follows that the Poisson structure on P pulls back to a smooth Dirac structure on N, and that $\mathbf{s}^{-1}(N) \cap \mathbf{t}^{-1}(N)$ is an over-pre-symplectic[18] groupoid inducing the same Dirac structure on N (Ex. 6.7 of [2]). $\mathbf{s}^{-1}(N) \cap \mathbf{t}^{-1}(N)$ has dimension equal to $2dimN + rk(N^*N \cap N^*\mathcal{O})$, where $\mathcal{O}$ the symplectic leaves of P intersecting C.
2) For a pre-Poisson submanifold C, the condition that $\sharp^{-1}TC$ have constant rank is equivalent to the characteristic distribution $TC \cap \sharp N^*C$ having constant rank[19].

Proposition 7.4. *Let C be a pre-Poisson submanifold with constant-rank characteristic distribution. Then for any cosymplectic submanifold $\tilde{P}$ in which C embeds coisotropically, $\mathbf{s}^{-1}(C) \cap \mathbf{t}^{-1}(C)$ is a coisotropic subgroupoid of $\mathbf{s}^{-1}(\tilde{P}) \cap \mathbf{t}^{-1}(\tilde{P})$.*

Proof. By the comments above we know that $\sharp^{-1}TC$ is a Lie subalgebroid, hence $\mathbf{s}^{-1}(C) \cap \mathbf{t}^{-1}(C)$ is a (smooth) subgroupoid of $\Gamma_s(P)$. We saw in Lemma 7.1 that $\mathbf{s}^{-1}(\tilde{P}) \cap \mathbf{t}^{-1}(\tilde{P})$ is endowed with a symplectic multiplicative 2-form for which its source and target maps are (anti-)Poisson maps onto $\tilde{P}$. Further its source and target fibers symplectic orthogonals of each other. Since $C \subset \tilde{P}$ is coisotropic, the above (together with the fact that the preimage of coisotropic submanifolds under Poisson maps are again coisotropic) implies that $\mathbf{s}^{-1}(C) \cap \mathbf{t}^{-1}(C)$ is coisotropic in $\mathbf{s}^{-1}(\tilde{P}) \cap \mathbf{t}^{-1}(\tilde{P})$. □

We now describe the subgroupoids corresponding to pre-Poisson manifolds.

Proposition 7.5. *Let C be any submanifold of P. Then $\mathbf{s}^{-1}(C) \cap \mathbf{t}^{-1}(C)$ is a (immersed) presymplectic submanifold iff C is pre-Poisson and its characteristic distribution has constant rank. In this case the characteristic distribution of $\mathbf{s}^{-1}(C) \cap \mathbf{t}^{-1}(C)$ has rank $2rk(\sharp N^*C \cap TC) + rk(N^*C \cap N^*\mathcal{O})$, where $\mathcal{O}$ denotes the symplectic leaves of P intersecting C.*

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[18]Recall from Def. 4.6 of [2] that an over-pre-symplectic groupoid is a Lie groupoid G over M equipped with a closed multiplicative 2-form ω such that $ker\omega_x \cap ker(ds)_x \cap ker(dt)_x$ has rank $dimG - 2dimM$ at all $x \in M$.

[19]Indeed more generally we have the following for any submanifold C of P: if any two of $\sharp^{-1}TC$, $\sharp N^*C + TC$ or $TC \cap \sharp N^*C$ have constant rank, then the remaining one also has constant rank. This follows trivially from $rk(\sharp N^*C + TC) = rk(\sharp N^*C) + \dim C - rk(TC \cap \sharp N^*C)$.

Proof. Assume that $\mathbf{s}^{-1}(C)\cap\mathbf{t}^{-1}(C)$ is a (immersed) presymplectic submanifold. We apply the same proof as in Prop. 8 of [14]: there is an isomorphism of vector bundles $T\Gamma_s(P)|_P \cong TP\oplus T^*P$, under which the non-degenerate bilinear form $\Omega|_P$ corresponds to $(X_1\oplus\xi_1, X_2\oplus\xi_2) := \langle\xi_1, X_2\rangle - \langle\xi_2, X_1\rangle + \Pi(\xi_1,\xi_2)$. Under the above isomorphism $T(\mathbf{s}^{-1}(C)\cap\mathbf{t}^{-1}(C))$ corresponds to $TC\oplus\sharp^{-1}TC$, and a short computation shows that the restriction of $(\bullet,\bullet)$ to $TC\oplus\sharp^{-1}TC$ has kernel $(TC\cap\sharp N^*C)\oplus(\sharp^{-1}TC\cap N^*C)$. From the smoothness of $\mathbf{s}^{-1}(C)\cap\mathbf{t}^{-1}(C)$ it follows that $(\sharp N^*C)^\circ = \sharp^{-1}TC$ has constant rank, so this kernel is a direct sum of two intersections of smooth subbundles. We deduce that $\mathbf{s}^{-1}(C)\cap\mathbf{t}^{-1}(C)$ is "presymplectic at points of C" (i.e. the pullback of Ω to $\mathbf{s}^{-1}(C)\cap\mathbf{t}^{-1}(C)$ has constant rank along C) iff $\sharp^{-1}TC\cap N^*C$ has constant rank, i.e. (taking annihilators) iff C is pre-Poisson. By the comments before Prop. 7.4 we also know that C has characteristic distribution of constant rank.

The other direction follows from Prop. 7.4. □

Remark 7.6. One can wonder whether any subgroupoid of a symplectic groupoid $(\Gamma_s(P),\Omega)$ which is a presymplectic submanifold (i.e. Ω pulls back to a constant rank 2-form) is contained coisotropically in some symplectic subgroupoid of $\Gamma_s(P)$. This would be exactly the "groupoid" version of Thm. 3.3. The above Prop. 7.4 and Prop. 7.5 together tell us that this is the case when the subgroupoid has the form $\mathbf{s}^{-1}(C)\cap\mathbf{t}^{-1}(C)$, where $C\subset P$ is its base. In general the answer to the above question is negative, as the following counterexample shows.

Let (P,ω) be some simply connected symplectic manifold, so that $\Gamma_s(P) = (P\times P, \omega_1-\omega_2)$ and the units are embedded diagonally. Take C to be any 1-dimensional closed submanifold of P. $C\rightrightarrows C$ is clearly a subgroupoid and a presymplectic submanifold; since $\omega_1-\omega_2$ there pulls back to zero, any subgroupoid G of $P\times P$ in which $C\rightrightarrows C$ embeds coisotropically must have dimension 2. If the base of G has dimension 2 then G is contained in the identity section of $P\times P$, which is Lagrangian. So let us assume that the base of G is C. Then G must be contained in $C\times C$, on which $\omega_1-\omega_2$ vanishes because $C\subset P$ is isotropic. So we conclude that there is no symplectic subgroupoid of $P\times P$ containing $C\rightrightarrows C$ as a coisotropic submanifold.

8 Existence of coisotropic embeddings of Dirac manifolds in Poisson manifolds

Let (M,L) be a (smooth) Dirac manifold. We ask when (M,L) can be embedded coisotropically in some Poisson manifold (P,Π), i.e. when there exists an embedding i such that $i^*L_P = L$ and $i(M)$ is a coisotropic submanifold of P [20].

When M consists of exactly one leaf, i.e. when M is a manifold endowed with a closed 2-form ω, the existence and uniqueness of coisotropic embeddings in symplectic manifolds was considered by Gotay in the short paper [15]: the coisotropic embedding exists iff $\ker\omega$ has constant rank, and in that case one has uniqueness up to neighborhood equivalence.

[20]Notice that when P is symplectic any coisotropic submanifold has an induced smooth presymplectic form, however when we take P to be Poisson the induced structure is generally not even continuous: for example the x-axis in $(\mathbb{R}^2, x\partial_x\wedge\partial_y)$ is coisotropic, but its pullback Dirac structure is not continuous at the origin.

Our strategy will be to check if we can apply Gotay's arguments "leaf by leaf" *smoothly* over M. Recall that $L \cap TM$ is the kernel of the 2-forms on the presymplectic leaves of (M, L).

Theorem 8.1. *(M, L) can be embedded coisotropically in a Poisson manifold iff $L \cap TM$ has constant rank.*

Proof. Suppose that an embedding $i\colon M \to P$ as above exists. Then $L \cap TM$ is equal $\sharp N^*C$ (where N^*C is the normalizer of C in P), the image of a vector bundle under a smooth bundle map, hence its rank can locally only increase. On the other hand the rank of $L \cap TM$, which is the intersection of two smooth bundles, can locally only decrease. Hence the rank of $L \cap TM$ must be constant on M.

Conversely, assume that the rank of $E := L \cap TM$ is constant and define P to be the total space of the vector bundle $\pi\colon E^* \to M$. We define the Poisson structure on P as follows. First take the pullback Dirac structure π^*L (which is smooth and integrable since π is a submersion). Then choose a smooth distribution V such that $E \oplus V = TM$. This choice gives an embedding $i_M\colon E^* \to T^*M$, which we can use to pull back the canonical symplectic form ω_{T^*M}. Our Poisson structure is $L_{E^*} := \tau_{i_M^*\omega_{T^*M}}\pi^*L$, i.e. it is obtained applying to π^*L the gauge transformation[21] by the closed 2-form $i_M^*\omega_{T^*M}$. It is clear that L_{E^*} is a smooth Dirac structure; we still have to show that it is actually Poisson, and that the zero section is coisotropic. In more concrete terms (E^*, L_{E^*}) can be described as follows: the leaves are all of the form $\pi^{-1}(F_\alpha)$ for $(F_\alpha, \omega_\alpha)$ a presymplectic leaf of M. The 2-form on the leaf is given by adding to $(\pi|_{\pi^{-1}(F_\alpha)})^*\omega_\alpha$ the 2-form $i_\alpha^*\omega_{T^*F_\alpha}$. The latter is defined considering the transverse distribution $V \cap TF_\alpha$ to $E|_{F_\alpha}$ in TF_α, the induced embedding $i_\alpha\colon \pi^{-1}(F_\alpha) = E^*|_{F_\alpha} \to T^*F_\alpha$, and pulling back the canonical symplectic form. One can check that $i_\alpha^*\omega_{T^*F_\alpha}$ is the pullback of $i_M^*\omega_{T^*M}$ via the inclusion of the leaf in E^*. But this is exactly Gotay's recipe to endow $\pi^{-1}(F_\alpha)$ with a symplectic form so that F_α is embedded as a coisotropic submanifold. Hence we conclude that a neighborhood of the zero section of E^*, with the above Dirac structure, is actually a Poisson manifold and that M is embedded as a coisotropic submanifold. □

We comment on how choices affect the construction of Thm. 8.1. We need the following version of Moser's theorem for Poisson structures (see Section 3.3. of [1]) : suppose we are given Poisson structures Π_t on some manifold P, $t \in [0,1]$. Assume that each Π_t is related to Π_0 via the gauge transformation by some closed 2-form B_t, i.e. $\Pi_t = \tau_{B_t}\Pi_0$. This means that the symplectic foliations agree and on each symplectic leaf $\mathcal{O}$ we have $\Omega_t = \Omega_0 + i_{\mathcal{O}}^*B_t$, where Ω_0,Ω_t are the symplectic forms on the leaf $\mathcal{O}$ and $i_{\mathcal{O}}$ the inclusion. Assume further that each $\frac{d}{dt}B_t$ be exact, and let α_t be a smooth family of primitives vanishing on some submanifold M. Then the time-1 flow of the Moser vector field[22] $\sharp_t\alpha_t$ is defined in a tubular neighborhood of M, it fixes M and maps Π_0 to Π_1.

Proposition 8.2. *Different choices of splitting V in the construction of Thm. 8.1 yield (canonically) isomorphic Poisson structures on E^*. Hence, given a Dirac manifold (M, L)*

[21] Given a Dirac structure L on a vector space W, the gauge transformation of L by a bilinear form $B \in \wedge^2 W^*$ is $\tau_B L := \{(X, \xi + i_X B) : (X, \xi) \in L\}$. Given a Dirac structure L on a manifold, the gauge-transformation $\tau_B L$ by closed 2-form B is again a Dirac structure (i.e. $\tau_B L$ is again closed under the Courant bracket).

[22] Here $\sharp_t$ denotes the map $T^*P \to TP$ induced by Π_t.

for which $L \cap TM$ has constant rank, there is a canonical (up to neighborhood equivalence) Poisson manifold in which M embeds coisotropically.

Proof. Let V_0, V_1 be two different splittings as in Thm. 8.1, i.e. $E \oplus V_i = TM$ for $i = 0, 1$. We can interpolate between them by defining the graphs $V_t := \{v + tAv : v \in V_0\}$ for $t \in [0, 1]$, where $A\colon V_0 \to E$ is determined by requiring that its graph be V_1. Obviously each V_t also gives a splitting $E \oplus V_t = TM$; denote by $i_t\colon E^* \to T^*M$ the corresponding embedding. We obtain Dirac structures $\tau_{i_t^*\omega_{T^*M}}\pi^*L$ on the total space of $\pi\colon E^* \to M$; by Thm. 8.1 they correspond to Poisson bivectors, which we denote by Π_t. These Poisson structures are related by a gauge transformation: $\Pi_t = \tau_{B_t}\Pi_0$ for $B_t := i_t^*\omega_{T^*M} - i_0^*\omega_{T^*M}$. A primitive of $\frac{d}{dt}B_t$ is given by $\frac{d}{dt}i_t^*\alpha_{T^*M}$; notice that this primitive vanishes at points of M, because the canonical 1-form α_{T^*M} on T^*M vanishes along the zero section. Hence the time-1 flow of $\sharp_t(\frac{d}{dt}i_t^*\alpha_{T^*M})$ fixes M and maps Π_0 to Π_1.

□

Assuming that (M, L) is integrable we describe the symplectic groupoid of (E^*, L_{E^*}), the Poisson manifold constructed in Thm. 8.1 with a choice of distribution V. It is $\pi^*(\Gamma_s(M))$, the pullback via $\pi\colon E^* \to M$ of the presymplectic groupoid of M, endowed with the following symplectic form: the pullback via $\pi^*(\Gamma_s(M)) \to \Gamma_s(M)$ of the presymplectic form on the groupoid $\Gamma_s(M)$, plus $\mathbf{s}^*(i_M^*\omega_{T^*M}) - \mathbf{t}^*(i_M^*\omega_{T^*M})$, where $i_M\colon E^* \to T^*M$ is the inclusion given by the choice of distribution V, ω_{T^*M} is the canonical symplectic form, and $\mathbf{s}, \mathbf{t}$ are the source and target maps of $\pi^*(\Gamma_s(M))$. This follows easily from Examples 6.3 and 6.6 in [2]. Notice that this groupoid is source simply connected when $\pi^*(\Gamma_s(M))$ is.

Now we can give an affirmative answer to the possibility raised in [14] (Remark (e) in Section 8.2), although we prove it “working backwards“; this is the “groupoid” version of Gotay's embedding theorem.

Proposition 8.3. *Any presymplectic groupoid in the sense*[23] *of [2] with constant rank characteristic distribution can be embedded coisotropically as a Lie subgroupoid in a symplectic groupoid.*

Proof. By Cor. 4.8 iv),v) of [2], a presymplectic groupoid $\Gamma_s(M)$ has characteristic distribution (the kernel of the multiplicative 2-form) of constant rank iff the Dirac structure L induced on its base M does. We can embed (M, L) coisotropically in the Poisson manifold (E^*, L_{E^*}) constructed in Thm. 8.1; we just showed that $\pi^*(\Gamma_s(M))$ is a symplectic groupoid for E^*. $\Gamma_s(M)$ embeds in $\pi^*(\Gamma_s(M))$ as $\mathbf{s}^{-1}(M) \cap \mathbf{t}^{-1}(M)$, and this embedding preserves both the groupoid structures and the 2-forms. $\mathbf{s}^{-1}(M) \cap \mathbf{t}^{-1}(M)$ is a coisotropic subgroupoid of $\pi^*(\Gamma_s(M))$ because M lies coisotropically in E^* and $\mathbf{s}, \mathbf{t}$ are (anti)Poisson maps. □

Remark 8.4. A partial converse to this proposition is given as follows: if $\mathbf{s}^{-1}(M) \cap \mathbf{t}^{-1}(M)$ is a coisotropic subgroupoid of a symplectic groupoid $\Gamma_s(P)$, then M is a coisotropic submanifold of the Poisson manifold P, it has an smooth Dirac structure (induced from P) with characteristic distribution of constant rank, and $\mathbf{s}^{-1}(M) \cap \mathbf{t}^{-1}(M)$ is a *over*-pre-symplectic groupoid over M inducing the same Dirac structure. This follows from our arguments in section 7.

[23] Recall from Def. 2.1 of [2] that a presymplectic groupoid is a Lie groupoid G over M equipped with a closed multiplicative 2-form ω such that $ker\omega_x \cap ker(ds)_x \cap ker(dt)_x = 0$ at all $x \in M$.

Now we draw the conclusions about deformation quantization. Recall that for any Dirac manifold (M, L) the set of admissible functions

$$C^{\infty}_{adm}(M) = \{f \in C^{\infty}(M) : \text{ there exists a smooth vector field } X_f \text{ s.t. } (X_f, df) \subset L\} \quad (7)$$

is naturally a Poisson algebra [13], with bracket $\{f, g\} = X_f(g)$.

Theorem 8.5. *Let (M, L) be a Dirac manifold such that $L \cap TM$ has constant rank, and denote by $\mathcal{F}$ the regular foliation integrating $L \cap TM$. If the first and second foliated de Rham cohomologies of the foliation $\mathcal{F}$ vanish then the Poisson algebra of admissible functions on (M, L) admits a deformation quantization.*

Proof. By Thm. 8.1 we can embed (M, L) coisotropically in a Poisson manifold P; hence we can apply again Corollary 3.3 of [10]: if the first and second Lie algebroid cohomology of the conormal bundle of a coisotropic submanifold vanish, then the Poisson algebra of basic functions on the coisotropic submanifold admits a deformation quantization. Since $L \cap TM$ has constant rank the inclusion $C^{\infty}_{adm}(M) \subset C^{\infty}_{bas}(M)$ is an equality[24]. Further the Poisson algebra structure on $C^{\infty}_{bas}(M)$ coming from (M, L) coincides with the one induced by M as a coisotropic submanifold of P, as follows from Prop. 6.1 and $i^* L_P = L$. So when the assumptions are satisfied we really deformation quantize $C^{\infty}_{adm}(M)$.

Notice that in Thm. 8.1 we constructed a Poisson manifold P of minimal dimension, i.e. of dimension $dim M + rk(L \cap TM)$. The anchor map $\sharp$ of the Lie algebroid N^*C is injective (see also Rem. 9.5 in the next section), hence the Lie algebroids N^*C and $L \cap TM$ are isomorphic. This allows us to state the assumptions of Corollary 3.3 of [10] in terms of the foliation $\mathcal{F}$ on M. □

Proposition 8.6. *Let (M, L) be a Dirac manifold such that $L \cap TM$ has constant rank, and denote by $\mathcal{F}$ the regular foliation integrating $L \cap TM$. Then the foliated de Rham complex $\Omega^{\bullet}_{\mathcal{F}}(M)$ admits the structure of an L_{∞}-algebra[25] $\{\lambda_n\}_{n \geq 1}$, the differential λ_1 being the foliated de Rham differential and the bracket λ_2 inducing on $H^0_{\lambda_1} = C^{\infty}_{bas}(M)$ the natural bracket (7). This L^{∞} structure is canonical up to L^{∞}-isomorphism.*

Proof. By the proof of Thm. 8.1 we know that M can be embedded coisotropically in a Poisson manifold P so that the Lie algebroids N^*M and $L \cap TM$ are isomorphic. After choosing an embedding of $NM := TP|_M / TM$ in a tubular neighborhood of M in P, Thm. 2.2 of [10] gives the desired L_{∞}-structure. By Prop. 8.2 the Poisson manifold P is canonical up to neighborhood equivalence, so the L_{∞}-structure depends only on the choice of embedding of NM in P; the first author and Schätz showed in [11] that different embeddings give the same structure up to L_{∞}-isomorphism. □

9 Uniqueness of coisotropic embeddings of Dirac manifolds

The coisotropic embedding of Gotay [15] is unique up to neighborhood equivalence, i.e. any two coisotropic embeddings of a fixed presymplectic manifold in symplectic manifolds

[24]Use that since $L \cap TM$ is the kernel of the projection $L \to T^*M$, the image of this projection has constant rank.

[25]The λ_n are derivations w.r.t. the wedge product, so one actually obtains what in [10] is called a P_{∞} algebra.

are intertwined by a symplectomorphism which is the identity on the coisotropic submanifold. It is natural to ask whether, given a Dirac manifold (M, L) such that $L \cap TM$ have constant rank, the coisotropic embedding constructed in Thm. 8.1 is the only one up to neighborhood equivalence. In general the answer will be negative: for example the origin is a coisotropic submanifold in $\mathbb{R}^2$ endowed either with the zero Poisson structure or with the Poisson structure $(x^2+y^2)\partial_x \wedge \partial_y$, and the two Poisson structures are clearly not equivalent. As Aissa Wade pointed out to us, it is necessary to require that the Poisson manifold in which we embed be of minimal dimension, i.e. of dimension $\dim M + rk(L \cap TM)$.

9.1 Infinitesimal uniqueness and global issues

We try to apply the construction of Gotay's uniqueness proof [15] on each presymplectic leaf of the Dirac manifold M; then we will show that under certain assumptions the resulting diffeomorphism varies smoothly from leaf to leaf.

We start establishing infinitesimal uniqueness.

Proposition 9.1. *Suppose we are given a Dirac manifold (M, L) for which $L \cap TM$ has constant rank k, and let (P_1, Π_1) and (P_2, Π_2) be Poisson manifolds of dimension $\dim M + k$ in which (M, L) embeds coisotropically. Then there is an isomorphism of Poisson vector bundles $\Phi\colon TP_1|_M \to TP_2|_M$ which is the identity on TM.*

Proof. Let P denote either of P_1 or P_2, by $(\mathcal{O}, \Omega)$ the symplectic leaf of P passing through some $x \in M$ and by F the presymplectic leaf of M passing through x. Since $T_xM \cap T_x\mathcal{O}$ is coisotropic in the symplectic vector space $T_x\mathcal{O}$, a simple dimension count shows that the assumption on the dimension of P is equivalent to $T_xM + T_x\mathcal{O} = T_xP$ [26]. Choose a distribution V such that $E \oplus V = TM$, where $E := L \cap TM$. We claim that $V \oplus \sharp V^\circ = TP|_M$: indeed $V_x \cap T_x\mathcal{O} = V_x \cap T_xF$ is a symplectic subspace of $(T_x\mathcal{O}, \Omega_x)$, being transverse to $E_x = ker(\Omega|_{T_xF})$. Hence $(V_x \cap T_x\mathcal{O})^{\Omega_x}$, which by section 2 is equal to $\sharp V_x^\circ$, is a complement to $V_x \cap T_x\mathcal{O}$ in $T_x\mathcal{O}$, so $V_x \oplus \sharp V_x^\circ = V_x + T_x\mathcal{O} = T_xP$ as we claimed. Now we repeat the construction of Gotay's uniqueness proof [15]: since E_x is Lagrangian in the symplectic subspace $\sharp V_x^\circ$, we can find a linear symplectomorphism $(\sharp V_x^\circ, \Omega|_{\sharp V_x^\circ}) \cong E_x \oplus E_x^*$, where the latter is equipped with the canonical antisymmetric pairing ω_E. This goes as follows: choose a complement to E_x in $\sharp V_x^\circ$, deform it canonically to a Lagrangian complement R_x (see [6]), and define the isomorphism $\sharp V_x^\circ = E_x \oplus R_x \to E_x \oplus E_x^*$ to be $(v_E, v_R) \mapsto (v_E, \Omega(v_R, \cdot)|_{E_x})$. Since V_x and the above linear symplectomorphism can be chosen to depend smoothly on $x \in M$ we obtain a smooth vector bundle isomorphism $TP|_M = V_x \oplus \sharp V_x^\circ \to V \oplus E \oplus E^*$. We equip the fibers of the latter vector bundle with bivectors as in Thm. 8.1, i.e. bivectors (depending only on the Dirac structure on M and V) so that the induced symplectic subspaces are $((V_x \cap T_xF) \oplus (E_x \oplus E_x^*), \Omega|_{V_x \cap T_xF} \oplus \omega_E)$. This isomorphism preserves the bivectors on the fibers because at each point it restricts to an isomorphism $T_x\mathcal{O} \to (V_x \cap T_xF) \oplus (E_x \oplus E_x^*)$ which matches the symplectic forms Ω_x and $\Omega|_{V_x \cap T_xF} \oplus \omega_E$. This shows that $TP_1|_M$ and $TP_2|_M$ are both isomorphic to the same Poisson vector bundle. ☐

Making a regularity assumption we can extend the infinitesimal uniqueness of Prop. 9.1 to a global statement.

[26] In particular M intersects cleanly the symplectic leaves of P and the intersections are the presymplectic leaves of M.

Proposition 9.2. *Let M,P_1 and P_2 be as in Proposition 9.1, and assume additionally that the presymplectic leaves of (M, L) have constant dimension. Then P_1 and P_2 are neighborhood equivalent.*

Proof. Since the presymplectic leaves of (M, L) have constant dimension, by the proof of Proposition 9.1 the symplectic leaves of each P_i also have constant dimension in a tubolar neighborhood of P_i. We can find[27] identifications ϕ_i between normal bundles $N_i \subset TP_i|_M$ and tubular neighborhoods of M in P_i which identify $N|_F$ and $\mathcal{O}$ in an neighborhood of M (for each presymplectic leaf F of M and corresponding symplectic leaf $\mathcal{O}$ of P_i, $i = 1, 2$).

Using the Poisson vector bundle isomorphism $\Phi\colon TP_1|_M \to TP_2|_M$ of Proposition 9.1 we obtain an identification $\phi_2 \circ \Phi \circ \phi_1^{-1}$ between tubular neighborhoods of M in P_1 and P_2. Using this identification can view Π_2 as a Poisson structure on $P := P_1$ with two properties: it induces exactly the same foliation as Π_1, and it coincides with Π_1 on $TP|_M$. We want to show that there is a diffeomorphism near M, fixing M, which maps Π_1 to Π_2.

To this aim we want to apply Moser's theorem on each symplectic leaf of P (Thm. 7.1 of [6]). Denote by Ω_i the symplectic form given by Π_i on a leaf $\mathcal{O}$. Since the convex linear combination $(1-t)\Omega_1 + t\Omega_2$ is symplectic (because Ω_1 and Ω_2 coincide at points of M) and lies in the same cohomology class, by Moser's theorem there is a diffeomorphism of ψ of $\mathcal{O}$ such that $\psi^*\Omega_2 = \Omega_1$. Concretely this goes as follows (see Chapter 6 of [6]). We identify a neighborhood of P with N (via ϕ_1) and consider $\rho_t\colon N \to N, v \mapsto tv$, where $t \in [0, 1]$. Denote by v_t the vector tangent to the curve $\rho_s(v)$ at time $s = t$. Now just consider $N|_F$, where F is the presymplectic leaf $\mathcal{O} \cap M$. The operator

$$Q : \Omega^\bullet(\mathcal{O}) \to \Omega^{\bullet - 1}(\mathcal{O}); \quad Q\omega = \int_0^1 \rho_t^*(i_{v_t}\omega)dt$$

has the property of providing primitives for closed differential forms whose pullback to M vanishes. So $\mu := Q(\Omega_2 - \Omega_1)$ is a primitive for $\Omega_2 - \Omega_1$. Consider the Moser vector field, obtained inverting via $(1-t)\Omega_1 + t\Omega_2$ the 1-form μ. Following from time 1 to time 0 the flow of the Moser vector field gives the desired diffeomorphism ψ (which keeps M fixed since μ vanishes at points of F).

This constructions varies smoothly from leaf to leaf: ρ_t and v_t are clearly smooth, and the foliated 2-forms $\Omega_2 - \Omega_1$ and $(1-t)\Omega_1 + t\Omega_2$ also are, as can be seen using coordinates adapted to the foliation. Hence we obtain a diffeomorphism ψ of a tubular neighborhood of M, fixing M, which maps Π_1 to Π_2. □

Since local uniqueness holds (see subsection 9.2) and since by Proposition 9.1 there is no topological obstruction, it seems that a global uniqueness statement should hold in the general case, i.e. when the presymplectic foliation of (M, L) is not necessarily regular. We conclude with some possible approaches to prove global uniqueness.

[27] Let P denote either of P_1 or P_2. Let V be a distribution on M such that $V \oplus (L \cap TM) = TM$. We saw in the proof of Proposition 9.1 that $V \oplus \sharp V^\circ = TP|_M$ and $L \cap TM \subset \sharp V^\circ$. Define N as a smooth complement to $L \cap TM$ in $\sharp V^\circ$; then $TP|_M = TM \oplus N$ and N is tangent to the symplectic leaves of P at points of M. Choose a Riemannian metric on P and define $\phi(v_x)$ to be $exp^{\mathcal{O}} v$, where $exp^{\mathcal{O}}$ is the exponential map of the symplectic leaf $\mathcal{O}$ passing through x (with the induced metric). The resulting map $\phi : N \to P$ is well-defined since v is tangent to $\mathcal{O}$, it maps $N|_F$ onto an open neighborhood in $\mathcal{O}$, and it is smooth because the symplectic leaves of P form a regular foliation.

The argument from [1] just before Prop. 8.2 shows that the uniqueness of (minimal dimensional) coisotropic embeddings of a given Dirac manifold (M,L) is equivalent to the following: whenever (P_1,Π_1) and (P_2,Π_2) are minimal Poisson manifolds in which (M,L) embeds coisotropically there exists a diffeomorphism $\phi\colon P_1 \to P_2$ near M so that Π_2 and $\phi_*\Pi_1$ differ by the gauge transformation by a closed 2-form B vanishing on M. One could hope that if $\phi\colon P_1 \to P_2$ is chosen to match symplectic leaves and to match $\Pi_1|_M$ and $\Pi_2|_M$ then a 2-form B as above automatically exist. This is not the case, as the following example shows.

Example 9.3. Take $M = \mathbb{R}^3$ with Dirac structure

$$L = span\{(-x_1^2\partial_{x_2}, dx_1), (x_1^2\partial_{x_1}, dx_2), (\partial_{x_3}, 0)\}.$$

There are two open presymplectic leaves $(\mathbb{R}_\pm \times \mathbb{R}^2, \frac{1}{x_1^2}dx_1 \wedge dx_2)$ and 1-dimensional presymplectic leaves $\{0\}\times\{c\}\times\mathbb{R}$ with zero presymplectic form (for every real number c); hence our Dirac structure is a product of the Poisson structure $x_1^2\partial_{x_1}\wedge\partial_{x_2}$ and of the zero presymplectic form on the x_3-axis. The characteristic distribution $L \cap TM$ is always $span\partial_{x_3}$. Clearly the construction of Thm. 8.1 gives

$$P_1 := (\mathbb{R}^4, x_1^2\partial_{x_1} \wedge \partial_{x_2} + \partial_{x_3} \wedge \partial_{y_3})$$

where y_3 is the coordinate on the fibers of $P_1 \to M$.

Another Poisson structure on $\mathbb{R}^4$ with the same foliation as Π_1 and which coincides with Π_1 along M is the following:

$$\Pi_2 := x_1^2\partial_{x_1} \wedge \partial_{x_2} + \partial_{x_3} \wedge \partial_{y_3} + x_1 y_3 \partial_{x_2} \wedge \partial_{x_3}.$$

On each of the two open symplectic leaves $\mathbb{R}_\pm \times \mathbb{R}^3$ the symplectic form corresponding to Π_1 is $\Omega_1 = \frac{1}{x_1^2}dx_1 \wedge dx_2 + dx_3 \wedge dy_3$, whereas the one corresponding to Π_2 is $\Omega_2 = \Omega_1 + \frac{y_3}{x_1}dx_1 \wedge dy_3$. Clearly the difference $\Omega_1 - \Omega_2$ does not extend to smooth a 2-form on the whole of $\mathbb{R}^4$. Hence there is no smooth 2-form on $\mathbb{R}^4$ relating Π_1 and Π_2.

Nevertheless Π_1 and Π_2 are Poisson diffeomorphic: Prop. 9.7 in subsection 9.2 will tell us that they are in neighborhoods of the origin, and the construction of Prop. 9.7 will provide a global coordinate change that maps Π_2 into Π_1, namely the coordinate change that transforms x_2 into $x_2 + \frac{y_3^2}{2}x_1$ and leaves the other coordinates untouched.

One could try to obtain a $\phi\colon P_1 \to P_2$ as above by integrating the isomorphisms Φ constructed in Prop. 9.1. Alternatively one could show the existence, for any minimal Poisson manifold (P,Π) in which (M,L) embeds coisotropically, of a projection $\pi : P \to M$ such that Π and the pullback Dirac structure π^*L be related by a (suitable) B-transformation (as happens for the Poisson manifold E^* of Thm. 8.1): by choosing a diffeomorphism $P_1 \to P_2$ intertwining the projections of P_1 and P_2 one would conclude that Π_1 and Π_2 are gauge equivalent. Another approach to prove global uniqueness is to construct a projection $\pi : P \to M$ with the weaker property that the Lie algebroids corresponding to Π and to π^*L be isomorphic, for then the symplectic groupoids of any P_1,P_2 as above will be isomorphic as Lie groupoid; then one would try to relate the corresponding symplectic forms by the flow of a multiplicative Moser vector field.

9.2 Local uniqueness

While we are not able to prove a global uniqueness statement in the general case, we prove in this subsection that a local uniqueness statements holds. We start with a normal form statement.

Proposition 9.4. *Suppose we are given a Dirac manifold (M^m, L) for which $L \cap TM$ have constant rank k, and let (P, Π) a Poisson manifold of dimension $m + k$ in which (M, L) embeds coisotropically. Then about any $x \in M$ there is a neighborhood $U \subset P$ and coordinates $\{q_1, \dots, q_k, p_1, \dots, p_k, y_1, \dots, y_{m-k}\}$ defined on U such that locally M is given by the constraints $p_1 = 0, \dots, p_k = 0$ and*

$$\Pi = \sum_{i=1}^{k} \partial_{q_i} \wedge \partial_{p_i} + \sum_{i,j=1}^{m-k} \varphi_{ij}(y)\partial_{y_i} \wedge \partial_{y_j} \tag{8}$$

for functions $\varphi_{ij} \colon \mathbb{R}^{m-k} \to \mathbb{R}$.

Remark 9.5. The existence of coordinates in which Π has the above split form is guaranteed by Weinstein's Splitting Theorem [21]; the point in the above proposition is that one can choose the coordinates (q, p, y) so that M is given by the constrains $p = 0$.
The assumption on the dimension of P is equivalent to $T_xM + T_x\mathcal{O} = T_xP$ at every $x \in M$, where $\mathcal{O}$ is the symplectic leaf through x, which in turn is equivalent to $\sharp|_{N^*M}$ being injective.

Proof. We adapt the proof of Weinstein's Splitting Theorem [21] to our setting. To simplify the notation we will often write P in place of U and M in place of $M \cap U$. Choose a function q_1 on P near x such that dq_1 does not annihilate $L \cap TM$. Then $X_{q_1}|_M$ doesn't vanish and is transverse to M, because there is a $\xi \in N^*M$ with $0 \neq \langle \sharp\xi, dq_1 \rangle = -\langle \xi, X_{q_1} \rangle$. Choose a hypersurface in P containing M and transverse to $X_{q_1}|_M$, and determine the function p_1 by requiring that it vanishes on the hypersurface and $dp_1(X_{q_1}) = -1$. Since $[X_{q_1}, X_{p_1}] = X_1 = 0$ the span of X_{p_1} and X_{q_1} is an integrable distribution giving rise to a foliation of P by surfaces. This foliation is transverse to P_1, the codimension two submanifold where p_1 and q_1 vanish. $M_1 := P_1 \cap M$ is clean intersection and is a codimension one submanifold of M. To proceed inductively we need

Lemma 9.6. *P_1 has an induced Poisson structure, $M_1 \subset P_1$ is coisotropic, and the sharp-map $\sharp_1$ of P_1 is injective on the conormal bundle to M_1.*

Proof. P_1 is cosymplectic because it is given by constraints whose matrix of Poisson brackets is non-degenerate: $\{q_1, p_1\} = 1$. Hence it has an induced Poisson structure, whose sharp map we denote by $\sharp_1$. Recall from section 2 that if $\xi_1 \in T_x^*P_1$ then $\sharp_1\xi_1 \in TP_1$ is given as follows: extend ξ_1 to a covector ξ of P by asking that it annihilates $\sharp N_x^*P_1$ and apply $\sharp$ to it. Now in particular let $x \in M_1$ and ξ_1 be an element of the conormal bundle of M_1 in P_1. We have $T_xM = T_xM_1 \oplus \mathbb{R}X_{p_1}(x) \subset T_xM_1 + \sharp N_x^*P_1$, so $\xi \in N_x^*M$, and since M is coisotropic in P we have $\sharp\xi \in T_xM$. Hence $\sharp_1\xi_1 \in T_xP_1 \cap T_xM = T_xM_1$, which shows the claimed coisotropicity. The injectivity of $\sharp_1$ on the conormal bundle follows by the above together with the injectivity of $\sharp|_{N^*M}$. $\square$

Thanks to Lemma 9.6 we are allowed to apply the above procedure to the codimension $k-1$ coisotropic submanifold M_1 of P_1. We obtain functions q_2, p_2 on P_1 such that[28] $\{q_2, p_2\}_1 = 1$, a codimension two submanifold P_2 of P_1 given by the points where q_2 and p_2 vanish, and a codimension one submanifold $M_2 := P_2 \cap M$ of P_2. After repeating this other $k-2$ times we get to M_k, a codimension zero submanifold of P_k which hence coincides with P_k.

Now we start working backwards: choose arbitrary functions $y_1, , y_{m-k}$ on P_k, extend them to P_{k-1} constantly along the surfaces integrating $span\{X_{q_k}, X_{p_k}\}$. The Poisson bracket on P_{k-1} satisfies $\{y_i, q_k\}_{k-1} = 0$ and $\{y_i, p_k\}_{k-1} = 0$, and using the Jacobi identity one sees that any $\{y_i, y_j\}_{k-1}$ Poisson commutes with q_k and p_k, and hence the $\{y_i, y_j\}_{k-1}$ are functions of the y's only. Now continue extending the y's and q_k, p_k to P_{k-2}. After k steps we obtain functions on P for which the non-trivial brackets are $\{q_i, p_i\} = 1$ and $\{y_i, y_j\} =: \varphi_{ij}(y)$. Hence formula (8) for the Poisson bivector Π follows.

To show that M is given by the constraints $p_1 = 0, \dots, p_k = 0$ we notice the following. We chose p_1 to vanish on M. We chose p_2 on P_1 to vanish on M_1, and we extended it to P asking that it be constant along the foliation tangent to the span of X_{p_1} and X_{q_1}. Since $X_{p_1}|_M$ is tangent to M and $TM|_{M_1} = TM_1 \oplus X_{p_1}|_{M_1}$, it follows that p_2 vanishes on the whole of M. Inductively one shows that all the p_i vanish on M, and by dimension counting one obtains that the p_i define exactly M. This concludes the proof of Prop. 9.5. $\square$

Using the normal forms derived above we can prove local uniqueness:

Proposition 9.7. *Let (P, Π) and $(\bar{P}, \bar{\Pi})$ be Poisson manifolds as in Prop. 9.4 in which (M, L) embeds coisotropically. Then about each $x \in M$ there are neighborhoods $U \subset P$, $\bar{U} \subset \bar{P}$ and a Poisson diffeomorphism $(U, \Pi) \cong (\bar{U}, \bar{\Pi})$ which is the identity on M.*

Proof. By integrating the vector bundle isomorphism Φ of Prop. 9.1 we may assume that $\bar{\Pi}$ is a Poisson bivector on P and that it coincides with Π at points of M. We will show below that we can make choices of coordinates $\{q_i, p_i, y_j\}$ on U and $\{\bar{q}_i, \bar{p}_i, \bar{y}_j\}$ on $\bar{U}$ which bring Π and $\bar{\Pi}$ respectively in the canonical form (8) and which are *compatible*, in the sense that these coordinate sets coincide once restricted to M. Then the diffeomorphism of P induced by the obvious coordinate change

$$q_1 \mapsto \bar{q}_i, p_i \mapsto \bar{p}_i, y_j \mapsto \bar{y}_j$$

is the identity on M. Further it is a Poisson diffeomorphism: one just has to check that the functions φ_{ij} appearing in (8), which are just $\{y_i, y_j\}_P$, coincide with $\bar{\varphi}_{ij} = \{\bar{y}_i, \bar{y}_j\}_{\bar{P}}$ when we consider them as functions of the $m-k$ variables y_i or $\bar{y}_i$. To this aim notice that $y_i|_M$ annihilates the characteristic distribution $L \cap TM$ of M, for $L \cap TM$ is spanned by $X_{p_1}|_M, \cdots, X_{p_k}|_M$. Hence $y_i|_M$ is an admissible function (7) for the Dirac manifold (M, L), and similarly $y_j|_M$, so we can apply to them the bracket $\{\bullet, \bullet\}_M$ of admissible functions on (M, L) which is of course determined only by the Dirac structure L on M. Since X_{y_i} is tangent to M it follows that $\{y_i|_M, y_j|_M\}_M$ is just the restriction to M of $\{y_i, y_j\}_P$. Similarly $\{\bar{y}_i|_M, \bar{y}_j|_M\}_M$ is the restriction to M of $\{\bar{y}_i, \bar{y}_j\}_{\bar{P}}$. Since as we saw $y_i|_M = \bar{y}_i|_M$, we deduce that φ_{ij} and $\bar{\varphi}_{ij}$ coincide on M, so $\varphi_{ij} = \bar{\varphi}_{ij}$ as functions $\mathbb{R}^{m-k} \to \mathbb{R}$.

[28] Here $\{q_2, p_2\}_1$ denotes the Poisson bracket on P_1, which coincides with (the restriction to P_1 of) the Poisson bracket on P of the functions obtained extending q_2, p_2 to P constantly along the surfaces tangent to $span\{X_{q_1}, X_{p_1}\}$.

In the rest of the proof we show that it is possible to perform the construction of the proof of Prop. 9.4 (which depended on several choices) to obtain *compatible* coordinates $\{q_i, p_i, y_j\}$ and $\{\bar{q}_i, \bar{p}_i, \bar{y}_j\}$. We refer to the proof of Prop. 9.4 for the notation and decorate with a bar the objects arising from Π. Choose functions $q_1, \bar{q}_1$ on P around x so that the functions and their differentials agree at points of M (of course here we could just take $q_1 = \bar{q}_1$). Then $M_1 = \{q_1 = 0\} \cap M$ coincides with $\bar{M}_1$. The conditions on the differentials, together with $\Pi|_M = \bar{\Pi}|_M$, imply $X_{q_1}|_M = \bar{X}_{\bar{q}_1}|_M$ (where the second hamiltonian vector field is taken w.r.t $\bar{\Pi}$). Choose two hypersurfaces of P containing M such that their tangent spaces at points of M coincide (of course we could take the hypersurfaces to be equal). This determines the functions $p_1, \bar{p}_1$ on P. Notice that $dp_1|_M$ (a section of the vector bundle $T^*P|_M \to M$) and $d\bar{p}_1|_M$ coincide, because they have the same kernel and both evaluate to -1 on $X_{q_1}|_M = X_{\bar{q}_1}|_M$. This has two consequences: first $X_{p_1}|_M = \bar{X}_{\bar{p}_1}|_M$. Second, even though $P_1 :=$ {points of P where $p_1 = 0, q_1 = 0$} and $\bar{P}_1$ do not coincide, they are tangent to each other along $M_1 = \bar{M}_1$, since the differentials of q_1 and p_1 coincide with their barred counterparts on M and in particular on $M_1 = \bar{M}_1$. Further the Poisson structures induced by Π on P_1 and $\bar{\Pi}$ on $\bar{P}_1$ coincide at points of $M_1 = \bar{M}_1$, because Π and $\bar{\Pi}$ there. To summarize we showed

$$M_1 = \bar{M}_1, \qquad TP_1|_{M_1} = T\bar{P}_1|_{\bar{M}_1} \text{ as Poisson vector bundles}, \qquad X_{p_1}|_M = \bar{X}_{\bar{p}_1}|_M. \tag{9}$$

Now we would like to apply the above procedure to $M_1 = \bar{M}_1$, which is coisotropic in the two Poisson manifolds P_1 and $\bar{P}_1$. The only difference to the above situation is that now we have two Poisson manifolds which do not agree as spaces. However since their tangent spaces along $M_1 = \bar{M}_1$ agree we can still proceed as above: we choose q_2 on P_1 and $\bar{q}_2$ on $\bar{P}_1$ so that they agree on $M_1 = \bar{M}_1$ together with their first derivates; we choose hypersurfaces in P_1 and $\bar{P}_1$ so that their tangent spaces along $M_1 = \bar{M}_1$ coincide, and these in turn determine p_2 and $\bar{p}_2$. Proceeding inductively we have

$$M_i = \bar{M}_i, \quad TP_i|_{M_i} = T\bar{P}_i|_{\bar{M}_i} \text{ as Poisson vector bundles}, \quad X_{p_i}|_{M_{i-1}} = \bar{X}_{\bar{p}_i}|_{\bar{M}_{i-1}} \text{ for } i \leq k.$$

Now we start working backwards. We choose arbitrary coordinates $\{y_1, \cdots, y_{m-k}\}$ on $P_k = \bar{P}_k$, and extend them to P_{k-1} constantly along the surfaces spanned by X_{q_k}, X_{p_k}, as well as to $\bar{P}_{k-1}$ constantly along the surfaces spanned by $\bar{X}_{\bar{q}_k}, \bar{X}_{\bar{p}_k}$. Since X_{p_k} and $\bar{X}_{\bar{p}_k}$ coincide on $M_{k-1} = \bar{M}_{k-1}$, we see that the resulting y and $\bar{y}$ coincide on $M_{k-1} = \bar{M}_{k-1}$. The coordinates q_k and $\bar{q}_k$ coincide there too by definition (p_k and $\bar{p}_k$ trivially too, because they vanish there). After k steps we see that the coordinates $q_1, \cdots, q_k, y_1, \cdots, y_{m-k}$ on P, once restricted to M, coincide with their barred counterparts. □

We refer to Example 9.3 for an example of the construction of Prop. 9.7

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Pre-Poisson submanifolds

by Alberto S. Cattaneo and Marco Zambon

Abstract

In this note we consider an arbitrary submanifold C of a Poisson manifold P and ask whether it can be embedded coisotropically in some bigger submanifold of P. We define the classes of submanifolds relevant to the question (coisotropic, Poisson-Dirac, pre-Poisson ones), present an answer to the above question and consider the corresponding picture at the level of Lie groupoids, making concrete examples in which P is the dual of a Lie algebra and C is an affine subspace.

Contents

1 Introduction

In this note we wish to give an analog in Poisson geometry to the following statement in symplectic geometry. Recall that (P,Ω) is a symplectic manifold if Ω is a closed, non-degenerate 2 form and that a submanifold $\hat{C}$ is called coisotropic if the symplectic orthogonal $T\hat{C}^{\Omega}$ of $T\hat{C}$ is contained in $T\hat{C}$. The statement is: if $i\colon C \to P$ is any submanifold of a symplectic manifold (P,Ω), then there exists some symplectic submanifold $\tilde{P}$ containing C as a coisotropic submanifold iff $i^*\Omega$ has constant rank. The submanifold $\tilde{P}$ is obtained taking any complement $R \subset TP|_C$ of $TC + TC^{\Omega}$ and "extending C along R". Further there is a uniqueness statement "to first order": if $\tilde{P}_1$ and $\tilde{P}_2$ are as above, then there is a symplectomorphism of P fixing C whose derivative at C maps $T\tilde{P}_1|_C$ to $T\tilde{P}_2|_C$. This result follows using techniques similar to those used by Marle in [13], and relies on a technique known as "Moser's path method".

The above result should not be confused with the theorem of Gotay [9] that states the following: any presymplectic manifold (i.e. a manifold endowed with a constant rank closed 2-form) can be embedded coisotropically in some symplectic manifold, which is moreover unique up to neighborhood equivalence. The difference is that Gotay considers an abstract presymplectic manifold and looks for an abstract symplectic manifold in which to embed; the problem above fixes a symplectic manifold (P,Ω) and considers only submanifolds of P.

In this note we ask:

1) Given an arbitrary submanifold C of a Poisson manifold (P, Π), under what conditions does there exist some submanifold $\tilde{P} \subset P$ such that

   a) $\tilde{P}$ has a Poisson structure induced from Π

   b) C is a coisotropic submanifold of $\tilde{P}$?

2) When the submanifold $\tilde{P}$ exists, is it unique up to neighborhood equivalence (i.e. up to a Poisson diffeomorphism on a tubular neighborhood which fixes C)?

We will see in Section 4 that a sufficient condition is that C belongs to a particular class of submanifolds called pre-Poisson submanifolds. In that case we also have uniqueness: if $\tilde{P}_1$ and $\tilde{P}_2$ are as above, then there is a Poisson diffeomorphism of (a tubular neighborhood of C in) P fixing C which takes $\tilde{P}_1$ to $\tilde{P}_2$. When the Poisson structure on P comes from a symplectic form Ω, the pre-Poisson submanifolds of P are exactly the submanifolds for which the pullback of Ω has constant rank; hence we improve the "uniqueness to first order" result in the symplectic setting mentioned above to uniqueness in a neighborhood of C.

Since the above question is essentially about when an arbitrary submanifold can be regarded as a coisotropic one, we want to motivate in Section 2 why coisotropic submanifolds are interesting at all. In Section 3 we will describe the submanifolds of P which inherit a Poisson structure; these are the "candidates" for $\tilde{P}$ as above. Then in Section 5 we will present a non-trivial example: we consider as Poisson manifold P the dual of a Lie algebra $\mathfrak{g}$, and as submanifold C either a translate of the annihilator of a Lie subalgebra or the annihilator of some subspace of $\mathfrak{g}$. Finally in Section 6 we recall how to a Poisson manifold one can associate symplectic groupoids and investigate what pre-Poisson submanifolds correspond to at the groupoid level, discussing again the example where P is the dual of a (finite dimensional) Lie algebra. All manifolds appearing in this note are assumed to be finite dimensional.

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2 Coisotropic submanifolds

A manifold P is called *Poisson manifold* if it is endowed with a bivector field $\Pi \in \Gamma(\Lambda^2 TP)$ satisfying $[\Pi, \Pi] = 0$, where $[\bullet, \bullet]$ denotes the Schouten bracket on multivector fields. Let us denote by $\sharp\colon T^*P \to TP$ the map given by contraction with Π. The image of $\sharp$ is a singular integrable distribution on P, whose leaves are endowed with a symplectic structure that encodes the bivector field Π. Hence one can think of a Poisson manifold as a manifold with a singular foliation by symplectic leaves.

Alternatively P is a Poisson manifold if there is a Lie bracket $\{\bullet, \bullet\}$ on the space of functions satisfying the Leibniz identity[1] $\{f, g\cdot h\} = \{f, g\}\cdot h + g\cdot\{f, h\}$. The Poisson bracket

[1]In this case one says that $(C^\infty(P), \{\bullet, \bullet\}, \cdot)$ forms a Poisson algebra.

$\{\bullet, \bullet\}$ and the bivector field Π determine each other by the formula $\{f, g\} = \Pi(df, dg)$. In this note we will use both the geometric and algebraic characterization of Poisson manifolds.

Symplectic manifolds (P, Ω) are examples of Poisson manifolds: the map $TP \to T^*P$ given by contracting with Ω is an isomorphism, and (the negative of) its inverse is the sharp map of the Poisson bivector field associated to Π. Connected symplectic manifolds are exactly the Poisson manifolds whose symplectic foliation consists of just one leaf.

A second standard example, which will be used in Section 5, is the dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$, as follows. A linear function v on $\mathfrak{g}^*$ can be regarded as an element of $\mathfrak{g}$; one defines the Poisson bracket on linear functions as $\{v_1, v_2\} := [v_1, v_2]$, and the bracket for arbitrary functions is determined by this in virtue of the Leibniz rule. Duals of Lie algebras are exactly the Poisson manifolds whose Poisson bivector field is linear. The symplectic foliation of $\mathfrak{g}^*$ is given by the orbits of the coadjoint action; the origin is a symplectic leaf, and unless $\mathfrak{g}$ is an abelian Lie algebra the symplectic foliation will be singular. We will discuss this example in more detail in Section 5.

A submanifold C of a Poisson manifold P is called *coisotropic* if $\sharp N^*C \subset TC$. Here N^*C (the conormal bundle of C) is defined as the annihilator of TC, and the singular distribution $\sharp N^*C$ on C is called the *characteristic distribution*. Notice that if the Poisson structure of P comes from a symplectic form Ω then the subbundle $\sharp N^*C$ is just the symplectic orthogonal of TC, so we are reduced to the usual definition of coisotropic submanifolds in the symplectic case. If a submanifold C intersects the symplectic leaves $\mathcal{O}$ of P cleanly, then C is coisotropic iff each intersection $C \cap \mathcal{O}$ is a coisotropic submanifold of the symplectic manifold $\mathcal{O}$. In algebraic terms we have the following characterization: a submanifold C is coisotropic iff $I_C\colon = \{f \in C^\infty(P)\colon f|_C = 0\}$ is a Poisson subalgebra of $(C^\infty(P), \{\bullet, \bullet\}, \cdot)$.

In the following we want to motivate the naturality and importance of coisotropic submanifolds.

- Graphs of Poisson maps are coisotropic:

  Proposition 2.1 (Cor. 2.2.3 of [15]). *Let $\Phi\colon (P_1, \Pi_1) \to (P_2, \Pi_2)$ be a map between Poisson manifolds. Φ is a Poisson map (i.e. $\Phi_*(\Pi_1) = \Pi_2$) iff its graph is a coisotropic submanifold of $(P_1 \times P_2, \Pi_1 - \Pi_2)$.*

- Certain canonical quotients of coisotropic submanifolds are Poisson manifolds: define $F_C\colon = \{f \in C^\infty(P)\colon \{f, I_C\} \subset I_C\}$, the Poisson normalizer of I_C. It is a Poisson subalgebra of $C^\infty(P)$, and $I_C \subset F_C$ is a Poisson ideal. Further notice that F_C consists exactly of the functions on P whose differentials annihilate the characteristic distribution $\sharp N^*C$. Hence we have the following statements about the quotient of C by the characteristic distribution:

  Proposition 2.2. *F_C/I_C inherits the structure of a Poisson algebra. Therefore $\underline{C}\colon = C/\sharp N^*C$, if smooth, inherits the structure of a Poisson manifold so that $C \to \underline{C}$ is a Poisson map.*

  Given any Poisson algebra A, one can ask whether it admits a deformation quantization, i.e. if it is possible to deform the commutative multiplication "in direction of the Poisson bracket" to obtain an associative product. Remarkable work of Kontsevich [11] showed that this is always possible if A is the algebra of functions on a smooth Poisson manifold. The Poisson algebras F_C/I_C provide natural and interesting instances of Poisson algebras which usually cannot be regarded as algebras of functions on a smooth manifold; the problem of their deformation quantization has been considered in [4, 5].

- Last, a coisotropic submanifold C gives rise to a Lie subalgebroid of the Lie algebroid associated to P. Recall that a *Lie algebroid* is a vector bundle $E \to P$ with a Lie bracket $[\bullet, \bullet]$ on its space of sections and a bracket preserving bundle map $\rho\colon E \to TP$ satisfying $[e_1, fe_2] = \rho(e_1)f \cdot e_2 + f[e_1, e_2]$; standard examples are tangent bundles and Lie algebras. Every Poisson manifold P induces the structure of a Lie algebroid on its cotangent bundle T^*P: the bracket is given by $[df, dg] = d\{f, g\}$ and the bundle map $T^*P \to TP$ by $-\sharp$. We have

  Proposition 2.3 (Cor. 3.1.5 of [15])**.** *If $C \subset P$ is coisotropic then the conormal bundle N^*C is a Lie subalgebroid of T^*P.*

3 Poisson-Dirac and cosymplectic submanifolds

In virtue of the question asked in the introduction it is necessary to determine which submanifolds $\tilde{P}$ of a Poisson manifold (P, Π) inherit a Poisson structure. Notice that, unlike symplectic forms, it is usually not possible to restrict a Poisson bivector field to a submanifold and obtain again a bivector field. However it is possible to view a Poisson bivector field as a Dirac structure [7], and Dirac structures restrict to (usually not smooth) Dirac structures on submanifolds. This point of view led to the definition below, which we phrase without reference to Dirac structures.

We first make the following remark, in which $(\mathcal{O}, \Omega)$ denotes a symplectic leaf of P and $\tilde{P} \subset P$ some submanifold: the linear subspace $T_p\tilde{P} \cap T_p\mathcal{O}$ of $(T_p\mathcal{O}, \Omega_p)$ is a symplectic subspace iff $\sharp N^*_p\tilde{P} \cap T_p\tilde{P} = \{0\}$. In this case $T\tilde{P}_p$ is endowed with a bivector field $\tilde{\Pi}_p$, obtained essentially by inverting the non-degenerate form $\Omega_p|_{T_p\tilde{P}\cap T_p\mathcal{O}}$. Now we can make sense of the following definition (Cor. 11 of [8]):

Definition 3.1. A submanifold $\tilde{P}$ of P is called *Poisson-Dirac submanifold* if $\sharp N^*\tilde{P} \cap T\tilde{P} = \{0\}$ and the induced bivector field $\tilde{\Pi}$ on $\tilde{P}$ is a smooth.

In this case the bivector field is automatically integrable (Prop. 6 of [8]), so that $(\tilde{P}, \tilde{\Pi})$ is a Poisson manifold. Equivalently (Def. 4 of [8]) $\tilde{P}$ is a Poisson-Dirac submanifold if it admits a Poisson structure for which the symplectic leaves are (connected) intersections with the symplectic leaves $\mathcal{O}$ of P and so that the former are symplectic submanifolds of the leaves $\mathcal{O}$. Notice that the inclusion $\tilde{P} \to P$ is usually not a Poisson map; it is iff $\tilde{P}$ is a Poisson submanifold, i.e. a smooth union of symplectic leaves.

A submanifold $\tilde{P}$ satisfying $T\tilde{P} \oplus \sharp N^*\tilde{P} = TP|_{\tilde{P}}$ is called a *cosymplectic submanifold*. In this case one can show that the induced bivector field $\tilde{\Pi}$ on $\tilde{P}$ is automatically smooth, hence cosymplectic submanifolds are Poisson-Dirac submanifolds. The Poisson bracket on a cosymplectic submanifold $\tilde{P}$ is computed as follows: $\{\tilde{f}_1, \tilde{f}_2\}_{\tilde{P}}$ is the restriction to $\tilde{P}$ of $\{f_1, f_2\}$, where the f_i are extensions of $\tilde{f}_i$ to P such that $df_i|_{\sharp N^*\tilde{P}} = 0$.

If the Poisson structure on P comes from a symplectic 2-form, then the Poisson-Dirac and cosymplectic submanifolds are just the symplectic submanifolds.

4 Coisotropic embeddings in Poisson-Dirac submanifolds

Now we determine under what conditions on a submanifold $i\colon C \to P$ there exists a Poisson-Dirac submanifold $\tilde{P} \subset P$ so that C is coisotropic in $\tilde{P}$. We saw in the introduction that, when the Poisson structure on P comes from a symplectic form Ω, a sufficient and necessary condition is that $ker(i^*\Omega)$, which in terms of the Poisson tensor is $TC \cap \sharp N^*C$, has constant rank. In the general Poisson case however $TC \cap \sharp N^*C$, even when it has constant rank, might not be a smooth distribution on C. In the symplectic case $ker(i^*\Omega)$ has constant rank iff $TC + TC^{\Omega}$ has constant rank, and it turns out that this is the right condition to generalize to the Poisson case. This motivates

Definition 4.1 (Def. 2.2 of [6])**.** A submanifold C of a Poisson manifold (P, Π) is called *pre-Poisson* if the rank of $TC + \sharp N^*C$ is constant along C.

Such submanifolds were first considered in [1, 2]. We have

Theorem 4.2. *[Thm. 3.3 of [6]] Let C be a pre-Poisson submanifold of a Poisson manifold (P, Π). Then there exists a cosymplectic submanifold $\tilde{P}$ containing C such that C is coisotropic in $\tilde{P}$.*

Sketch of the proof. Because of the rank condition on C we can choose a smooth subbundle R of $TP|_C$ which is a complement to $TC + \sharp N^*C$. By linear algebra, at every point p of C, $T_pC \oplus R_p$ is a cosymplectic subspace of T_pP in which T_pC sits coisotropically. Now we "thicken" C to a smooth submanifold $\tilde{P}$ of P satisfying $T\tilde{P}|_C = TC \oplus R$. One can show that in a neighborhood of C $\tilde{P}$ is a cosymplectic submanifold, so shrinking $\tilde{P}$ if necessary we are done. □

Remark 4.3. The cosymplectic submanifold $\tilde{P}$ above is constructed by taking any complement $R \subset TP|_C$ of $TC + \sharp N^*C$ and "extending C along R".

There are submanifolds C which are not pre-Poisson but still admit some Poisson-Dirac submanifold $\tilde{P}$ in which they embed coisotropically. This happens for example if C has trivial intersection with the symplectic leaves of P (and the symplectic foliation of P is not regular): in this case $\tilde{P} := C$ is a Poisson-Dirac submanifold, the induced Poisson bivector field being zero.

However, if we ask that the submanifold $\tilde{P}$ be not just Poisson-Dirac but actually cosymplectic, then C is necessarily a pre-Poisson submanifold, and $\tilde{P}$ is constructed as described above (Lemma 4.1 of [6]).

The following are elementary examples of pre-Poisson submanifolds and of cosymplectic submanifolds in which they embed coisotropically. In section 5 we will give less trivial examples; see also Section 5 of [6].

Example 4.4. When C is a coisotropic submanifold of P, the construction of Thm. 4.2 delivers $\tilde{P} = P$ (or more precisely, a tubular neighborhood of C in P).

Example 4.5. When C is just a point x then the construction of Thm. 4.2 delivers as $\tilde{P}$ any slice through x transversal to the symplectic leaf $\mathcal{O}_x$.

Example 4.6. If $C_1 \subset P_1$ and $C_2 \subset P_2$ are pre-Poisson submanifolds of Poisson manifolds, the cartesian product $C_1 \times C_2 \subset P_1 \times P_2$ also is, and if the construction of Thm. 4.2 gives cosymplectic submanifolds $\tilde{P}_1 \subset P_1$ and $\tilde{P}_2 \subset P_2$, the same construction applied to $C_1 \times C_2$ (upon suitable choices of complementary subbundles) delivers the cosymplectic submanifold $\tilde{P}_1 \times \tilde{P}_2$ of $P_1 \times P_2$.

The following lemma will be useful in Section 5:

Lemma 4.7. *Let P_1, P_2 be Poisson manifolds and $f\colon P_1 \to P_2$ be a submersive Poisson morphism. If $C \subset P_2$ is a pre-Poisson submanifold then $f^{-1}(C)$ is a pre-Poisson submanifold of P_1. Further, if $\tilde{P}_2$ is a cosymplectic submanifold containing C as a coisotropic submanifold, then $f^{-1}(\tilde{P}_2)$ is a cosymplectic submanifold containing $f^{-1}(C)$ as a coisotropic submanifold.*

Proof. Let $y \in C$ and $x \in f^{-1}(y)$. Since

$$f_*(\sharp N_x^*(f^{-1}(C))) = f_*(\sharp f^*(N_y^*C)) = \sharp N_y^*C$$

it follows that the restriction of f_* to $T_x(f^{-1}(C)) + \sharp N_x^*(f^{-1}(C))$ has image $T_yC + \sharp N_y^*C$, whose rank is independent of $y \in C$ by assumption. Since the kernel of this restriction, being $T_x(f^{-1}(y))$, also has constant rank, it follows that $f^{-1}(C)$ is pre-Poisson.

Further it is clear that f_* maps a complement R_x of $T_x(f^{-1}(C)) + \sharp N_x^*(f^{-1}(C))$ in T_xP_1 isomorphically onto a complement R_y of $T_yC + \sharp N_y^*C$ in T_yP_2, so that $R_x + T_x(f^{-1}(C))$ is the pre-image of $R_y + T_yC$ under f_*. Using Remark 4.3 this proves the second assertion. □

The answer to the problem of uniqueness is given by

Theorem 4.8. *[Thm. 4.4 of [6]] Let C be a pre-Poisson submanifold (P, Π), and $\tilde{P}_0$, $\tilde{P}_1$ cosymplectic submanifolds that contain C as a coisotropic submanifold. Then, shrinking $\tilde{P}_0$ and $\tilde{P}_1$ to a smaller tubular neighborhood of C if necessary, there is a Poisson diffeomorphism Φ of P taking $\tilde{P}_0$ to $\tilde{P}_1$ and which is the identity on C.*

Sketch of proof. In a neighborhood U of $\tilde{P}_0$ take a projection $\pi\colon U \to \tilde{P}_0$. Applying Thm. 4.2 one can construct a curve of cosymplectic submanifolds $\tilde{P}_t$ containing C which, at points of C, are all transverse to the fibers of π. Using the cosymplectic submanifolds $\tilde{P}_t$ one can construct a hamiltonian time-dependent vector field X_{H_t} whose time-t flow maps $\tilde{P}_0$ to $\tilde{P}_t$. Further X_{H_t} vanishes on C, hence its time-1 flow is the identity on C. □

5 Duals of Lie algebras

In this subsection $\mathfrak{g}$ will always denote a finite dimensional Lie algebra. We saw in Section 2 that its dual $\mathfrak{g}^*$ is a Poisson manifold, whose Poisson bracket on linear functions (which can be identified with elements of $\mathfrak{g}$) is given by $\{g_1, g_2\} := [g_1, g_2]$. In what follows we will need the notion of adjoint action of G on $\mathfrak{g}$, which is $Ad_g v := \frac{d}{dt}|_0 g \cdot exp(tv) \cdot g^{-1}$. Its derivative at the identity gives the Lie algebra action of $\mathfrak{g}$ on itself by $ad_w v := \frac{d}{dt}|_0 Ad_{exp(tw)} v = [w, v]$. We will also consider the (left) actions Ad^* and ad^* on $\mathfrak{g}^*$ obtained by dualizing; the orbits of the coadjoint action Ad^* are exactly the symplectic leaves of the Poisson manifold $\mathfrak{g}^*$.

It is known that if $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, then its annihilator $\mathfrak{h}^\circ$ is a coisotropic submanifold of $\mathfrak{g}^*$ (also see Prop. 5.1 below). We shall look at two generalizations: the first considers affine subspaces obtained translating $\mathfrak{h}^\circ$; the second is obtained by weakening the condition that $\mathfrak{h}$ be a subalgebra.

Proposition 5.1. *Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$ and fix $\lambda \in \mathfrak{g}^*$. Then the affine subspace $C := \mathfrak{h}^\circ + \lambda$ is always pre-Poisson, and it is coisotropic iff λ is a character of $\mathfrak{h}$ (i.e. by definition $\lambda \in [\mathfrak{h}, \mathfrak{h}]^\circ$).*

Proof. The restriction $f : \mathfrak{g}^* \to \mathfrak{h}^*$ is a Poisson map because $\mathfrak{h}$ is a Lie subalgebra. Every point ν of $\mathfrak{h}^*$ is a pre-Poisson submanifold (see Ex. 4.5), hence by Lemma 4.7 its pre-image $f^{-1}(\nu)$ (which will be a translate of $\mathfrak{h}^\circ$) is pre-Poisson. Notice that by Lemma 4.7 we also know that, for any slice $S \subset \mathfrak{h}^*$ transverse to the H-coadjoint orbit through ν, $f^{-1}(S)$ is a cosymplectic submanifold containing coisotropically $f^{-1}(\nu)$. Further from the proof of Lemma 4.7 it is clear that $f^{-1}(\nu)$ is coisotropic in $\mathfrak{g}^*$ iff $\{\nu\}$ is coisotropic in $\mathfrak{h}^*$, i.e. if ν is a fixed-point of the H-coadjoint action or equivalently $\nu|_{[\mathfrak{h},\mathfrak{h}]} = 0$. □

Example 5.2. Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$. In a suitable basis the Lie algebra structure is given by $[e_1, e_2] = -e_3, [e_2, e_3] = e_1, [e_3, e_1] = e_2$. The symplectic leaves of $\mathfrak{g}^*$ are given essentially by the connected components of level sets of the Casimir function $\nu_1^2 + \nu_2^2 - \nu_3^2$ (where ν_i is just e_i viewed as a linear function on $\mathfrak{g}^*$), and they consist of a family of two-sheeted hyperboloids, the cone[2] $\nu_1^2 + \nu_2^2 - \nu_3^2 = 0$ and a family of one-sheeted hyperboloids [3]. $C := \{(0, t, t) : t \in \mathbb{R}\} \subset \mathfrak{g}^*$ is contained in the cone and is clearly a coisotropic submanifold; indeed it is the annihilator of the Lie subalgebra $\mathfrak{h} := span\{e_1, e_2 - e_3\}$ of $\mathfrak{g}$. If we translate C by an element in the annihilator of $[\mathfrak{h}, \mathfrak{h}] = \mathbb{R}(e_2 - e_3)$ we obtain an affine line contained in one of the hyperboloids, which hence is lagrangian there, therefore coisotropic in $\mathfrak{g}^*$. If we translate C by any other $\lambda \in \mathfrak{g}^*$ we obtain a line that intersects transversely the hyperboloids, so at every point of such a line C' we have $TC' + \sharp N^*C' = T\mathfrak{g}^*$, showing that C' is pre-Poisson.

Before considering the case when $\mathfrak{h}$ is *not* a subalgebra of $\mathfrak{g}$ we need the

Lemma 5.3. *Let $C \subset \mathfrak{g}^*$ be an affine subspace obtained by translating the annihilator of a linear subspace $\mathfrak{h} \subset \mathfrak{g}$. Then $\sharp N_x^*C = ad_{\mathfrak{h}}^*(x) := \{ad_h^*(x) : h \in \mathfrak{h}\}$ for all $x \in C$.*

Proof. N_x^*C is given by the differentials at x of the functions $h \in \mathfrak{h} \subset C^\infty(\mathfrak{g}^*)$. Now for any $g \in \mathfrak{g}$ we have

$$\langle \sharp d_x h, g\rangle = d_x g(\sharp d_x h) = \{h, g\}(x) = \langle [h, g], x\rangle = \langle ad_h^*(x), g\rangle,$$

i.e. $\sharp d_x h = ad_h^*(x)$. □

Remark 5.4. An alternative proof of Prop. 5.1 can be given using Lemma 5.3. Indeed any $x \in C$ can be written uniquely as $y + \lambda$ where $y \in \mathfrak{h}^\circ$. Notice that $ad_h^*(y) \in \mathfrak{h}^\circ$ for all $h \in \mathfrak{h}$, because $\langle ad_h^*(y), \mathfrak{h}\rangle = \langle y, [h, \mathfrak{h}]\rangle$ vanishes since $\mathfrak{h}$ is a subalgebra. Hence

$$T_xC + \sharp N_x^*C = \mathfrak{h}^\circ + \{ad_h^*(y) + ad_h^*(\lambda) : h \in \mathfrak{h}\} = \mathfrak{h}^\circ + ad_{\mathfrak{h}}^*(\lambda),$$

which is independent on the point x. From the first computation above (applied to λ instead of y) it is clear that $ad_{\mathfrak{h}}^*(\lambda) \in \mathfrak{h}^\circ$ iff $\lambda \in [\mathfrak{h}, \mathfrak{h}]^\circ$.

[2]The cone is the union of 3 leaves, one being the origin.

Now we consider the case when $\mathfrak{h}$ is just a linear subspace of $\mathfrak{g}$ and $\mathfrak{h}^\circ \subset \mathfrak{g}^*$ its dual. Since the Poisson tensor of $\mathfrak{g}^*$ vanishes at the origin we have $T(\mathfrak{h}^\circ) + \sharp N^*(\mathfrak{h}^\circ) = T(\mathfrak{h}^\circ)$ at the origin, so $\mathfrak{h}^\circ$ is pre-Poisson iff it is coisotropic (i.e. if $\mathfrak{h}$ is a Lie subalgebra). The open subset C of $\mathfrak{h}$ on which $T(\mathfrak{h}^\circ) + \sharp N^*(\mathfrak{h}^\circ)$ has maximal rank will be pre-Poisson. Then, shrinking C if necessary, we can find a subspace $R \subset \mathfrak{g}^*$ (independent of $x \in C$) with $R \oplus (T_xC + \sharp N_x^*C) = \mathfrak{g}^*$ for all $x \in C$. For example we can construct such an R at one point $\bar{x}$ of C, and since transversality is an open condition, R will be transverse to $TC + \sharp N^*C$ in a neighborhood of $\bar{x}$ in C. By Thm. 4.2 an open subset $\tilde{P}$ of the subspace $\mathfrak{p}^\circ := R \oplus C$ (containing C) is cosymplectic. If we *assume* that $\sharp N_y^* \tilde{P}$ is independent of the point $y \in \tilde{P}$ we are in the situation of the following proposition.

Proposition 5.5. *Let $\mathfrak{p}$ be a linear subspace of $\mathfrak{g}$ such that an open subset $\tilde{P} \subset \mathfrak{p}^\circ$ is cosymplectic and $\mathfrak{k}^\circ := \sharp N_y^* \tilde{P}$ is independent of $y \in \tilde{P}$. Then $\mathfrak{k} \oplus \mathfrak{p} = \mathfrak{g}$, $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$ and $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$. Hence, whenever $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, $(\mathfrak{k}, \mathfrak{p})$ forms a symmetric pair [10].*

Proof. The fact that $\mathfrak{k} \oplus \mathfrak{p} = \mathfrak{g}$ follows from $\mathfrak{k}^\circ \oplus \mathfrak{p}^\circ = \mathfrak{g}^*$, which holds because $\tilde{P}$ is cosymplectic. Recall that given functions f_1, f_2 on $\tilde{P}$, the bracket $\{f_1, f_2\}_{\tilde{P}}$ is obtained by extending the functions in a constant way along $\mathfrak{k}^\circ$ to obtain functions $\hat{f}_1, \hat{f}_2$ on $\mathfrak{g}^*$, taking their Poisson bracket and restricting to $\tilde{P}$. Further (see Cor. 2.11 of [16]) the differential of $\{\hat{f}_1, \hat{f}_2\}$ at any point of $\tilde{P}$ kills $\mathfrak{k}^\circ$. So if the f_i are restrictions of linear functions on $\mathfrak{p}^\circ$ then $\hat{f}_i$ will be linear functions on $\mathfrak{g}^*$ corresponding to elements of $\mathfrak{k}$, and $\{\hat{f}_1, \hat{f}_2\}$, which is a linear function on $\mathfrak{g}^*$, will also correspond to an element of $\mathfrak{k}$. We deduce that $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$ (and that the Poisson structure on $\tilde{P}$ induced from $\mathfrak{g}^*$ is the restriction of a linear Poisson structure on $\mathfrak{p}^\circ$).

To show $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ pick any $k \in \mathfrak{k}, p \in \mathfrak{p}$ and $y \in \tilde{P}$. Then $\langle [k,p], y\rangle = -\langle k, ad_p^*(y)\rangle = \langle k, \sharp d_y p\rangle = 0$, using Lemma 5.3 in the second equality, because $\sharp d_y p \subset \sharp N_y^* \tilde{P} = \mathfrak{k}^\circ$. This shows that $[k,p]$ annihilates $\tilde{P}$, hence it must annihilate its span $\mathfrak{p}^\circ$. □

Remark 5.6. The text preceding Prop. 5.5 and the proposition itself give a way to start with a simple piece of data (a subspace of $\mathfrak{g}$) and, in favorable cases, obtain a decomposition $\mathfrak{k} \oplus \mathfrak{p} = \mathfrak{g}$ where $\mathfrak{k}$ is a Lie subalgebra and $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$. If $\mathfrak{g}$ admits a non-degenerate Ad-invariant bilinear form B, then the B-orthogonal $\mathfrak{p}$ of any subalgebra $\mathfrak{k}$ satisfies $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, because for any $k, k' \in \mathfrak{k}$ and $p \in \mathfrak{p}$ we have $B([k,p], k') = -B(p, [k,k']) = 0$. If B is positive-definite (such a B exists for example if the simply connected Lie group integrating $\mathfrak{g}$ is compact), then we clearly also have $\mathfrak{k} \oplus \mathfrak{p} = \mathfrak{g}$. Hence for such Lie algebras one obtains the kind of decomposition of Prop. 5.5 in a much easier way.

A converse statement to Prop. 5.5 is given by

Proposition 5.7. *Assume that $\mathfrak{k} \oplus \mathfrak{p} = \mathfrak{g}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and there exists a point $y \in \mathfrak{p}^\circ$ at which none of the fundamental vector fields $\frac{d}{dt}|_0 Ad^*_{exp(tp)}(y)$ vanish, where p ranges over $\mathfrak{p} \setminus \{0\}$. Then there is an open subset $\tilde{P} \subset \mathfrak{p}^\circ$ which is cosymplectic and $\mathfrak{k}^\circ := \sharp N_x^* \tilde{P}$ is independent of $x \in \tilde{P}$. (Hence applying Prop. 5.5 it follows that $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$).*

Proof. For all $x \in \mathfrak{p}^\circ$ we have $\sharp N_x^*(\mathfrak{p}^\circ) = ad_{\mathfrak{p}}^*(x) \subset \mathfrak{k}^\circ$, as can be seen using $\langle ad_p^*(x), \mathfrak{k}\rangle = \langle x, [p, \mathfrak{k}]\rangle = 0$ for all $p \in \mathfrak{p}$ (which holds because of $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$). The assumption on the coadjoint action at y means that the map $\mathfrak{p} \to \mathfrak{g}^*, p \mapsto ad_p^*(y)$ is injective; by continuity it is injective also on an open subset $\tilde{P} \subset \mathfrak{p}^\circ$, and by dimension counting we get $\sharp N_x^*(\mathfrak{p}^\circ) = \mathfrak{k}^\circ$ on $\tilde{P}$. $\square$

Now we display an example for Prop. 5.5

Example 5.8. Let $\mathfrak{g} = \mathfrak{gl}(2, \mathbb{R})$. We identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via the non-degenerate (indefinite) inner product $(A, B) = Tr(A \cdot B)$. Since it is Ad-invariant, the action of ad_X and ad_X^* on $\mathfrak{g}$ and $\mathfrak{g}^*$ are intertwined (up to sign).

Now take $\mathfrak{h} = \left\{ \left(\begin{smallmatrix} 0 & b \\ c & d \end{smallmatrix}\right) : b, c, d \in \mathbb{R} \right\}$, which is not a subalgebra. Its annihilator is identified with the line C spanned by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Since C is one-dimensional and the Poisson structure on $\mathfrak{g}^*$ linear it is clear that $\sharp N_x^* C$ is independent of $x \in C \backslash \{0\}$ and $C \backslash \{0\}$ is pre-Poisson. Using Lemma 5.3 we compute $\sharp N_x^* C = \left\{ \left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right) : b, c \in \mathbb{R} \right\}$, so as complement R to $T_x C + \sharp N_x^* C$ we can take the line spanned by $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then $\mathfrak{p}^\circ := R \oplus C$ is given by the diagonal matrices, and $\mathfrak{p} \subset \mathfrak{g}$ is given by matrices with only zeros on the diagonal. For any $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \in \mathfrak{p}^\circ$ we compute $\sharp N^*_{\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)} \mathfrak{p}^\circ$ using Lemma 5.3 and obtain the set of matrices with only zeros on the diagonal if $a \neq d$ and $\{0\}$ otherwise. So the open set $\tilde{P}$ on which $\mathfrak{p}^\circ$ is cosymplectic is a plane with a line removed, and $\mathfrak{k}^\circ := \sharp N^*_{\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)} \tilde{P}$ is independent of the footpoint $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \in \tilde{P}$. $\mathfrak{k} \subset \mathfrak{g}$ coincides hence with the set of diagonal matrices. As predicted by Lemma 5.5 $\mathfrak{k}$ is a Lie subalgebra and $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$; one can check easily that $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ too.

Since $\mathfrak{k}$ is abelian, the linear Poisson structure induced on $\tilde{P}$ is the zero Poisson structure. This can be seen also looking at the explicit Poisson structure on $\mathfrak{g}^*$, which with respect to the coordinates given by the basis $a = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $b = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $c = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ and $d = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ of $\mathfrak{g}^*$ is

$$-b\partial_a \wedge \partial_b + c\partial_a \wedge \partial_c + (d-a)\partial_b \wedge \partial_c - b\partial_b \wedge \partial_d + c\partial_c \wedge \partial_d.$$

Indeed at a point $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$ of $\mathfrak{p}^\circ$ the bivector field reduces to $(d-a)\partial_b \wedge \partial_c$. Finally remark that if we had chosen R to be spanned by $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ instead we would have obtained as $\sharp N^*_{\left(\begin{smallmatrix} a & b \\ 0 & b \end{smallmatrix}\right)} \mathfrak{p}^\circ$ the span of $\left(\begin{smallmatrix} -b & b \\ a-b & b \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & b-a \\ 0 & 0 \end{smallmatrix}\right)$, which obviously is not constant on any open subset of $\mathfrak{p}^\circ$

6 Subgroupoids associated to pre-Poisson submanifolds

In Section 2 we defined Lie algebroids and recalled that for every Poisson manifold P there is an associated Lie algebroid, namely the cotangent bundle T^*P.

In analogy to the fact that finite dimensional Lie algebras integrate to Lie groups (uniquely if required to be simply connected), Lie algebroids - when integrable - integrate to objects called *Lie groupoids*. Recall that a Lie groupoid over P is given by a manifold Γ endowed with surjective submersions $\mathbf{s}$,$\mathbf{t}$ (called source and target) to the base manifold P, a smooth associative multiplication defined on elements $g, h \in \Gamma$ satisfying $\mathbf{s}(g) = \mathbf{t}(h)$, an embedding of P into Γ as the spaces of "identities" and a smooth inversion map $\Gamma \to \Gamma$; see for example [14] for the precise definition. The total space of the Lie algebroid associated to the Lie groupoid Γ is $ker(\mathbf{t}_*|_P) \subset T\Gamma|_P$, with a bracket on sections defined using left invariant vector fields on Γ and $\mathbf{s}_*|_P$ as anchor. A Lie algebroid A is said to be integrable if there exists a Lie groupoid whose associated Lie algebroid is isomorphic to A; in this case there is a unique (up to isomorphism) integrating Lie groupoid with simply connected source fibers.

The cotangent bundle T^*P of a Poisson manifold P carries more data then just a Lie algebroid structure; when it is integrable, the corresponding Lie groupoid Γ is actually a *symplectic groupoid* [12], i.e. [14] there is a symplectic form Ω on Γ such that the graph of the multiplication is a lagrangian submanifold of $(\Gamma\times\Gamma\times\Gamma, \Omega\times\Omega\times(-\Omega))$. Ω is uniquely determined (up to symplectic groupoid automorphism) by the requirement that $\mathbf{t}\colon \Gamma\to P$ be a Poisson map; a canonical Lie algebroid isomorphism between T^*P and $ker(\mathbf{t}_*|_P)$ is then given by mapping du (for u a function on P) to the hamiltonian vector field $-X_{\mathbf{s}^*u}$. For example, if P carries the zero Poisson structure, then the symplectic groupoid is T^*P with the canonical symplectic structure and fiberwise addition as multiplication. We will describe in Example 6.2 below the symplectic groupoid of the dual of a Lie algebra.

In this Section we want to investigate how a pre-Poisson submanifold C of a Poisson manifold (P,Π) gives rise to subgroupoids of the source simply connected symplectic groupoid Γ (assuming that T^*P is an integrable Lie algebroid). By Prop. 3.6 of [6] $N^*C\cap\sharp^{-1}TC$ is a Lie subalgebroid of T^*P. When $\sharp N^*C$ has constant rank there is another Lie subalgebroid associated to C, namely $\sharp^{-1}TC=(\sharp N^*C)^\circ$. We want to describe the subgroupoids[3] of Γ integrating $N^*C\cap\sharp^{-1}TC$ and $\sharp^{-1}TC$.

Proposition 6.1. *[Prop. 7.2 of [6]] Let C be a pre-Poisson submanifold of (P,Π). Then the subgroupoid of (Γ,Ω) integrating $N^*C\cap\sharp^{-1}TC$ is an isotropic subgroupoid.*

We exemplify Prop. 6.1 considering the dual of a Lie algebra $\mathfrak{g}$ as a Poisson manifold, as in Section 5. The symplectic groupoid of $\mathfrak{g}^*$ (see Ex. 3.1 of [14]) is T^*G with its canonical symplectic form, where G is the simply connected Lie group integrating $\mathfrak{g}$. To describe the groupoid structure we identify T^*G with $\mathfrak{g}^*\times G$ by (the cotangent lift of) right translation. Then the target map $\mathfrak{g}^*\times G\to\mathfrak{g}^*$ is $\mathbf{t}(\xi,g)=\xi$ and the source map is $\mathbf{s}(\xi,g)=Ad^*_{g^{-1}}\xi$, and the multiplication is $(\xi,g_1)\cdot(Ad^*_{g^{-1}}\xi,g_2)=(\xi,g_1g_2)$.

Example 6.2. Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$ and $\lambda\in\mathfrak{g}^*$. By Prop. 5.1 we know that $C:=\mathfrak{h}^\circ+\lambda$ is a pre-Poisson submanifold of $\mathfrak{g}^*$. We claim here that the subgroupoid of $\mathfrak{g}^*\times G$ integrating the Lie subalgebroid $N^*C\cap\sharp^{-1}TC$ is $C\times D$, where the subgroup $D\subset G$ is the connected component of the identity of $\{g\in H : (Ad^*_g\lambda)|_{\mathfrak{h}}=\lambda|_{\mathfrak{h}}\}$. By Prop. 6.1 we know that it is an isotropic subgroupoid.

To prove our claim, we first make the Lie subalgebroid more explicit: for all $x\in C$ using Remark 5.4 we have

$$N^*_xC\cap\sharp^{-1}T_xC=(\mathfrak{h}^\circ+ad^*_{\mathfrak{h}}(\lambda))^\circ=\mathfrak{h}\cap\{v\in\mathfrak{g}:(ad^*_v\lambda)|_{\mathfrak{h}}=0\}=:\mathfrak{d},$$

so that the Lie subalgebroid $N^*C\cap\sharp^{-1}TC\subset T^*\mathfrak{g}^*=\mathfrak{g}^*\times\mathfrak{g}$ is just the product $C\times\mathfrak{d}$. The canonical Lie algebroid isomorphism $T^*P\cong ker(\mathbf{t}_*|_P), du\mapsto -X_{\mathbf{s}^*u}$ is just the identity on $\mathfrak{g}^*\times\mathfrak{g}$, as can be checked using the explicit formula for the symplectic form on the groupoid $\mathfrak{g}^*\times G$ given in Ex. 3.1 of [14]. Now notice that the Lie subalgebra $\mathfrak{d}$ integrates to the connected subgroup D defined above. Using the definition of D one checks that $\mathbf{t}$ and $\mathbf{s}$ map $C\times D$ into C, and the fact that D is a subgroup allows us to check that $C\times D$ is actually a Lie subgroupoid of $\mathfrak{g}^*\times G$, proving our claim.

[3]Here, for any Lie subalgebroid A of T^*P integrating to a source simply connected Lie groupoid H, we take "subgroupoid" to mean the (usually just immersed) image of the (usually not injective) morphism $H\to\Gamma$ induced from the inclusion $A\to T^*P$.

Now we consider $\sharp^{-1}TC$ and assume that it has constant rank, or equivalently that the characteristic distribution $TC \cap \sharp N^*C$ have constant rank[4]. Then $\sharp^{-1}TC$ is a Lie subalgebroid of T^*P, and quoting part of Prop. 7.2 of [6]:

Proposition 6.3. *The subgroupoid of* Γ *integrating* $\sharp^{-1}TC$ *is* $\mathbf{s}^{-1}(C) \cap \mathbf{t}^{-1}(C)$*, and it is a presymplectic submanifold of* (Γ, Ω).

Remark 6.4. In this case the foliation integrating the characteristic distribution of $\mathbf{s}^{-1}(C) \cap \mathbf{t}^{-1}(C)$ (i.e. the kernel of the pullback of Ω) is given by the orbits of the action by right and left multiplication of the source-connected isotropic subgroupoid integrating $N^*C \cap \sharp^{-1}TC$.

Example 6.5. Let C be a submanifold of $\mathfrak{g}^*$ such that $T_xC \cap T_x\mathcal{O} = \{0\}$ at every point x where C intersects a coadjoint orbit $\mathcal{O}$. Then C is pre-Poisson iff $\sharp^{-1}TC$ has constant rank, which in this case just means that the coadjoint orbits that C intersects all have the same dimension. By the above proposition the source connected subgroupoid of $\mathfrak{g}^* \times G$ integrating $\sharp^{-1}TC$ is $\{(x,g) : x \in C, Ad^*_{g^{-1}}(x) = x\}$, a bundle of groups integrating a bundle of isotropy Lie algebras of the coadjoint action. We also have the following alternative description for this bundle of Lie algebras, which sometimes is more convenient for computations: $\sharp^{-1}T_xC = (\sharp N_x^*C)^\circ$ can be described as $N_x^*\mathcal{O}$, for $\mathcal{O}$ the coadjoint orbit through x.

If $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$ and $\lambda \in \mathfrak{g}^*$, we know that $C := \mathfrak{h}^\circ + \lambda$ is a pre-Poisson submanifold of $\mathfrak{g}^*$, but generally $\sharp^{-1}TC$ does not have constant rank. A case where it has a constant rank is the following. As in Example 5.2 consider $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$ and the pre-Poisson submanifold $C := \{(0,t,t+1) : t \in \mathbb{R}\}$. As remarked there C intersects transversely the symplectic leaves of $\mathfrak{g}^*$, which are the level sets of the Casimir function $\nu_1^2 + \nu_2^2 - \nu_3^2$. At $x = (0,t,t+1)$ we have $N_x^*\mathcal{O} = \mathbb{R}(td\nu_2 - (t+1)d\nu_3)$, which in terms of the basis $e_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $e_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $e_3 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ of $\mathfrak{sl}(2,\mathbb{R})$ used in Example 5.2 is $\mathbb{R}\begin{pmatrix} t & -(t+1) \\ t+1 & -t \end{pmatrix}$. As seen above, integrating these Lie algebras to subgroups of G (the simply connected Lie group integrating $\mathfrak{sl}(2,\mathbb{R})$) we obtain the source connected subgroupoid of $\mathfrak{g}^* \times G$ integrating $\sharp^{-1}TC$.

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[4] Indeed more generally we have the following for any submanifold C of P: if any two of $\sharp^{-1}TC$, $\sharp N^*C + TC$ or $TC \cap \sharp N^*C$ have constant rank, then the remaining one also has constant rank. This follows trivially from $rk(\sharp N^*C + TC) = rk(\sharp N^*C) + \dim C - rk(TC \cap \sharp N^*C)$.

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Reduction of branes in generalized complex geometry

Marco Zambon

Abstract

We show that certain submanifolds of generalized complex manifolds ("weak branes") admit a natural quotient which inherits a generalized complex structure. This is analog to quotienting coisotropic submanifolds of symplectic manifolds. In particular Gualtieri's generalized complex submanifolds ("branes") quotient to space-filling branes. Along the way we perform reductions by foliations (i.e. no group action is involved) for exact Courant algebroids - interpreting the reduced Ševera class - and for Dirac structures.

Contents

1 Introduction

Consider the following setup in ordinary geometry: a manifold M and a submanifold C endowed with some integrable distribution $\mathcal{F}$ so that $\underline{C} := C/\mathcal{F}$ be smooth. Then we have a projection $pr : C \to \underline{C}$ which induces a vector bundle morphism $pr_* : TC \to T\underline{C}$. If M is endowed with some geometric structure, such as a symplectic 2-form ω, one can ask when ω induces a symplectic form on $\underline{C}$.

This happens for example when C is a coisotropic submanifold[1]. Indeed in this case the pullback $i^*\omega$ of ω to C has a kernel $\mathcal{F}$ which is of constant rank and integrable, and the closeness of ω ensures that if p and q lie in the same $\mathcal{F}$-leaf then $(i^*\omega)_p$ and $(i^*\omega)_q$ project to the same linear symplectic form at $pr(p) = pr(q)$, so that one obtains a well-defined symplectic form on $\underline{C}$. An instance of the above is when there is a Lie group G acting hamiltonianly on M with moment map $\nu : M \to \mathfrak{g}^*$ and C is the zero level set of ν (Marsden-Weinstein reduction [17]).

In this paper we consider the geometry that arises when one replaces the tangent bundle TM with an exact Courant algebroid E over M (any such E is non-canonically isomorphic to $TM \oplus T^*M$). In this context reduction by the action of a Lie group has been considered by several authors (Burstzyn-Cavalcanti-Gualtieri [3], Hu [11, 12], Stienon-Xu [19], Tolman-Lin [15, 16]); in this paper we do not assume any group action. Unlike the tangent bundle case, knowing $\underline{C}$ does not automatically determine the exact Courant algebroid over it. We have to replace the foliation $\mathcal{F}$ by more data, namely a suitable subbundle K of $E|_C$ (projecting to $\mathcal{F}$ under the anchor map $\pi : E \to TM$); we determine conditions on K that allow to construct by a quotienting procedure a Courant algebroid $\underline{E}$ on $\underline{C}$ (Theorem 3.7) endowed with a morphism from E to $\underline{E}$ (Remark 3.9). Our construction follows closely the one of Bursztyn-Cavalcanti-Gualtieri [3], in which a suitable group action on E is assumed. In [3] the group action provides an identification between fibers of E at different points; in our case we make up for this asking that there exist enough "basic sections" (Def. 3.3). We also describe how a submanifold C with a foliation $\mathcal{F}$, once equipped with a suitable maximal isotropic subbundle L of $E|_C$, naturally has a reduced Courant algebroid over its leaf-space $\underline{C}$ (see Prop. 3.14). We describe in a simple way (see Def. 3.11) which splittings of E induce 3-forms on M (representing the Ševera class of E) which descend to 3-forms on $\underline{C}$ (representing the Ševera class of $\underline{E}$). Finally, in the case when the exact Courant algebroid E is split, we give an explicit and simple description of the reduction procedure of Thm. 3.7 in terms of differential forms (Prop. 3.18).

Once we know how to reduce an exact Courant algebroid, we can ask when geometric structures defined on them descend to the reduced exact Courant algebroid. We consider Dirac structures (suitable subbundles of E) and generalized complex structures (suitable endomorphisms of E). We give sufficient conditions for these structures to descend in Prop. 4.1 and Prop. 5.1 respectively. The ideas and techniques are borrowed the literature cited above, in particular from [3] and [19] (however our proof differs from these two references in that we reduce generalized complex structures directly and not viewing them as Dirac structures in the complexification of E).

The heart of this paper is Section 6, where we identify the objects that automatically satisfy the assumptions needed to perform generalized complex reduction. When M is a generalized complex manifold we consider pairs consisting of a submanifold C of M and suitable maximal isotropic subbundle L of $E|_C$ (we call them "weak branes" in Def. 6.9). We show in Prop. 6.10 that weak branes admit a canonical quotient $\underline{C}$ which is endowed with an exact Courant algebroid and a generalized complex structure; this construction is inspired by Thm. 2.1 of Vaisman's work [21] in the setting of the standard Courant algebroid.

[1] This means that the symplectic orthogonal of TC is contained in TC.

Particular cases of weak branes are generalized complex submanifolds (C, L) (also known as "branes", see Def. 6.3), which were first introduced by Gualtieri [9] and are relevant to physics [14]. Using our reduction of Dirac structures we show in Thm. 6.4 that the quotients $\underline{C}$ of branes, which by the above are generalized complex manifolds, are also endowed with the structure of a space-filling brane (i.e. $\underline{C}$ together with the reduction of L forms a brane). This is interesting also because space filling branes induce an honest complex structure on the underlying manifold [8].

The reduction statements we had to develop in order to prove the results of Section 6 are versions "without group action" of statements that already appeared in the literature [3][11, 12] [2, 19] [15, 16] [21]. Consequently many ideas and techniques are borrowed from the existing literature; we make appropriate references in the text whenever possible. In particular we followed closely [3] (also as far as notation and conventions are concerned).

Plan of the paper: in Section 2 we review exact Courant algebroids, mainly following [3]. In Section 3 we perform the reduction of exact Courant algebroids, determine objects that naturally satisfy the assumptions needed for the reduction, and comment on the reduced Ševera class. In Section 4 we perform the reduction of Dirac structures, and present as an example the coisotropic reduction in Poisson manifolds. In Section 5 we reduce generalized complex structures and comment briefly on generalized Kähler reduction. The main section of this paper is Section 6: we reduce branes and weak branes, providing few examples. We also give a criteria that allows to obtain weak branes by restricting to cosymplectic submanifolds.

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2 Review of Courant algebroids

We review the notion of exact Courant algebroid; see [3] and [11] for more details.

Definition 2.1. A *Courant algebroid* over a manifold M is a vector bundle $E \to M$ equipped with a fibrewise non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$, a bilinear bracket $[\cdot, \cdot]$ on the smooth sections $\Gamma(E)$, and a bundle map $\pi : E \to TM$ called the *anchor*, which satisfy the following conditions for all $e_1, e_2, e_3 \in \Gamma(E)$ and $f \in C^\infty(M)$:

C1) $[e_1, [e_2, e_3]] = [[e_1, e_2], e_3] + [e_2, [e_1, e_3]]$, C2) $\pi([e_1, e_2]) = [\pi(e_1), \pi(e_2)]$,

C3) $[e_1, fe_2] = f[e_1, e_2] + (\pi(e_1)f)e_2$,

C4) $\pi(e_1)\langle e_2, e_3\rangle = \langle [e_1, e_2], e_3\rangle + \langle e_2, [e_1, e_3]\rangle$,

C5) $[e_1, e_1] = \mathcal{D}\langle e_1, e_1\rangle$,

where $\mathcal{D} = \frac{1}{2}\pi^* \circ d : C^\infty(M) \to \Gamma(E)$ (using $\langle\cdot,\cdot\rangle$ to identify E with E^*).

We see from axiom $C5$) that the bracket is not skew-symmetric:

$$[e_1, e_2] = -[e_2, e_1] + 2\mathcal{D}\langle e_1, e_2\rangle.$$

Hence we have the following "Leibniz rule for the first entry": $[fe_1, e_2] = f[e_1, e_2] - (\pi(e_2)f)e_1 + 2\langle e_1, e_2\rangle\mathcal{D}f$.

Definition 2.2. A Courant algebroid is *exact* if the following sequence is exact:

$$0 \longrightarrow T^*M \xrightarrow{\pi^*} E \xrightarrow{\pi} TM \longrightarrow 0 \tag{1}$$

To simplify the notation, in the sequel we will often omit the map $T^*M \overset{\pi^*}{\to} E^* \cong E$ and think of T^*M as being a subbundle of E. Given an exact Courant algebroid, we may always choose a right splitting $\sigma : TM \to E$ whose image in E is isotropic with respect to $\langle\cdot,\cdot\rangle$. Such a splitting induces the closed 3-form on M given by

$$H(X, Y, Z) = 2\langle[\sigma X, \sigma Y], \sigma Z\rangle.$$

Using the bundle isomorphism $\nabla + \frac{1}{2}\pi^* : TM \oplus T^*M \to E$, one can transport the Courant algebroid structure onto $TM \oplus T^*M$. The resulting structure is as follows (where $X_i + \xi_i \in \Gamma(TM \oplus TM^*)$): the bilinear pairing is

$$\langle X_1 + \xi_1, X_2 + \xi_2\rangle = \frac{1}{2}(\xi_2(X_1) + \xi_1(X_2)), \tag{2}$$

and the bracket is

$$[X_1 + \xi_1, X_2 + \xi_2]_H = [X_1, X_2] + \mathcal{L}_{X_1}\xi_2 - i_{X_2}d\xi_1 + i_{X_2}i_{X_1}H, \tag{3}$$

which is the H-twisted Courant bracket on $TM \oplus T^*M$ [18]. Isotropic splittings of (1) differ by 2-forms $b \in \Omega^2(M)$, and a change of splitting modifies the curvature H by the exact form db. Hence there is a well-defined cohomology class $[H] \in H^3(M, \mathbb{R})$ attached to the exact Courant algebroid structure on E; $[H]$ is called the *Ševera class* of E.

We refer to [3] and [11] for information on the group of automorphisms $Aut(E)$ and its Lie algebra $Der(E)$. Here we just mention few facts, the first of which underlies many of our constructions: for any $e \in \Gamma(E)$, $[e, \cdot]$ is an element of $Der(E)$ and hence integrates to an automorphism of the Courant algebroid E. Notice that for closed 1-forms ξ (seen as sections of $T^*M \subset E$) we have $[\xi, \cdot] = 0$ by (3). Further, any 2-form B on M determines a vector bundle map $TM \oplus TM^* \to TM \oplus TM^*$ by $e^B : X + \xi \mapsto X + \xi + i_X B$ [9] and these "gauge transformations" satisfy

$$[e^B\ \cdot, e^B\ \cdot]_H = e^B[\cdot,\cdot]_{H+dB}. \tag{4}$$

3 The case of exact Courant algebroids

In this section we reduce exact Courant algebroids (see Thm. 3.7 and Prop. 3.18), display objects whose quotient is naturally endowed with a reduced exact Courant algebroid, and then comment on the relation between a Courant algebroid and its reduction, as well as on the relation between the Ševera classes.

3.1 Reducing exact Courant algebroids

Let M be a manifold and E an exact Courant algebroid over M. We fix a submanifold C.

Lemma 3.1. *Let $D \to C$ be a subbundle of E such that $\pi(D^\perp) \subset TC$ (where $D^\perp$ denotes the orthogonal to D w.r.t. the symmetric pairing), and e_1, e_2 sections of $D^\perp$. Then the expression $[\tilde{e}_1, \tilde{e}_2]|_C$, where $\tilde{e}_i$ are extensions of e_i to sections of $E \to M$, depends on the extensions only up to sections of D.*

Proof. Fix extensions $\tilde{e}_i$ of e_i ($i = 1, 2$). We have to show that for functions f_i vanishing on C and sections $\hat{e}_i$ of E we have $[\tilde{e}_1 + f_1\hat{e}_1, \tilde{e}_2 + f_2\hat{e}_2]|_C = [\tilde{e}_1, \tilde{e}_2]|_C$ up to sections of D. By the Leibniz rule $C3$) and since $\pi(e_1) \subset TC$ we have $[\tilde{e}_1, f_2\hat{e}_2]|_C = 0$. Also $[f_1\hat{e}_1, \tilde{e}_2]|_C = 2\langle \hat{e}_1, \tilde{e}_2 \rangle (\mathcal{D} f_1)|_C \subset N^*C \subset (\pi(D^\perp))^\circ = D \cap T^*M$ [2]. The term $[f_1\hat{e}_1, f_2\hat{e}_2]|_C$ vanishes by the above since $(f_1\hat{e}_1)|_C$ is a section of D. □

Remark 3.2. If $D \to C$ is a subbundle of E such that $\pi(D^\perp) \subset TC$ we can make sense of a statement like "$[e_1, e_2] \in \Gamma(D)$" for $e_1, e_2 \in \Gamma(D^\perp)$: it means that $[\tilde{e}_1, \tilde{e}_2]|_C \in \Gamma(D)$ for one (or equivalently, by Lemma 3.1, for all) extensions $\tilde{e}_i$ to sections of $E \to M$. Similarly, we take $[\Gamma(D^\perp), \Gamma(D^\perp)] \subset \Gamma(D)$ to mean $[e_1, e_2] \in \Gamma(D)$ for all $e_1, e_2 \in \Gamma(D^\perp)$.

Now fix an *isotropic* subbundle $K \to C$ of E, i.e. $K \subset K^\perp$, such that $\pi(K_x^\perp) = T_xC$ at each $x \in C$.

Definition 3.3. We define the space of sections of $K^\perp$ which are *basic w.r.t.* K as

$$\Gamma_{bas}(K^\perp) := \{e \in \Gamma(K^\perp) : [\Gamma(K), e] \subset \Gamma(K)\}. \tag{5}$$

Remark 3.4. To ensure that a section e of $K^\perp$ be basic it suffices to consider locally defined sections of K that span K point-wise. That is, it suffices to show that for every point of C there is a neighborhood $U \subset C$ and a subset $S \subset \Gamma(K|_U)$ with $span\{k_p : k \in S\} = K_p$ (for every $p \in U$) so that $[S, e|_U] \subset \Gamma(K|_U)$. Indeed from the "Leibniz rule in the first entry" it follows that $[\Gamma(K), e] \subset \Gamma(K)$.

Lemma 3.5. *Assume that the sections of $\Gamma_{bas}(K^\perp)$ span $K^\perp$ at every point, i.e. that $span\{e_p : e \in \Gamma_{bas}(K^\perp)\} = K_p^\perp$ for every $p \in C$. Then*

1) $[\Gamma(K), \Gamma(K^\perp)] \subset \Gamma(K^\perp)$

2) $[\Gamma(K), \Gamma(K)] \subset \Gamma(K)$.

[2] Indeed for any subspace D of a vector space $T \oplus T^*$, denoting by π the projection onto T, we have $D \cap T^* = (\pi(D^\perp))^\circ$. This follows from $(D \cap T^*)^\perp = D^\perp + T^* = \pi^{-1}(\pi(D^\perp))$.

Proof. Fix a subset of sections $\{e_i\} \subset \Gamma_{bas}(K^\perp)$ that spans point-wise $K^\perp$. For any section k of K and functions f_i (so that the sum $\sum f_i e_i$ is locally finite) by the Leibniz rule we have $[k, \sum f_i e_i] \subset K^\perp$, proving 1). Now 1) is equivalent to 2), as can be seen using axiom C4) in the definition of Courant algebroid: let k_1, k_2 be sections of K and e a section of $K^\perp$. Then $\langle [k_1, e], k_2\rangle + \langle e, [k_1, k_2]\rangle = \pi(k_1)\langle e, k_2\rangle = 0$ because $\pi(K) \subset \pi(K^\perp) = TC$. □

Remark 3.6. A converse to Lemma 3.5 for local sections is given in [4].

The proof of the following theorem is modeled on Thm. 3.3 of [3].

Theorem 3.7 (Exact Courant algebroid reduction). *Let E be an exact Courant algebroid over M, C a submanifold of M, and K an isotropic subbundle of E over C such that $\pi(K^\perp) = TC$. Assume that the space of (global) sections $\Gamma_{bas}(K^\perp)$ spans point-wise $K^\perp$ (i.e. that $span\{e_p : e \in \Gamma_{bas}(K^\perp)\} = K_p^\perp$ for every $p \in C$) and that the quotient $\underline{C}$ of C by the foliation integrating $\pi(K)$ be a smooth manifold. Then there is an exact Courant algebroid $\underline{E}$ over $\underline{C}$ that fits in the following pullback diagram of vector bundles:*

$$\begin{tikzcd} K^\perp/K \arrow[r] \arrow[d] & \underline{E} \arrow[d] \\ C \arrow[r] & \underline{C} \end{tikzcd}$$

Proof. Notice that since $\pi(K)$ has constant rank iff $\pi(D^\perp)$ does (use the previous footnote or eq. (2.17) of [20]) it follows that $\pi(K)$ is a regular distribution on C. Further, by the assumption on basic sections and item 2) of Lemma 3.5, $\pi(K)$ is an integrable distribution, so there exists a regular foliation integrating $\pi(K)$. We divide the proof in 3 steps.

Step 1 To describe the vector bundle $\underline{E}$ we have to explain how we identify fibers of $K^\perp/K$ over two points p, q lying in the same leaf F of $\pi(K)$. We do this as follows: we identify two elements $\hat{e}(p) \in (K^\perp/K)_p$ and $\hat{e}(q) \in (K^\perp/K)_q$ iff there is a section $e \in \Gamma_{bas}(K^\perp)$ which under the projection $K^\perp \to K^\perp/K$ maps[3] to $\hat{e}(p)$ at p and $\hat{e}(q)$ at q. To show that this procedure gives a well-defined identification of $(K^\perp/K)_p$ and $(K^\perp/K)_q$, we need to show that if e_1 and e_2 are sections of $\Gamma_{bas}(K^\perp)$ such that $e_1(p)$ and $e_2(p)$ map to $\hat{e}(p)$, then $e_1(q)$ and $e_2(q)$ map to the same element of $(K^\perp/K)_q$.

Pick a finite sequence of local sections $k_1, \dots, k_n$ of K that join p to q, i.e. such that following successively the vector fields $\pi(k_i)$ for times t_i the point p is mapped to q. Extend each k_i to a section $\tilde{k}_i$ of E. Denote by $e^{ad_{\tilde{k}_i}}$ the Courant algebroid automorphism of E obtained integrating $ad_{\tilde{k}_i} = [\tilde{k}_i, \cdot]$, and by Φ the composition $e^{ad_{t_n \tilde{k}_n}} \circ \dots \circ e^{ad_{t_1 \tilde{k}_1}}$. Since e_1 is a basic section we have $[k_i, e_1] \subset K$ for all i. So $\Phi(e_1(p)) - e_1(q) \in K_q$, and similarly for e_2. Now $e_1(p) - e_2(p) \in K_p$ by assumption, so because of item 2) of Lemma 3.5 we have $\Phi(e_1(p) - e_2(p)) \in K_q$. We deduce that $e_1(q) - e_2(q)$ also belong to K_q and therefore project to the zero vector in $(K^\perp/K)_q$.

It is clear that $\underline{E}$, obtained from $K^\perp/K$ by identifying the fibers over each leaf of $\pi(K)$ as above, is endowed with a projection $\underline{pr}$ onto $\underline{C}$ (induced from the projection $pr : K^\perp/K \to C$). $\underline{E}$ is indeed a smooth vector bundle: given any point $\underline{p}$ of $\underline{C}$ choose a preimage $p \in C$

[3]In other words, we give a canonical trivialization of $(K^\perp/K)|_F$ by projecting into it a frame for $K^\perp|_F$ consisting of basic sections; by assumptions we have enough basic sections to really get a frame for $(K^\perp/K)|_F$.

and a submanifold $S \subset C$ through p transverse to the leaves of $\pi(K)$. S provides a chart around $\underline{p}$ for the manifold $\underline{C}$, and $pr^{-1}(S)$ is a vector subbundle of $K^{\perp}/K$ proving a chart for $\underline{E}$ around $\underline{p}$.

Notice that pulling back by the vector bundle epimorphism $K^{\perp}/K \to \underline{E}$ we can embed the space of sections of $\underline{E}$ into the space of sections of $K^{\perp}/K$, the image being the image of $\Gamma_{bas}(K^{\perp})$ under the map $K^{\perp} \to K^{\perp}/K$. In other words, we have a canonical identification $\Gamma(\underline{E}) \cong \Gamma_{bas}(K^{\perp})/\Gamma(K)$.

Step 2 The pairing $\langle\cdot,\cdot\rangle$ on the fibers of E induces a symmetric bilinear form on each fiber of $K^{\perp}/K$, which is moreover non-degenerate because it is obtained by "odd linear symplectic reduction". This pairing descends to $\underline{E}$, because for any two given sections $e_1, e_2 \in \Gamma_{bas}(K^{\perp})$ the expression $\langle e_1, e_2\rangle$ is constant along each leaf of $\pi(K)$: for any section k of K we have $\pi(k)\langle e_1, e_2\rangle = 0$ using C4).

For the bracket of sections of $\underline{E}$, first notice that $\Gamma_{bas}(K^{\perp})$ is closed (in the sense of Remark 3.2) under the bracket $[\cdot,\cdot]$ of E: if $e_1, e_2 \in \Gamma_{bas}(K^{\perp})$, then for any section k of K we have by C4) $\langle [e_1, e_2], k\rangle = -\langle e_2, [e_1, k]\rangle + \pi(e_1)\langle e_2, k\rangle$. This vanishes since e_1 is basic and $\pi(e_1)$ is tangent to C, so $[e_1, e_2]$ is a section of $K^{\perp}$. Further it is basic again by the "Jacobi identity" C1): for any section k of K we have $[k, [e_1, e_2]] = [[k, e_1], e_2] + [e_1, [k, e_2]]$. Now by definition of basic section each $[k, e_i]$ lies in K, and applying once more the definition of basic section[4] we see that $[k, [e_1, e_2]] \subset K$, i.e. that $[e_1, e_2]$ is basic. In the light of Lemma 3.1, what we really have a well-defined bilinear form $\Gamma_{bas}(K^{\perp}) \times \Gamma_{bas}(K^{\perp}) \to \Gamma_{bas}(K^{\perp})/\Gamma(K)$. Using the definition of basic section we then have an induced bracket on $\Gamma_{bas}(K^{\perp})/\Gamma(K)$, which as we saw is canonically isomorphic to $\Gamma(\underline{E})$.

We define the anchor $\underline{\pi} : \underline{E} \to T\underline{C}$ to make the following diagram of vector bundle morphisms commute:

$$\begin{tikzcd} K^{\perp} \arrow[r] \arrow[d, "\pi"] & K^{\perp}/K \arrow[r] & \underline{E} \arrow[d, "\underline{\pi}"] \\ TC \arrow[rr] & & T\underline{C} \end{tikzcd} .$$

To show that $\underline{\pi}$ is well-defined we choose an element v of $\underline{E}_{\underline{p}}$ and view it as a section $\tilde{e}$ of $(K^{\perp}/K)|_F$, where F is the leaf of $\pi(K)$ corresponding to $\underline{p}$. We define $\underline{\pi}(v)$ as $\pi(\tilde{e}_p) \in T_pC/\pi(K_p) \cong T_{\underline{p}}\underline{C}$, for $p \in F$ and abusing notation by calling π the map $(K^{\perp}/K)_p \to T_pC/\pi(K_p)$. We have to show that the above definition is independent of the point $p \in F$: take any basic section $e \in \Gamma_{bas}(K^{\perp})$ defined near F and mapping to $\tilde{e}$ under $K^{\perp} \to K^{\perp}/K$. We have to show that $\pi(e)$ is a projectable vector field; this is the case since for any vector field Y on C tangent to the leaves of $\pi(K)$ we can write $Y = \pi(k)$ for a smooth section of K, and by C2) and the definition of basic section $[Y, \pi(e)] = \pi([k, e]) \subset \pi(K)$.

Step 3 Up to now we have defined the vector bundle $\underline{E} \to \underline{C}$ and endowed it with a fiber-wise non-degenerate symmetric pairing, with a bilinear bracket on $\Gamma(\underline{E})$ and an anchor $\underline{\pi}$. It is straightforward to check that the axioms C1)-C5) in the definition of Courant algebroid (Def. 2.1) are fulfilled.

We are left with showing that $\underline{E}$ is an exact Courant algebroid. To this aim it suffices to show that $rk(\underline{E}) = 2dim(\underline{C})$ and that the kernel of the anchor $\underline{\pi}$ is isotropic in $\underline{E}$[5].

[4] Together with the fact that for any section $\hat{k}$ of K we have $[e_1, \hat{k}] = -[\hat{k}, e_1] + 2\mathcal{D}\langle e_1, \hat{k}\rangle$ and $\mathcal{D}\langle e_1, \hat{k}\rangle \subset N^*C = K \cap T^*M$.

[5] Any Courant algebroid satisfying these two conditions is exact, as we show now (sticking to our previous

The dimension of $\underline{C}$ is equal to the rank of $\pi(K^\perp)/\pi(K)$, which is $dim(M) - rk(K)$ as can be seen using $K \cap T^*M = (\pi(K^\perp))^\circ$. The rank of $\underline{E}$ is the rank of $K^\perp/K$, which is $2(dim(M) - rk(K))$. The kernel of $\underline{\pi}$ is the image under $K^\perp \to K^\perp/K \to \underline{E}$ of $(\pi|_{K^\perp})^{-1}(\pi(K))$ by the commutativity of the above diagram, and it's isotropic iff the latter is. Now $(\pi|_{K^\perp})^{-1}(\pi(K)) = K + (K^\perp \cap ker(\pi))$, which is isotropic because both K and $ker(\pi)$ are. □

Remark 3.8. We give an alternative way to describe the identification (see Step 1 of the above proof) between fibers of $K^\perp/K$ over two points p, q lying in the same leaf of $\pi(K)$: they are identified by the action of any sequence of sections of K joining p to q. More precisely, pick a finite sequence of local sections $k_1, \dots, k_n$ of K that join p to q, pick extensions $\tilde{k}_i \in \Gamma(E)$ and denote again by Φ the induced Courant algebroid automorphism of E. By item 1) of Lemma 3.5 Φ preserves the subbundle $K^\perp$ of E. By item 2) of the same lemma Φ preserves K, hence it induces a linear map $(K^\perp/K)_p \to (K^\perp/K)_q$. For any $e \in \Gamma_{bas}(K^\perp)$ we have $\Phi(e(p)) - e(q) \in K_q$. So, when $\Gamma_{bas}(K^\perp)$ spans point-wise $K^\perp$, the map $(K^\perp/K)_p \to (K^\perp/K)_q$ gives the same identification as in the proof of Thm. 3.7 (hence it is independent of the choices of $\tilde{k}_i$'s and their extensions).

Remark 3.9. The Courant algebroids E and $\underline{E}$ in Thm. 3.7 give rise to two pieces of data: the submanifold $S := \{(p, \underline{p}) | p \in C\}$ of $M \times \underline{C}$ and a subbundle $F := \{(e, \underline{e}) | e \in K^\perp\}$ of $(E \times \underline{E})|_S$. The subbundle F is maximal isotropic in $E \times \underline{E}^-$ (where the superscript "$-$" denotes that we invert the sign of the symmetric pairing on $\underline{E}$), we have $(\pi \times \underline{\pi})(F) = TS$, and F is closed under the Courant bracket on the product Courant algebroid $E \times \underline{E}^-$ (in the sense of Remark 3.2). These three statements are easily checked using the Courant algebroid structure on $\underline{E}$ as defined in the proof of Thm. 3.7. Hence the subbundle $F \to S$ provides a morphism of Courant algebroids from E to $\underline{E}$ as defined [6] in Def. 6.12 of [1] or in Def. 3.5.1 of [13].

We present a simple example:

Example 3.10 (Quotients of submanifolds). Take E to be $TM \oplus T^*M$ with the untwisted bracket, i.e. the one given by setting $H = 0$ in (3). Let C be a submanifold endowed with a regular distribution $\mathcal{F}$, and assume that the quotient $\underline{C} = C/\mathcal{F}$ be smooth. Take $K := \mathcal{F} \oplus N^*C$ (so $K^\perp = TC \oplus \mathcal{F}^\circ$). We want to check that the basic sections of $K^\perp$ span $K^\perp$. $\Gamma(K)$ is spanned by vector fields on C lying in $\mathcal{F}$ and differentials of functions vanishing on C. Since the latter (as all closed 1-forms) act trivially, it is enough to consider the action of a vector field $X \subset \mathcal{F}$. Let $Y \oplus df|_C$ be a section of $K^\perp$, where Y is a projectable vector field and f is the extension to M of the pullback of a function on $\underline{C}$. The action of

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notation for $\underline{E}$). By dimension counting it follows that $\underline{\pi}$ is surjective and that $ker(\underline{\pi})$ is maximal isotropic. Fix a covector $\xi \in T^*\underline{C}$. Then for all $e \in ker(\underline{\pi})$ we have

$$0 = \langle \underline{\pi}(e), \xi \rangle_{T\underline{C}, T^*\underline{C}} = \langle e, \underline{\pi}^*(\xi) \rangle_{\underline{E}, \underline{E}^*} = \langle e, \Psi(\underline{\pi}^*(\xi)) \rangle_{\underline{E}}.$$

Here the subscripts indicate that the first two are pairings of a vector space with its dual and the third one the non-degenerate symmetric bilinear form on $\underline{E}$; $\Psi : \underline{E}^* \to \underline{E}$ is the induced isomorphism. Now since $ker(\underline{\pi})$ is maximal isotropic it follows that $\Psi(\underline{\pi}^*(\xi)) \in ker(\underline{\pi})$. Since this holds for all covectors ξ we obtain $\underline{\pi}^*(T^*\underline{C}) \subset \ker(\underline{\pi})$, and since $\underline{\pi}^*$ is injective because $\underline{\pi}$ is surjective, we deduce that $\underline{E}$ is exact.

[6] We actually use a slight modification of the definitions of [1] and [13], for in these two references S is required to be the graph of an honest map $M \to \underline{C}$. Further in [1] Courant algebroids are endowed with the skew-symmetric Courant bracket.

X on this sections is just $[X, Y] \oplus (\mathcal{L}_X df)|_C$, which lies again in K. Since such $Y \oplus df|_C$ span $K^\perp$ we can apply Thm. 3.7 and obtain a reduced Courant algebroid on $\underline{C}$. Of course this is just $T\underline{C} \oplus T^*\underline{C}$ with the untwisted bracket.

The above example can be also easily recovered from Prop. 3.14 below (choosing $L = TC \oplus N^*C$ there) or from Prop. 3.18 (choosing $F \in \Omega^2(C)$ to be zero).

3.2 Adapted splittings

In this subsection we consider "good" splittings of an exact Courant algebroid $E \to M$, and using their existence we determine simple data on a foliated submanifold that induce an exact Courant algebroid on the leaf space.

Let E be an exact Courant algebroid over M and let C be a submanifold endowed with a coisotropic subbundle $K^\perp$ of E satisfying $\pi(K^\perp) = TC$. Assume that $\pi(K)$ is integrable and $\underline{C} := C/\pi(K)$ smooth.

Definition 3.11. We call a splitting $\sigma : TM \to E$ of the sequence (1) *adapted to* K if

a) The image of σ is isotropic

b) $\sigma(TC) \subset K^\perp$

c) for any vector field X on C which is projectable to $\underline{C}$ we have $\sigma(X) \in \Gamma_{bas}(K^\perp)$.

Remark 3.12. For such a splitting it follows automatically that $\sigma(\pi(K)) \subset K$. Indeed by $\pi(K^\perp) = TC$, b) in the definition above and $K^\perp \cap T^*M = (\pi(K))^\circ$ we have $K^\perp = \sigma(TC) + (\pi(K))^\circ$. Now $\langle \sigma(\pi(K)), \sigma(TC)\rangle = 0$ by a) in the definition above and $\langle \sigma(\pi(K)), (\pi(K))^\circ\rangle$ is equal to one-half the pairing of $\pi(K)$ and $(\pi(K))^\circ$, which is zero. Hence $\sigma(\pi(K))$ has zero symmetric pairing with $K^\perp$, so $\sigma(\pi(K)) \subset K$

The following proposition says that splittings adapted to K exist if and only if the reduced exact Courant algebroid $\underline{E}$ as in Thm. 3.7 exists.

Proposition 3.13. *Let $K \to C$ be an isotropic subbundle of E with $\pi(K^\perp) = TC$ and assume that $\pi(K)$ is integrable and $\underline{C} := C/\pi(K)$ smooth. Then splittings adapted to K exist if and only if $\Gamma_{bas}(K^\perp)$ spans $K^\perp$ at every point of C.*

Proof. Assume first that a splitting σ adapted to K exists. Let X be a projectable vector field on C. By c) of Def. 3.11 $\sigma(X)$ we will lie in $\Gamma_{bas}(K^\perp)$. Take a function on $\underline{C}$, pull it back to a function on C and extend it to a function f on M. Then $df|_C$ is a section of $(\pi(K))^\circ = T^*M \cap K^\perp$. Further it lies in $\Gamma_{bas}(K^\perp)$: for any $k \in \Gamma(K)$ we have

$$[k, df|_C] = -[df|_C, k] + d\langle k, df|_C\rangle \subset N^*C \subset K$$

because df as a closed 1-form acts trivially and it annihilates $\pi(K)$. Since $K^\perp = \sigma(TC) + (T^*M \cap K^\perp)$, taking all projectable vector fields X and functions f as above we see that $\Gamma_{bas}(K^\perp)$ spans $K^\perp$ at every point of M.

Conversely, assume now that $\Gamma_{bas}(K^\perp)$ spans $K^\perp$ at every point of M. Then by Thm. 3.7 the Courant algebroid $\underline{E}$ over $\underline{C}$ exists. We show that any isotropic splitting $\underline{\sigma} : T\underline{C} \to \underline{E}$

can be "lifted" to a splitting of E adapted to K. For any point $p \in C$ (and its image $\underline{p} \in \underline{C}$) consider the commutative diagram

$$\begin{tikzcd} K_p^\perp \arrow[r, "\pi"] \arrow[d] & T_pC \arrow[d] \\ (K^\perp/K)_p \cong \underline{E}_{\underline{p}} \arrow[r, "\pi"] & T_{\underline{p}}\underline{C} \end{tikzcd} .$$

To simplify the notation we leave out the footpoints until the end of this paragraph. Choose any subspace S such that $S \oplus K = K^\perp$, and use the isomorphism $\pi|_S : S \to K^\perp/K$ to obtain from $\underline{\sigma}(T\underline{C}) \subset \underline{E} \cong K^\perp/K$ a subspace $A \subset S$. By construction A maps isomorphically onto $T\underline{C}$ by the left and bottom map of the diagram, hence the same is true also using the top and right map. In particular $\pi(A)$ is a complement to $\pi(K)$ in TC, allowing us to define $\sigma : \pi(A) \to A \subset K^\perp$. We summarize the situation in the following commutative diagram:

$$\begin{tikzcd} K_p^\perp & \supset & A_p \arrow[r, "\pi"] \arrow[d] & \pi(A_p) \arrow[d] & \subset & T_pC \\ \underline{E}_{\underline{p}} \cong (K^\perp/K)_p & \supset & \underline{\sigma}(T_{\underline{p}}\underline{C}) \arrow[r, "\pi"] & T_{\underline{p}}\underline{C} & & \end{tikzcd} .$$

Notice that A is isotropic in E because $\underline{\sigma}$ was an isotropic splitting of $\underline{E}$. Extend $\sigma : \pi(A) \to A$ to TC so that the resulting $\sigma : TC \to K^\perp$ maps $\pi(K)$ to K. Then b) of Def. 3.11 is satisfied, and since $\sigma(TC)$ is isotropic condition a) is also satisfied.

We are left with showing that c) of Def. 3.11 holds, i.e. that if X is a basic vector field on C, then $\sigma(X) \in \Gamma_{bas}(K^\perp)$. Writing $X = X_{\pi(A)} + X_{\pi(K)}$ we see that $\sigma(X)_p$ is the sum of a section of K_p and the lift to A_p of $\underline{\sigma}(\underline{X})_{\underline{p}} \in \underline{E}_{\underline{p}} \cong (K^\perp/K)_p$. The projection of $\sigma(X)_p$ to $(K^\perp/K)_p$ is just $\underline{\sigma}(\underline{X})_{\underline{p}}$, i.e. it does not depend on p but just on its image $\underline{p} \in \underline{C}$. This shows that $\sigma(X)$ induces a well-defined section of $\underline{E}$ and hence lies in $\Gamma_{bas}(K^\perp)$. Now one can extend[7] $\sigma : TC \to K^\perp$ to the whole of M and obtain an isotropic splitting $TC \to E$. $\square$

Now we are able to determine a class of objects[8] that admit a reduced exact Courant algebroid on their quotient:

Proposition 3.14. *Let E be an exact Courant algebroid over a manifold M, C a submanifold endowed with a regular integrable foliation $\mathcal{F}$ so that $C/\mathcal{F}$ be smooth, and L a maximal isotropic subbundle $L \subset E|_C$ with $\pi(L) = TC$ such that $[\Gamma(K), \Gamma(L)] \subset \Gamma(L)$ where $K := L \cap \pi^{-1}(\mathcal{F})$. Then the assumptions of Thm. 3.7 are satisfied, hence E descends to an exact Courant algebroid on $C/\mathcal{F}$.*

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[7] At any point p of C first extend σ from T_pC to T_pM as follows. Again we suppress the index "p". Since $\sigma(TC) \cap T^*M = \{0\}$ it follows that $(\sigma(TC))^\perp$ maps surjectively onto TM under π; choose a subspace W with $\sigma(TC) \subset W \subset (\sigma(TC))^\perp$ which maps isomorphically onto TM. W is a complement in E to the (maximal) isotropic subspace $ker(\pi)$, hence we can deform it canonically to a (maximal) isotropic subspace $\hat{W}$ of E as one does in symplectic linear algebra (see Chapter 8 of [5]; here we think of $\langle \cdot, \cdot \rangle$ as an odd linear symplectic form). Explicitly, we define a map $\phi : W \to ker(\pi)$ by $\langle \phi w, \cdot \rangle|_W = -\frac{1}{2}\langle w, \cdot \rangle|_W$ and define $\hat{W}$ as the graph of ϕ. Since ϕ maps $\sigma(TC)$ to zero we have $\sigma(TC) \subset \hat{W}$, and $\hat{W}$ is still transverse to the kernel of π, allowing us to define $\sigma : TM \to \hat{W} \subset E$. Now we just extend $\hat{W} \subset E|_M$ in any way to a subbundle of $E \to M$ and apply the same construction as above to deform it into an isotropic subbundle.

[8] Compare also with Def. 6.1 and Def. 6.9.

Proof. Notice that K is isotropic and has constant rank, because $ker(\pi|_K) = K \cap T^*M = L \cap T^*M = N^*C$ has constant rank and $\pi(K) = \mathcal{F}$ has constant rank by assumption. Also $K^\perp = L + \mathcal{F}^\circ$, so $\pi(K^\perp) = TC$. Let $\sigma : TM \to E$ be an isotropic splitting such that $\sigma(TC) \subset L$. Since $\pi(L) = TC$ such a splitting always exists. We claim that σ is automatically a splitting adapted to K (Def. 3.11). Since $L \subset K^\perp$ we just have to check that if $k \in \Gamma(K)$ and X is a projectable vector field on C then $[k, \sigma(X)] \in \Gamma(K)$. By assumption this bracket is a section of L, and $\pi([k, \sigma(X)]) = [\pi(k), X] \subset \mathcal{F}$ since $\pi(k)$ lies in $\mathcal{F}$ and X is projectable, so altogether it follows that σ is a splitting adapted to K. By Prop. 3.13 the existence of a splitting adapted to K implies that $\Gamma_{bas}(K^\perp)$ spans pointwise $K^\perp$, and we are done. □

3.3 The Ševera class of the reduced Courant algebroid

As in the previous subsection let E be an exact Courant algebroid over M and let C be a submanifold endowed with a coisotropic subbundle $K^\perp$ of E satisfying $\pi(K^\perp) = TC$. In Theorem 3.7 we showed that, when certain assumptions are met, one obtains an exact Courant algebroid $\underline{E}$ over the quotient $\underline{C}$ of C by the distribution $\pi(K)$. In this subsection we will discuss how to obtain the Ševera class of $\underline{E}$ from the one of E.

Assume that $\pi(K)$ is integrable, $\underline{C} := C/\pi(K)$ smooth, and σ a splitting adapted to K. We start observing that j^*H_σ descends to a 3-form on $\underline{C}$, where j is the inclusion of C in M. We need to check that $i_X(j^*H_\sigma) = 0$ and $\mathcal{L}_X(j^*H_\sigma) = 0$ for any vector field X on C tangent to $\pi(K)$. Since H_σ is closed by Cartan's formula for the Lie derivative we just need to check the first condition: take a vector $X \in \pi(K_p)$ and extend it to a vector field tangent to $\pi(K)$; take vectors $Y, Z \in T_pC$ and extend them locally to projectable vector fields of C. Since σ is an splitting adapted to K we know that $\sigma(Y) \in \Gamma_{bas}(K^\perp)$, and since $\sigma(X) \subset K$ (by Remark 3.12) we have $[\sigma(X), \sigma(Y)] \subset K$. Therefore

$$H_\sigma(X, Y, Z) \quad = \quad 2\langle[\sigma(X), \sigma(Y)], \sigma(Z)\rangle = 0, \tag{6}$$

which is what we needed to prove. Even more is true by the following, which is an analog of Prop. 3.6 of [3] (but unlike that proposition does not involve equivariant cohomology; see also [16, 15]).

Proposition 3.15. *Assume that $\underline{C}$ is a smooth manifold. If σ is a splitting adapted to K then $j^*(H_\sigma)$ descends to a closed 3-form on $\underline{C}$ which represents the Ševera class of $\underline{E}$.*

Proof. We first describe an isotropic splitting $\underline{\sigma}$ of $\underline{E}$ induced by σ. Fix a distribution B on C such that $\pi(K) \oplus B = TC$. Fix a point $p \in C$ and define the subspace D_p as the image of $\sigma(B_p)$ under $K_p^\perp \to (K^\perp/K)_p$ (here we use $\sigma(B_p) \subset K_p^\perp$ by b) in Def. 3.11). Notice that since $\sigma(B_p) \cap K_p = \{0\}$ all four arrows of this commutative diagram are isomorphisms:

$$\begin{tikzcd}
K_p^\perp & \supset & \sigma(B_p) \arrow[r, "\pi"] \arrow[d] & B_p \arrow[d] & \subset & T_pC \,. \\
\underline{E}_{\underline{p}} \cong (K^\perp/K)_p & \supset & D_p \arrow[r, "\pi"] & T_{\underline{p}}\underline{C} & &
\end{tikzcd}$$

It is clear that D_p is isotropic because $\sigma(B_p)$ is. Reversing the bottom isomorphism we obtain a linear map $\underline{\sigma}_p : T_{\underline{p}}\underline{C} \to D_p \subset \underline{E}_{\underline{p}}$. We want to show that this map depends only on $\underline{p}$ and not on the choice of point p in the $\pi(K)$-leaf F sitting over $\underline{p}$. To this aim take $\underline{X} \in T_{\underline{p}}\underline{C}$, lift it (using the Ehresmann connection B for the submersion $C \to \underline{C}$) to $X \in \Gamma(B|_F)$. $\underline{\sigma}_p(\underline{X})$ is by definition the image of $\sigma(X_p)$ under the left vertical isomorphism, and it depends only on $\underline{p}$ because by c) in Def. 3.11 $\sigma(X)$ is a section of $\Gamma_{bas}(K^\perp)$ (defined over F). So we obtain a well-defined splitting $\underline{\sigma}$ of the Courant algebroid $\underline{E}$ over $\underline{C}$.

To compute the 3-form on $\underline{C}$ induced by j^*H_σ pick three tangent vectors on $\underline{C}$ at some point $\underline{p}$, which by abuse by notation we denote by $\underline{X}, \underline{Y}, \underline{Z}$. Extend them to vector fields on $\underline{C}$ and lift them to obtain vector fields X, Y, Z which are projectable. $\sigma(Z)$ lies in $\Gamma_{bas}(K^\perp)$, and using the commutativity of the above diagram we see that it is a lift of $\underline{\sigma}(\underline{Z}) \in \Gamma(\underline{E})$. The same holds for X and Y, therefore, by the definition of Courant bracket on $\underline{E}$, we know that $[\sigma(X), \sigma(Y)] \in \Gamma_{bas}(K^\perp)$ is a lift of $[\underline{\sigma}(\underline{X}), \underline{\sigma}(\underline{Y})] \in \Gamma(\underline{E})$. Hence

$$\begin{aligned} H_\sigma(X,Y,Z) &= 2\langle[\sigma(X),\sigma(Y)],\sigma(Z)\rangle && (7)\\ &= 2\langle[\underline{\sigma}(\underline{X}),\underline{\sigma}(\underline{Y})],\underline{\sigma}(\underline{Z})\rangle. && (8)\end{aligned}$$

This shows that H_σ descends to the curvature 3-form of $\underline{E}$ induced by the isotropic splitting $\underline{\sigma}$. □

Remark 3.16. If σ and $\hat{\sigma}$ are any two isotropic splittings for $E \to TM$ then there is a 2-form $b \in \Omega^2(M)$ for which $\sigma(X) - \hat{\sigma}(X) = b(X, \cdot) \in T^*M$ for all $X \in TM$. It is also known that H_σ and $H_{\hat{\sigma}}$ differ by db. Now let σ and $\hat{\sigma}$ be adapted to K (Def. 3.11). Then the interior product of a vector X tangent to $\pi(K)$ with $d(j^*b)$ vanishes, because $d(j^*b)$ is the difference of 3-forms which descend to $\underline{C}$. Also, $b(X,\cdot) = \sigma(X) - \hat{\sigma}(X) \in K \cap T^*M = N^*C$. So the interior product of X with j^*b vanishes too and j^*b descends to a 2-form on $\underline{C}$. This is consistent with the fact that by Prop. 3.15 H_σ and $H_{\hat{\sigma}}$ descend to 3-forms that represent the same element of $H^3(\underline{C}, \mathbb{R})$ (namely the Ševera class of $\underline{E}$).

As an instance of how a splitting adapted to K is used to compute the Ševera class of the reduced Courant algebroid we revisit Example 3.12 of [3], because it is simple and displays how a reduced Courant algebroid can have non-trivial Ševera class even though the original one has trivial Ševera class. We will reconsider this example in Ex. 3.20 below.

Example 3.17. Let $M = C = S^3 \times S^1$, denote by ∂_t the infinitesimal generator of the action of the circle on S^3 giving rise to the Hopf bundle $p : S^3 \to S^2$, and by s the coordinate on the second factor S^1. Let $E = TM \oplus T^*M$ the untwisted (i.e. $H = 0$) Courant algebroid on M. We choose the rank-one subbundle K to be spanned by $\partial_t + ds$. Choose a connection one form α for the circle bundle $S^3 \to S^2$, and denote by $X^H \in TS^3$ the horizontal lift of a vector X on S^2. $K^\perp$ is spanned by $\{\partial_t, \partial_s - \alpha, X^H, p^*\xi, ds\}$ where X (resp. ξ) runs over all vectors (resp. covectors) on S^2. Since ds is closed the adjoint action of $\partial_t + ds$ is just the Lie derivative w.r.t. ∂_t, which kills any of $\partial_t, \alpha, X^H, p^*\xi, \partial_s, ds$. In particular $\Gamma_{bas}(K^\perp)$ spans $K^\perp$. Hence the assumptions of Thm. 3.7 are satisfied, and on $S^2 \times S^1$ we have a reduced exact Courant algebroid. Now we choose the splitting $\sigma : TM \to K^\perp$ as follows:

$$\sigma(\partial_t) = \partial_t + ds, \quad \sigma(X^H) = X^H + 0 \text{ for all } X \in TS^2, \quad \sigma(\partial_s) = \partial_s - \alpha.$$

This splitting is isotropic, its image lies in $K^\perp$ and it maps projectable vector fields to elements of $\Gamma_{bas}(K^\perp)$ as one checks directly using $[\partial_t + ds, \cdot] = \mathcal{L}_{\partial_t}$. Hence σ satisfies the conditions of Def. 3.11, i.e. it is a splitting adapted to K.

Now we compute H_σ. If X, Y are vector fields on S^2 we have $[\sigma(X^H), \sigma(Y^H)] = [X^H, Y^H] + 0 = ([X,Y]^H - F(X,Y)\partial_t) + 0$ where $F \in \Omega^2(S^2)$ is the curvature of α. Also $[\sigma(\partial_s), \sigma(X^H)] = 0 + p^*(i_X F)$, and the analog computation for other other combinations of pairs of $\sigma(\partial_t), \sigma(X^H), \sigma(\partial_s)$ is zero. From this we deduce that $H_\sigma = p^*F \wedge ds$. This form descends to the 3-form $F \wedge ds$ on $S^2 \times S^1$, and by Prop. 3.15 it represents the Ševera class of the reduced Courant algebroid $\underline{E}$.

As pointed out in [3] $F \wedge ds$ defines a non-trivial cohomology class. An "explanation" for this fact is that by Prop. 3.15 to obtain a 3-form on C that descends to a representative of the Ševera class of $\underline{E}$ we need to choose a splitting adapted to K; the trivial splitting $\hat{\sigma}: TM \to TM \oplus T^*M$, which delivers $H_{\hat{\sigma}} = 0$, fails to be one because it does not map into $K^\perp$.

3.4 Explicit formulae in the split case

In this subsection we consider a split exact Courant algebroid and write down in explicit terms our reduction procedure for exact Courant algebroids (Thm. 3.7).

Let E be an exact Courant algebroid over M and let C be a submanifold endowed with a coisotropic subbundle $K^\perp$ of E satisfying $\pi(K^\perp) = TC$. Assume that $\mathcal{F} := \pi(K)$ is integrable and $C/\mathcal{F}$ smooth. Now consider the case that E is equal to $(TM \oplus T^*M, [\cdot,\cdot]_H)$, where H is some closed 3-form on M. Then there is a unique bilinear form $\hat{F}: TC \times \mathcal{F} \to \mathbb{R}$ with

$$K^\perp = \{(X,\xi) : X \in TC, \xi|_{\mathcal{F}} = \hat{F}(X,\cdot)\},$$

and the restriction of $\hat{F}$ to $\mathcal{F} \times \mathcal{F}$ is skew-symmetric (Prop. 2.2 of [20]). Since the subbundle $K^\perp$ and the bilinear form $\hat{F}$ determine each other, in the following we will use interchangeably the one or the other.

Proposition 3.18. *Consider the Courant algebroid $(TM \oplus T^*M, [\cdot,\cdot]_H)$ where H is a closed 3-form on M, and let $j: C \to M$ be a submanifold endowed with a regular integrable foliation $\mathcal{F}$ so that $\underline{C} := C/\mathcal{F}$ be smooth. Let $\hat{F}: TC \times \mathcal{F} \to \mathbb{R}$ be a bilinear form which is skew-symmetric on $\mathcal{F} \times \mathcal{F}$.*

Then $\hat{F}$ induces an exact Courant algebroid on $\underline{C}$ as in Thm. 3.7 iff there exists an extension $F \in \Omega^2(C)$ of $\hat{F}$ so that $dF + j^*H$ descends to $\underline{C}$. For any $F \in \Omega^2(C)$ as above, $dF + j^*H$ descends to a 3-form representing the Ševera class of the reduced Courant algebroid.

Remark 3.19. In the course of the proof and later on we will use the following fact which holds for any 2-form F on C and follows by a straight-forward computation using eq. (3): if $X_i + \xi_i$ are sections of the maximal isotropic subbundle $\tau_C^F := \{(X,\xi) \in TC \oplus T^*M|_C : \xi|_{TC} = i_X F\}$ then

$$2\langle [X_1 + \xi_1, X_2 + \xi_2]_H, X_3 + \xi_3\rangle = (j^*H + dF)(X_1, X_2, X_3). \tag{9}$$

Proof. Suppose that there exists an extension $F \in \Omega^2(C)$ of $\hat{F}$ so that $dF + j^*H$ descends to $\underline{C}$. Let $B \in \Omega^2(M)$ any extension of F, and σ the induced splitting of $TM \oplus T^*M$ (so $\sigma(Y) = (Y, i_Y B)$ for $Y \in TM$). We show now that σ is a splitting adapted to K; then by Prop. 3.13 we can conclude that $\hat{F}$ induces an exact Courant algebroid over $\underline{C}$. To check that σ is an adapted splitting, notice first that $\sigma(TC) \subset K^\perp$ because B extends $\hat{F}$.

Pick a projectable vector field Y on C; we want to show that $\langle [k,(Y,i_Y B)],e\rangle = 0$ for all $k \in \Gamma(K), e \in \Gamma(K^\perp)$. Since $K = \sigma(\mathcal{F}) + N^*C$ by Remark 3.12 and N^*C is spanned by closed 1-forms (which act trivially via the bracket), we may assume that $k = (X, i_X B)$ for some vector field $X \subset \mathcal{F}$. Since $K^\perp = \sigma(TC) + \mathcal{F}^\circ$ we can write $e = (Z, i_Z B) + \xi$ where $Z \in TC$ and $\xi \in \mathcal{F}^\circ$. Since the tangent component of $[(X, i_X B), (Y, i_Y B)]$ lies in $\mathcal{F}$ (because Y is a projectable vector field), we are left with showing that $\langle [(X,i_X B),(Y,i_Y B)],(Z,i_Z B)\rangle$ vanishes. Since $B \in \Omega^2(M)$ is an extension of $F \in \Omega^2(C)$, by (9) this expression is equal to $\frac{1}{2}(dF + j^*H)(X,Y,Z)$, which vanishes since we assumed that $dF + j^*H$ descends to $\underline{C}$. So we showed that σ is a splitting adapted to K.

Conversely, let us assume that $\hat{F}$ induces an exact Courant algebroid $\underline{E}$ on $\underline{C}$ as in Thm. 3.7. By Prop. 3.13 there exists a splitting $\sigma : TM \to TM \oplus T^*M$ adapted to K, which is necessarily of the form $\sigma(Y) = (Y, i_Y B)$ for some $B \in \Omega^2(M)$. As above, the fact that $\sigma(TC) \subset K^\perp$ means that B extends $\hat{F}$. Since σ is an adapted splitting, by Prop. 3.15 $j^*(H_\sigma) \in \Omega^3(C)$ descends (to a representative of the Ševera class of $\underline{E}$). By definition of H_σ we have

$$j^*(H_\sigma)(X,Y,Z) = 2\langle [(X,i_X B),(Y,i_Y B)],(Z,i_Z B)\rangle,$$

which together with eq. (9) shows that $j^*(H_\sigma)$ is equal to $dF + j^*H$, where $F = j^*B$; hence $dF + j^*H$ descends.

To conclude the proof of the theorem notice that, as we showed in the first half of the proof, any extension $F \in \Omega^2(C)$ of $\hat{F}$ so that $dF + j^*H$ descends to $\underline{C}$ is the restriction of a $B \in \Omega^2(M)$ corresponding to a splitting adapted to K. □

Example 3.20. Consider again Example 3.17: $M = C = S^3 \times S^1$, $H = 0$, and K is spanned by $\partial_t + ds$ where ∂_t is the infinitesimal generator of the action of the circle on S^3 (giving rise to the Hopf bundle $S^3 \to S^2$) and s the coordinate on the second factor S^1. $K^\perp$ corresponds to $\hat{F} : TM \times \mathbb{R}\partial_t \to \mathbb{R}$ given by $-ds \otimes (\alpha|_{\mathbb{R}\partial_t})$, where α is a connection one form for the circle bundle $S^3 \to S^2$. $\hat{F}$ extends to $F = \alpha \wedge ds \in \Omega^2(M)$, and dF descends to $F_\alpha \wedge ds$ on $S^2 \times S^1$ (where $F_\alpha \in \Omega^2(S^2)$ is the curvature of α), which by Prop. 3.18 represents the Ševera class of the the Courant algebroid obtained reducing $(TM \oplus T^*M, [\cdot,\cdot]_0)$ via the subbundle K.

In the above example one sees easily that any exact Courant algebroid on $S^2 \times S^1$ can be obtained from $(TM \oplus T^*M, [\cdot,\cdot]_0)$ via reduction, where $M = S^3 \times S^1$. Indeed (adopting the notation of the example above) any class $[\tilde{H}]$ in $H^3(S^2 \times S^1, \mathbb{R})$ has a representative of the form $\lambda F_\alpha \wedge ds$ for some $\lambda \in \mathbb{R}$, and restricting $F := \lambda\alpha \wedge ds \in \Omega^2(M)$ to $TM \times \mathbb{R}\partial_t$ gives rise to a subbundle $K^\perp \subset TM \oplus T^*M$ which by Prop. 3.18 produces by reduction the desired $[\tilde{H}]$-twisted Courant algebroid.

This is an instance of the following

Proposition 3.21. *Let M be a manifold endowed with an integrable distribution $\mathcal{F}$ so that $\underline{M} := M/\mathcal{F}$ is smooth, denote by p the projection, and let $H \in \Omega^3(M)$. The Ševera classes of the Courant algebroids on $\underline{M}$ obtained from $(TM \oplus T^*M, [\cdot,\cdot]_H)$ by reduction (as in Thm. 3.7) are exactly the preimages of $[H]$ under $p^* : H^3(\underline{M}, \mathbb{R}) \to H^3(M, \mathbb{R})$.*

Proof. Given any isotropic subbundle K of $(TM \oplus T^*M, [\cdot,\cdot]_H)$ with $\pi(K) = \mathcal{F}$ and satisfying the assumption of Thm. 3.7, choose an adapted splitting σ. By Prop. 3.15 the curvature H_σ of the splitting descends to a 3-form $\underline{H_\sigma}$ representing the Ševera class of the reduced Courant algebroid of $\underline{M}$, and $p^*[\underline{H_\sigma}] = [H_\sigma] = [H]$.

Conversely let $\tilde{H}$ be a 3-form on $\underline{M}$ so that $p^*[\tilde{H}] = [H]$. This means that there exists a 2-form F on M so that $dF + H = p^*\tilde{H}$. In particular $dF + H$ descends, and by Prop. 3.18 $\hat{F} := F|_{TC\times\mathcal{F}}$ corresponds to a coisotropic subbundle $K^\perp$ of $(TM \oplus T^*M, [\cdot,\cdot]_H)$ which by reduction produces an exact Courant algebroid on $\underline{M}$ with Ševera class $[\tilde{H}]$. □

4 The case of Dirac structures

Let E be an exact Courant algebroid over M. Recall [7] that a *Dirac structure* is a maximal isotropic subbundle of E which is closed under the Courant bracket. Now we let C be a submanifold of M and consider a maximal isotropic subbundle $L \subset E$ defined over C (not necessarily satisfying $\pi(L) \subset TC$).

The following is analog to Thm. 4.2 of [3].

Proposition 4.1 (Dirac reduction)**.** *Let $E \to M$ and $K \to C$ satisfy the assumptions of Thm. 3.7, so that we have an exact Courant algebroid $\underline{E} \to \underline{C}$. Let L be a maximal isotropic subbundle of $E|_C$ such that $L \cap K^\perp$ has constant rank, and assume that*

$$[\Gamma(K), \Gamma(L \cap K^\perp)] \subset \Gamma(L + K). \tag{10}$$

Then L descends to a maximal isotropic subbundle $\underline{L}$ of $\underline{E} \to \underline{C}$. If furthermore

$$[\Gamma_{bas}(L \cap K^\perp), \Gamma_{bas}(L \cap K^\perp)] \subset \Gamma(L + K). \tag{11}$$

then $\underline{L}$ is an (integrable) Dirac structure. Here $\Gamma_{bas}(L \cap K^\perp) := \Gamma(L) \cap \Gamma_{bas}(K^\perp)$

Proof. At every $p \in C$ we have a Lagrangian relation[9] between E_p and $(K^\perp/K)_p$ given by $\{(e, e + K_p) : e \in K_p^\perp\}$. The image of L_p under this relation, which we denote by $\underline{L}(p)$, is maximal isotropic because L_p is. Doing this at every point of C we obtain a maximal isotropic subbundle of $K^\perp/K$, which is furthermore smooth because $\underline{L}(p)$ is the image of $(L\cap K^\perp)_p$, which has constant rank by assumption, under the projection $K_p^\perp \to (K^\perp/K)_p$.

Recall that in Thm. 3.7 we identified $(K^\perp/K)_p$ and $(K^\perp/K)_q$ when p and q lie in the same leaf of $\pi(K)$, and that the identification was induced by the Courant algebroid automorphism Φ of E obtained integrating any sequence of locally defined sections $k_1, \dots, k_n$ of K that join p to q (see Remark 3.8). Assumption (10) (together with Lemma 3.5 1)) is exactly what is needed to ensure that Φ maps $L \cap K^\perp$ into $(L + K) \cap K^\perp = (L \cap K^\perp) + K$, so that $\underline{L}(p)$ gets identified with $\underline{L}(q)$. As a consequence we obtain a well-defined smooth maximal isotropic subbundle $\underline{L}$ of the reduced Courant algebroid $\underline{E}$, i.e. an almost Dirac structure for $\underline{E}$. Now assume that (11) holds, and take two sections of $\underline{L}$, which by abuse of notation we denote $\underline{e}_1, \underline{e}_2$. Since $L\cap K^\perp$ has constant rank we can lift them to sections e_1, e_2 of $\Gamma_{bas}(L \cap K^\perp)$. As for all elements of $\Gamma_{bas}(K^\perp)$ their bracket lies in $\Gamma_{bas}(K^\perp)$, and by assumption it also lies in $L+K$, so $[e_1, e_2]$ is a basic section of $(L+K)\cap K^\perp = (L\cap K^\perp)+K$. Its projection under $K^\perp/K \to \underline{E}$, which is by definition the bracket of $\underline{e}_1$ and $\underline{e}_2$, lies then in $\underline{L}$. □

[9] A lagrangian (or canonical) relation between two vector spaces V, W endowed with (ever or odd) symplectic forms σ_V, σ_W is a maximal isotropic subspace of $(V, \sigma_V) \times (W, -\sigma_W)$.

Example 4.2 (Coisotropic reduction). Let (M,Π) be a Poisson manifold and C a coisotropic submanifold[10]. It is known [7] that the characteristic distribution $\mathcal{F} := \sharp N^*C$ is a singular integrable distribution; assume that it is regular and the quotient $\underline{C} = C/\mathcal{F}$ be smooth. It is known that $D = \{(\sharp\xi,\xi) : \xi \in T^*P\}$ is a Dirac structure for the standard (i.e. $H = 0$) Courant algebroid $TM \oplus T^*M$. By Example 3.10, choosing $K = \mathcal{F} \oplus N^*C$, we know that we can reduce this Courant algebroid and obtain the standard Courant algebroid on $\underline{C}$.

Using Prop. 4.1 now we show that $L := D|_C$ also descends. $L \cap K^\perp$ has constant rank since it's isomorphic to $\mathcal{F}^\circ$. To check (10) we use the fact that K is spanned by closed 1-forms and hamiltonian vector fields of functions vanishing on C. The former act trivially, the latter (acting by Lie derivative) map $\Gamma(L)$ to itself because hamiltonian vector fields preserve the Poisson structure. An arbitrary section of K maps $\Gamma(L\cap K^\perp)]$ to $\Gamma(L+K)$ by the "Leibniz rule in the first entry" (see Section 2), so (10) is satisfied. Further it's known [7] that the integrability of Π is equivalent to $\Gamma(D)$ being closed under the Courant bracket, so (11) holds. Hence Prop. 4.1 tells us that Π descends to a Dirac structure on $\underline{C}$. This of course is the well-known Poisson structure on $\underline{C}$ determined by $pr^*\{\underline{f}_1,\underline{f}_2\} = \{f_1,f_2\}|_C$, where $pr : C \to \underline{C}$ and f_i is any extension on $pr^*\underline{f}_i$ to M.

Prop. 4.1 allows us to interpret some results of Section 3 in a more conceptual way.

Remark 4.3. Let E be an exact Courant algebroid over M, and $K \to C$ a subbundle satisfying the assumptions of Thm. 3.7. Then, by the prescription $\sigma \mapsto L := \sigma(TM)$, splittings σ adapted to K correspond exactly to subbundles $L \subset E$ with $\pi(L) = TM$ satisfying (compare with Def. 3.11)

a) L is maximal isotropic

b) $\pi(L\cap K^\perp) = TC$

c) $[\Gamma(K),\Gamma(L\cap K^\perp)] \subset \Gamma(L+K)$, which is just eq. (10).

In particular L satisfies the assumptions of Prop. 4.1 and therefore descends to a maximal isotropic subbundle $\underline{L}$ of $\underline{E}$. Because of $\pi(L\cap K^\perp) = TC$ it is clear that the anchor maps $\underline{L}$ onto $T\underline{C}$, hence $\underline{L}$ corresponds to an isotropic splitting of the reduced Courant algebroid $\underline{E}$. This splitting is just the splitting $\underline{\sigma}$ constructed n Prop. 3.15. Remark 4.4 below will make clear that the induced splitting $\underline{\sigma}$ doesn't depend on the whole of L but actually depends only on j^*L, the pullback of L to C. This explains also why Prop. 3.18 involves only a 2-form F on C (which encodes j^*L).

Now we comment on why we chose to perform our reductions (Thm. 3.7 and Prop. 4.1) directly and not by first pulling back our subbundles to the submanifold C.

Remark 4.4. Let E be an exact Courant algebroid over M and C a submanifold of M. Then with $\hat{K} = N^*C$ (and $\hat{K}^\perp = \pi^{-1}(TC)$) the assumptions of Thm. 3.7 are satisfied; indeed all the sections of $\hat{K}^\perp$ are basic. Hence we recover Lemma 3.7 of [3], which says that $E_C := \hat{K}^\perp/\hat{K}$ is an exact Courant algebroid over C.

Now let K be an isotropic subbundle of E over C such that $\pi(K^\perp) = TC$. For any $p \in C$ we have the inclusion of coisotropic subspaces $K_p^\perp \subset \hat{K}_p^\perp$. Hence, applying (the odd version of) symplectic reduction in stages we know that $K_p^\perp/K_p \cong i^*K_p^\perp/i^*K_p$ as vector spaces with non-degenerate symmetric pairing, where i^*K_p denotes the (isotropic) subspace of $(E_C)_p$ given by the image of K_p under $\hat{K}_p^\perp \to \hat{K}_p^\perp/\hat{K}_p$. Now assume that the quotient

[10] This means that $\sharp N^*C \subset TC$, where $\sharp : T^*M \to TM$ is the contraction with Π.

$\underline{C}$ of C by the foliation integrating $\pi(K)$ be a smooth manifold. One can check that the assumptions of Thm. 3.7 for the coisotropic subbundle $K^{\perp} \subset E$ and for the coisotropic subbundle $\hat{K}^{\perp} \subset E_C$ are equivalent, and that when they are satisfied the two reduced exact Courant algebroids over $\underline{C}$ obtained via Thm. 3.7 coincide. Hence, reducing directly $K^{\perp}$ or first restricting to C and then reducing amounts exactly to the same thing.

Now we introduce a new piece of data, namely a maximal isotropic subbundle $L \subset E|_C$ such that $L \cap K^{\perp}$ has constant rank. The restriction i^*L of L to C generally is not a smooth subbundle. If we *assume* that $L \cap N^*C$ has constant rank then i^*L is smooth. In this case using $i^*(L \cap K^{\perp}) = i^*L \cap i^*K^{\perp}$ one can show that $L \cap K^{\perp}$ has constant rank iff $i^*L \cap i^*K^{\perp}$ does, that the remaining assumptions of Prop. 4.1 (i.e. (10) and (11)) for $L \subset E$ and $i^*L \subset E_C$ are equivalent, and that the reduced Dirac structures on $\underline{C}$ coincide. Since restricting to C forces an extra assumption on L, altogether it is preferable to reduce L directly than first restricting to C.

In the next section we will consider a generalized complex structure on M, which is in particular an endomorphism $\mathcal{J}$ of E which leaves invariant $\langle \cdot, \cdot \rangle$, and ask when it descends to the Courant algebroid $\underline{E}$ induced by $K^{\perp} \subset E$. The endomorphism $\mathcal{J}$ can not generally be pulled back to C: as the composition of three Lagrangian relations[11] $E_C \sim E \sim E \sim E_C$ the endomorphism $\mathcal{J}$ will induce a Lagrangian relation from E_C to itself, but this will usually not be the graph of an honest endomorphism[12]. As it is easier to induce an endomorphism of $\underline{E}$ from one on E rather than from a Lagrangian relation on E_C , we made our constructions so to reduce directly rather than first restrict to C.

An alternative description of generalized complex structures on M is given in terms of a Dirac structure $L_{\mathbb{C}}$ in the complexification of E; however even from this perspective it is preferable to reduce directly $L_{\mathbb{C}}$ rather than than first restrict to C, in order to avoid extra assumptions on $L_{\mathbb{C}}$.

5 The case of generalized complex structures

Let E be an exact Courant algebroid over M. Recall that a generalized complex structure is a vector bundle endomorphism $\mathcal{J}$ of E which preserves $\langle \cdot, \cdot \rangle$, squares to $-Id_E$ and for which the Nijenhuis tensor

$$N_{\mathcal{J}}(e_1, e_2) := [\mathcal{J}e_1, \mathcal{J}e_2] - [e_1, e_2] - \mathcal{J}([e_1, \mathcal{J}e_2] + [\mathcal{J}e_1, e_2]). \tag{12}$$

vanishes[13].

The analog of the following proposition when a group action is present is Thm. 4.8 of [19]; we borrow the first part of our proof from them, but use different arguments to prove the integrability of the reduced generalized complex structure.

Proposition 5.1 (Generalized complex reduction)**.** *Let $E \to M$ and $K \to C$ satisfy the assumptions of Thm. 3.7, so that we have an exact Courant algebroid $\underline{E} \to \underline{C}$. Let $\mathcal{J}$ be a*

[11] The second relation is the graph of $\mathcal{J}$, the third one is given by $\{(e, e + \hat{K}) : e \in \hat{K}^{\perp}\}$, and similarly the first one.

[12] It is exactly when $\mathcal{J}\hat{K} \cap \hat{K}^{\perp} \subset \hat{K}$.

[13] $N_{\mathcal{J}}$ coincides with the Nijenhuis tensor of $\mathcal{J}$ written with the skew-symmetrized Courant bracket, more commonly found in the literature, as can be seen using that $\mathcal{J}$ preserves the symmetric pairing and squares to -1.

generalized complex structure on M such that $\mathcal{J}K \cap K^{\perp}$ has constant rank and is contained in K. Assume further that $[\Gamma(K), \mathcal{J}(\Gamma_{bas}(K^{\perp} \cap \mathcal{J}K^{\perp}))] \subset \Gamma(K)$ (i.e. that $\mathcal{J}$ applied to any basic section of $\mathcal{J}K^{\perp} \cap K^{\perp}$ is again a basic section). Then $\mathcal{J}$ descends to a generalized complex structure $\underline{\mathcal{J}}$ on $\underline{E} \to \underline{C}$.

Remark 5.2. The linear algebra conditions on $\mathcal{J}K \cap K^{\perp}$ are in particular satisfied when $\mathcal{J}$ preserves K (in which case the proof below simplifies quite a bit as well), for in that case $\mathcal{J}K \cap K^{\perp} = K$. The opposite extreme case is when $\mathcal{J}K \cap K^{\perp} = \{0\}$.

Proof. First we show that $\underline{J}$ induces a smooth[14] endomorphism of the vector bundle $K^{\perp}/K$ over C. Indeed $\mathcal{J}K \cap K^{\perp} \subset K$ is equivalent to $\mathcal{J}K^{\perp} + K \supset K^{\perp}$, so that $K^{\perp} = K^{\perp} \cap (\mathcal{J}K^{\perp} + K) = (K^{\perp} \cap \mathcal{J}K^{\perp}) + K$. From this it is clear that $K^{\perp} \cap \mathcal{J}K^{\perp}$ maps surjectively under $\Pi : K^{\perp} \to K^{\perp}/K$. Since $ker(\Pi|_{K^{\perp} \cap \mathcal{J}K^{\perp}}) = (K^{\perp} \cap \mathcal{J}K^{\perp}) \cap K = K \cap \mathcal{J}K^{\perp}$, by our constant rank assumption we obtain a smooth vector bundle $K^{\perp} \cap \mathcal{J}K^{\perp}/ker(\Pi|_{K^{\perp} \cap \mathcal{J}K^{\perp}})$ canonically isomorphic to $K^{\perp}/K$.

We use again the assumption $\mathcal{J}K \cap K^{\perp} \subset K$, interpreting it as follows: if e lies in the kernel of $\Pi : K^{\perp} \to K^{\perp}/K$ and $\mathcal{J}e \in K^{\perp}$ then $\mathcal{J}e$ is still in the kernel. This applies in particular to all $e \in ker(\Pi|_{K^{\perp} \cap \mathcal{J}K^{\perp}})$ (since $K^{\perp} \cap \mathcal{J}K^{\perp}$ is $\mathcal{J}$-invariant), so we deduce that $\mathcal{J}$ leaves $ker(\Pi_{K^{\perp} \cap \mathcal{J}K^{\perp}})$ invariant, i.e. $\mathcal{J}$ induces a well-defined endomorphism on $K^{\perp} \cap \mathcal{J}K^{\perp}/ker(\Pi|_{K^{\perp} \cap \mathcal{J}K^{\perp}}) \cong K^{\perp}/K$. Further it is clear that it squares to -1 and preserves the induced symmetric pairing on $K^{\perp}/K$.

Now take a section $\underline{e}$ of $\underline{E}$, lift it to a (automatically basic) section e of $K^{\perp} \cap \mathcal{J}K^{\perp}$. Then by assumption $\mathcal{J}e$ is again a basic section; this shows that the endomorphism on $K^{\perp} \cap \mathcal{J}K^{\perp}/ker(\Pi|_{K^{\perp} \cap \mathcal{J}K^{\perp}})$ descends to an endomorphism $\underline{\mathcal{J}}$ of $\underline{E}$.

We are left with showing that $\underline{\mathcal{J}}$ is integrable, i.e. with showing that the Nijenhuis tensor $N_{\underline{\mathcal{J}}}$ vanishes. Let $\underline{e}_1, \underline{e}_2$ be elements of $\underline{E}$, extend them to local sections and pull them back to basic sections e_1, e_2 of $K^{\perp} \cap \mathcal{J}K^{\perp}$. We claim that $N_{\mathcal{J}}(e_1, e_2)$ is a lift of $N_{\underline{\mathcal{J}}}(\underline{e}_1, \underline{e}_2)$; since the former vanishes, the latter vanishes too and we are done.

To prove our claim we reason as follows. By the definition of $\underline{\mathcal{J}}$ we known that $\mathcal{J}e_i \in \Gamma_{bas}(K^{\perp} \cap \mathcal{J}K^{\perp})$ is a lift of $\underline{\mathcal{J}}(\underline{e}_i)$, hence the four Courant brackets of sections appearing on the r.h.s. of (12) are lifts of the analogous brackets in $\underline{E}$. Since $\Gamma_{bas}(K^{\perp})$ is closed under the Courant bracket we know that the term

$$([e_1, \mathcal{J}e_2] + [\mathcal{J}e_1, e_2]) \tag{13}$$

of (12) lies[15] in $\Gamma_{bas}(K^{\perp})$. However, to conclude that applying $\underline{\mathcal{J}}$ to (13) we obtain a lift of the analogous term in $\underline{E}$ (and hence that $N_{\mathcal{J}}(e_1, e_2)$ is a lift of $N_{\underline{\mathcal{J}}}(\underline{e}_1, \underline{e}_2)$), we still need to show that (13) is a section of $\mathcal{J}K^{\perp}$, because then it will lie in $\mathcal{J}K^{\perp} \cap K^{\perp}$ which is where we let $\mathcal{J}$ act to define $\underline{\mathcal{J}}$. To this aim pick a section k of K, and apply the Leibniz rule C4) to $\pi(e_1)\langle \mathcal{J}e_2, \mathcal{J}k \rangle$ and to $\pi(\mathcal{J}e_1)\langle e_2, \mathcal{J}k \rangle$, both of which vanish because $e_2, \mathcal{J}e_2 \subset \mathcal{J}K^{\perp} \cap K^{\perp} = (K + \mathcal{J}K)^{\perp}$ and $\pi(e_1), \pi(\mathcal{J}e_1) \subset \pi(K^{\perp}) = TC$. Taking the sum of the two equations we obtain

$$0 = \langle [e_1, \mathcal{J}e_2] + [\mathcal{J}e_1, e_2], \mathcal{J}k \rangle + \langle e_2, -\mathcal{J}[e_1, \mathcal{J}k] + [\mathcal{J}e_1, \mathcal{J}k] \rangle. \tag{14}$$

[14]This is clear when $\mathcal{J}$ preserves $K^{\perp}$.
[15]This concludes the proof in the case $\mathcal{J}K^{\perp} = K^{\perp}$.

Now the vanishing of $N_{\mathcal{J}}(e_1, k)$ means that $-\mathcal{J}[e_1, \mathcal{J}k] + [\mathcal{J}e_1, \mathcal{J}k] = [e_1, k] + \mathcal{J}[\mathcal{J}e_1, k]$, and the latter lies in $K + \mathcal{J}K$ because e_1 and $\mathcal{J}e_1$ are in particular basic sections of $K^{\perp}$. Hence the last term in (14) vanishes, and we deduce that $[e_1, \mathcal{J}e_2] + [\mathcal{J}e_1, e_2]$ has zero symmetric pairing with $\mathcal{J}K$, i.e. that it lies in $\mathcal{J}K^{\perp}$. $\square$

In Prop. 5.1 the condition $[\Gamma(K), \mathcal{J}(\Gamma_{bas}(K^{\perp} \cap \mathcal{J}K^{\perp}))] \subset \Gamma(K)$ does not follow from the integrability of $\mathcal{J}$ (see Ex. 5.3 below for an explicit example); this is not surprising. In Section 6 we will consider submanifolds C for which the integrability of $\mathcal{J}$ does imply all the assumptions of Prop. 5.1, in analogy to the case of coisotropic submanifolds in the Poisson setting (see also Example 4.2).

Example 5.3 (Complex foliations). Take E to be the standard Courant algebroid and $\mathcal{J}$ be given by a complex structure J on M. Take $\mathcal{F}$ to be a real integrable distribution on M preserved by J (so J induces the structure of a complex manifold on each leaf of $\mathcal{F}$) and $K = \mathcal{F} \oplus 0$, so that $\underline{M} := M/\pi(K) = M/\mathcal{F}$ be smooth. The generalized complex structure $\mathcal{J}$ preserves K. If $\mathcal{J}$ mapped $\Gamma_{bas}(K^{\perp})$ into itself[16] then by Prop. 5.1 it would follow that $\underline{M}$ would have an induced generalized complex structure. Further, it would necessarily correspond to an honest complex structure on $\underline{M}$ that makes $M \to \underline{M}$ into a holomorphic map. However there are examples for which such a complex structure on $\underline{M}$ does not exist; in [22] Winkelmann quotes an example where M is a twistor space of real dimension 6 and $\underline{M}$ is the 4-dimensional torus.

Example 5.4 (Symplectic foliations). Take again E to be the standard Courant algebroid and $\mathcal{J}$ be given by a symplectic form ω on M. Take $K = \mathcal{F}$ to be a real integrable distribution on M. One checks that $\mathcal{J}K \cap K^{\perp}$ is contained in K only if it is trivial, which is equivalent to saying that the leaves of $\mathcal{F}$ are symplectic submanifolds. $\mathcal{J}$ maps basic sections of $\mathcal{J}K^{\perp} \cap K^{\perp} = \mathcal{F}^{\omega} \oplus \mathcal{F}^{\circ}$ into basic sections iff the hamiltonian vector field $X_{\pi^* f}$ is a projectable vector field for any function f, where $pr : M \to \underline{M} := M/\mathcal{F}$. When this is the case the induced generalized complex structure on $\underline{M}$ is the symplectic structure given by the isomorphism of vector spaces $\mathcal{F}^{\omega}_x \cong T_{pr(x)}\underline{M}$ (where $x \in M$).

Remark 5.5. We recall that a submanifold C of a Poisson manifold (M, Π) is called *coisotropic* if $\sharp N^*C \subset TC$, where $\sharp : T^*M \to TM$ is given by contraction with Π. In this case $\sharp N^*C$ is a singular integrable distribution on C, called *characteristic distribution*, and it is well-known that when it is regular and the quotient $C/\sharp N^*C$ is a smooth manifold then it has an induced Poisson structure.

It is known that a generalized complex manifold $(M, \mathcal{J})$ comes with a canonical Poisson structure Π, whose sharp map $\sharp$ is given by the composition $T^*M \hookrightarrow E \xrightarrow{\mathcal{J}} E \xrightarrow{\pi} TM$. If in Prop. 5.1 we assume that $\mathcal{J}$ preserves K, then C is a necessarily a coisotropic submanifold, because from $N^*C = (\pi(K^{\perp}))^{\circ} = K \cap ker(\pi) \subset K$ we have $\sharp(N^*C) = \pi(\mathcal{J}N^*C) \subset \pi(K) \subset \pi(K^{\perp}) = TC$. So $C/\sharp N^*C$ (if smooth) has an induced Poisson structure. We know that also $\underline{C} := C/\pi(K)$ has a Poisson structure, induced from the reduced generalized complex structure. In general $\pi(K)$ is *not* the characteristic distribution of C; we just have an inclusion $\sharp N^*C \subset \pi(K)$[17]. The Poisson structure on $C/\pi(K)$ is induced from the one on

[16] This is equivalent to saying that for any vector field X on M which is projectable the vector field $J(X)$ is also projectable.

[17] A case in which this inclusion is strict is when $\mathcal{J}$ corresponds to the standard complex structure on $M = \mathbb{C}^n$ (with complex coordinates $z_k = x_k + iy_k$) and $K = span\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}\}$.

M in a natural way, namely $pr^*\{\underline{f},\underline{g}\}_{\underline{C}} = \{f,g\}_M|_C$ where $pr : C \to \underline{C}$ and f, g are any extensions to M of $pr^*(\underline{f})$ and $pr^*(\underline{g})$. Indeed $df \in \Gamma(K^\perp)$, the commutativity of diagram in the proof of Thm. 3.7, and the non-degeneracy of the symmetric pairing on $\underline{E}$ imply that df is basic and indeed a lift of $d\underline{f} \in \Gamma(\underline{E})$. Hence $\mathcal{J}df$ is a lift of $\underline{\mathcal{J}}(d\underline{f})$ and

$$pr^*\{\underline{f},\underline{g}\}_{\underline{C}} = pr^*\langle \underline{\mathcal{J}}(d\underline{f}), d\underline{g}\rangle = \langle \mathcal{J}df, dg\rangle|_C = \{f,g\}_M|_C.$$

Given an exact Courant algebroid E on M, recall that a *generalized Kähler structure* consists of two commuting generalized complex structures $\mathcal{J}_1, \mathcal{J}_2$ such that the symmetric bilinear form on E given by $\langle \mathcal{J}_1\mathcal{J}_2\cdot,\cdot\rangle$ be positive definite. The following result borrows the proof of Thm. 6.1 of [3], except for the fact that the integrability of the reduced generalized complex structures is automatic by Prop. 5.1.

Proposition 5.6 (Generalized Kähler reduction)**.** *Let $E \to M$ and $K \to C$ satisfy the assumptions of Prop. 3.7, so that we have an exact Courant algebroid $\underline{E} \to \underline{C}$. Let $\mathcal{J}_1, \mathcal{J}_2$ be a generalized Kähler structure on M such that $\mathcal{J}_1 K = K$. Assume further that $\mathcal{J}_1$ maps $\Gamma_{bas}(K^\perp)$ into itself and that $\mathcal{J}_2$ maps $\Gamma_{bas}(\mathcal{J}_2K^\perp \cap K^\perp)$ into itself. Then $\mathcal{J}_1, \mathcal{J}_2$ descend to a generalized Kähler structure on $\underline{E} \to \underline{C}$.*

Proof. By Thm. 5.1 $\mathcal{J}_1$ induces a generalized complex structure $\underline{\mathcal{J}_1}$ on $\underline{E}$. The orthogonal $K^{\mathcal{G}}$ of K w.r.t. $\langle \mathcal{J}_1\mathcal{J}_2\cdot,\cdot\rangle$ is $(\mathcal{J}_2\mathcal{J}_1K)^\perp = \mathcal{J}_2K^\perp$. Because of the identity $K^\perp = K \oplus (K^{\mathcal{G}} \cap K^\perp)$ the restriction to $\mathcal{J}_2K^\perp \cap K^\perp$ of the projection $K^\perp \to K^\perp/K$ is an isomorphism. So we can apply Prop. 5.1 to $\mathcal{J}_2$ and obtain a generalized complex structure $\underline{\mathcal{J}_2}$ on $\underline{E}$. Notice that both $\mathcal{J}_1$ and $\mathcal{J}_2$ preserve $\mathcal{J}_2K^\perp \cap K^\perp$; pulling back sections of $\underline{E}$ to basic sections of $\mathcal{J}_2K^\perp \cap K^\perp$ one sees that $\underline{\mathcal{J}_1}, \underline{\mathcal{J}_2}$ form a generalized Kähler structure on $\underline{E}$. □

6 The case of (weak) branes

In this section we define branes and show that they admit a natural quotient which is a generalized complex manifold endowed with a space-filling brane. Then we notice that quotients of more general objects, which we call "weak branes", also inherit a generalized complex structure; examples of weak branes are coisotropic submanifolds in symplectic manifolds. Finally we show how weak branes can be obtained by passing from a generalized complex manifold to a suitable submanifold.

6.1 Reducing branes

Definition 6.1. Let E be an exact Courant algebroid over a manifold M. A *generalized submanifold* is a pair (C, L) consisting of a submanifold $C \subset M$ and a maximal isotropic subbundle $L \subset E$ over C with $\pi(L) = TC$ which is closed under the Courant bracket (i.e. $[\Gamma(L), \Gamma(L)] \subset \Gamma(L)$ with the conventions of Remark 3.2).

We show that this definition, which already appeared in the literature[18], is just a splitting-independent rephrasing[19] of Gualtieri's original definition (Def. 7.4 of [9]). See also Lemma 3.2.3 of [13].

[18]It appeared in Def. 3.2.2 of [13] with the name "maximally isotropic extended submanifold". Also, a subbundle L as above but for which we just ask $\pi(L) \subset TC$ is called generalized Dirac structure in Def. 6.8 of [1] (in the setting of the skew-symmetric Courant bracket).

[19]Up to a sign, since Def. 7.4 of [9] requires $i^*H_\sigma = dF$ (in the notation of this lemma).

Lemma 6.2. *Let E be an exact Courant algebroid over M. Choose an isotropic splitting σ for E, giving rise to an isomorphism of Courant algebroids $(E, [\cdot,\cdot]) \cong (TM \oplus T^*M, [\cdot,\cdot]_{H_\sigma})$ where H_σ is the curvature 3-form of the splitting (see Section 2). Then pairs (C, L) as in Def. 6.1 correspond bijectively to pairs (C, F), where $F \in \Omega^2(C)$ satisfies $-i^*H_\sigma = dF$ (for i the inclusion of C in M).*

Proof. The fact that $L \subset E$ is maximal isotropic and $\pi(L) = TC$ means that under the isomorphism it maps to

$$\tau_C^F := \{(X, \xi) \in TC \oplus T^*M|_C : \xi|_{TC} = i_X F\}$$

for some 2-form F on C. The correspondence $L \leftrightarrow F$ is clearly bijective. Equation (9) shows that the integrability conditions also correspond.

□

By Lemma 6.2 the following definition is equivalent to Gualtieri's original one (i.e. to Def. 7.6 of [9], again up to a sign):

Definition 6.3. Let E be an exact Courant algebroid over a manifold M and $\mathcal{J}$ be a generalized complex structure on E. A *generalized complex submanifold* or *brane* is a generalized submanifold (C, L) satisfying $\mathcal{J}(L) = L$.

Now we state the main theorem of this paper. Recall that we gave the definition of coisotropic submanifold in Remark 5.5.

Theorem 6.4 (Brane reduction)**.** *Let E be an exact Courant algebroid over a manifold M, $\mathcal{J}$ a generalized complex structure on E, and (C, L) a brane. Then C is coisotropic w.r.t. the Poisson structure induced by $\mathcal{J}$ on M. If the quotient $\underline{C}$ of C by its characteristic foliation is smooth, then*

a) E induces an exact Courant algebroid $\underline{E}$ over $\underline{C}$

b) $\mathcal{J}$ induces a generalized complex structure $\underline{\mathcal{J}}$ on $\underline{E} \to \underline{C}$

c) L induces the structures of a space-filling brane on $\underline{C}$ and the Ševera class of $\underline{E}$ is trivial.

Proof. Recall that the Poisson structure Π induced by $\mathcal{J}$ on M (or rather its sharp map $\sharp$) is given by the composition $T^*M \hookrightarrow E \xrightarrow{\mathcal{J}} E \xrightarrow{\pi} TM$. Since $N^*C = (\pi(L))^\circ = L \cap ker(\pi) \subset L$ we have $\sharp(N^*C) = \pi(\mathcal{J}N^*C) \subset \pi(L) = TC$, so C is a coisotropic submanifold. As above we let $\mathcal{F} := \sharp N^*C$, assume that it be a regular distribution and that $\underline{C} := C/\mathcal{F}$ be a smooth manifold.

a) C, L and $\mathcal{F}$ satisfy the assumptions of Prop. 3.14. Hence we can apply Thm. 3.7 with $K := L \cap \pi^{-1}(\mathcal{F})$ and obtain an exact Courant algebroid $\underline{E}$ over $\underline{C}$. Notice that we have not made use of the integrability of $\mathcal{J}$ here, if not for the fact that the induced bivector Π is integrable and hence the distribution $\mathcal{F}$ is involutive.

b) Now we check that the assumptions of Prop. 5.1 are satisfied. From $L \cap T^*M = N^*C$, the fact that $\mathcal{J}N^*C$ is contained in L and that it projects onto $\mathcal{F}$ we deduce that $K = N^*C + \mathcal{J}N^*C$, which is clearly preserved by $\mathcal{J}$. So we just need to check that, for any basic section e of $K^{\perp}$, $\mathcal{J}e$ is again basic. Locally we can write $K = span\{(dg_i)|_C, \mathcal{J}(dg_i)|_C\}$ where $g_1, \ldots, g_{codim(C)}$ are local functions on M vanishing on C. Since each dg_i is a closed one form, $[(dg_i)|_C, \mathcal{J}e] \subset K$. Using the fact that the Nijenhuis tensor $N_{\mathcal{J}}$ vanishes (12) we have

$$[\mathcal{J}(dg_i)|_C, \mathcal{J}e] = \mathcal{J}[\mathcal{J}(dg_i)|_C, e] + \mathcal{J}[(dg_i)|_C, \mathcal{J}e] + [(dg_i)|_C, e].$$

The first term on the r.h.s. lies in K because e is a basic section, and the last two because dg_i is a closed 1-form. So $[\mathcal{J}(dg_i)|_C, \mathcal{J}e] \subset K$, hence e is again a basic section. Hence the assumptions of Prop. 5.1 are satisfied, concluding the proof of b).

c) We want to apply Prop. 4.1 to obtain a brane on $\underline{C}$. Since $L \subset K^{\perp}$ the assumption (10) needed for L to descend reads $[\Gamma(K), \Gamma(L)] \subset \Gamma(L)$, and the integrability assumption (11) reads $[\Gamma_{bas}(L), \Gamma_{bas}(L)] \subset \Gamma(L)$. As L is closed under the bracket both assumptions hold, and we obtain an (integrable) Dirac structure $\underline{L}$ on $\underline{C}$. Furthermore from the fact that $\mathcal{J}$ preserves L we see that $\underline{\mathcal{J}}$ preserves $\underline{L}$. Hence $(\underline{C}, \underline{L})$ is a brane for the generalized complex structure $\underline{\mathcal{J}}$ on $\underline{E}$.

If we chose any isotropic splitting for $\underline{E}$, as discussed in Lemma 6.2, then $\underline{L}$ gives rise to a 2-form $\hat{F}$ on $\underline{C}$ such that $-d\hat{F}$ equals the curvature of the splitting, which hence is an exact 3-form. This concludes the proof of c) and of the theorem.

□

Remark 6.5. We saw in Thm. 6.4 that branes C are coisotropic and their quotient by the characteristic foliation is endowed with a generalized complex structure. As pointed out in Remark 5.5, if one starts with a $\mathcal{J}$-invariant coisotropic subbundle $K^{\perp}$ of $E|_C$ (instead of constructing one from the brane (C, L) as in Thm. 6.4) in general it is a different quotient of C that is endowed with a generalized complex structure (via Prop. 5.1). If one picks just any arbitrary coisotropic submanifold C, its quotient by the characteristic foliation inherits a Poisson structure, but in general it does not inherit a generalized complex structure: take for example any odd dimensional submanifold of a complex manifold.

Remark 6.6. When the characteristic foliation of a brane $(C, L) \subset M$ is regular, using coordinates adapted to the foliation one sees that the quotient of small enough open sets U of C by the characteristic foliation is smooth, and Thm. 6.4 gives a local statement. However in general the characteristic foliation is singular, as the following example shows. Take $M = \mathbb{C}^2$, the untwisted exact Courant algebroid as E, and as $\mathcal{J}$ take $\begin{pmatrix} I & \Pi \\ 0 & -I^* \end{pmatrix}$. Here $I(\partial_{x_i}) = \partial_{y_i}$ is the canonical complex structure on $\mathbb{C}^2$ and $\Pi = y_1(\partial_{x_1} \wedge \partial_{x_2} - \partial_{y_1} \wedge \partial_{y_2}) - x_1(\partial_{y_1} \wedge \partial_{x_2} + \partial_{x_1} \wedge \partial_{y_2})$ is the imaginary part of the holomorphic Poisson bivector (see [8][10]) $z_1\partial_{z_1} \wedge \partial_{z_2}$. It is easy to check that $C = \{z_2 = 0\}$ with $F = 0$ define a brane for $\mathcal{J}$, and that the characteristic distribution of C has rank zero at the origin and rank 2 elsewhere.

Example 6.7 (Branes in symplectic manifolds [9]). Consider a symplectic manifold (M, ω) and view it as a generalized complex structure on the standard Courant algebroid. Example 7.8 of [9] states that if a generalized submanifold (C, F) (so F is a closed 2-form on C) is a brane then F descends to the quotient $\underline{C}$ (which we assume to be smooth), and $\underline{F} + i\underline{\omega}$ is a holomorphic symplectic form on $\underline{C}$.

Remark 6.8. Suppose that in the setting of Thm. 6.4 E is additionally endowed with some $\mathcal{J}_2$ so that $\mathcal{J}_1, \mathcal{J}_2$ form a generalized Kähler structure. Then using Prop. 5.6 we see that *if*

$\mathcal{J}_2$ descends to $\underline{E}$ then $\underline{E}$ is endowed with a generalized Kähler structure too.

6.2 Reducing weak branes

We weaken the conditions in the definition of brane; at least for the time being, we refer to resulting object as "weak branes".

Definition 6.9. Let E be an exact Courant algebroid over a manifold M, $\mathcal{J}$ a generalized complex structure on E. We will call *weak brane* a pair (C, L) consisting of a submanifold C and a maximal isotropic subbundle $L \subset E|_C$ with $\pi(L) = TC$ such that

$$\mathcal{J}(N^*C) \subset L, \qquad [\Gamma(K), \Gamma(L)] \subset \Gamma(L) \tag{15}$$

(where $K := L \cap \pi^{-1}(\mathcal{F})$ and $\mathcal{F} := \sharp N^*C$, or equivalently $K = N^*C + \mathcal{J}N^*C$.)

Notice that weak branes for which $\mathcal{F}$ has constant rank automatically satisfy the assumptions of Prop. 3.14. Also notice that in the proof of Thm. 6.4 (except for c)) we just used properties of weak branes, hence we obtain

Proposition 6.10. *If in Thm. 6.4 we let (C, L) be a weak brane then C is a coisotropic submanifold and a) and b) of Thm. 6.4 still hold, i.e. there is a reduced Courant algebroid and a reduced generalized complex structure on $\underline{C}$ (when it is a smooth manifold).*

We describe how weak branes look like in the split case, i.e. when $E = (TM \oplus T^*M, [\cdot,\cdot]_H)$. We write $\mathcal{J}$ in matrix form as $\begin{pmatrix} A & \Pi \\ \omega & -A^* \end{pmatrix}$ where A is an endomorphism of TM, Π the Poisson bivector canonically associated to $\mathcal{J}$, and ω a 2-form on M.

Corollary 6.11. *Let C be a submanifold of M and $F \in \Omega^2(C)$. Fix an extension $B \in \Omega^2(M)$ of F. Then (C, τ_C^F) is a weak brane (with smooth quotient $\underline{C}$) iff C is coisotropic (with smooth quotient $\underline{C}$), $A + \Pi B : TM \to TM$ preserves TC , and the 3-form $dF + i^*H$ on C descends to $\underline{C}$.*

In this case the Ševera class of the reduced Courant algebroid $\underline{E}$ is represented by the pushforward of $dF + i^*H$. Further there is a splitting of $\underline{E}$ in which the reduced generalized complex structure is

$$\underline{\tilde{\mathcal{J}}} = \begin{pmatrix} \underline{\tilde{A}} & \underline{\tilde{\Pi}} \\ \underline{\tilde{\omega}} & -\underline{\tilde{A}}^* \end{pmatrix},$$

where the endomorphism $\underline{\tilde{A}}$ is the pushforward of $(A + \Pi B)|_{TC}$, the Poisson bivector $\underline{\tilde{\Pi}}$ is induced by Π, and the 2-form $\underline{\tilde{\omega}}$ is the pushforward of $i^*(\omega - B\Pi B - BA - A^*B)$.

Proof. Since K is $\tau_C^F \cap \pi^{-1}(\mathcal{F})$ equation (9) shows that $[\Gamma(K), \Gamma(\tau_C^F)] \subset \Gamma(\tau_C^F))$ is equivalent to the fact that the closed 3-form $i^*H + dF$ descend to $\underline{C}$. Now perform a $-B$-transformation; the transformed objects are $\tilde{L} = TC \oplus N^*C$ and $\tilde{\mathcal{J}} = \begin{pmatrix} \tilde{A} & \tilde{\Pi} \\ \tilde{\omega} & -\tilde{A}^* \end{pmatrix}$, with components $\tilde{A} = A + \Pi B$, $\tilde{\Pi} = \Pi$ and $\tilde{\omega} = \omega - B\Pi B - BA - A^*B$ (see for example [21]). Hence we see that the first condition in (15) is equivalent to C begin coisotropic and $A + \Pi B$ preserving TC (a condition independent of the extension B). Further, since by the proof of Thm. 6.4 $\mathcal{J}$ preserves $TC \oplus \mathcal{F}^\circ$ and $\mathcal{F} \oplus N^*C$, it is clear that in the induced splitting of $\underline{E}$ the components of $\underline{\tilde{\mathcal{J}}}$ are induced from those of $\tilde{\mathcal{J}}$.

Since we saw that $dF+i^*H$ descend to $\underline{C}$, by Prop. 3.18 the Ševera class of the reduced Courant algebroid $\underline{E}$ is represented by the pushforward of $dF+i^*H$. □

We use the characterization of Cor. 6.11 in the following examples.

Example 6.12 (Coisotropic reduction). If $\mathcal{J}$ corresponds to a symplectic structure on M, then any coisotropic submanifold C endowed with $F=0$ is a weak brane. The generalized complex structure on $\underline{C}$ (assumed to be a smooth manifold) corresponds to the reduced symplectic form.

If $\mathcal{J}$ corresponds to a complex structure, then any weak brane is necessarily a complex submanifold. If $\mathcal{J}$ is obtained deforming a complex structure in direction of a holomorphic Poisson structure [8][10] this is no longer the case, as in the following two examples. In both cases however the reduced generalized complex structures we obtain are quite trivial.

Example 6.13. Similarly to Remark 6.6 take M to be the open halfspace $\{(x_1,y_1,x_2,y_2): y_1>0\}\subset\mathbb{C}^2$, the untwisted exact Courant algebroid as E, and as $\mathcal{J}$ take $\begin{pmatrix} I & \Pi \\ 0 & -I^* \end{pmatrix}$ where $I(\partial_{x_i})=\partial_{y_i}$ is the canonical complex structure on $\mathbb{C}^2$ and $\Pi=y_1(\partial_{x_1}\wedge\partial_{x_2}-\partial_{y_1}\wedge\partial_{y_2})-x_1(\partial_{y_1}\wedge\partial_{x_2}+\partial_{x_1}\wedge\partial_{y_2})$ is the imaginary part of the holomorphic Poisson bivector $z_1\partial_{z_1}\wedge\partial_{z_2}$. We now take $C=\{(x_1,y_1,x_2,0): y_1>0\}$ and on C the closed 2-form $F:=-\frac{1}{y_1}dy_1\wedge dx_2$. We show that the pair (C,F) forms a weak brane. By dimension reasons C is coisotropic (the characteristic distribution is regular and spanned by $x_1\partial_{x_1}+y_1\partial_{y_1}$), so we just have to check that $I+\Pi B$ preserves TC, where B the 2-form on M given by the same formula as F. This is true as one computes $I+\Pi B:\partial_{x_1}\mapsto\partial_{y_1},\ \ \partial_{y_1}\mapsto-\frac{x_1}{y_1}\partial_{y_1},\ \ \partial_{x_2}\mapsto-\frac{x_1}{y_1}\partial_{x_2}$.

Now we want to compute the generalized complex structure on $\underline{C}$ given by Prop. 6.10, We do so by first applying the gauge transformation by $-B$ to obtain a generalized complex structure $\tilde{\mathcal{J}}$ and then using the diffeomorphism $\underline{C}\cong(-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{R}$ induced by $C\to(-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{R},(x_1,y_1,x_2)\mapsto(\theta:=arctg(\frac{x_1}{y_1}),x_2)$. The Poisson bracket of the coordinate functions θ and x_2 on $\underline{C}$ is computed by pulling back the two functions to C, extending them to the whole of M and taking their Poisson bracket there. This gives the constant function 1. Next the coordinate vector field ∂_θ on $\underline{C}$ is lifted by the vector field $\frac{x_1^2+y_1^2}{y_1}\partial_{x_1}$ on C, and of course ∂_{x_2} on $\underline{C}$ is lifted by ∂_{x_2} on C. Applying the endomorphism $I+\Pi B$ of TC we see the induced endomorphism on $T\underline{C}$ is just multiplication by $-tg(\theta)$. Finally, the component $\tilde{\omega}$ of $\tilde{\mathcal{J}}$ is given by $-BI-B\Pi B-I^*B$, which on C restricts to the 2-form $\frac{1}{y_1^2}(y_1dx_1-x_1dy_1)\wedge dx_2$, which in turn is the pullback of the 2-form $(1+tg^2(\theta))d\theta\wedge dx_2$ on $\underline{C}$. Hence the induced generalized complex structure on $\underline{C}$ is

$$\begin{pmatrix} -tg(\theta)\cdot Id & \partial_\theta\wedge\partial_{x_2} \\ (1+tg^2(\theta))d\theta\wedge dx_2 & tg(\theta)\cdot Id \end{pmatrix}.$$

This is just the gauge transformation by the closed 2-form $tg(\theta)d\theta\wedge dx_2$ of the generalized complex structure on $(-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{R}$ that corresponds to the symplectic form $d\theta\wedge dx_2$.

Example 6.14. Similarly to the previous example we take $M=\mathbb{C}^2$, the untwisted exact Courant algebroid as E, and as $\mathcal{J}$ we take $\begin{pmatrix} I & \Pi \\ 0 & -I^* \end{pmatrix}$ where $I(\partial_{x_i})=\partial_{y_i}$ is the canonical complex structure on $\mathbb{C}^2$ and $\Pi=y_1(\partial_{x_1}\wedge\partial_{x_2}-\partial_{y_1}\wedge\partial_{y_2})-x_1(\partial_{y_1}\wedge\partial_{x_2}+\partial_{x_1}\wedge\partial_{y_2})$. Now we let C be the hypersurface $\{x_1^2+y_1^2=1\}$. The characteristic distribution is generated by

∂_{y_2}, so the quotient $\underline{C}$ is a cylinder. Let $a, b, c \in C^\infty(C)$ so that, denoting by $F_{(a,b,c)}$ the pullback to C of

$$B_{(a,b,c)} := a \cdot dx_1 \wedge dy_1 + b \cdot dx_1 \wedge dx_2 + c \cdot dy_1 \wedge dx_2 - y_1 \cdot dx_1 \wedge dy_2 + +x_1 \cdot dy_1 \wedge dy_2,$$

$dF_{(a,b,c)}$ descends[20] to $\underline{C}$. One checks that $I^* + B_{(a,b,c)}\Pi$ preserves N^*C, so that $(C, F_{(a,b,c)})$ is a weak brane. A computation analog to the one of the previous example shows that the reduced generalized complex structure on $\underline{C} = S^1 \times \mathbb{R}$ with coordinates θ and x_2 is given by

$$\begin{pmatrix} \lambda_{(a,b)} \cdot Id & \partial_\theta \wedge \partial_{x_2} \\ (1+\lambda^2_{(a,b)}) d\theta \wedge dx_2 & -\lambda_{(a,b)} \cdot Id \end{pmatrix}$$

where $\lambda_{(a,b)} \in C^\infty(\underline{C})$ is the function that lifts to $-by_1 + cx_1 \in C^\infty(C)$ via $C \to \underline{C}$. Again this is a gauge transformation of the standard symplectic structure on $S^1 \times \mathbb{R}$.

A consequence is that for no choice of a, b, c as above the weak brane $(C, F_{(a,b,c)})$ is actually a brane. Indeed if this was the case by Thm. 6.4 we would obtain a space-filling brane for a symplectic structure on $S^1 \times \mathbb{R}$; applying again Thm. 6.4, by Example 7.8 of [9], we would obtain the structure of a holomorphic symplectic manifold on $S^1 \times \mathbb{R}$, which can not exist because holomorphic symplectic manifolds have real dimension $4k$ for some integer k.

6.3 Cosymplectic submanifolds

Recall that a submanifold $\tilde{M}$ of a Poisson manifold (M, Π) is *cosymplectic* if $\sharp N^*\tilde{M} \oplus T\tilde{M} = TM|_{\tilde{M}}$. It is known (see for example [23]) that a cosymplectic submanifold inherits canonically a Poisson structure. The following lemma, which follows also from more general results of [2], says that generalized complex structures are also inherited by cosymplectic submanifolds:

Lemma 6.15. *Let E be an exact Courant algebroid over a manifold M, $\mathcal{J}$ a generalized complex structure on E and $\tilde{M}$ a cosymplectic submanifold of M (w.r.t. the natural Poisson structure on M induced by $\mathcal{J}$). Then $\tilde{M}$ is naturally endowed with a generalized complex structure.*

Proof. We want apply Prop. 5.1 with $K = N^*\tilde{M}$ (so $K^\perp = \pi^{-1}(T\tilde{M})$). The intersection $\mathcal{J}K \cap K^\perp$ is trivial. Indeed if $\xi \in N^*\tilde{M}$ and $\pi(\mathcal{J}\xi) \in T\tilde{M}$ then by the definition of cosymplectic submanifold $\pi(\mathcal{J}\xi) = 0$ (recall that $\sharp = \pi\mathcal{J}|_{T^*M}$) and the restriction $\sharp$ to $N^*\tilde{M}$ is injective, so that $\xi = 0$. Further, as seen in Remark 4.4, all sections of $K^\perp$ are basic, so $\mathcal{J}$ maps the set of basic sections of $\mathcal{J}K^\perp \cap K^\perp$ into itself. Hence the assumptions of Prop. 5.1 are satisfied and we obtain a generalized complex structure on $\tilde{M}$. □

Now we describe how a pair (C, L) which doesn't quite satisfy the conditions of Def. 6.9 can be regarded as a weak brane by passing to a cosymplectic submanifold.

Proposition 6.16. *Let E be an exact Courant algebroid over a manifold M, $\mathcal{J}$ a generalized complex structure on E, C a submanifold and L a maximal isotropic subbundle of $E|_C$ with $\pi(L) = TC$. Suppose that $\mathcal{J}(N^*C) \cap \pi^{-1}(TC)$ is contained in L and has constant rank.*

[20] This happens exactly when $F_{(a,b,c)}$ is closed.

Then there exists a submanifold $\tilde{M}$ (containing C) which inherits a generalized complex structure $\tilde{\mathcal{J}}$ from M, and so that $\tilde{L}$ satisfies $\tilde{\mathcal{J}}(\tilde{N}^*C) \subset \tilde{L}$. Here $\tilde{L}$ is the pullback of L to $\tilde{M}$ and $\tilde{N}^*C$ the conormal bundle of C in $\tilde{M}$.

Further assume that $[\Gamma(L \cap \pi^{-1}(\mathcal{F})), \Gamma(L)] \subset \Gamma(L)$ where $\mathcal{F} := \sharp N^*C \cap TC$ is the characteristic distribution of C. Then $[\Gamma(\tilde{L} \cap \tilde{\pi}^{-1}(\mathcal{F})), \Gamma(\tilde{L})] \subset \Gamma(\tilde{L})$. Hence $(C, \tilde{L})$ is a weak brane in $(\tilde{M}, \tilde{\mathcal{J}})$.

Proof. Since the intersection of $\mathcal{J}(N^*C)$ and $\pi^{-1}(TC)$ has constant rank the same holds for their sum and for $\pi(\mathcal{J}(N^*C) + \pi^{-1}(TC)) = \sharp N^*C + TC$. Hence C is a *pre-Poisson* submanifold [6] of (M, Π). Fix any complement R of $\sharp N^*C + TC$ in $TM|_C$; by Theorem 3.3 of [6], "extending" C in direction of R we obtain a submanifold $\tilde{M}$ of M which is cosymplectic. By Lemma 6.15 we know that $\tilde{M}$ is endowed with a generalized complex structure $\tilde{\mathcal{J}}$. Further by the same lemma $\mathcal{J}K \cap K^\perp$ is trivial. The projection $K^\perp \to K^\perp/K$ (for $K = N^*\tilde{M}$) maps $\mathcal{J}K^\perp \cap K^\perp$ isomorphically onto $K^\perp/K$, and $\tilde{\mathcal{J}}$ is induced by the action of $\mathcal{J}$ on $\mathcal{J}K^\perp \cap K^\perp$. Therefore, denoting by $\tilde{L} := L/K$ the pullback of L to $\tilde{M}$, requiring $\tilde{\mathcal{J}}(\tilde{N}^*C) \subset \tilde{L}$ is equivalent to requiring that $\mathcal{J}(N^*C \cap (\mathcal{J}K^\perp \cap K^\perp))$ maps into $\tilde{L}$ under $K^\perp \to K^\perp/K$, which in turn means $\mathcal{J}(N^*C) \cap K^\perp \subset L$. Now using $K^\perp = \pi^{-1}(T\tilde{M})$, $T\tilde{M}|_C = R \oplus TC$ and recalling that R was chosen so that $R \oplus (\sharp N^*C + TC) = TM|_C$, it follows that $\mathcal{J}(N^*C) \cap K^\perp = \mathcal{J}(N^*C) \cap \pi^{-1}(TC)$. So our assumption ensures that $\tilde{\mathcal{J}}(\tilde{N}^*C) \subset \tilde{L}$.

Finally notice that the projection $K^\perp \to K^\perp/K$ maps L onto $\tilde{L}$. Since $\pi^{-1}(\mathcal{F})$ is mapped onto $\tilde{\pi}^{-1}(\mathcal{F})$ we also have that $L \cap \pi^{-1}(\mathcal{F})$ is mapped onto $\tilde{L} \cap \tilde{\pi}^{-1}(\mathcal{F})$. Hence our assumption $[\Gamma(L \cap \pi^{-1}(\mathcal{F})), \Gamma(L)] \subset \Gamma(L)$ implies $[\Gamma(\tilde{L} \cap \tilde{\pi}^{-1}(\mathcal{F})), \Gamma(\tilde{L})] \subset \Gamma(\tilde{L})$. □

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Reduction of Dirac structures along isotropic subbundles

Iván Calvo, Fernando Falceto and Marco Zambon

Abstract

Given a Dirac subbundle and an isotropic subbundle, we provide a canonical method to obtain a new Dirac subbundle. When the original Dirac subbundle is Courant involutive this construction has interesting applications, unifying and generalizing some results on the reduction of Dirac structures previously found in the literature.

Contents

1 Introduction

The structure underlying the reduction of a Poisson or symplectic manifold M is a Poisson algebra. Such algebraic data can be encoded in geometric terms through the concept of *Dirac structure*, which generalizes the Poisson and presymplectic geometries by embedding them in the framework of the geometry of $TM \oplus T^*M$. Dirac structures were introduced in a remarkable paper by T. Courant [5]. Therein, they are related to the Marsden-Weinstein reduction [11] and to the Dirac bracket [7] on a submanifold of a Poisson manifold. Recently, Dirac subbundles have been considered in connection to the reduction of implicit Hamiltonian systems (see [2],[1]). This simple but powerful construction allows to deal with mechanical situations in which we have both gauge symmetries and Casimir functions.

In the most general setup, Dirac structures are lagrangian subbundles of exact Courant algebroids. In a recent work H. Burzstyn, G. R. Cavalcanti and M. Gualtieri [3] have considered the natural generalization of group actions to the context of Courant algebroids and the reduction induced by them. If the generalized action is a *symmetry* in the sense that it preserves the Dirac structure the latter may be transported to the reduced Courant algebroid.

We deal with the same problem of reducing Dirac structures but our strategy is somehow different. We perform the reduction in two steps. First we deform (stretch) canonically the Dirac structure along the *symmetry*, obtaining a Dirac subbundle which is interesting in its own right. Then we argue that the stretched structure reduces in a natural way. Our procedure does not require the symmetry to preserve the original Dirac structure, but just the stretched Dirac subbundle (a strictly weaker requirement). As a byproduct of our construction we obtain the analogue of the Dirac bracket for this setup. In our approach the rôle of symmetry is played by integrable isotropic subbundles of the Courant algebroid. These objects and their applications in the context of constrained dynamical systems have been extensively studied recently by I. Vaisman (see [13] and [14]).

The relation between our work and [3] is exactly parallel to the relation between Marsden-Ratiu reduction by distributions [10] and Marsden-Weinstein reduction by symmetries [11]. Actually, our original motivation was to generalize for any Dirac structure the Marsden-Ratiu reduction of Poisson manifolds; we present here a variation of the Marsden-Ratiu reduction, and we will deal with a proper generalization in a subsequent paper.

The paper is organized as follows. In Section 2 we fix the notation and give the basic definitions. Section 3 contains the main results of the paper. Examples and applications are described in Section 4. Section 5 is devoted to the conclusions.

2 Courant algebroids and Dirac structures

We define a *Courant algebroid* [9] over a manifold M as a vector bundle $E \to M$ equipped with an $\mathbb{R}$-bilinear bracket $[\cdot,\cdot]$ on $\Gamma(E)$, a non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$ on the fibers and a bundle map $\pi : E \to TM$ (the *anchor*) satisfying, for any $e_1, e_2, e_3 \in \Gamma(E)$ and $f \in C^\infty(M)$:

(i) $[e_1,[e_2,e_3]] = [[e_1,e_2],e_3] + [e_2,[e_1,e_3]]$

(ii) $\pi([e_1,e_2]) = [\pi(e_1),\pi(e_2)]$

(iii) $[e_1, fe_2] = f[e_1,e_2] + (\pi(e_1)f)e_2$

(iv) $\pi(e_1)\langle e_2,e_3\rangle = \langle [e_1,e_2],e_3\rangle + \langle e_2,[e_1,e_3]\rangle$

(v) $[e,e] = \mathcal{D}\langle e,e\rangle$

where $\mathcal{D} : C^\infty(M) \to \Gamma(E)$ is defined by $\mathcal{D} = \frac{1}{2}\pi^* \circ \mathrm{d}$, using the bilinear form to identify E and its dual. We see from axiom (v) that the bracket is not skew-symmetric, but rather satisfies $[e_1,e_2] = -[e_2,e_1] + 2\mathcal{D}\langle e_1,e_2\rangle$.

A Courant algebroid is called *exact* noteEvery Courant algebroid in this paper is assumed to be exact. if

$$0 \longrightarrow T^*M \xrightarrow{\pi^*} E \xrightarrow{\pi} TM \longrightarrow 0 \tag{2.1}$$

is an exact sequence. Choosing a splitting $TM \to E$ of the above sequence with isotropic image every exact Courant algebroid is identified with $TM \oplus T^*M$ endowed with the natural symmetric pairing

$$\langle (X,\xi),(X',\xi')\rangle = \frac{1}{2}(i_{X'}\xi + i_X\xi') \tag{2.2}$$

and the Courant bracket

$$[(X,\xi),(X',\xi')] = ([X,X'],\mathcal{L}_X\xi' - i_{X'}\mathrm{d}\xi + i_{X'}i_X H) \tag{2.3}$$

for some closed 3-form H. In fact, the Courant algebroid uniquely determines the cohomology class of H, called Ševera class. The anchor π is given by the projection onto the first component. When it is important to stress the value of the 3-form H we shall use the notation E_H for $TM \oplus T^*M$ equipped with this Courant algebroid structure.

A diffeomorphism φ in M transforms E_H into $E_{\varphi^* H}$. Also for $B \in \Omega^2(M)$ the bundle map given by $\tau_B : (X,\xi) \mapsto (X, \xi + i_X B)$ is a morphism of Courant algebroids between E_H and $E_{H-\mathrm{d}B}$. The most general symmetry of E_H (i.e. the most general orthogonal bundle automorphism preserving the Courant bracket (2.3)) is the product of a B-transform and a diffeomorphism such that $H = \varphi^* H - \mathrm{d}B$ [3]. The B-transforms associated to closed forms $\Omega^2_{\mathrm{closed}}(M)$ constitute an abelian normal subgroup of the group of symmetries. For every section e of E_H contraction with $[e,\cdot]$ is an infinitesimal symmetry of E_H, which integrates to a symmetry of E_H.

A *Dirac subbundle* D in an exact Courant algebroid is a maximal isotropic subbundle with respect to $\langle\cdot,\cdot\rangle$. Maximal isotropy implies that $D^\perp = D$, where $D^\perp$ stands for the orthogonal subspace of D. In particular, $\mathrm{rank}(D) = \dim(M)$.

A *Dirac structure* D is an integrable Dirac subbundle, i.e. a Dirac subbundle whose sections close under the Courant bracket. In this case the restriction to D of the Courant bracket is skew-symmetric and D with anchor π is a Lie algebroid.

The two basic examples of Dirac structures are:

Example 2.1. For any 2-form ω, the graph L_ω of $\omega^\flat : TM \to T^*M$ is a Dirac subbundle such that $\pi(L_\omega) = TM$ at every point of M. L_ω is a Dirac structure in E_H if and only if $\mathrm{d}\omega = -H$. In particular, L_ω is a Dirac structure in E_0 if and only if ω is closed.

Example 2.2. Let Π be a bivector field on M. The graph L_Π of the map $\Pi^\sharp : T^*M \to TM$ is always a Dirac subbundle. In this case the natural projection from L_Π to T^*M is one-to-one. L_Π is a Dirac structure in E_H if and only if Π is a twisted Poisson structure. In particular, L_Π is a Dirac structure in E_0 if and only Π is a Poisson structure.

Definition 2.1. Let V be a subbundle of a Courant algebroid E and take $e \in \Gamma(E)$. We say that e is *V-invariant* if $[v,e] \in \Gamma(V), \forall v \in \Gamma(V)$. The set of V-invariant sections of E will be denoted by $\Gamma(E)_V$.

Based on [3] we will say that $W \subset E$ is *preserved by V*, or *V-preserved* for shortness, if $[\Gamma(V),\Gamma(W)] \subset \Gamma(W)$.

Note that from the previous definition if e is V-invariant, then for every $p \in M$, either $\pi^*(T^*_pM) \subset V$ or $e_p \in V_p^\perp$. The second possibility will be the one we meet in the paper.

3 Stretched Dirac structures

Take a Dirac subbundle $D \subset E$ and an isotropic subbundle $S \subset E$, i.e. $S \subset S^\perp$. We always assume that $D \cap S$ (or equivalently $D \cap S^\perp$) has constant rank. It is not difficult to show that we can "stretch" D along S and obtain another Dirac subbundle (see Remark 3.1 below for an interpretation in terms of reduced Dirac structures), namely

$$D^S := (D \cap S^\perp) + S.$$

We call this the *stretching of D along S*. We must show that D^S is maximal isotropic, but this is immediate since

$$\begin{aligned}(D^S)^\perp &= (D^\perp + S) \cap S^\perp \\ &= (D \cap S^\perp) + S = D^S,\end{aligned}$$

where in the last line we have used that D is maximal isotropic and S is a subset of $S^\perp$. It is also clear that D^S, as the sum of two subbundles, is a (smooth) subbundle.

This construction is canonical, for D^S is the Dirac subbundle closest to D among those containing S, as stated in the following

Theorem 3.1. *Let D, S and D^S be as above and let D' be a Dirac subbundle such that $S \subset D'$. Then, $D' \cap D \subset D^S \cap D$. In addition, $D' \cap D = D^S \cap D$ if and only if $D' = D^S$.*

Proof: From the isotropy of D' and given that $S \subset D'$ we deduce that $D' \subset S^\perp$. Hence,

$$D' \cap D \subset S^\perp \cap D = D^S \cap D.$$

If the equality $D' \cap D = D^S \cap D$ holds, then $D' \supset D' \cap D = S^\perp \cap D$. Since $S \subset D'$, we find that $D^S = (D \cap S^\perp) + S \subset D'$. But D^S and D' have the same dimension, so that they are equal. $\square$

The following propositions, whose proofs are immediate, will be useful in the sequel.

Proposition 3.1. *For any 2-form B we have $\tau_B(D^S) = \tau_B(D)^{\tau_B(S)}$*

Proposition 3.2. *Let S be of the form $S = K \oplus L$, with $K, L \subset E$ and $K \subset L^\perp$. Then, $D^S = (D^K)^L = (D^L)^K$.*

Now assume that D is a Dirac structure. In general, even if D^S is a smooth subbundle its sections do not close under the Courant bracket (see Example 2 in the next section). In order to obtain some meaningful and interesting results on the closeness properties of D^S we shall assume from now on that S is closed under the Courant bracket, or equivalently that S is S-preserved. In this situation we have the following

Theorem 3.2. *The set of S-invariant sections of D^S is closed under the Courant bracket.*

Proof: Consider $e_1, e_2 \in \Gamma(D^S)_S = \{e \in D^S : [s, e] \in \Gamma(S), \forall s \in \Gamma(S)\}$. First, let us prove that $[e_1, e_2]$ is an S-invariant section. Take $s \in \Gamma(S)$ and write

$$[s, [e_1, e_2]] = [e_1, [s, e_2]] - [[e_1, s], e_2].$$

Recall that $[e, s] = -[s, e]$ for $e \in \Gamma(D^S)$ and $s \in \Gamma(S)$ because $D^S = (D + S) \cap S^\perp \subset S^\perp$. The S-invariance of $[e_1, e_2]$ follows immediately.

It remains to show that $[e_1, e_2] \in \Gamma(D^S)$. Since we assumed that both $D \cap S^\perp$ and S are subbundles, every section $e \in \Gamma(D^S)$ can be written as $e = v + w$ with $v \in \Gamma(D \cap S^\perp)$

and $w \in \Gamma(S)$. Notice that if e is S-invariant, v is also S-invariant because S is Courant involutive. The expression

$$[e_1, e_2] = [v_1 + w_1, v_2 + w_2] = [v_1, v_2] + [v_1, w_2] + [w_1, v_2] + [w_1, w_2] \tag{3.1}$$

makes clear that $[e_1, e_2] \in \Gamma(D + S)$, since $[v_1, v_2] \in \Gamma(D)$ and the remaining terms on the right-hand side of (3.1) are sections of S. Finally, let us prove that $[e_1, e_2] \in \Gamma(S^\perp)$. For any $s \in \Gamma(S)$,

$$\langle s, [e_1, e_2]\rangle \;\; = \;\; \pi(e_1)\langle s, e_2\rangle - \langle [e_1, s], e_2\rangle = 0$$

where we have used axiom (iv) in the definition of a Courant algebroid and the orthogonality of s and e_i, $i = 1, 2$. $\square$

Remark 3.1. We give an interpretation of the definition of stretching in terms of reduced structures. As S is assumed to be isotropic, by odd linear symplectic reduction $S^\perp/S$ is endowed with a nondegenerate symmetric bilinear form, and D can be pushed forward[1] to a maximal isotropic (smooth) subbundle of $S^\perp/S$, namely the image of $D \cap S^\perp$ under $S \to S^\perp/S$. Pulling back this we obtain a maximal isotropic (smooth) subbundle of E, precisely $(D \cap S^\perp) + S = D^S$. In this sense the definition of stretching is very natural.

Next we give an interpretation of Theorem 3.2, where we assumed that both S and D are closed under the Courant bracket. Using axiom (iv) in the definition of Courant algebroid one checks that the closedness of S is equivalent to $[\Gamma(S), \Gamma(S^\perp)] \subset \Gamma(S^\perp)$, i.e. to $S^\perp$ being preserved by S. Now *assume* also that $\pi(S^\perp)$ is a regular integrable distribution. Then by Corollary 3.2 below (applied to $\hat{D} := S^\perp$) for every point of $S^\perp$ there exists a local S-invariant section of $S^\perp$ passing through it. Then (see Theorem 3.7 of [15]), for any small open set $U \subset M$, there is a reduced Courant algebroid E_{red} on $U/\pi(S)$. *If* $D \cap S^\perp$ is S-preserved then D descends to a Dirac structure for E_{red} (see Proposition 4.1 of [15]), and from its closedness and the definition of the reduced Courant bracket it follows that the S-invariant sections of $(D \cap S^\perp) + S = D^S$ are closed under the Courant bracket.

Notice that Theorem 3.2 shows the closedness of the S-invariant sections of D^S avoiding the constant rank and invariance assumptions that are necessary when one resorts to the reduction procedure because it works directly on subbundles of E.

The next proposition is obvious

Proposition 3.3. *For any 2-form B,*
$\tau_B(\Gamma(D^S)_S) = \Gamma(\tau_B(D^S))_{\tau_B(S)} = \Gamma(\tau_B(D)^{\tau_B(S)})_{\tau_B(S)}$.

Here in the second and third members of the equality the invariance is meant with respect to the Courant bracket of $E_{H-\mathrm{d}B}$.

Inspired by [12] we will give the following

Definition 3.1. Given a Dirac subbundle D and an isotropic S-preserved subbundle $S \subset E$, we say that S *is canonical for* D if there exists a local S-invariant section of D^S passing through any of its points.

[1] Pushforwards and pullbacks of subspaces are best defined in terms of $\{(x, [x]) : x \in S^\perp\}$, which is an odd lagrangian relation [3] (i.e. a maximal isotropic subbundle of $E \times (S^\perp/S)$, with the symmetric bilinear form in the second factor multiplied by -1).

Recall that by Theorem 3.2, given a Dirac structure D the S-invariant sections of D^S are closed under the Courant bracket.

If D comes from a Poisson structure and $S \subset TM$ our definition reduces to Definition 10.4.2 of ref. [12]. In this reference the authors analyze the relation between canonical distributions and infinitesimal Poisson automorphisms. The discussion can be repeated in this context where the rôle of the infinitesimal Poisson automorphisms is played by subbundles that preserve the Dirac structure. The main result in this direction is given by the following theorem.

Theorem 3.3. *Let D be a Dirac subbundle in an exact Courant algebroid E and let $S \subset E$ be isotropic, Courant involutive (S-preserved) and such that $\pi(S^{\perp})$ is an integrable regular distribution. Then, S is canonical for D if and only if D^S is preserved by S. i.e. $[\Gamma(S), \Gamma(D^S)] \subset \Gamma(D^S)$.*

Proof: Observe that only properties of D^S are involved in Definition 3.1; hence, without loss of generality, we can assume in the proof that $D^S = D$, or equivalently, $S \subset D$.

Assuming S is canonical for D and $D^S = D$ we know that we have, at any point, a local basis of S-invariant sections for D. Then, any section $e \in \Gamma(D)$ can be written as a linear combination of the elements of this basis and then it follows from (iii) in the definition of Courant algebroid that $[\Gamma(S), e] \subset \Gamma(D)$.

For the other implication notice first that $\pi(S)$ is a regular integrable distribution since

$$\mathrm{Ker}(\pi) \cap S = \pi^*(\pi(S^{\perp})^0)$$

and given that $\pi(S^{\perp})$ is regular and π^* is injective for exact Courant algebroids, then $\mathrm{Ker}(\pi) \cap S$ is a subbundle. Now, the fact that S is also a subbundle implies the regularity of $\pi(S)$. Integrability follows from the S-preservation of S.

Under these conditions we can take a commuting basis of sections in $\pi(S)$ denoted by $\{\partial_i\}$. Let us denote by s_i an arbitrary lift of ∂_i to S, i.e. $s_i \in \Gamma(S)$ and $\pi(s_i) = \partial_i$. We define now a connection on D (actually on D restricted to any leaf of $\pi(S)$) by

$$\nabla_i e = [s_i, e].$$

The curvature of the connection, with components F_{ij}, is given by

$$F_{ij} e = \nabla_i \nabla_j e - \nabla_j \nabla_i e = [s_i [s_j, e]] - [s_j [s_i, e]] = [[s_i, s_j], e]$$

and given that ∂_i and ∂_j commute and S is S-preserved we have $[s_i, s_j] \in \mathrm{Ker}(\pi) \cap S$.

Next we want to show that

$$[\Gamma(\mathrm{Ker}(\pi) \cap S), \Gamma(D)] \subset \Gamma(\mathrm{Ker}(\pi) \cap S). \tag{3.2}$$

For that, take a section $s \in \Gamma(\mathrm{Ker}(\pi) \cap S)$ and write it as $\pi^*(\eta)$ with $\eta \in \Gamma(\pi(S^{\perp})^0)$. Also take arbitrary sections $e \in \Gamma(D)$ and $s^{\perp} \in \Gamma(S^{\perp})$. Now

$$\begin{aligned} \langle [s, e], s^{\perp} \rangle &= \langle \pi^*(\eta), [e, s^{\perp}] \rangle - i_{\pi(e)} \mathrm{d} \langle s, s^{\perp} \rangle \\ &= i_{\pi([e, s^{\perp}])} \eta \\ &= i_{[\pi(e), \pi(s^{\perp})]} \eta = 0, \end{aligned}$$

where in the last equality we have used that $D \subset S^{\perp}$ and $\pi(S^{\perp})$ is integrable.

From (3.2) it follows that $[\Gamma(S), \Gamma(\mathrm{Ker}(\pi) \cap S)] \subset \Gamma(\mathrm{Ker}(\pi) \cap S)$, so we can use ∇ to define a connection $\widetilde{\nabla}$ on $D/(\mathrm{Ker}(\pi) \cap S)$. Furthermore $\widetilde{\nabla}$ is a flat connection. The local horizontal sections for $\widetilde{\nabla}$ (that exist through any point) can be lifted to sections e_h in D that satisfy $\nabla e_h \in \Gamma(\mathrm{Ker}(\pi) \cap S)$. But the sections $\{s_i\}$ used to build the connection together with $\Gamma(\mathrm{Ker}(\pi) \cap S)$ span $\Gamma(S)$. Then,

$$[\Gamma(S), e_h] \subset \Gamma(S),$$

completing the proof. □

From the previous theorem we may derive other results:

Corollary 3.1. With the hypotheses of Theorem 3.3, if D is preserved by S then S is canonical for D.

The converse is not true, see e. g. [12] for a counterexample in the context of Poisson manifolds.

Proof of Corollary 3.1: As a consequence of axiom (iv) in the definition of Courant algebroid, S is closed under the Dirac bracket if and only if $S^{\perp}$ is S-preserved. Then, if D is preserved by S so is D^S, and invoking Theorem 3.3 the result follows. □

Actually, the proof of Theorem 3.3 shows a more general statement:

Corollary 3.2. Let $\hat{D}$ be a subbundle in an exact Courant algebroid E and S with $S \subset \hat{D} \subset S^{\perp}$ be S-preserved and such that $\pi(S^{\perp})$ is an integrable regular distribution. Then for every point of $\hat{D}$ there exists a local S-invariant section of $\hat{D}$ passing through it if and only if $\hat{D}$ is preserved by S.

4 Examples and applications of the stretching to the reduction of Dirac structures

In this section we work with exact Courant algebroids of the form E_H for some closed 3-form H.

1. Dirac bracket (or Dirac Dirac structure). The first example deals with a natural generalization to Dirac structures of the Dirac bracket for constrained Poisson manifolds, which also gives a clear geometric interpretation to the classical Dirac bracket. We will see in Remark 4.3 that, contrary to the Poisson case, in our situation no additional properties for the constraints are required.

Consider an integrable distribution $\Upsilon \subset TM$ and let $\Upsilon^0 \subset T^*M$ be its annihilator, i.e. sections of Υ^0 are the one-forms that kill all sections of Υ. Then, for any Dirac structure D on $E_H \to M$ so that $D \cap \Upsilon^0$ has constant rank, D^{Υ^0} is a Dirac subbundle such that $\pi(D^{\Upsilon^0})$ is everywhere tangent to the foliation (i.e. $\pi(D^{\Upsilon^0}) \subset \Upsilon$). That is,

$$(D^{\Upsilon^0})_p = \{(X_p, \xi_p + \nu_p) | (X_p, \xi_p) \in D_p, X_p \in \Upsilon_p, \nu_p \in {\Upsilon^0}_p\}.$$

Let us work out the Υ^0-invariant sections of D^{Υ^0}. Taking $(X, \xi + \nu) \in \Gamma(D^{\Upsilon^0})$ and $(0, \nu') \in \Gamma(\Upsilon^0)$,

$$[(0, \nu'), (X, \xi + \nu)] = (0, -i_X \mathrm{d}\nu'). \tag{4.1}$$

It is easy to show that the right-hand side of (4.1) is always a section of Υ^0. Namely, for any $X' \in \Gamma(\Upsilon)$,

$$i_{X'} i_X \mathrm{d}\nu' = \mathrm{d}\nu'(X, X') = X\nu'(X') - X'\nu'(X) - \nu'([X, X']) = 0 \tag{4.2}$$

Thus every section of D^{Υ^0} is Υ^0-invariant and due to Theorem 3.2 (which applies since Υ^0 is closed under the Courant bracket) we deduce that D^{Υ^0} is closed under the Courant bracket. In other words, D^{Υ^0} is a Dirac structure.

The Dirac structure D^{Υ^0} can be restricted to any leaf N of the foliation induced by Υ ((see [5] for the case $H = 0$, [4] for the twisted case). Let $\iota : N \to M$ be the inclusion. The image of the bundle map

$$\iota_*^{-1} \oplus \iota^* : D^{\Upsilon^0}|_N \to TN \oplus T^*N$$

defines a Dirac structure $D_N^{\Upsilon^0}$ in $TN \oplus T^*N$ twisted by ι^*H, referred to in the literature as the *pullback* of D^{Υ^0} along the inclusion ι. The isotropy of $D_N^{\Upsilon^0}$ is obviously inherited from the isotropy of D^{Υ^0}. Now, using that $\mathrm{Ker}(\iota_*^{-1} \oplus \iota^*) = \Upsilon^0|_N$ and that $\dim(D_N^{\Upsilon^0}) = \dim(D^{\Upsilon^0}) - \mathrm{Ker}(\iota_*^{-1} \oplus \iota^*)$ we deduce that $D_N^{\Upsilon^0}$ is maximal isotropic in $TN \oplus T^*N$. The proof of the closedness of the Courant bracket in $D_N^{\Upsilon^0}$ is just the proof of the closedness in D^{Υ^0} given above (recall at this point that we are assuming that D^{Υ^0} is a subbundle).

Remark 4.1. The stretched Dirac structure D^{Υ^0} can be also described as follows: for any leaf $\iota : N \to M$ take the pullback of D (along ι) and then its pushforward [4](again along ι). However in this description the smoothness of D^{Υ^0} is less transparent.

Remark 4.2. The construction of the stretched Dirac structure D^{Υ^0} and its reduction to the leaves of the foliation integrating Υ generalize the contruction of the Dirac bracket in a Poisson manifold and its reduction to the submanifolds of constraints. In the next paragraph we shall discuss this as well as the conditions for having a Poisson structure on M after the stretching process.

We recall the construction of the Dirac bracket on a Poisson manifold (M, Π). Let us consider a regular foliation on M with cosymplectic leaves, i.e. for any leaf N of the foliation $\Pi^\sharp TN^0 \oplus TN = TM|_N$. This implies that on an open set $U \subset M$ the leaves of the foliation can be obtained as the level sets of a family of second class constraints $\varphi^1, \ldots, \varphi^m$ for which the matrix $C^{ab} := \{\varphi^a, \varphi^b\}_\Pi$ is invertible (with inverse C_{ab}). The formula

$$\{f, g\}_{Dirac} := \{f, g\}_\Pi - \{f, \varphi^a\}_\Pi C_{ab} \{\varphi^b, g\}_\Pi \tag{4.3}$$

defines a new Poisson bracket on U (which depends on the foliation but not on the choice of constraints) called the Dirac bracket; we denote by Π_{Dirac} the corresponding Poisson bivector. One checks easily that $\{\varphi^i, g\}_{Dirac} = 0$ for all $g \in C^\infty(U)$, i.e. that the φ^i are Casimir functions for Π_{Dirac}, hence the level sets of $(\varphi^1, \ldots, \varphi^m)$ are Poisson submanifolds (i.e. unions of symplectic leaves) w.r.t. Π_{Dirac}. As we shall see below Π and Π_{Dirac} are related by the fact that they induce the same Poisson structure on every level set of the constraints.

Now we consider the general setup where D is a Dirac structure and Υ an integrable distribution on M, and show that D^{Υ^0} has properties analogous to those of the Dirac bracket. Indeed from the explicit formula for D^{Υ^0} it is clear that $\pi(D^{\Upsilon^0}) \subset \Upsilon$, so the leaves of Υ are unions of presymplectic leaves of D^{Υ^0}. The formula for D^{Υ^0} also shows that the reduced Dirac structure $D_N^{\Upsilon^0}$ on a leaf N of Υ is equal to D_N, the structure induced by the original Dirac structure D.

The Dirac structure $D^{\Upsilon^0} \subset E_H$ is the graph of a bivector field (which will be a twisted Poisson structure due to Courant involutivity) if and only if at every point of M

$$D^{\Upsilon^0} + TM = TM \oplus T^*M. \tag{4.4}$$

Taking the orthogonal of (4.4) we get the more familiar (and equivalent) condition

$$(D + \Upsilon^0) \cap \Upsilon = \{0\}. \tag{4.5}$$

If D itself comes from a Poisson structure Π, (4.5) can be rewritten as $\Pi^\sharp(\Upsilon^0) \cap \Upsilon = \{0\}$. This last condition means that the leaves of the distribution Υ are (pointwise) Poisson-Dirac submanifolds of (M, Π) (see Proposition 6 of [6]). Now assume $H = 0$ and the stronger condition $\Pi^\sharp(\Upsilon^0) \oplus \Upsilon = TM$ (on some open subset $U \subset M$), which means that the leaves of Υ are cosymplectic submanifolds of (M, Π), and let $\varphi^1, \dots, \varphi^m$ be functions whose level sets are the leaves of Υ. With this data the Dirac bracket (4.3) can be defined, giving rise to a Poisson structure Π_{Dirac} on U.

We claim that Π_{Dirac} is given exactly by the stretched Dirac structure D^{Υ^0}, and therefore our construction generalizes the classical Dirac bracket. We saw that the leaves of Υ are Poisson submanifolds for both D^{Υ^0} and Π_{Dirac}, so it is sufficient to make sure that pulling back D^{Υ^0} and $graph(\Pi_{Dirac})$ to each leaf N gives identical Poisson structures. We saw above that the pullback of D^{Υ^0} agrees with the pullback of D, which (since N is cosymplectic w.r.t. Π) is characterized as follows [6]: the Poisson bracket of functions f, g on N is the restriction to N of $\{\tilde{f}, \tilde{g}\}_\Pi$, where we take extensions of our functions to M and $d\tilde{f}$ is required to annihilate $\Pi^\sharp(TN^0)$ at points of N. This agrees with the Poisson bracket on N induced by the Dirac bracket (4.3) since, for an extension $\tilde{f}$ as above we have $\{\tilde{f}, \varphi^a\}_\Pi = 0$ for all constraints φ^a. This concludes the proof that D^{Υ^0} and $graph(\Pi_{Dirac})$ agree, and also proofs the claim made just after (4.3).

Remark 4.3. Even within the framework of Poisson geometry, i.e. in the case that both D and D^{Υ^0} correspond to Poisson structures, our construction is more general than the classical Dirac bracket: we do not need to assume that the constraints be second class (but just that they define Poisson-Dirac submanifolds), and in the case of second class constraints what we use are not the constraints themselves but just their level sets.

2. Projection along an integrable distribution. Now, let $\Theta \subset TM$ be an integrable distribution. We assume that H descends to M/Θ, the space of leaves of the foliation defined by Θ (assuming that M/Θ is a manifold). Given that H is closed, our assumption amounts to demand, $i_Y H = 0, \forall Y \in \Gamma(\Theta)$. This, in turn, ensures that Θ is Courant involutive.

D^Θ is not Courant involutive in general. This is not strange since one would expect to be able to define a Dirac structure only on M/Θ. Objects on M which descend suitably to M/Θ will be said *projectable along* Θ. Functions on M/Θ, $C^\infty(M)_{\mathrm{pr}}$, can be viewed as the set

$$C^\infty(M)_{\mathrm{pr}} = \{f \in C^\infty(M) \mid X(f) = 0, \forall X \in \Gamma(\Theta)\}.$$

Vector fields on M/Θ are then defined as derivations on $C^\infty(M)_{\mathrm{pr}}$, i.e. $\mathfrak{X}(M)_{\mathrm{pr}} = \{X \in \mathfrak{X}(M) \mid X(C^\infty(M)_{\mathrm{pr}}) \subset C^\infty(M)_{\mathrm{pr}}\}$. Notice that X belongs to $\mathfrak{X}(M)_{\mathrm{pr}}$ if and only if $Z(X(f)) = 0, \forall f \in C^\infty(M)_{\mathrm{pr}}, \forall Z \in \Gamma(\Theta)$. Or equivalently, $[Z, X](f) = (\mathcal{L}_Z X)(f) = 0$ using that $Z(f) = 0$. Hence, we obtain the more useful characterization

$$\mathfrak{X}(M)_{\mathrm{pr}} = \{X \in \mathfrak{X}(M) \mid \mathcal{L}_Z X \in \Gamma(\Theta), \forall Z \in \Gamma(\Theta)\}.$$

Analogously, 1-forms on M/Θ, $\Omega^1(M)_{\mathrm{pr}}$, are linear maps from $\mathfrak{X}(M)_{\mathrm{pr}}$ to $C^\infty(M)_{\mathrm{pr}}$. An analogous argument to that followed for vector fields yields:

$$\Omega^1(M)_{\mathrm{pr}} = \{\xi \in \Gamma(\Theta^0) \mid \mathcal{L}_Z \xi = 0, \forall Z \in \Gamma(\Theta)\}.$$

Now, we are ready to prove that the set of sections $(X + Y, \xi) \in \Gamma(D^\Theta)$ which are projectable along Θ, i.e.

a) ξ is a section of Θ^0,
b) $\forall Y' \in \Gamma(\Theta)$, $(\mathcal{L}_{Y'}(X + Y), \mathcal{L}_{Y'}\xi) \in \Gamma(\Theta)$

coincides with the set of Θ-invariant sections of D^Θ. Let $(X+Y, \xi)$ be a section of D^Θ and $(Y', 0)$ be a section of Θ. It is clear that

$$[(Y', 0), (X + Y, \xi)] = ([Y', X + Y], \mathcal{L}_{Y'}\xi)$$

belongs to $\Gamma(\Theta)$ if and only if $(X + Y, \xi)$ is projectable along Θ. Invoking again Theorem 3.2 we automatically obtain that the set of projectable sections of D^Θ is closed under the Courant bracket.

Now if Θ is canonical for D, i.e. if for any point $p \in M$ and any $(X_p, \xi_p) \in (D^\Theta)_p$ there exists a (local) projectable section of D^Θ such that it coincides with (X_p, ξ_p) at p, we can project the Dirac subbundle onto M/Θ. Given the projection $\rho : M \to M/\Theta$ the image of

$$\rho_* \oplus (\rho^*)^{-1} : D^\Theta \to T(M/\Theta) \oplus T^*(M/\Theta),$$

which is independent of the point of the fiber, correctly defines a Dirac structure on M/Θ. The twist in this case is given by the only three form $\tilde{H}$ in M/Θ such that $\rho^*\tilde{H} = H$, whose existence is guaranteed by the properties of H.

The Dirac structure on M/Θ is obtained pointwise as the pushforward of D (or D^Θ) along the projection $M \to M/\Theta$. The above discussions shows that, if Θ is canonical for D, the pushforward gives a *welldefined* Dirac structure on the quotient.

For a study of this construction in the context of abstract exact Courant algebroids (i.e. without making a choice of isotropic splitting) see [15].

3. Restriction and projection. The previous two examples can be combined. Take two integrable distributions Θ and Υ such that $\Theta \subset \Upsilon$ and $i_Y H = 0$, $\forall Y \in \Gamma(\Theta)$. Then, $S = \Theta \oplus \Upsilon^0$ is isotropic and closed under the Courant bracket. Using the result of Proposition 3.2, the stretching of D can be made in any order of Θ and Υ^0 or all at once and we get the same result:

$$D_p^S = \{(X_p + Y_p, \xi_p + \eta_p) | (X_p, \xi_p) \in D_p \cap (\Upsilon_p \oplus \Theta_p^0), Y_p \in \Theta_p, \eta_p \in \Upsilon_p^0\}.$$

On the other hand, S-invariant sections (X,ξ) of D^S are characterized by:

$$\mathcal{L}_Y X \in \Gamma(\Theta), \quad \mathcal{L}_Y \xi \in \Gamma(\Upsilon^0), \forall Y \in \Gamma(\Theta).$$

Note that due to the second condition above, S-invariant sections are not, in general, projectable onto M/Θ. Instead we can pullback the sections in M to a leaf N of the foliation induced by Υ like in Example 1, and the resulting sections are indeed projectable along Θ_N (the restriction of Θ to N; recall that $\Theta \subset \Upsilon$) onto N/Θ_N.

It is interesting to note that although the stretching of the Dirac structure by Θ and Υ^0 can be made in any order, the interpretation of the S-invariant sections suggests a definite order in the reduction procedure: first reduce to a leaf of the foliation induced by Υ and then project to the orbit space induced on the leaf.

4. B-transform and projection. We consider now a different generalization of Example 2. Given $\omega \in \Omega^2(M)$ let the fibers of $S \subset E_H$ be given by

$$S_p = \{(Y_p, i_{Y_p}\omega_p) |\ Y_p \in \Theta_p\}$$

for every $p \in M$, where Θ is an integrable distribution and $H + \mathrm{d}\omega$ is projectable along Θ fibers, i.e. $i_Y(H + \mathrm{d}\omega) = 0$ for any $Y \in \Gamma(\Theta)$.

Under these assumptions sections of S close under the Courant bracket. The orthogonal space of S is easily seen to be

$$S_p^{\perp} = \{(X_p, \xi_p) |\ X_p \in T_pM,\ \xi_p - i_{X_p}\omega_p \in \Theta_p^0\}$$

so that

$$D_p^S = \{(X_p, \xi_p) + (Y_p, i_{Y_p}\omega) |\ (X_p, \xi_p) \in D_p,\ \xi_p - i_{X_p}\omega_p \in \Theta_p^0,\ Y_p \in \Theta_p\}$$

A section $(X,\xi) \in \Gamma(D^S)$ is S-invariant if, for any vector field $Y \in \Gamma(\Theta)$, $[(Y, i_Y\omega), (X,\xi)] = (Y', i_{Y'}\omega)$ for some $Y' \in \Gamma(\Theta)$. On the other hand, it is clear that

$$\begin{aligned}
[(Y, i_Y\omega), (X,\xi)] &= ([Y,X], \mathcal{L}_Y\xi - i_X\mathrm{d}(i_Y\omega) + i_X i_Y H) \\
&= ([Y,X], \mathcal{L}_Y\xi - \mathcal{L}_X(i_Y\omega) + \mathrm{d}(i_X i_Y\omega) + i_X i_Y H) \\
&= ([Y,X], \mathcal{L}_Y\xi + i_{[Y,X]}\omega - i_Y\mathrm{d}(i_X\omega) - \mathrm{d}(i_Y i_X\omega)) \\
&= ([Y,X], i_{[Y,X]}\omega) + (0, \mathcal{L}_Y(\xi - i_X\omega))
\end{aligned}$$

where we have used that $i_X i_Y(\mathrm{d}\omega + H) = 0$ for any $Y \in \Gamma(\Theta)$. Hence, $(X,\xi) \in \Gamma(D^S)$ is S-invariant if and only if

$$\mathcal{L}_Y X \in \Gamma(\Theta), \quad \mathcal{L}_Y(\xi - i_X\omega) = 0, \forall Y \in \Gamma(\Theta).$$

An alternative way to perform these computations is to apply a $-\omega$-transformation to E_H, and use $\tau_{-\omega}S = \Theta$, $\tau_{-\omega}(D^S) = (\tau_{-\omega}D)^{\Theta}$ (see Proposition 3.1) to reduce the computation to Example 2 above.

The geometric meaning of these conditions is the following: Given a S-invariant section (X,ξ) then $\tau_{-\omega}(X,\xi)$ is projectable along the distribution Θ onto M/Θ. Then if S is canonical for D one can project $\tau_{-\omega}(D^S) \subset E_{H+\mathrm{d}\omega}$ onto M/Θ defining a Dirac structure on the orbit space.

5. A variant of Marsden-Ratiu reduction. Our last example is actually the motivation of the present work. It is a version of the Poisson reduction by distributions of Marsden and Ratiu [10] extended to the case of Dirac structures. The Marsden and Ratiu procedure can be expressed in words as the reduction to a submanifold N along a distribution Θ which is canonical for the Poisson structure.

In our case we start with an integrable distribution Θ and a second integrable distribution Υ s. t. one of its leaves is the submanifold N. We assume that $\Phi = \Theta \cap TN$ is a regular distribution (its integrability is a consequence of that of Θ) and the space of leaves N/Φ is a regular manifold.

The reduction then proceeds by considering first

$$D_N = \iota_*^{-1} \oplus \iota^*((D^\Theta)^{\Upsilon^0})$$

where $\iota : N \to M$ is the inclusion map. This produces a Dirac subbundle in N that contains Φ. Assuming that Φ is canonical for D_N one can project the latter to N/Φ to obtain a Dirac subbundle $D_{N/\Phi}$ in the quotient. The problem now is under which conditions $D_{N/\Phi}$ is integrable; the arguments used in Example 2 do not apply because D_N is not necessarily involutive. We will show below that the integrability is guaranteed if Θ is canonical for D. Before addressing this problem we would like to remark that although the initial data include the distributions Θ and Υ, the final Dirac subbundle depends only on the submanifold N and on the restriction of Θ to N; we need the distributions Θ and Υ in order to phrase in terms in D the integrability condition for Dirac subbundle $D_{N/\Phi}$.

In order to study the question of integrability of $D_{N/\Phi}$ we can use a combination of the previous examples. Let us assume that M/Θ is smooth and that there exists an smooth embedding ι' that makes the following diagram

$$\begin{tikzcd} N \arrow[r, "\iota"] \arrow[d, "\rho'"] & M \arrow[d, "\rho"] \\ N/\Phi \arrow[r, "\iota'"'] & M/\Theta \end{tikzcd} \tag{4.6}$$

commutative. The plan is to perform the reduction going the other way: we start form D^Θ, project it to M/Θ and then reduce to N/Φ using ι'. We have to show first that fiberwise the two procedures give the same result.

Given a point $m \in N$ the fiber over it in $(D^\Theta)^{\Upsilon^0}$ is given by

$$\begin{aligned} ((D^\Theta)^{\Upsilon^0})_m &= \{(X+Y, \xi+\eta) | (X,\xi) \in D_m \cap (TM + \Theta^0)_m, \\ &\qquad Y \in \Theta_m, \eta \in \Upsilon^0_m, X+Y \in \Upsilon_m\} \end{aligned}$$

and the reduction to N

$$\begin{aligned} (D_N)_m &= \{(X+Y, \xi|_{\Upsilon_m}) | (X,\xi) \in D_m \cap (TM + \Theta^0)_m, \\ &\qquad Y \in \Theta_m, X+Y \in \Upsilon_m\}. \end{aligned}$$

Now the projection by ρ' is

$$\begin{aligned}\rho'_* \oplus (\rho'^*)^{-1}(D_N)_m &= \{(X+Y+\Phi_m, \xi|_{\Upsilon_m}) | (X,\xi) \in D_m \cap (TM+\Theta^0)_m, \\ &\qquad Y \in \Theta_m, X+Y \in \Upsilon_m\}\end{aligned}$$

where vectors in $T_{\rho'(m)}(N/\Phi)$ are identified with elements of the quotient space $(T_m N)/\Phi_m$.

Going the other way in the commutative diagram, we first project the fiber $(D^\Theta)_m$ by ρ to obtain

$$\rho_* \oplus (\rho^*)^{-1}(D^\Theta)_m = \{(X+Y_m, \xi) | (X,\xi) \in D_m \cap (TM+\Theta^0)_m, Y \in \Theta_m\}.$$

And the pullback by ι' to N/Φ gives

$$\{((X+\Theta_m) \cap \Upsilon_m, \xi|_{\Upsilon_m}) | (X,\xi) \in D_m \cap (TM+\Theta^0)_m, (X+\Theta_m) \cap \Upsilon_m \neq \emptyset\}$$

which is exactly $\rho'_* \oplus (\rho'^*)^{-1}(D_N)_m$ as defined above.

We consider now the case in which Θ is canonical for D, i.e. the projection of D^Θ (which is smooth if we assume $D \cap \Theta$ to have constant rank) by ρ is well defined (the projected fiber does not depend on the point m we start with). In this case the reduction sketched in the previous paragraph produces a Dirac structure on M/Θ by Example 2, and the pullback to N/Φ (if smooth) also defines a Dirac structure in the final space. Thus we have shown that the first construction also provides a well-defined Dirac structure, i.e. that the Dirac subbundle $D_{N/\Phi}$ is actually integrable.

Remark 4.4. A similar construction in the setting of Poisson structures appeared in [8].

Remark 4.5. It is interesting to note that the final Dirac structure only depends on the restriction of Θ to the submanifold N, as it is clear in the first reduction procedure. This is actually the spirit of the original approach to the problem in ref. [10] in the Poisson context (see also ref. [12] for a recent review). Therein, instead of a distribution in M one starts with a subbundle of $T_N M$. A more detailed study of the possible generalizations of the original Marsden and Ratiu construction to the context of Dirac structures will appear elsewhere.

5 Conclusions.

To summarize, the stretching of Dirac structures is an interesting type of deformation which, on the one hand, generalizes the Dirac bracket to any Dirac structure when the stretching is made along a subbundle of T^*M (concretely, the annihilator of an integrable distribution). On the other hand, if the stretching is performed along an integrable distribution which is canonical for the Dirac structure, then our construction corresponds to its projection along the integrable distribution. Using these tools we present a way to perform a reduction of the Marsden-Ratiu type in the context of Dirac structures. In Theorem 3.3 and its corollaries, we have clarified the relation between the different types of 'symmetries' in this context.

As pointed out throughout the paper, our work is closely related to ref. [3] even though we only deal with Dirac structures whereas [3] considers more general situations. Our approach to the subject is, however, different, because it involves two steps: first we deform the Dirac structure D inside the initial Courant algebroid using the symmetry S, then we construct the reduced Dirac structure from the deformed one. In addition, we require the existence of sections invariant under the symmetry only at points of $D \cap S^\perp$ (see Def. 3.1), and not at all points of D. Another benefit of our approach is that we obtain a new Dirac subbundle (the deformation of the original one) which not only reduces in a natural way, but may also have interest by itself. A paradigmatic case is when the deformation is performed along a subbundle of the cotangent bundle, yielding a generalization of the construction of the Dirac bracket. Also the presentation of the Marsden-Ratiu reduction as the combination of two consecutive deformations is particularly clean and simple.

Regarding this last point, it should be noted that our approach to Marsden-Ratiu reduction differs from the original one [10] in an important aspect: for our reduction to work we need the distribution to be canonical in a neighborhood of the submanifold, not just on the submanifold itself as is assumed in [10] (see also [12]). It is natural to guess that it should be possible to perform our reduction with weaker assumptions so that the final Dirac subbundle is integrable. We will try to elucidate this issue in our future research.

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Contact reduction and groupoid actions

Marco Zambon and Chenchang Zhu

Abstract

We introduce a new method to perform reduction of contact manifolds that extends Willett's and Albert's results. To carry out our reduction procedure all we need is a complete Jacobi map $J : M \to \Gamma_0$ from a contact manifold to a Jacobi manifold. This naturally generates the action of the contact groupoid of Γ_0 on M, and we show that the quotients of fibers $J^{-1}(x)$ by suitable Lie subgroups Γ_x are either contact or locally conformal symplectic manifolds with structures induced by the one on M.

We show that Willett's reduced spaces are prequantizations of our reduced spaces; hence the former are completely determined by the latter. Since a symplectic manifold is prequantizable iff the symplectic form is integral, this explains why Willett's reduction can be performed only at distinguished points. As an application we obtain Kostant's prequantizations of coadjoint orbits [13]. Finally we present several examples where we obtain classical contact manifolds as reduced spaces.

Contents

1 Introduction

Marsden and Weinstein introduced symplectic reduction in 1974 [15]. Since then, the idea of reduction has been applied in many geometric contexts. In the realm of contact geometry, two different reduction procedures for contact Hamiltonian actions were developed by Albert [1] in 1989 and Willett [19] in 2002. However neither method is as natural as the classical Marsden-Weinstein reduction: the contact structure of Albert's reduction depends on the choice of the contact 1-form; Willett's requires additional conditions on the reduction points. In this paper we perform contact reduction via contact groupoids, following the idea of Mikami and Weinstein [16] who generalized the classical symplectic reduction to reduction via so-called symplectic groupoids.

Our approach not only puts both Albert's and Willett's reduction into one unified framework, but also delivers a structure on the reduced space which is independent of the choice of the contact 1-form and can be performed at all points. Moreover, to carry out our reduction, we only need a "complete Jacobi map". We will elaborate below.

We first describe the way to recover Willett's reduction from ours. Given a Hamiltonian action of a group G on a contact manifold (M, θ_M) as in [19], we can associate the action of a contact groupoid on M, for which we are able to perform reduction . If for simplicity we assume that G is compact then our reduced spaces are always symplectic manifolds, and we have

> **Result I:** (Theorem 5.4) Willett's reduced spaces are prequantizations of our reduced (via groupoids) spaces.

Since we can realize coadjoint orbits as our reduced spaces, this allows us to construct prequantizations of coadjoint orbits, hence reproducing the results of Kostant's construction [13]. As an example with $G = U(2)$, by our reduction, we obtain certain lens spaces as prequantizations of S^2.

Let us now outline our reduction procedure via groupoids. We first have to introduce some terminology, which will be defined rigurously in Section 2.

Groupoids are generalizations of groups and are suitable to describe geometric situations in a global fashion.

Jacobi manifolds [14] arise as generalizations of Poisson manifolds and include contact manifolds. Exactly as Poisson manifolds are naturally foliated by symplectic leaves, Jacobi manifolds are foliated by two kinds of leaves: the odd dimensional ones are contact manifolds, and the even dimensional ones are so-called locally conformal symplectic (l.c.s.) manifolds.

Given a Jacobi manifold, one can associate to it a *contact groupoid* (i.e. a groupoid with a compatible contact structure), which one can view as the "global object" corresponding to the Jacobi structure.

In analogy to the well-known fact in symplectic geometry that the moment map allows one to reconstruct the corresponding Hamiltonian action, we have the following result:

> **Result II** (Theorem 3.8): Any complete Jacobi map J which is a surjective submersion from a contact manifold (M, θ_M) to a Jacobi manifold Γ_0 naturally induces a contact groupoid action of the contact groupoid Γ of Γ_0 on M.

Using the notation above our main result on reduction is:

> **Result III** (Theorem 4.1): Let the contact groupoid Γ act on (M, θ_M) by contact groupoid action. Suppose that $x \in \Gamma_0$ is a regular value of J and that Γ_x acts freely and properly on $J^{-1}(x)$ (here $\Gamma_x \subset \Gamma$ is the isotropy group at x). Then the reduced space $M_x := J^{-1}(x)/\Gamma_x$ has an induced
>
> 1. contact structure, if x belongs to a contact leaf
> 2. conformal l.c.s. structure, if x belongs to a l.c.s. leaf.

This is the point-wise version of a result about global reduction: the quotient of a contact manifold by the action of a contact groupoid is naturally a Jacobi manifold, the leaves of which are the above reduced spaces M_x (therefore not necessarily contact). This shows that performing any natural reduction procedure on a contact manifold one should not expect to obtain contact manifolds in general.

Notice that combining the two results above we are able to obtain contact manifolds by reduction starting with a simple piece of data, namely a complete Jacobi map, without even mentioning groupoids.

The paper is structured as follows: in Section 2 we introduce the basic terminology. In Section 3 we prove Result II and in Section 4 we prove our point-wise reduction procedure (Result III) as well as our global reduction.

Section 5 contains the results about Willett's and Albert's reduced spaces and prequantization, and can be read independently[1] of the previous sections. Finally, in Section 6 we give some simple concrete examples (such as cosphere-bundles) of contact manifolds obtained via groupoid reduction.

In Appendix I we show that the structures on our reduced spaces do not depend on the choice of contact form θ_M on M but only on the corresponding contact structure, and in Appendix II we explain how the conventions we adopt relate to other conventions found in the literature. We hope this will make the literature on Jacobi manifolds and contact groupoids more easily accessible.

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2 Basic Terminology

In this section we introduce Jacobi manifolds and their global counterparts, namely contact groupoids.

[1] More precisely: Section 5 requires only the definition of contact groupoid together with two examples (Section 2.2), the definition of contact groupoid action (Definition 3.1) and the statement of our point-wise reduction result (Theorem 4.1).

2.1 Jacobi manifolds

A *Jacobi manifold* is a smooth manifold M with a bivector field Λ and a vector field E such that

$$[\Lambda, \Lambda] = 2E \wedge \Lambda, \quad [\Lambda, E] = 0, \tag{1}$$

where $[\cdot,\cdot]$ is the usual Schouten-Nijenhuis brackets. A Jacobi structure on M is equivalent to a "local Lie algebra" structure on $C^\infty(M)$ in the sense of Kirillov [**?**], with the bracket,

$$\{f, g\} = \sharp\Lambda(df, dg) + fE(g) - gE(f) \quad \forall f, g \in C^\infty(M).$$

We call it a Jacobi bracket on $C^\infty(M)$. It is a Lie bracket satisfying the following equation (instead of the Leibniz rule, as Poisson brackets):

$$\{f_1 f_2, g\} = f_1\{f_2, g\} + f_2\{f_1, g\} - f_1 f_2\{1, g\}, \tag{2}$$

i.e. it is a first order differential operator on each of its arguments. If $E = 0$, (M, Λ) is a Poisson manifold.

Recall that a *contact manifold*[2] is a $2n+1$-dimensional manifold equipped with a 1-form θ such that $\theta \wedge (d\theta)^n$ is a volume form. If (M, Λ, E) is a Jacobi manifold such that $\Lambda^n \wedge E$ is nowhere 0, then M is a contact manifold with the contact 1-form θ determined by

$$\iota(\theta)\Lambda = 0, \quad \iota(E)\theta = 1,$$

where ι is the contraction between differential forms and vector fields. On the other hand, given a contact manifold (M, θ), let E be the Reeb vector field of θ, i.e. the unique vector field satisfying

$$\iota(E)d\theta = 0, \quad \iota(E)\theta = 1.$$

Let μ be the map $TM \to T^*M$, $\mu(X) = -\iota(X)d\theta$. Then μ is an isomorphism between $\ker(\theta)$ and $\ker(E)$, and can be extended to their exterior algebras. Let $\Lambda = \mu^{-1}(d\theta)$. (Note that if $\iota(E)d\theta = 0$, then $d\theta$ can be written as $\alpha \wedge \beta$ and $\iota(E)\alpha = \iota(E)\beta = 0$.) Then E and Λ satisfy (1). So a contact manifold is always a Jacobi manifold [14]. Notice that in this case the map $\sharp\Lambda : T^*M \longrightarrow TM$ given by $\sharp\Lambda(X) = \Lambda(X, \cdot)$ and the map μ above are inverses of each other when restricted to $\ker(\theta)$ and $\ker(E)$.

A *locally conformal symplectic manifold* (*l.c.s. manifold* for short) is a $2n$-dimensional manifold equipped with a non-degenerate two-form Ω and a closed one-form ω such that $d\Omega = \omega \wedge \Omega$. To justify the terminology notice that locally $\omega = df$ for some function f, and that the local conformal change $\Omega \mapsto e^{-f}\Omega$ produces a symplectic form. If (M, Λ, E) is a Jacobi manifold such that Λ^n is nowhere 0, then M is a l.c.s. manifold: the two-form Ω is defined so that the corresponding map $TM \longrightarrow T^*M$ is the negative inverse of $\sharp\Lambda : T^*M \longrightarrow TM$, and the one-form is given by $\omega = \Omega(E, \cdot)$. Conversely, if (Ω, ω) is a l.c.s. structure on M, then defining E and Λ in terms of Ω and ω as above, (1) will be satisfied.

A Jacobi manifold is always foliated by contact and locally conformal symplectic (l.c.s.) leaves [10]. In fact, like a Poisson manifold, the foliation of a Jacobi manifold is also given by the distribution of the Hamiltonian vector fields

$$X_u := uE + \sharp\Lambda(du).$$

[2]A related concept is the following: a *contact structure* on the manifold M is a choice of hyperplane $\mathcal{H} \subset TM$ such that locally $\mathcal{H} = \ker(\theta)$ for some one-form θ satisfying $\theta \wedge (d\theta)^n \neq 0$. In this paper all contact structures will be co-orientable, so that $\mathcal{H}$ will be the kernel of some globally defined contact one form θ.

The leaf through a point will be a l.c.s. (contact) leaf when E lies (does not lie) in the image of $\sharp\Lambda$ at that point.

Given a nowhere vanishing smooth function u on a Jacobi manifold (M, Λ, E), a conformal change by u defines a new Jacobi structure:

$$\Lambda_u = u\Lambda, \qquad E_u = uE + \sharp\Lambda(du) = X_u.$$

We call two Jacobi structures equivalent if they differ by a conformal change. A *conformal Jacobi structure* on a manifold is just an equivalence class of Jacobi structures[3]. The relation between the Jacobi brackets induced by the u-twisted and the original Jacobi structures is given by

$$\{f, g\}_u = u^{-1}\{uf, ug\}.$$

The relation between the Hamiltonian vector fields is given by

$$X_f^u = X_{u \cdot f}.$$

A smooth map ϕ between Jacobi manifolds (M_1, Λ_1, E_1) and (M_2, Λ_2, E_2) is a *Jacobi morphism* if

$$\phi_*\Lambda_1 = \Lambda_2, \qquad \phi_* E_1 = E_2,$$

or equivalently if $\phi_*(X_{\phi^* f}) = X_f$ for all functions f on M_2. Given $u \in C^\infty(M_1)$, a u *conformal Jacobi morphism* from a Jacobi manifold (M_1, Λ_1, E_1) to (M_2, Λ_2, E_2) is a Jacobi morphism from $(M_1, (\Lambda_1)_u, (E_1)_u)$ to (M_2, Λ_2, E_2).

2.2 Contact groupoids

Before introducing contact groupoids, let us fix our conventions about Lie groupoids [6] [17]. Throughout the paper $\Gamma \overset{\mathbf{s}}{\underset{\mathbf{t}}{\rightrightarrows}} \Gamma_0$ will be a Lie (contact) groupoid, its Lie algebroid will be identified with $\ker(d\mathbf{t})$, and the multiplication will be defined on the fiber-product $\Gamma_{\mathbf{s}} \times_{\mathbf{t}} \Gamma := \{(g, h) | \mathbf{s}(g) = \mathbf{t}(h), g, h \in \Gamma\}$[4].

Definition 2.1. A *contact groupoid* [12] is a Lie groupoid $\Gamma \rightrightarrows \Gamma_0$ equipped with a contact 1-form θ and a smooth non-vanishing function f, such that on $\Gamma_{\mathbf{s}} \times_{\mathbf{t}} \Gamma$ we have

$$^*\theta = pr_2^* f \cdot pr_1^*\theta + pr_2^*\theta, \tag{3}$$

where pr_j is the projection from $\Gamma_{\mathbf{s}} \times_{\mathbf{t}} \Gamma \subset \Gamma \times \Gamma$ onto the j-th factor.

Remark 2.2. Let us recall some useful facts from [12], [9], and [8] about contact groupoids:

a) A contact groupoid $\Gamma \rightrightarrows \Gamma_0$ induces a Jacobi structure on its base manifold. We denote the vector fields and bivector fields defining the Jacobi structures by E_Γ, E_0 and $\Lambda_\Gamma, \Lambda_0$ respectively.

b) With respect to this Jacobi structure the source map $\mathbf{s}$ is Jacobi morphism and the target $\mathbf{t}$ is $-f$-conformal Jacobi (See also Appendix II).

[3] Clearly a conformal contact manifold is just a manifold with a coorientable contact structure.

[4] Also see Definition 3.1.

c) On the other hand, for certain Jacobi manifolds Γ_0, there is a unique contact groupoid $\Gamma \rightrightarrows \Gamma_0$ with connected, simply connected $\mathbf{t}$-fibers (or equivalently, $\mathbf{s}$-fibres) satisfying b). In this case, we call Γ_0 *integrable*. Integrability conditions of Jacobi manifolds are studied in detail in [8].

d) Furthermore, at any $g \in \Gamma$, the kernels of $T\mathbf{s}$ and $T\mathbf{t}$ are given by ([9])

$$\ker T_g\mathbf{t} = \{X_{\mathbf{s}^*u}(g) : u \in C^\infty(\Gamma_0)\}$$
$$\ker T_g\mathbf{s} = \{X_{f\cdot\mathbf{t}^*u}(g) : u \in C^\infty(\Gamma_0)\}.$$

e) The function f in Definition 2.1 is automatically multiplicative, i.e. $f(gh) = f(g)f(h)$ for all composable $g, h \in \Gamma$. Furthemore, f satisfies $df(E_\Gamma) = 0$.

f) The constructions of this paper admit a version that involves only contact structures and is independent of contact forms. Interested readers are referred to Appendix I.

Example 2.3. [Contact groupoid of $S(\mathfrak{g}^*)$] For a Lie group G, let $\mathfrak{g}^*$ be the dual of its Lie algebra $\mathfrak{g}$. Choose any Riemannian metric on it, then the quotient space $S(\mathfrak{g}^*) := (\mathfrak{g}^* - 0)/\mathbb{R}^+$ is a Jacobi manifold[5] ([14] and [10]). The "Poissonization" of $S(\mathfrak{g}^*)$ is the Poisson manifold $\mathfrak{g}^* - 0$.

In particular, when G is compact, we can choose a bi-invariant metric, then $S(\mathfrak{g}^*)$ can be embedded in $\mathfrak{g}^*$ as the unit sphere which is Poisson with the restricted Poisson structure because all the symplectic leaves—the coadjoint orbits— will stay in the sphere. In this case, the contact groupoid of $S(\mathfrak{g}^*)$ is $(U^*G, \theta_c, 1)$, where U^*G is the set of covectors of length one and θ_c is the restriction of the canonical 1-form to the cosphere bundle (see Example 6.8 of [4]). Recall that the groupoid structure is given by

$$\mathbf{t}(\bar\eta) = R_g^*\bar\eta, \quad \mathbf{s}(\bar\eta) = L_g^*\bar\eta,$$
$$\bar\eta_1 \cdot \bar\eta_2 = \tfrac{1}{2}(R^*_{g_2^{-1}}\bar\eta_1 + L^*_{g_1^{-1}}\bar\eta_2) \in U^*_{g_1g_2}G$$

where $\bar\eta \in U_g^*G$, $\bar\eta_i \in U^*_{g_i}G$, and R_g,L_g we denote the right and left translations by g. Identifying U^*G and $S(\mathfrak{g}^*) \rtimes G$ by right translations, i.e. identifying a covector $R^*_{g^{-1}}\xi$ at g with (ξ, g), the contact groupoid structure is given by

$$\mathbf{t}(\xi, g) = \xi, \qquad \mathbf{s}(\xi, g) = L_g^*R^*_{g^{-1}}\xi,$$
$$(\xi_1, g_1)\cdot(\xi_2, g_2) = (\xi_1, g_1g_2), \qquad \theta_c(\delta\xi, \delta g)_{(\xi,g)} = \langle \xi, R_{g^{-1}*}\delta g\rangle.$$

For a general Lie group G, the symplectification of the quotient cosphere bundle $S^*G := (T^*G - G)/\mathbb{R}^+$ is $T^*G - G$, which is exactly the symplectic groupoid of $\mathfrak{g}^* - 0$—the Poissonization of $S(\mathfrak{g}^*)$. By the main result in [8] $(T^*G - G)/\mathbb{R}^+$ is the contact groupoid of $S(\mathfrak{g}^*)$ with contact 1-form θ and function f which, using the trivilization by right translations, are given by

$$\theta(\delta\xi, \delta g)_{([\xi],g)} = \frac{\langle \xi, R_{g^{-1}*}\delta g\rangle}{\|L_g^*R^*_{g^{-1}}\xi\|}, \quad f([\xi], g) = \frac{\|\xi\|}{\|L_g^*R^*_{g^{-1}}\xi\|},$$

where $[\cdot]$ denotes the equivalence class under the $\mathbb{R}^+$ action. The groupoid structure is inherited from T^*G (very similar to the compact case we have just presented and also see the examples in [8]).

[5]Its structure depends on the metric.

Example 2.4. [Contact groupoid of $\mathfrak{g}^*$] Using the same notation as the last example, we view the Poisson manifold $\mathfrak{g}^*$ as a Jacobi manifold. Then the contact groupoid of $\mathfrak{g}^*$ is $(T^*G \times \mathbb{R}, 1, \theta_c + dr)$, where θ_c is the canonical 1-form on T^*G and dr is the 1-form on $\mathbb{R}$. (The proof is similar to the one of Theorem 4.8 in [8]).

Identifying $T^*G \times \mathbb{R}$ with $\mathfrak{g}^* \rtimes G \times \mathbb{R}$ by right translation the groupoid structure is given by

$$\mathbf{t}(\xi, g, r) = \xi, \quad \mathbf{s}(\xi, g, r) = L_g^* R_{g^{-1}}^* \xi,$$
$$(\xi_1, g_1, r_1) \cdot (\xi_2, g_2, r_2) = (\xi_1, g_1 g_2, r_1 + r_2).$$

3 Contact groupoid actions and contact realizations

In this section, we introduce contact groupoid action and show that they can be encoded by their “moment maps”. To this aim we present a new concept—contact realizations. At the end of this section we introduce the f-multiplicative functions, which are also called reduction functions to allow us to perform reductions in the next section.

3.1 Contact groupoid actions and moment maps

Just as groups, groupoids can also act on a manifold, though in a more subtle way:

Definition 3.1. [(Contact) Groupoid Action] Let $\Gamma \rightrightarrows \Gamma_0$ be a Lie groupoid, M a manifold equipped with a moment map $J : M \to \Gamma_0$. A *groupoid (right) action* of Γ on M is a map

$$\Phi : M_J \times_t \Gamma \to M, \quad (m, g) \mapsto \Phi(m, g) := m \cdot g$$

such that

i) $J(m \cdot g) = \mathbf{s}(g)$,

ii) $(m \cdot g) \cdot h = m \cdot gh$,

iii) $m \cdot J(m) = m$, with the identification $\Gamma_0 \hookrightarrow \Gamma$ as the unit elements.

Here $M_J \times_{\mathbf{t}} \Gamma$ is the fibre product over Γ_0, that is, the pre-image of the diagonal under the map $(J, \mathbf{t}) : M \times \Gamma \to \Gamma_0 \times \Gamma_0$. Since $\mathbf{t}$ is a submersion (because Γ is a Lie groupoid), $M_J \times_{\mathbf{t}} \Gamma$ is a smooth manifold.

Given a contact groupoid $(\Gamma, \theta_\Gamma, f)$ and a contact manifold (M, θ_M), Φ is a *contact groupoid (right) action* if it is a groupoid action and additionally satisfies

$$\Phi^*(\theta_M) = pr_\Gamma^*(f) pr_M^*(\theta_M) + pr_\Gamma^*(\theta_\Gamma), \tag{4}$$

where pr_Γ and pr_M are projections from $M_J \times_{\mathbf{t}} \Gamma$ to Γ and M respectively. This definition is modelled so that the action of a contact groupoid on itself by right multiplication is a contact groupoid action (see equation (3)).

Remark 3.2.

i) The moment map $J : \Gamma \to M$ of any groupoid action is equivariant ([16]).

ii) A groupoid action is *free* if there is no fixed point; a groupoid action is *proper* if the following map is proper:

$$M_J \times_{\mathbf{t}} \Gamma \to M \times M \quad \text{given by} \quad (m, g) \mapsto (m, m \cdot g). \tag{5}$$

The following Lemma gives an alternative, more geometrical characterization of contact groupoid action.

Lemma 3.3. *Let Φ be an action of the contact groupoid $(\Gamma, \theta_\Gamma, f)$ on the contact manifold (M, θ_M). Then Φ is a contact groupoid action if and only if the graph of Φ is a Legendrian submanifold of the contact manifold*

$$(M \times \mathbb{R} \times \Gamma \times \mathbb{R} \times M, -fe^{-a}\theta_1 - e^{-b}\theta_\Gamma + \theta_3),$$

where a and b denote the coordinates on the first and second copy of $\mathbb{R}$ respectively, θ_1 and θ_3 are the contact forms on the first and last copy of M respectively.

Proof. We denote the one form on $M \times \mathbb{R} \times \Gamma \times \mathbb{R} \times M$ by Θ. Then

$$\begin{aligned} d\Theta &= -e^{-a} df \wedge \theta_1 + fe^{-a} da \wedge \theta_1 - fe^{-a} d\theta_1 \\ &\quad + e^{-b} db \wedge \theta_\Gamma - e^{-b} d\theta_\Gamma + d\theta_3. \end{aligned}$$

One can easily check that the Reeb vector field E_3 of the last copy of M lies in the kernel of $d\Theta$, and that on the tangent space at any point of $M \times \mathbb{R} \times \Gamma \times \mathbb{R} \times M$, the form $d\Theta$ is non-degenerate on a complement of $span\{E_3\}$. Therefore Θ is indeed a contact form (with Reeb vector field E_3).

Denote the graph of Φ by $\mathcal{A}$, then the natural embedding of $\mathcal{A}$ into $M \times \mathbb{R} \times \Gamma \times \mathbb{R} \times M$ is given by $(m, g, \Phi(m, g)) \mapsto (m, 0, g, 0, \Phi(m, g))$. Suppose Γ has dimension $2n + 1$ and M dimension k. Since $\mathbf{t} : \Gamma \to \Gamma_0$ is a submersion, by a simple dimension counting, $\mathcal{A}$ has the same dimension as $\Gamma_J \times_{\mathbf{t}} M$, which has dimension $n + k + 1$. Since $M \times \mathbb{R} \times \Gamma \times \mathbb{R} \times M$ has dimension $2n + 2k + 3$, the embedding of $\mathcal{A}$ is Legendrian if and only if $\mathcal{A}$ is tangent to the contact distribution $\ker \Theta$. It is not hard to see that this condition is equivalent to Φ being a contact groupoid action from the equation

$$\Theta(Y, 0, V, 0, \Phi_*(Y, V)) = -f(g)\theta_M(Y) - \theta_\Gamma(V) + \theta_M(\Phi_*(Y, V)),$$

where $Y \in T_m M$ and $V \in T_g \Gamma$ for which $\Phi_*(Y, V)$ is defined. $\square$

The moment map of a contact groupoid action has the following nice property:

Proposition 3.4. *The moment map $J : M \to \Gamma_0$ of any contact groupoid action is a Jacobi map.*

Proof. We claim that it is enough to show that $(0, X_{\mathbf{s}^* u}, X_{J^* u})$ is in $T\mathcal{A}$, where $\mathcal{A}$ denotes the graph of Φ and we identify it as its natural embedding as in Lemma 3.3. This is equivalent to

$$0(m) \cdot X_{\mathbf{s}^* u}(g) = X_{J^* u}(m \cdot g) \tag{6}$$

for all $(m, g) \in M_J \times_{\mathbf{t}} \Gamma$, and $u \in C^\infty(\Gamma_0)$, where $0(m)$ denotes the zero vector in $T_m M$. By the definition of groupoid action and since $\mathbf{s}$ is a Jacobi map, it follows that

$$J_*(X_{J^* u}(m \cdot g)) = \mathbf{s}_*(X_{\mathbf{s}^* u}(g)) = X_u(\mathbf{s}(g)) = X_u(J(m \cdot g)).$$

Therefore we have $J_*(X_{J^*u}) = X_u$ for all $u \in C^\infty(\Gamma_0)$, which is equivalent to J being a Jacobi map.

Let $(Y, V) \in T_{(m,g)}(M_{\mathbf{t}} \times_J \Gamma)$. Using the 2-form $d\Theta$ from Lemma 3.3, we have at point $(m, 0, g, 0, m \cdot g)$,

$$\begin{aligned} & d\Theta\Big((0(m), 0, X_{\mathbf{s}^*u}(g), 0, X_{J^*u}(m \cdot g))\ ,\ (Y, 0, V, 0, Y \cdot V)\Big) \\ &= -X_{\mathbf{s}^*u}(f)\theta_M(Y) - d\theta_\Gamma(X_{\mathbf{s}^*u}, V) + d\theta_M(X_{J^*u}, Y \cdot V) \\ &= \big(f(g)\theta_M(Y) + \theta_\Gamma(V)\big) \cdot du\big((J_*E_M) - E_0\big). \end{aligned} \tag{7}$$

In the last equation, we use the fact from [9] that $\{X_f, X_{\mathbf{s}^*u}\} = 0$, and the fact that $J_*(Y \cdot V) = \mathbf{s}_*V$, and finally the fact that for a Hamiltonian vector field X_h and a vector field W, $d\theta_\Gamma(X_h, W) = -dh(W_{\mathcal{H}})$, where $W_{\mathcal{H}} = W - \theta_\Gamma(W)E_\Gamma$ is the projection of W onto $\mathcal{H} = \ker(\theta_\Gamma)$. It is easy to see that $(0, 0, X_{\mathbf{s}^*u}, 0, X_{J^*u}) \in \ker\Theta$, because

$$-\mathbf{s}^*u(g) + J^*u(m \cdot g) = 0.$$

$\mathcal{A}$ is embedded as a Legendrian submanifold by Lemma 3.3 and the vector field $(0, 0, X_{\mathbf{s}^*u}, 0, X_{J^*u})$ along $\mathcal{A}$ lies in $\ker(\Theta)$, so if it lies in $(T\mathcal{A})^{d\Theta}$—as we will show below—then it automatically lies in $T\mathcal{A}$.

Now, if $u = 1$, then (7) is clearly zero. Notice that $X_{\mathbf{s}^*1} = E_\Gamma$ and $X_{J^*1} = E_M$. So $(0, 0, E_\Gamma, 0, E_M)$ lies in $(T\mathcal{A})^{d\Theta}$, and hence in $T\mathcal{A}$. Therefore

$$J_*(E_M) = \mathbf{s}_*(E_\Gamma) = E_0,$$

which implies that (7) is 0 for all $u \in C^\infty(\Gamma_0)$. Repeating verbatim the above reasoning we conclude that $(0, 0, X_{\mathbf{s}^*u}, 0, X_{J^*u}) \in T\mathcal{A}$, as claimed.

□

With the same set-up as the last two statements, we have the following lemma.

Lemma 3.5. *The contact groupoid action is locally free at $m \in M$ iff J is a submersion at m and $T_mJ^{-1}(J(m)) \not\subset \ker(\theta_M)_m$.*

Remark 3.6. This differs from the corresponding statement for symplectic groupoid actions. In that case J is a submersion iff the action is locally free. Example 6.2 and Remark 6.3 show that the two conditions above in the contact case are both neccessary.

Proof. J being a submersion at m is equivalent to the fact that the set $\{J^*du_i(m)\}$ is linearly independent, where $u_1, \cdots, u_n$ are coordinate functions on Γ_0 vanishing at $x = J(m)$. By equation (6) the Γ-action is locally free if and only if $span\{X_{J^*1} = E_M, X_{J^*u_1}, \cdots, X_{J^*u_n}\}$ at m has dimension equal to the one of the $\mathbf{t}$-fibers, which is $n + 1$.

If we assume that J is a submersion, then the $J^*du_i(m)$'s are linearly independent. If we assume that $TJ^{-1}(x) \not\subset \ker(\theta_M)_m$, then no nontrivial linear combination $\sum a_i \cdot J^*du_i(m)$ lies in $\ker(\sharp\Lambda_M)_m = span(\theta_M)_m$ (because $TJ^{-1}(x)$ is contained in the kernel of $\sum a_i \cdot J^*du_i(m)$ but not in the kernel of θ_M). But this means that $\{X_{J^*u_1}, \cdots, X_{J^*u_n}\}$ is linearly independent at m. The independence is preserved after we add $X_{J^*1} = E_M$ to this set, so the action is free there.

Conversely, let us assume that the action is locally free at m, i.e. that $\{E_M, X_{J^*u_1}, \cdots, X_{J^*u_n}\}$ is a linearly independent set at m. Since $\sharp\Lambda_M J^* du_i = X_{J^*u_i}$, this implies that the $\{J^*u_i(m)\}$'s are linearly independent, i.e. that J is a submersion at m. This also implies that no nontrivial linear combination of the $J^*du_i(m)$ lies in $\ker(\sharp\Lambda_M)_m = span(\theta_M)_m$. Since J is a submersion, we have $\{J^*du_i\} = (T_m J^{-1}(x))^0$, so this is possible only if $T_m J^{-1}(x) \not\subset \ker(\theta_M)_m$.
□

3.2 Contact realizations and moment maps

When exactly can a map from a contact manifold M to a Jacobi manifold Γ_0 be realized as a moment map of some contact groupoid action? From Proposition 3.4, we know that the map must necessarily be a Jacobi map. To determine the remaining necessary conditions we introduce complete contact realizations.

Definition 3.7. A *contact realization* of a Jacobi manifold Γ_0 consists of a contact manifold M together with a surjective Jacobi submersion $J : M \to \Gamma_0$. A contact realization is called *complete* if X_{J^*u} is a complete vector field on M whenever u is a compactly supported function on Γ_0.

The remainder of this subsection is devoted to the proof of the following theorem:

Theorem 3.8. *Let M be a contact manifold and Γ_0 an integrable Jacobi manifold, and let $J : M \to \Gamma_0$ be a complete contact realization. Then J induces a (right) contact groupoid action of Γ on M, where Γ is the unique contact groupoid integrating Γ_0 with connected, simply connected* $\mathbf{t}$*-fibers.*

Remark 3.9. One can remove the above integrability condition on the Jacobi manifold. In fact, the existence of a complete contact realization for a Jacobi manifold is equivalent to its integrability. This will be explored in a future work.

Proof. In the first part of the proof[6], we will construct a suitable subset L of $M \times \Gamma \times M$ and show that (a natural embedding of) it is Legendrian. In the second part, we will show that L is the graph of a contact groupoid action.

Let $K = M \times \Gamma_{\mathbf{s}} \times_J M$, which is $n+2k+1$-dimensional[7]. Consider the $(n+1)$-dimensional distribution

$$D := \{(0, X_{\mathbf{s}^*u}, X_{J^*u}) | u \in C^\infty(\Gamma_0)\}.$$

Since both $\mathbf{s}$ and J are Jacobi maps, $(\mathbf{s}, J)_*(0, X_{\mathbf{s}^*u}, X_{J^*u})|_K$ is tangent to the diagonal in $\Gamma_0 \times \Gamma_0$. So $D|_K$ is tangent to K.

Claim 1: $D|_K$ defines an integrable distribution on K. We denote by F the $(n+1)$-dimensional foliation of K integrating it.
Proof: Denote by $\hat{K}$ the natural inclusion of K into the $(2k+2n+3)$-dimensional manifold $M \times \mathbb{R} \times \Gamma \times \mathbb{R} \times M$, and let $\hat{J} = \{(m, a, g, 0, m') | m \in M, a \in \mathbb{R}, \mathbf{s}(g) = J(m')\}$ (so $\dim \hat{J} = n + 2k + 2$ and $\hat{K} \subset \hat{J}$). Denote by $\hat{D}$ the distribution $\{(0, 0, X_{\mathbf{s}^*u}, 0, X_{J^*u})\}$ on $M \times \mathbb{R} \times \Gamma \times \mathbb{R} \times M$. Now we adopt the notation of Lemma 3.3 and claim that

$$\hat{D}|_{\hat{J}} = (T\hat{J} \cap \ker\Theta)^{d\Theta} \cap \ker\Theta. \tag{8}$$

[6] We adapt the proofs of the analogous statements for symplectic realizations from [7] and [20].
[7] Here as usual $\dim M = k$ and $\dim \Gamma = 2n + 1$.

Both are distributions of dimension $n+1$, so we just need to show the inclusion "$\subset$". A computation shows that for any tangent vector Y we have

$$\begin{aligned} d\Theta((0,0,X_{\mathbf{s}^*u},0,X_{J^*u})\ ,\ Y) &= du(E_0)\cdot\Theta(Y) - e^{-b}\cdot db(Y_b)\cdot \mathbf{s}^*u \\ &- J^*du(Y_3) + e^{-b}\mathbf{s}^*du(Y_\Gamma), \end{aligned}$$

where the subscripts denote the components of Y analogously to the notation of Lemma 3.3. Clearly this vanishes if $Y \in T\hat{J}\cap\ker\Theta$. Together with the fact that $\hat{D}|_{\hat{J}}$ is annihilated by Θ, this proves equation (8). To complete the proof, we need to recall the following fact:
Fact: If (C,θ) is a contact manifold and S a submanifold which satisfies the "coisotropicity" condition

$$(TS\cap\ker\theta)^{d\theta}\cap\ker\theta\ \subset\ TS\cap\ker\theta$$

then the subbundle $(TS\cap\ker\theta)^{d\theta}\cap\ker\theta$ is integrable.
Proof: The proof is a straightforward computation using $d^2\theta = 0$ to show that$[X,Y]\in (TS\cap\ker\theta)^{d\theta\cap\ker\theta}$ whenever $X,Y\in(TS\cap\ker\theta)^{d\theta}\cap\ker\theta\subset TS\cap\ker\theta$.

Since $\mathbf{s}$ and J are both Jacobi maps, $\hat{D}|_{\hat{J}}\subset T\hat{J}\cap\ker\Theta$. Therefore our distribution $\hat{D}|_{\hat{J}}$ is integrable. Since $\hat{D}|_{\hat{K}}$ is tangent to $\hat{K}$, it is also integrable and the integrability of $\hat{D}|_{\hat{K}}$ is clearly equivalent to the integrability of D_K. $\triangledown$

Now define $I := \{(m,J(m),m)|m\in M\}$, a k-dimensional submanifold of K. Notice that I is transversal to the foliation F. We define

$$L := \prod_{x\in I} F_x,$$

where F_x is the leaf of F through x. As in Appendix 3 of [7] one shows that$L$ is an immersed $(n+k+1)$-dimensional submanifold of K.

Claim 2: $\hat{L}$ is an immersed Legendrian submanifold of $M\times\mathbb{R}\times\Gamma\times\mathbb{R}\times M$, endowed with the contact form Θ as in Lemma 3.3.
Proof: Denote by $\hat{I}$ and $\hat{L}$ respectively the natural inclusions of $I,L\subset M\times\Gamma\times M$ into $M\times\mathbb{R}\times\Gamma\times\mathbb{R}\times M$. By contracting with Θ and $d\Theta$, one can show that the vector fields $(0,0,X_{\mathbf{s}^*u},0,X_{J^*u})$ and the Hamiltonian vector field $\hat{X}_{J^*u-e^{-b}\mathbf{s}^*u}$ on $M\times\mathbb{R}\times\Gamma\times\mathbb{R}\times M$ coincide. Therefore the tangent spaces to the foliation $\hat{F}$ of $\hat{K}$ are actually spanned by Hamiltonian vector fields.

It is clear that at all points $\hat{x}$ in $\hat{I}$ the tangent space $T_{\hat{x}}\hat{L}$ is annihilated by Θ: for vectors tangent to $\hat{I}$ we have $(-\theta_1-\theta_\Gamma+\theta_3)(\delta m,0,J_*(\delta m),0,\delta m)=0$ because $\Gamma_0\subset\Gamma$ is Legendrian for θ_Γ, for vectors tangent to the foliation $\hat{F}$ we clearly have $(-\theta_1-\theta_\Gamma+\theta_3)(0,0,X_{\mathbf{s}^*u},0,X_{J^*u})=0$. A general point $\hat{y}$ of $\hat{L}$ can be joint to some $\hat{x}\in\hat{I}$ by finitely many segments of flows of vector fields of the form $(0,0,X_{\mathbf{s}^*u},0,X_{J^*u})$. Since we just showed that these are Hamiltonian vector fields, their flows will preserve $\ker\Theta$. Furthermore, since these vector fields are tangent to $\hat{L}$, they will preserve tangent spaces to $\hat{L}$, so we can conclude that since $T\hat{L}\subset\ker\Theta$ at $\hat{x}$ the same must be true at $\hat{y}$. The argument is finished by a simple dimension counting. $\triangledown$

Claim 3: L is the graph of a contact groupoid action
Proof: Recall that L was defined in such a way that any $(m,g,m')\in L$ can be reached from

$(m, J(m), m)$ following the flows of vector fields of the form $(0, X_{\mathbf{s}^*u}, X_{J^*u})$. Since $X_{\mathbf{s}^*u}$ is tangent to the $\mathbf{t}$-fibers we have $J(m) = \mathbf{t}(g)$; in the next claim we will show that L is the graph of a map $M_J \times_{\mathbf{t}} \Gamma \to M$. Now we check that conditions i)-iii) and equation (4) in Definition 3.1 are satisfied.

Since both $\mathbf{s}$ and J are Jacobi maps, so from the above remark about L we have $\mathbf{s}(g) = J(m')$, i.e. i). Condition iii) is trivially satisfied, and equation (4) is satisfied because $\hat{L}$ is Legendrian in $M \times \mathbb{R} \times \Gamma \times \mathbb{R} \times M$ using Lemma 3.3.

To establish ii) we have to show that, if (m, g, m') and (m', g', m'') lie in L, then (m, gg', m'') also lies in L. We have $g = \phi_{t_0}^{\mathbf{s}^*u_0}(J(m))$, where by the symbol $\phi_{t_0}^{\mathbf{s}^*u_0}$ we denote a suitable flow of a collection $u_0 \in C^\infty(\Gamma_0)$ at time t_0, and similarly for $\phi_{t_0}^{J^*u_0}(m)$ and $g' = \phi_{t_1}^{\mathbf{s}^*u_1}(J(m'))$. Therefore we must have $m'' = \phi_{t_1}^{J^*u_1} \circ \phi_{t_0}^{J^*u_0}(m)$. But $gg' = g\phi_{t_1}^{\mathbf{s}^*u_1}(\mathbf{s}(g)) = \phi_{t_1}^{\mathbf{s}^*u_1}(g)$ since vector fields of the form $X_{\mathbf{s}^*u}$ are left invariant (see Proposition 4.3 in [9]), therefore $(m, gg', m'') \in L$. $\triangledown$

To end the proof we still need

Claim 4: L is the graph of a map $M_J \times_{\mathbf{t}} \Gamma \to M$.

Proof: Restrict to L the obvious projections pr_1 (onto the first copy of M) and pr_Γ, originally defined on $M \times \Gamma \times M$, and denote them by the same symbols. We need to show that (pr_1, pr_Γ) is a diffeomorphism of L onto $M_J \times_{\mathbf{t}} \Gamma$, or equivalently that, for any $m \in M$, the map

$$pr_\Gamma : pr_1^{-1}(m) \to \mathbf{t}^{-1}(J(m))$$

is a diffeomorphism. Since $pr_1 : L \to M$ is a submersion and $dimL = n + k + 1$ one sees that the domain of pr_Γ has dimension $n + 1$, which is the dimension of the target space. We claim that pr_Γ is surjective. Let $g \in \mathbf{t}^{-1}(J(m))$. Since $\mathbf{t}^{-1}(J(m))$ is connected and its tangent spaces are spanned by vector fields of the form $X_{\mathbf{s}^*u}$, we can find functions (collectively denoted u_0) such that a composition $\phi^{\mathbf{s}^*u_0}$ of their Hamiltonian flows maps g to $\mathbf{t}(g)$, i.e. for some t_0 we have $\phi_{t_0}^{\mathbf{s}^*u_0}(\mathbf{t}(g)) = g$. Let us denote by ϕ^{u_0} and $\phi^{J^*u_0}$ the analogously defined Hamiltonian flows on Γ_0 and M. The image of the curve $[0, t_0] \to \Gamma_0, t \mapsto \phi_t^{u_0}(\mathbf{t}(g))$ lies in a compact subset of Γ_0, so we may assume that all the functions that we collectively denote by u_0 have compact support. By the completeness assumption on J we conclude that $\phi_t^{J^*u_0}(m)$ is well defined for all time. In particular it is at time t_0, and clearly $(m, g, \phi_{t_0}^{J^*u_0}(m))$ is an element of L that projects to g via pr_Γ.

Now we show that $pr_\Gamma : pr_1^{-1}(m) \to \mathbf{t}^{-1}(J(m))$ is a covering map using again the path-lifting property of J. Given g as above, it is easy to see that we can parametrize a small neighborhood $U^{\mathbf{s}}$ of g in $\mathbf{t}^{-1}(J(m))$ by functions u on Γ_0 (where the u's lie in the $(n+1)$-dimensional span of coordinate functions centered at $\mathbf{s}(g)$ and a constant function) simply by writing every point in $U^{\mathbf{s}}$ as $\phi_1^{\mathbf{s}^*(u)}(g)$, the time-1 flow of the integral curve to $X_{\mathbf{s}^*u}$ starting at g. If m' is any point such that $(m, g, m') \in L$ (so in particular $J(m') = \mathbf{s}(g)$), denote by $\phi_1^{J^*(u)}(m')$ the time-1 flow of the integral curve to X_{J^*u} starting at m', which is well defined by the completeness of J. Then, again because $\mathbf{s}$ and J are Jacobi maps, $\{(m, \phi_1^{\mathbf{s}^*(u)}(g), \phi_1^{J^*(u)}(m')) : u \in P\}$ is a neighborhood of (m, g, m') in $pr_1^{-1}(m)$, and it is clearly mapped diffeomorphically onto $U^{\mathbf{s}}$ by pr_Γ.

Since $pr_\Gamma : pr_1^{-1}(m) \to \mathbf{t}^{-1}(J(m))$ is a covering map and $\mathbf{t}^{-1}(J(m))$ is simply connected we conclude that pr_Γ is a diffeomorphism. $\triangledown$

□

3.3 f-multiplicative functions

Given a free and proper contact groupoid action, we automatically have "f-multiplicative functions", which will play an important role in our reduction. So we also call them "reduction functions".

Proposition-Definition 3.10. If a contact groupoid action of Γ on M is free and proper, there exists a non-vanishing function F on M such that

$$F(m \cdot g) = F(m) f(g).$$

We call such a function *f-multiplicative.*

To prove the above we need a technical result about general groupoid actions:

Lemma 3.11. *If the action of any Lie groupoid Γ on any manifold M is free and proper then through every point $m \in M$ there exists a disk that meets each Γ orbit at most once and transversely.*

Proof of Proposition-Definition 3.10: Slices $\{D_i\}$ as in Lemma 3.11 provide manifold charts for the quotient M/Γ, and the quotient is Hausdorff because the Γ-action is proper (see Proposition B.8 in [11]). Now choose a subordinate partition of unity, and pull it back to obtain a Γ-invariant partition of unity $\{(U_i, \rho_i)\}$ on M. On each U_i construct an f-multiplicative function by letting F_i be an arbitrary positive function on the slice $D_i \subset U_i$ and extending F_i to U_i by $F_i(mg) = F_i(m) f(g)$. Then

$$F = \sum \rho_i F_i$$

is an f-multiplicative function on M.

□

Proof of Lemma 3.11: The proof is analogous to the one of the slice theorem for group actions (see Theorem B.24 in [11]). Choose a disk D that intersects the orbit $m \cdot \Gamma$ transversely, and consider the map

$$\phi : D_J \times_{\mathbf{t}} \Gamma \to M \ , \ (u, g) \mapsto ug. \tag{9}$$

This map is an immersion at $(m, 1_{J(m)})$ since the Γ-action is free at m. Here $1_{J(m)}$ denotes $J(m)$ as an element of the space of units.

The above map is injective (one may eventually need to make D smaller), as follows: take sequences (u_n, g_n) and (v_n, h_n) in $D_J \times_{\mathbf{t}} \Gamma$ such that u_n and v_n converge to m and $u_n g_n = v_n h_n$. We may assume that $h_n \equiv 1_{J(v_n)}$ (otherwise act by h_n^{-1}), so $u_n g_n = v_n$. The map $M_J \times_{\mathbf{t}} \Gamma \to M \times M, (m, g) \mapsto (m, m \cdot g)$ is proper because the action is proper, and since the sequence (u_n, v_n) converges, the sequence (u_n, g_n) also converges, say to (m, g) for some $g \in \Gamma$. Since the action is free, it follows that $g = 1_{J(m)}$, and since the map ϕ is injective in a neighborhood of $(m, 1_{J(m)})$ it follows that the two sequences we started with must agree for n big enough. So the map ϕ is injective, and by dimension counting we see that it is a diffeomorphism. Since ϕ is Γ-equivariant and each orbit on the left hand side of (9) intersects the disk $\{(u, 1_{J(u)}) | u \in D\}$ exactly once, D is a slice at m for the Γ-action.

□

The next two lemmas are technical and are necessary in the proofs of the reduction theorems. In both lemmas we consider a contact groupoid action of a contact groupoid Γ on a contact manifold (M, θ_M) with moment map $J : M \to \Gamma_0$.

Lemma 3.12. *For any f-multiplicative function F on M and any function $\hat{u}$ constant on the Γ-orbits we have*

$$d(F\hat{u})(E_M) = 0.$$

Proof. By equation (6) (choosing $u = 1$ there) we know that at any point $m \in M$ we have

$$0(m) \cdot E_\Gamma(x) = E_M(m), \tag{10}$$

where $x = J(m)$. Denoting by $\gamma(\epsilon)$ an integral curve of E_Γ in $T\mathbf{t}^{-1}(x)$ we have

$$\begin{aligned} dF(E_M(m)) &= \frac{d}{d\epsilon}\Big|_0 F(m \cdot \gamma(\epsilon)) = \frac{d}{d\epsilon}\Big|_0 F(m) \cdot f(\gamma(\epsilon)) \\ &= F(m) \cdot df(E_\Gamma(x)) = 0, \end{aligned}$$

where we used $df(E_\Gamma) = 0$ (see e) in Remark 2.2). The lemma follows since $\hat{u}$ is constant along the Γ-orbits and by equation (10) E_M is tangent to these orbits. □

Lemma 3.13. *For any f-multiplicative function F and any function $\hat{u}$ constant along the Γ-orbits the Hamiltonian vector field $X_{F\hat{u}}$ lies in $TJ^{-1}(x)$. In particular $TJ^{-1}(x)$ is not contained in* $\ker(\theta_M)$ *if the action admits an f-multiplicative function.*

Proof. We will show that

$$X_{F\hat{u}} \cdot E_\Gamma = X_{F\hat{u}} + E_M, \tag{11}$$

and the fact that $X_{F\hat{u}}$ and E_Γ are multipliable implies that $J_*(X_{F\hat{u}}) = t_*(E_\Gamma) = 0$ as desired. To show (11) we use the same method as in Lemma 3.3 and adapt the notation there too. We only have to show that $(X_{F\hat{u}}, 0, E_\Gamma, 0, X_{F\hat{u}} + E_M)$ lies in $T\mathcal{A}$.

Evaluation of $d\Theta$ on this vector and on $(Y, 0, V, 0, Y \cdot V)$ gives zero, as one can see using $df(E_\Gamma) = 0$, Lemma 3.12 and the f-multiplicativity of F. Therefore $(X_{F\hat{u}}, 0, E_\Gamma, 0, X_{F\hat{u}} + E_M)$ lies in the $d\Theta$-orthogonal to $T\mathcal{A}$. Since evaluation of Θ on this vector also gives zero and $\mathcal{A}$ is Legendrian by Lemma 3.3, the above vector lies in $T\mathcal{A}$. □

4 Reductions

In this section, we will first prove the main result using a classical method, i.e. without using groupoid. Then, with a slightly stronger assumption, we can prove the same result with the help of groupoids in a much simpler and illustrative way. Finally, we will establish the relation between the two reductions and explain why they yield the same reduced spaces.

4.1 Classical reduction

We recall that $\Gamma_x := \mathbf{t}^{-1}(x) \cap \mathbf{s}^{-1}(x)$ is the isotropy group of Γ at x.

Theorem 4.1. *Let $(\Gamma, \theta_\Gamma, f)$ act on (M, θ_M) by a contact groupoid action. Suppose that $x \in \Gamma_0$ is a regular value of J and that Γ_x acts freely and properly on $J^{-1}(x)$, and let F be a f-multiplicative function defined on $J^{-1}(x)$. Then the reduced space $M_x := J^{-1}(x)/\Gamma_x$ has an induced*

1. *contact structure, a representative of which is induced by the restriction of $J^{-1}(x)$ of* [8] *$-F^{-1}\theta_M$, if x belongs to a contact leaf of the Jacobi manifold Γ_0,*
2. *conformal l.c.s. structure, a representative of which is induced by the restriction of $J^{-1}(x)$ of $(-F^{-1}d\theta_M, -F^{-1}dF)$, if x belongs to a l.c.s. leaf.*

Before beginning the proof we need a lemma that involves only the contact groupoid $(\Gamma, \theta_\Gamma, f)$ and not the action:

Lemma 4.2. *Consider the isotropy group Γ_x for some $x \in \Gamma_0$. If x lies in a contact leaf then θ_Γ vanishes on vectors tangent to Γ_x. If x lies is a l.c.s. leaf then df vanishes on vectors tangent to Γ_x, i.e. $f|_{\Gamma_x}$ is locally constant.*

Proof. Let $g \in \Gamma_x$. We will first determine explicitly a basis for $T_g\Gamma_x = T_g\mathbf{t}^{-1}(x) \cap T_g\mathbf{s}^{-1}(x)$. To this aim choose functions $\{u_1, \cdots, u_n\}$ on Γ_0 vanishing at x such that their differentials at x are linearly independent. We may assume that $\{du_1(x), \cdots, du_\sigma(x)\}$ span $\ker(\sharp\Lambda_0)$ Recall that a basis for $T_g\mathbf{t}^{-1}(x)$ is given by $\{X_{\mathbf{s}^*u_1}, \cdots, X_{\mathbf{s}^*u_n}, E_\Gamma\}$. We have $\mathbf{s}_*(\sum a_i X_{\mathbf{s}^*u_i} + cE_\Gamma) = \sum a_i \#\Lambda_0(du_i) + cE_0$.

If the leaf through x is a contact leaf, then E_0 does not lie in the image of $\#\Lambda_0$, therefore the above vanishes iff $a_{\sigma+1} = \cdots = a_n = c = 0$. So in this case a basis for $T_g\Gamma_x$ is

$$\{X_{\mathbf{s}^*u_1}, \cdots, X_{\mathbf{s}^*u_\sigma}\},$$

and clearly $\theta_\Gamma(X_{\mathbf{s}^*u_i}(g)) = u_i(x) = 0$.

If the leaf through x is a l.c.s. leaf, then E_0 lies in the image of $\#\Lambda_0$, therefore there exists exactly one linear combination $u(x)$ of $u_{\sigma+1}, \cdots, u_\sigma$ such that $\#\Lambda_0(du) + E_0 = 0$. So in this case a basis for $T_g\Gamma_x$ is

$$\{X_{\mathbf{s}^*u_1}, \cdots, X_{\mathbf{s}^*u_\sigma}, X_{\mathbf{s}^*u} + E_\Gamma\}.$$

We have

$$df(X_{\mathbf{s}^*u_i}) = f(g)du_i(E_0)$$

using d) and e) in Remark 2.2. So, since for $i = 1, \cdots, \sigma$ we have $du_i \in \ker(\sharp\Lambda_0) = Im(\sharp\Lambda_0)^\circ$ and $E_0 \in Im(\sharp\Lambda_0)$, we have $df(X_{\mathbf{s}^*u_i}) = 0$. Also,

$$df(X_{\mathbf{s}^*u} + E_\Gamma) = df(X_{\mathbf{s}^*u}) = f(g)du(E_0) = f(g)du(-\#\Lambda_0(du)) = 0.$$

$\square$

Remark 4.3. One can show that θ_Γ vanishes on the tangent space of Γ_x iff x lies in a contact leaf and that df vanishes there iff x lies in a l.c.s leaf.

[8] The presence of the minus sign here and in Theorem 4.4 will be explained in Example 4.7 below.

Now we are ready to prove Theorem 4.1. We will consider separately the cases when x belongs to a contact or l.c.s. leaf. The steps in the proofs that apply to only one of these two situations are those where Lemma 4.2 is used, i.e. Claim 2 for the contact case and Claims 2 and 4 for the l.c.s. case.

Proof of the contact case. Choose an f-multiplicative function F on $J^{-1}(x)$. Such a function always exists (the proof is the same as for Lemma 3.10). Denote by $\tilde{\theta}_M$ the pullback of θ_M to $J^{-1}(x)$. We will show that $-F^{-1}\tilde{\theta}_M$ descends to a contact form α_F on the reduced space M_x, and that the corresponding contact structure is independent of the choice of F.

Claim 1: $F^{-1}\tilde{\theta}_M$ is invariant under the action of Γ_x on $J^{-1}(x)$.
Proof: Let $Y_m \in T_m J^{-1}(x)$ and $g \in \Gamma_x$. From the definition of contact groupoid action it follows immediately that $\theta_M(Y_m \cdot 0_g) = f(g)\theta_M(Y_m)$. This means that $g^*(\tilde{\theta}_M) = f(g)\cdot\tilde{\theta}_M$. So

$$g^*(F^{-1}\tilde{\theta}_M)_m = F^{-1}(m)f^{-1}(g)(g^*\tilde{\theta}_M)_m = F^{-1}(m)(\tilde{\theta}_M)_m = (F^{-1}\tilde{\theta}_M)_m.$$

$\bigtriangledown$

Claim 2: The orbits of the Γ_x-action are tangent to the kernel of $\tilde{\theta}_M$.
Proof: To see this, let $m \in J^{-1}(x)$ and let $V_x \in T_x\Gamma_x$. Again from the definition of contact groupoid action we infer that $\theta_M(0_m \cdot V_x) = \theta_\Gamma(V_x)$, which vanishes by Lemma 4.2. $\bigtriangledown$

Claim 3: $-F^{-1}\tilde{\theta}_M$ descends to a contact form α_F on $J^{-1}(x)/\Gamma_x$.
Proof: It is clear by the above two claims that $-F^{-1}\tilde{\theta}_M$ descends, so we only have to ensure that it gives rise to a contact form. To this aim we first extend F arbitrarily to an open neighborhood of $J^{-1}(x)$ in M and we determine explicitly $\ker(d(F^{-1}\tilde{\theta}_M))$, i.e. $T_mJ^{-1}(x) \cap T_mJ^{-1}(x)^{d(F^{-1}\theta_M)}$. Notice that

$$d(F^{-1}\theta_M)(X_{J^*u}, X) = F^{-2}du_x(J_*X_F)\theta_M(X) - F^{-1}du_x(J_*X) + F^{-2}dF(X)J^*u. \quad (12)$$

This together with the fact that X_F is the Reeb vector field of $F^{-1}\theta_M$ implies that,

$$T_mJ^{-1}(x)^{d(F^{-1}\theta_M)} \supset \{X_{J^*u} | u(x) = 0, du_x(J_*X_F) = 0\} \oplus X_F, \quad (13)$$

and

$$\{X_{J^*u} | u(x) = 0, du_x(J_*X_F) = 0\}^{d(F^{-1}\theta_M)} \subset T_mJ^{-1}(x) + X_F. \quad (14)$$

Since $\ker(dF^{-1}\theta_M) = span\{X_F\}$, by taking the orthogonals with respect to $dF^{-1}\theta_M$ on both sides of the above two equations, we obtain the opposite inclusions. Therefore we actually have equality in (13) and (14).

By Lemma 4.2 and (6), and the fact that $d(F^{-1}\theta_M)$ descends, we have

$$\begin{aligned} \{X_{J^*u} | u(x) = 0, du_x \in \ker(\sharp\Lambda_0)\} = 0_m \cdot T_x\Gamma_x \quad &\subset \quad \ker d(F^{-1}\tilde{\theta}_M) \\ &\subset \quad T_mJ^{-1}(x)^{d(F^{-1}\theta_M)}. \end{aligned}$$

Combining with (13), this says that if $u(x) = 0$ and $du_x \in \ker(\sharp\Lambda_0) = \operatorname{im}(\sharp\Lambda_0)^0$ (the annihilator of the image of $\sharp\Lambda_0$) then $du_x(J_*X_F) = 0$. This means that $J_*X_F \in \operatorname{im}(\sharp\Lambda_0)$[9].

[9]Notice that in Lemma 3.13 we showed that if F is f-multiplicative on the whole of M then $J_*X_F = 0$.

Therefore there exists some function u_0 vanishing at x such that $J_*X_F(m) = (\sharp\Lambda_0 du_0)(x)$. Since $X_{F-J^*u_0}$ lies in $T_mJ^{-1}(x)$ but not in $\ker(\theta_M)$ we conclude that $T_mJ^{-1}(x) \not\subset \ker(\theta_M)_m$.

Now set $J_*(X_{J^*u} + cX_F) = \sharp\Lambda_0 du + cJ_*X_F$ equal to zero, by (13) we conclude that,

$$\ker(d(F^{-1}\tilde{\theta}_M)) = T_mJ^{-1}(x) \cap T_mJ^{-1}(x)^{d(F^{-1}\theta_M)} = 0_m \cdot T_x\Gamma_x \oplus (X_F - X_{J^*v}),$$

where v is the unique function vanishing at x (could be 0) on Γ_0 such that $\sharp\Lambda_0 dv_x = J_*X_F$. Uniqueness and existence are ensured by the facts that $T_mJ^{-1}(x) \not\subset \ker(\theta_M)_m$ and $J_*X_F \in \mathrm{im}(\sharp\Lambda_0)$. Therefore $d\alpha_F$ induced on M_x by $F^{-1}\tilde{\theta}_M$ has one-dimensional kernel spanned by the image of $X_F - X_{J^*v}$, and since $F^{-1}\tilde{\theta}_M(X_F - X_{J^*v}) = 1 \neq 0$ it follows that α_F is a contact form. $\triangledown$

Claim 4: The contact structure on M_x given by $\ker(\alpha_F)$ is independent of the chosen f-multiplicative function F.

Proof: From the construction of the contact form α_F, it is easy to see that, for another f-multiplicative function $\hat{F}$ on $J^{-1}(x)$,

$$\pi^*(\alpha_F) = \frac{\hat{F}}{F} \cdot \pi^*(\alpha_{\hat{F}}),$$

where $\pi : J^{-1}(x) \to M_x$ is the projection. By the f-multiplicativity, $\frac{\hat{F}}{F}$ is Γ_x-invariant, so it descends to a function Q on M_x. Since π^* is injective, we have $\alpha_F = Q\alpha_{\hat{F}}$. $\triangledown$

$\square$

Now we prove the locally conformal symplectic case:

Proof of the l.c.s. case. Adapt the same notation as above. We will show that the two-form $-F^{-1}d\tilde{\theta}_M$ and the one-form $-F^{-1}dF$ descend to forms Ω_F and ω_F respectively on M_x. The reduced space M_x together with the pair (Ω_F, ω_F) will be a l.c.s. manifold, i.e. Ω_F is non-degenerate, ω_F closed, and $d\Omega_F = \omega_F \wedge \Omega_F$. Furthermore, a different choice of f-multiplicative function will give a conformally equivalent l.c.s. structure on M_x.

Claim 1: $F^{-1}d\tilde{\theta}_M$ is invariant under the Γ_x action on $J^{-1}(x)$.

Proof: Let $g \in \Gamma_x$ and $m \in J^{-1}(x)$. Notice that $g^*(\tilde{\theta}_M) = f(g) \cdot \tilde{\theta}_M$, hence $g^*(d\tilde{\theta}_M) = f(g) \cdot d\tilde{\theta}_M$. A calculation analogous to the one presented in Claim 1 of the proof of the contact case allows us to conclude that $g^*(F^{-1}d\tilde{\theta}_M) = F^{-1}d\tilde{\theta}_M$. $\triangledown$

Claim 2: $-F^{-1}d\tilde{\theta}_M$ descends to a non-degenerate two-form Ω_F on M_x.

Proof: Since $-F^{-1}d\tilde{\theta}_M$ is a non-vanishing multiple of $d\tilde{\theta}_M$, the above claim will be true if and only if at all $m \in J^{-1}(x)$

$$0_m \cdot T_x\Gamma_x = \ker(d\tilde{\theta}_M)(= T_mJ^{-1}(x) \cap (T_mJ^{-1}(x))^{d\theta_M}).$$

For the inclusion "$\subset$" we compute for any $V \in T_x\Gamma_x$ and $Y \in T_mJ^{-1}(x)$ that $d\theta_M(0_m \cdot V, Y) = 0$ by taking the exterior derivative of (4) in Definition 3.1 and using Lemma 4.2. So $0_m \cdot V \in T_m(J^{-1}(x))^{d\theta_M}$, and since Γ_x acts on $J^{-1}(x)$ the first inclusion is proven.

For the opposite inclusion "$\supset$" we will show below that

$$0_m \cdot T_x\mathbf{t}^{-1}(x) = (T_mJ^{-1}(x) \cap \mathcal{H}_m)^{d\theta_M} \tag{15}$$

where $\mathcal{H}_m$ denotes the kernel of $(\theta_M)_m$. Then, taking the $d\theta_M$-complement of the relation $T_mJ^{-1}(x) \cap \mathcal{H}_m \subset T_mJ^{-1}(x)$, we obtain

$$0_m \cdot T_x\mathbf{t}^{-1}(x) \supset (T_mJ^{-1}(x))^{d\theta_M}.$$

Clearly we preserve the inclusion if we intersect both sides with $T_mJ^{-1}(x)$. Now, since for any $V \in T_x\mathbf{t}^{-1}(x)$ we have $0_m \cdot V \in T_mJ^{-1}(x) \Leftrightarrow V \in T_x\mathbf{s}^{-1}(x)$, we obtain

$$0_m \cdot T_x\Gamma_x = 0_m \cdot T_x\mathbf{t}^{-1}(x) \cap T_mJ^{-1}(x) \supset T_mJ^{-1}(x) \cap (T_mJ^{-1}(x))^{d\theta_M}$$

and we are done.

To complete the proof of "$\supset$" we still have to show equation (15). By d) in Remark 2.2 and (6), we have $0_m \cdot T_x\mathbf{t}^{-1}(x) = 0_m \cdot \{X_{\mathbf{s}^*u}(x)\} = \{X_{J^*u}(m)\}$, where u ranges over all functions on Γ_0. Notice that for $Y \in \mathcal{H}_m$ we have $d\theta_M(X_{J^*u}, Y) = -du(J_*Y)$, so that

$$\{X_{J^*u}(m)\}^{d\theta_M} \cap \mathcal{H}_m = T_mJ^{-1}(x) \cap \mathcal{H}_m.$$

Since the Reeb vector field E_M lies in $\{X_{J^*u}\}$, taking orthogonals of the above, we are done.

Claim 3: $F^{-1}dF$ is invariant under the Γ_x action on $J^{-1}(x)$.
Proof: The f-multiplicativity of F implies $(g^*dF) = f(g) \cdot dF$. The rest of the proof is analogous to the one of Claim 1 of the proof of the contact case.

Claim 4: $-F^{-1}dF$ descends to a one-form ω_F on M_x.
Proof: We have to check that if $V \in T_x\Gamma_x$ then $0_m \cdot V$ lies in the kernel of $-F^{-1}dF$. This is satisfied because $dF(0_m \cdot V) = F(m)df(V) = 0$ by the f-multiplicativity of F and by the second part of Lemma 4.2. $\triangledown$

Claim 5: The two-form Ω_F induced by $-F^{-1}d\tilde{\theta}_M$ and the one-form ω_F induced by $-F^{-1}dF$ endow M_x with a l.c.s. structure.
Proof: We have to show that ω_F is closed and that $d\Omega_F = \omega_F \wedge \Omega_F$. Since $\pi : J^{-1}(x) \to J^{-1}(x)/\Gamma_x$ is a submersion, it suffices to show $\pi^*(d\omega_F) = 0$ and $\pi^*d\Omega_F = \pi^*(\omega_F \wedge \Omega_F)$. The former is clear since $\pi^*\omega_F = -d(\ln|F|)$ is exact, the latter follows by a short computation. $\triangledown$

Claim 6: The conformal class of the l.c.s. structure on M_x given by ω_F and Ω_F is independent of the choice of F.
Proof: Let $\hat{F}$ be another f-multiplicative function on $J^{-1}(x)$ and denote by Q the function on M_x induced by $\frac{\hat{F}}{F}$. We have $\Omega_F = Q\Omega_{\hat{F}}$ because

$$\pi^*\Omega_F = -F^{-1}d\tilde{\theta}_M = -\frac{\hat{F}}{F}\hat{F}^{-1}d\theta_M = \pi^*(Q \cdot \Omega_{\hat{F}}),$$

and similarly we obtain $\omega_F = d(\ln|Q|) + \omega_{\hat{F}}$. Now a standard computation shows that the identity $Id : (M_x, \Omega_F, \omega_F) \to (M_x, \Omega_{\hat{F}}, \omega_{\hat{F}})$ is a Q-conformal Jacobi map. $\triangledown$

$\square$

4.2 Global reduction

In this subsection, we will achieve the desired reduction result through a global reduction procedure. It is technically easier and also suggests that the reduced spaces "glue well together".

The key observation (see [16]) is the following: if a contact groupoid Γ acts (say from the right) on a manifold M with moment map J, then the orbit space of the action is

$$M/\Gamma = \coprod_{\mathcal{O}} J^{-1}(\mathcal{O})/\Gamma,$$

where the disjoint union ranges over all orbits $\mathcal{O}$ of the groupoid Γ, i.e. over all leaves of the Jacobi manifold Γ_0.

Also, for each $x \in \mathcal{O}$, by the equivariance of J we have

$$J^{-1}(x)/\Gamma_x = J^{-1}(\mathcal{O})/\Gamma.$$

So topologically M/Γ is equal to a disjoint union of reduced spaces, one for each leaf $\mathcal{O}$ of Γ_0. This suggests that the reduced space is a Jacobi manifold with foliation given by these individual reduced spaces. Indeed we have:

Theorem 4.4. *Let $(\Gamma, \theta_\Gamma, f)$ act on (M, θ_M) freely and properly, F an f-multiplicative function on M. Then there is an induced Jacobi structure on M/Γ such that the projection $pr : M \to M/\Gamma$ is a $-F$-conformal Jacobi map* [10].

Moreover, the Jacobi foliation is given exactly by (the connected components of) the decomposition

$$M/\Gamma = \coprod_{\mathcal{O}, x\in\mathcal{O}} J^{-1}(x)/\Gamma_x,$$

and the reduced manifolds $J^{-1}(x)/\Gamma_x$ are contact or l.c.s. manifolds exactly when the leaves $\mathcal{O}$ through x are.

The conformal class of the Jacobi structure on M/Γ is independent of the choice of F.

We first determine that the Γ-action on M preserves the contact form up to a factor of f:

Lemma 4.5. *Let Σ be a Legendrian bisection of $(\Gamma, f, \theta_\Gamma)$ and r_Σ: $M \to M$, $m \mapsto m \cdot \Sigma(J(m))$ the induced diffeomorphism of M, where Σ is viewed as a section of* $\mathbf{t}$. *Then*

$$r_\Sigma^* \theta_M = f(\Sigma \circ J) \cdot \theta_M.$$

Furthermore, through any given point of Γ there exists a local Legendrian bisection.

Proof. Let $m \in M$, $V \in T_m M$, $g := \Sigma(J(m))$ and $Y := \Sigma_* J_* V \in T_g\Gamma$. Then since Y is tangent to a Legendrian bisection

$$r_\Sigma^* \theta_M(V) = \theta_M(V \cdot Y) = f(g) \cdot \theta_M(V) + \theta_\Gamma(Y) = f(g) \cdot \theta_M(V).$$

This establishes the first part of the Lemma.

[10] The presence of the minus sign here will be explained in Example 4.7 below.

Now we show that there exists a local Legendrian bisection of Γ through every $g \in \Gamma$. By a generalized Darboux theorem we can assume that a neighborhood of g in (Γ, θ_Γ) is equal to a neighborhood of the origin in $(\mathbb{R}^{2n+1}, dz - \sum x_i dy_i)$. Consider the natural projection $\mathbb{R}^{2n+1} \to \mathbb{R}^{2n}$ with kernel the z-axis. By [9], the $(n+1)$-dimensional subspaces $T_g\mathbf{s}^{-1}$ and $T_g\mathbf{t}^{-1}$ are both not contained in $\ker(\theta_M)_g$, so the derivative at the origin ($=g$) of the above projection maps $T_g\mathbf{s}^{-1} \cap \ker(\theta_M)_g$ and $T_g\mathbf{t}^{-1} \cap \ker(\theta_M)_g$ to subspaces of $\mathbb{R}^{2n}$ of dimension n. Therefore we can find a Lagrangian subspace of $\mathbb{R}^{2n}$ which is transversal to both. It is known (see [sw], p. 186) that any Lagrangian submanifold of $\mathbb{R}^{2n}$ through the origin which is exact (this condition is always satisfied locally) can be lifted to a Legendrian submanifold of $\mathbb{R}^{2n+1}$ through the origin. The lift of this Lagrangian subspace will be a Legendrian bisection nearby g, because it will be transversal to both $T_g\mathbf{s}^{-1}$ and $T_g\mathbf{t}^{-1}$. □

Proof of Theorem 4.4. We fix an f-multiplicative function F. It follows from Lemma 4.5 that for any Legendrian bisection Σ the induced map r_Σ on M preserves $-F^{-1}\theta_M$, which corresponds to the Jacobi structure on M obtained by $-F$-conformal change of the original one[11]. Therefore r_Σ preserves the corresponding Jacobi bracket $\{\cdot,\cdot\}_{-F} = -F^{-1}\{-F\,\cdot\,, -F\,\cdot\,\}$, and for any functions $\hat{h}$ and $\hat{k}$ on M which are constant along the Γ-orbits we have

$$r_\Sigma^*\{\hat{h},\hat{k}\}_{-F} = \{r_\Sigma^*\hat{h}, r_\Sigma^*\hat{k}\}_{-F} = \{\hat{h},\hat{k}\}_{-F}.$$

So, by the existence of local Legendrian bisections in Lemma 4.5, $\{\hat{h},\hat{k}\}_{-F}$ is also a function constant along the orbits. Hence such functions are closed under the new bracket $\{\cdot,\cdot\}_{-F}$.

By Lemma 3.11 M/Γ is a manifold. The bracket $\{\cdot,\cdot\}_{-F}$ induces a bracket on $C^\infty(M/\Gamma)$: for any functions h, k on M/Γ we define

$$\{h,k\}_{M/\Gamma} = \{pr^*h, pr^*k\}_{-F}.$$

The induced bracket still satisfies the Jacobi identity and (2). That is, $C^\infty(M/\Gamma)$ is endowed with a structure of local Lie algebra in the sense of Kirillov, therefore M/Γ is endowed with the structure of a Jacobi manifold with Jacobi bracket $\{\cdot,\cdot\}_{M/\Gamma}$ (see [Da], p. 434). The map $pr : M \to M/\Gamma$ is $-F$-conformal Jacobi by construction.

Now we will show that for $x \in \Gamma_0$ (any connected component of) $J^{-1}(x)/\Gamma$ is a leaf of M/Γ, i.e. that $span_{h\in C^\infty(M/\Gamma)}\{X_h\} = T(J^{-1}(x)/\Gamma)$. It is enough to show that at any $m \in J^{-1}(x)$

$$span_{\{\hat{h} \text{ is } \Gamma\text{-invariant}\}}\{X_{\hat{h}}^{-F}(m)\} = T_m J^{-1}(x), \tag{16}$$

since $pr|_{J^{-1}(x)} : J^{-1}(x) \longrightarrow J^{-1}(x)/\Gamma$ is a submersion and for any Γ-invariant function $\hat{h} = pr^*(h)$ we have $pr_*(X_{\hat{h}}^{-F}) = X_h$. Here X^{-F} denotes the Hamiltonian vector field with respect to the new $-F$-twisted Jacobi structure on M.

The inclusion "$\subset$" in Equation (16) is clearly implied by Lemma 3.13.

The inclusion "$\supset$" can be seen by a simple dimension counting. Suppose $\dim M = k$ and $\dim\Gamma = 2n+1$. Since the action is free, each Γ-orbit has dimension $n+1$, so the space $\{d\hat{h}_m\}$ has dimension $k-n-1$. Choose a basis $\{d\hat{h}_1, \ldots, d\hat{h}_{k-n-1}\}$ of this space where the $\hat{h}_i$'s are functions vanishing at m. The corresponding vectors $X_{\hat{h}_i}^{-F}(m)$ are linearly independent, because none of them lies in $\ker(-\sharp F\Lambda_M) = span\{\theta_M\}$ (this is true since

[11] This follows from the general fact that if (N,θ) is any contact manifold and φ a non-vanishing function on N, then the Jacobi structure corresponding to $\varphi\theta$ is $(\varphi^{-1}\Lambda, X_{\varphi^{-1}})$.

each $d\hat{h}_i$ annihilates E_M by equation (6) but θ_M does not). Adding $X_1^{-F}(m)$ we obtain a basis for $\{X_{\hat{h}}^{-F}(m)\}$ consisting of $k-n$ elements. Since by Lemma 3.5 J is a submersion, $\dim J^{-1}(x)$ is also $k-n$, so (16) is proven.

A similar dimension counting shows that the reduced manifold $J^{-1}(x)/\Gamma_x$ is a contact (l.c.s.) manifolds exactly when the leaf $\mathcal{O}$ through x is: $J^{-1}(\mathcal{O})/\Gamma$ has dimension $k-2n-1+\dim(\mathcal{O})$, which has the same parity as $\dim(\mathcal{O})$ because k is always odd.

If we take another f-multiplicative function G, then $\frac{G}{F}$ is constant along the orbits, therefore it defines a function Q on M/Γ. It is easy to see that the bracket on M/Γ induced by $\{\cdot,\cdot\}_{-G}$ is given by a Q-conformal change of the bracked induced by $\{\cdot,\cdot\}_{-F}$.

□

Remark 4.6. It turns out that the global reduction can be carried out via symplectification, namely, one can go to the symplecticification of the contact groupoid and use reduction via symplectic groupoids in the sense of [16]. But the local reduction which requires weaker condition is not obvious to be carried out using symplectictification.

Example 4.7. [Groupoid multiplication] If $(M,\theta_M)=(\Gamma,\theta_\Gamma)$ and the action Φ is by right multiplication (so $J=\mathbf{s}$), then the map $\mathbf{t}: M\to\Gamma_0$ gives an identification $M/\Gamma\cong\Gamma_0$. Under this identification the map $pr: M\to M/\Gamma$ corresponds exactly to $\mathbf{t}$. Endow $M/\Gamma\cong\Gamma_0$ with the Jacobi structure as by Theorem 4.4 using the function $F:=f$. Since $\mathbf{t}$ is a $-f$-Jacobi map for the original Jacobi structure on Γ_0, the induced Jacobi structure on Γ_0 is exactly the original one.

4.3 Relation between the two reductions

Next we show that the classical reduction procedure (Theorem 4.1) and the groupoid reduction procedure (Theorem 4.4) both yield the same contact or l.c.s. structures on the reduces spaces $J^{-1}(x)/\Gamma_x$. It is enough to show:

Theorem 4.8. *Let (Γ,θ_Γ,f) act on (M,θ_M) by a contact groupoid action freely and properly. Choose an f-multiplicative function F and endow M/Γ with a Jacobi structure as in Theorem 4.4. Then the contact or l.c.s structures on $M_x:=J^{-1}(x)/\Gamma_x$ are induced by the restrictions to $J^{-1}(x)$ of the following forms:*

1. $-F^{-1}\theta_M$ *if M_x is a contact leaf,*
2. $(-F^{-1}d\theta_M,-F^{-1}dF)$ *if M_x is a l.c.s. leaf.*

Proof. Case 1: M_x is a contact leaf. Denote by α_F the contact form on $J^{-1}(x)/\Gamma_x$ given by the Jacobi structure on M/Γ. We consider $pr|_{J^{-1}(x)}: J^{-1}(x)\to J^{-1}(x)/\Gamma_x$ and want to show that at $m\in J^{-1}(x)$ we have $(pr|_{J^{-1}(x)})^*\alpha_F=-F^{-1}\tilde{\theta}_M$, where $\tilde{\theta}_M$ denotes the restriction of θ_M to $J^{-1}(x)$. By equation (16) and $pr_*(X_{pr^*h}^{-F})=X_h$, we only have to show that

$$\alpha_F(X_h)=-F^{-1}\tilde{\theta}_M(X_{pr^*h}^{-F}),$$

which is obvious since both sides are equal to $h(x)$.

Case 2: M_x is an l.c.s. leaf. Denote by ω_F and Ω_F the one-form and two-form defining the l.c.s. structure on $J^{-1}(x)/\Gamma_x$. As above we want to show that $(pr|_{J^{-1}(x)})^*\omega_F=-F^{-1}dF$

and $(pr|_{J^{-1}(x)})^*\Omega_F = -F^{-1}d\tilde{\theta}_M$. A computation using $dF(E_M) = 0$ (by Lemma 3.12) and $d\hat{h}(E_M) = 0$ (since E_M is tangent to the Γ-orbits by equation (6) shows that for all $h \in C^\infty(M/\Gamma)$ we have

$$\omega_F(X_h) = dh(E_0) = -F^{-1}dF(X^{-F}_{pr^*h})$$

and

$$\begin{aligned} \Omega_F(X_h, X_k) &= -k \cdot dh(E_0) + h \cdot dk(E_0) - dh(\#\Lambda_0 dk) \\ &= -F^{-1}d\tilde{\theta}_M(X^{-F}_{pr^*h}, X^{-F}_{pr^*k}), \end{aligned}$$

so we are done. □

5 Relation with other contact reductions and prequantization

In this section, which can be read independently of the previous ones, we clarify Willett's procedure for contact reduction and point out the relation between the reduced spaces by contact groupoid reduction on one hand and Willett's and Albert's reduced spaces on the other hand.

5.1 Relation with Willett's reduction

Suppose G is a Lie group acting on a contact manifold (M, θ_M) from the right preserving the contact one form θ_M. A moment map [1] [19] is a map ϕ from the manifold M to $\mathfrak{g}^*$ (the dual of the Lie algebra) such that for all v in the Lie algebra $\mathfrak{g}$:

$$\langle \phi, v \rangle = \theta_M(v_M), \tag{17}$$

where v_M is the infinitesimal generator of the action on M given by v. The moment map ϕ is automatically equivariant with respect to the (right) coadjoint action of G on $\mathfrak{g}^*$ given by $\xi \cdot g = L_g^* R_{g^{-1}}^* \xi$. A group action as above together with its moment map is called Hamiltonian action. In [19], Willett defines the contact reduction at the point $\xi \in \mathfrak{g}^*$ to be

$$M_\xi^W := \phi^{-1}(\mathbb{R}^+ \cdot \xi)/K_\xi,$$

where K_ξ is the unique connected subgroup of G_ξ (the stabilizer group at ξ of the coadjoint action) such that its Lie algebra is the intersection of $\ker \xi$ and $\mathfrak{g}_\xi$ (the Lie algebra of G_ξ). If the following three conditions hold:

a) $\ker \xi + \mathfrak{g}_\xi = \mathfrak{g}$,

b) ϕ is transverse to $\mathbb{R}^+ \cdot \xi$,

c) the K_ξ action is proper,

then the reduced space M_ξ^W is a contact orbifold. It is a manifold if the K_ξ action is free. When $\xi = 0$, Willett's reduced space is the same as the one obtained by Albert [1].

It turns out that Willett's reduction is strongly related to (the prequantization of) our reduction.

First of all, given a contact Hamiltonian action, we naturally have a groupoid action. Using the notation of Example 2.3, we have

Proposition 5.1. *Identify S^*G and $S(\mathfrak{g}^*) \rtimes G$ by right translation, then a Hamiltonian G action on (M, θ_M) gives rise to a contact groupoid action of S^*G on $(M, \frac{\theta_M}{\|\phi\|})$ by*

$$m \cdot ([\xi], g) := m \cdot g$$

with moment map $J = [\phi]$, if 0 is not in the image of ϕ. Here $[\,\cdot\,]$ denotes the equivalence class under the $\mathbb{R}^+$ action.

Proof. Let m be in M and $([\xi], g)$ in $S(\mathfrak{g}^*) \rtimes G$ with $J(m) = \mathfrak{t}([\xi], g) = [\xi]$. Since the coadjoint action on $\mathfrak{g}^*$ is linear and using the equivariance of ϕ, one can easily check that the given action is a groupoid action (Definition 3.1).

To see whether this is a contact groupoid action, we only have to verify (4). Suppose $(Y, (\delta\xi, R_{g_*}v)) \in T_{(m,(\xi,g))}(M_J \times_{\mathfrak{t}} \mathfrak{g}^* \rtimes G)$, where v is an element in $\mathfrak{g}$ and R_g denotes right translation by g. Notice that the image of $(Y, R_{g_*}v)$ under the derivative of the group action map $M \times G \to M$ is $(v_M + Y) \cdot g$. Here by $\cdot g$ we denote the lift action of G on TM. Then (4) follows from (17). □

Remark 5.2. If we are given a free Hamiltonian contact action, from this claim, we can see that we can perform our reduction at every point except for 0. For $\xi = 0$, one can use another groupoid (See Claim 5.11) to make up this deficiency.

Now we give another characterization of the conditions a), b), c) above which ensure that Willett's reduced space be a contact orbifold.

Lemma 5.3. *Given a free Hamiltonian action of a compact group G on a contact manifold M, Willett's conditions for contact reduction a), b) and c) are equivalent to the following two conditions:*

1. *$[\xi]$ is a regular value of J;*

2. *ξ is conjugate to a multiple of an integer point.*

For any Lie algebra $\mathfrak{t}$ of a maximal torus in G we call a point of $\mathfrak{t}^*$ integer if it has integral pairing with all elements of $\ker(\exp|_{\mathfrak{t}})$.

Proof. We identify $\mathfrak{g}$ and $\mathfrak{g}^*$, $\mathfrak{t}$ and $\mathfrak{t}^*$ using a bi-invariant metric on G, where $\mathfrak{t}$ is the Lie algebra of a maximal torus T of G. We may assume ξ is inside $\mathfrak{t}$ since the statement is invariant under coadjoint actions. Then condition a) is automatically satisfied, since regarding ξ as an element of $\mathfrak{g}$ we have $\ker \xi = \xi^\perp$. Clearly, (1) is equivalent to the transversality condition b). So we only have to show that (2) is equivalent to condition c).

In general, if a compact group G acts on a manifold N, then the induced action of a subgroup K is proper if and only if K is also compact. This can be easily seen through the definition of properness (cf. (5)): an action Φ of K on N is proper iff the map $\Phi \times id : K \times N \to N \times N$ is proper. Let O be an orbit of the action of G on N. Then the compactness of O implies the compactness of $(\Phi \times id)^{-1}(O \times O) = K \times O$, hence of K. In particular, applying this to our case, we see that c) is equivalent to K_ξ being compact.

Notice that the Lie algebra of G_ξ is $\mathfrak{g}_\xi = \{a : [a, \xi] = 0\}$ and the Lie algebra of K_ξ is $\mathfrak{k}_\xi = \xi^\perp \cap \mathfrak{g}_\xi$. So we have $\mathfrak{g}_\xi = \mathfrak{k}_\xi \oplus \xi \cdot \mathbb{R}$.

If ξ is not a multiple of any integer point, $\mathfrak{k}_\xi$ will contain a vector whose coordinates are linearly independent over $\mathbb{Z}$, hence the Lie algebra of an irrational flow. This is not hard to see because the set of vectors with $\mathbb{Z}$-linearly dependent coordinates is the union of countably many hyperplanes indexed by $\mathbb{Z}^n$ and $\mathfrak{k}_\xi$ is not one of these, so the vectors of $\mathfrak{k}_\xi$ with $\mathbb{Z}$-linearly dependentent coordinates are contained in countably many hyperplanes of $\mathfrak{k}_\xi$. The fact that this vector has $\mathbb{Z}$-linearly independent coordinates exactly means that it is not contained in any subtorus. So the Lie group $K_\xi \cap T$ integrating $\mathfrak{k}_\xi \cap \mathfrak{t}$ is dense in T. If K_ξ is compact, then $K_\xi \cap T$ is compact too; hence $K_\xi \cap T = T$. But this is impossible because its Lie algebra $\mathfrak{k}_\xi \cap \mathfrak{t}$ doesn't contain ξ.

On the other hand, if ξ is a multiple of some integer point, then the Lie group $K_\xi \cap T$ integrating $\mathfrak{k}_\xi \cap \mathfrak{t}$ is compact. According to [19], $\mathfrak{k}_\xi$ is a Lie ideal of $\mathfrak{g}_\xi$, therefore K_ξ is a normal subgroup of G_ξ. Since G_ξ is compact, $K_\xi = \cup_{g \in G_\xi} \big(g(K_\xi \cap T)g^{-1}\big)$ is compact too. So c) is equivalent to (2). □

Theorem 5.4. *Suppose we are given a free Hamiltonian action of a compact group G on a contact manifold (M, θ_M) and a non-zero element $\xi \in \mathfrak{g}^*$ satisfying a), b) and c) and suppose that the isotropy group G_ξ is connected. Then Willett's reduced space M_ξ^W (with a suitable choice of contact 1-form) is the prequantization of the reduced space $M_{[\xi]}$ obtained from the contact groupoid action of S^*G with a suitable choice of reduction function F.*

Proof. By Claim 5.1, given a Hamiltonian action of G on (M, θ_M), there is automatically a contact groupoid action of S^*G on (M, θ_M). Since G is compact, the function f on the groupoid S^*G is 1 (see Example 2.3). So we can choose as reduction function F a constant function. We adopt the same notation as in Lemma 5.3. Then the reduction space

$$M_{[\xi]} = J^{-1}([\xi])/G_\xi = \phi^{-1}(\xi \cdot \mathbb{R}^+)/G_\xi,$$

is a symplectic manifold by Theorem 4.8, since F is constant and $S(\mathfrak{g}^*)$ only has even dimensional leaves.

Since K_ξ is compact, the right action of K_ξ on G_ξ is proper. Notice that G_ξ is connected and K_ξ is a normal subgroup, so G_ξ/K_ξ is a 1-dimensional compact connected group, therefore S^1. Let the quotient group G_ξ/K_ξ act on M_ξ^W by $[x] \cdot [g] = [x \cdot g]$. This action is free, and

$$M_\xi^W/(G_\xi/K_\xi) = \phi^{-1}(\xi \cdot \mathbb{R}^+)/G_\xi = M_{[\xi]}.$$

So M_ξ^W is an S^1-principal bundle over $M_{[\xi]}$.

Now we claim that the S^1-principal bundle M_ξ^W is furthermore a prequantization of $M_{[\xi]}$. >From the construction in Section 4, the symplectic form ω on $M_{[\xi]}$ is induced by the restriction of $-F^{-1}d(\|\phi\|^{-1}\theta_M)$ on $\phi^{-1}(\xi \cdot \mathbb{R}^+)$. We choose the contact 1-form θ_W on M_ξ^W to be the one induced by the restriction of $-(F\|\phi\|)^{-1}\theta_M$ on $\phi^{-1}(\xi \cdot \mathbb{R}^+)$. Since Willett's reduction only depends on contact structures, we can choose any G-invariant contact form representing the same structure to do reduction. Here, by the equivariance of ϕ, the new form $-(F\|\phi\|)^{-1}\theta_M$ is G-invariant and it is just a rescaling to θ_M, so the level set of the new moment map is unchanged. Notice that the pullback of ω by $\pi : M_\xi^W \to M_{[\xi]}$ is exactly $d\theta_W$.

On $\phi^{-1}(\xi \cdot \mathbb{R}^+)$ we have

$$\theta_M(\xi_M) = \langle \phi, \xi \rangle = \|\phi\| \cdot \|\xi\|, \quad L_{\xi_M}\theta_M = 0,$$

where ξ_M is the infinitesimal action generated by ξ. Using $d\theta_M(v_M, \cdot) = -d\langle \phi, v \rangle$ (see Proposition 3.1 in [19]) we see that $\phi_*\xi_M = 0$, so ξ_M is tangent to $\phi^{-1}(\xi \cdot \mathbb{R}^+)$. This and the fact that the function $\|\phi\|$ is invariant under the flow of ξ_M imply that, on the quotient space M_ξ^W, the induced vector field $[-F\frac{\xi_M}{\|\xi\|}]$ is the Reeb vector field of θ_W. However, in general, $[-F\frac{\xi_M}{\|\xi\|}]$ is not the generator of the S^1 action (cf. Example 5.10). Let

$$t_0 = \min_{t>0}\{\exp t\xi \in K_\xi\}. \tag{18}$$

Then the generator of the G_ξ/K_ξ action is $t_0[\xi_M]$. Therefore, to finish the proof, we can just choose $F = -t_0\|\xi\|$, which only depends on G and ξ but not the action.

In fact, it is not hard to determine t_0, hence F. We might assume $\xi \in \mathfrak{t}^*$ and write ξ as a multiple of an integer point,

$$\xi = \frac{\|\xi\|}{\sqrt{n_1^2 + ... + n_k^2}} \cdot (n_1, ..., n_k), \quad gcd(n_1, ..., n_k) = 1.$$

Let $T = \min_{t>0}\{\exp t\xi = 1\}$ and S_ξ^1 be the circle generated by ξ. Then S_ξ^1 intersects K_ξ at finitely many points since they are both compact and the intersection of their Lie algebras is trivial. Then t_0 is

$$t_0 = \frac{T}{\sharp(S_\xi^1 \cap K_\xi)}.$$

It is not hard to see that T is the smallest positive number for which $T \cdot \xi$ is integer, hence $T = \sqrt{n_1^2 + ... + n_k^2}/\|\xi\|$. And since $\xi \perp \mathfrak{k}_\xi$, by simple combinatorics, S_ξ^1 and K_ξ intersect at $n_1^2 + ... + n_k^2$ points. Therefore

$$t_0 = (\|\xi\|\sqrt{n_1^2 + ... + n_k^2})^{-1}. \tag{19}$$

So $F = -(\sqrt{n_1^2 + ... + n_k^2})^{-1}$.

□

Remark 5.5.

i) When G is not a compact group it is harder to predict what statements hold in place of Lemma 5.3 and Theorem 5.4. Indeed, in that case one can have the noncompact subgroup K_ξ acting properly on $\Phi^{-1}(\mathbb{R}^+\xi)$ (see the proof of Lemma 5.3), and furthermore the isotropy group of the groupoid at ξ might no longer be G_ξ. (See [19], also see Example 6.5).

ii) If G_ξ is not connected we can prove a statement analogous to Theorem 5.4 by modifying suitably Willett's reduction procedure (see Theorem 5.7 and Remark 5.9.).

Remark 5.6.

i) We also have a direct proof that the manifold $M_{[\xi]}$ of Theorem 5.4 is symplectic, as follows. Let a Lie group G act freely on a contact manifold (M, θ_M) with moment map ϕ, and assume that ϕ be transverse to $\xi \cdot \mathbb{R}^+$ (here $\xi \in \mathfrak{g}^*$ is non-zero) and G_ξ act properly on $\phi^{-1}(\xi \cdot \mathbb{R}^+)$. The lifted action to the symplectization $(M \times \mathbb{R}, -d(e^s\theta_M)$ is Hamiltonian with moment map $\tilde{\phi} = e^s\phi$. Since the actions of G_ξ on $\tilde{\phi}^{-1}(\xi)$ and $\phi^{-1}(\xi \cdot \mathbb{R}^+)$ are intertwined, by taking the Marsden-Weinstein reduction at ξ we see that $(\phi^{-1}(\xi \cdot \mathbb{R}^+)/G_\xi, d(\theta_M/\|\phi\|)$ is a symplectic manifold.

As a consequence of this, we obtain a quick proof of Willett's reduction result. Indeed, assume additionally that Willett's conditions a) and c) are satisfied, and consider

$$\pi : \phi^{-1}(\xi \cdot \mathbb{R}^+)/K_\xi \to \phi^{-1}(\xi \cdot \mathbb{R}^+)/G_\xi.$$

The pullback of $d(\theta_M/\|\phi\|)$ via π is non-degenerate on hyper-distributions transverse to $\ker \pi_*$, showing that $\theta_M/\|\phi\|$ is a contact 1-form on $\phi^{-1}(\xi \cdot \mathbb{R}^+)/K_\xi$.

ii) In spite of the existence of a direct proof, the use of contact groupoids allows us to work in a general framework. It provides a unified treatment for both Willett's and Albert's (see Section 5.3) reduction and makes it possible to do reduction at a general point even in the case when G is non-compact (see Example 6.5).

5.2 Application to the prequantization of coadjoint orbits

Kostant constructed prequantizations of coadjoint orbits for applications in representation theory, using tools from Lie theory [13]. Here, using Theorem 5.4, we can give a different description of Kostant's prequatization.

Let G be a compact Lie group and M be S^*G endowed with the contact form as in Example 2.3, which using left translation to identify M with $S(\mathfrak{g}^*) \times G$ reads

$$\theta_M(\delta\xi, \delta g)_{([\xi],g)} = \langle \frac{\xi}{\|\xi\|}, L_{g^{-1}*}\delta g \rangle.$$

Consider the right action of G on M obtained by taking the cotangent lift of the action of G on itself by right multiplication. The action of G and the infinitesimal action of $\mathfrak{g}$, using the above identification, read[12]

$$([\xi], g)h = ([Ad_h^*\xi], gh), \qquad v_M([\xi], g) = ([ad_v^*\xi], L_{g*}v).$$

Since $\theta_M([ad_v^*\xi], L_{g*}v)_{([\xi],g)} = \|\xi\|^{-1}\langle \xi, v\rangle$, this action is Hamiltonian in the sense of (17) with moment map $\phi([\xi], g) = \|\xi\|^{-1}\xi$. According to Claim 5.1, there is automatically a contact groupoid action of S^*G on M, given by the moment map $J = [\phi]$ and $([\xi], g) \cdot ([\eta], h) = ([Ad_h^*\xi], gh)$. This action is actually the right action of S^*G on itself by groupoid multiplication.

Before stating the theorem, let us recall Kostant's construction of prequantizations of coadjoint orbits [13], where the coadjoint orbits are endowed with the *negative* of the usual KKS (Kostant-Kirillov-Souriau, see [5]) symplectic form. View $\mathbb{R}$ as a Lie algebra with the zero structure, then

$$2\pi i\xi|_{\mathfrak{g}_\xi} : \mathfrak{g}_\xi \to \mathbb{R}$$

is a Lie algebra homomorphism. Kostant [13] has proved that it can be integrated into a group homomorphism $\chi : G_\xi \to S^1$ iff the KKS symplectic form ω_ξ on the coadjoint orbit O_ξ is integral. In this case, the prequantization bundle L is simply

$$G \times S^1/G_\xi, \text{ by identifying } (g, s) \sim (gh, \chi(h)^{-1}s).$$

There is a natural 1-form $(\alpha_\xi, \frac{ds}{2\pi})$ on $G \times S^1$, where α_ξ is the left translation of ξ on G and s is the coordinate on S^1. It turns out that it descends to a 1-form θ_L on L, and that θ_L is exactly the connection 1-form.

[12] Here $Ad_h^* = L_h^* R_{h^{-1}}^*$ is a right action of G on $\mathfrak{g}^*$ and so is ad^*. It preserves the bi-invariant metric, therefore it is a right action on $S(\mathfrak{g}^*)$ too.

Theorem 5.7. *Let G be a compact Lie group, $\xi \in \mathfrak{g}^*$, and assume that G_ξ is connected. Then*

i) the KKS symplectic form ω_ξ on the coadjoint orbit O_ξ is integral iff ξ is conjugate to an integer point $(d_1, ...d_k)$;

ii) the contact reduction via groupoids $M_{[\xi]}$ is the coadjoint orbit O_ξ through ξ with the standard KKS symplectic form, with a suitable choice of the reduction function F;

iii) in the case of i), the quotient of the S^1-bundle $M_\xi^W \to O_\xi$ by $\mathbb{Z}_n$ is exactly Kostant's prequantization bundle L, where $n = gcd(d_1, ..., d_k)$.

Remark 5.8. Statement i) above is well known and follows easily from the main construction of the proof.

Proof. Choose a bi-invariant metric on $\mathfrak{g}^*$ and choose a maximal torus as in Theorem 5.4. We adapt the notation used there too. Then we might assume that $\xi \in \mathfrak{t}^*$ since all statements dependent only on the conjugacy class of ξ.

The reduced space at ξ of the contact groupoid action of S^*G on M is

$$M_{[\xi]} = J^{-1}([\xi])/G_\xi = G/G_\xi = O_\xi.$$

Since the action of S^*G on M is the right action of S^*G on itself, if we performed reduction using $F = 1$ then by Example 4.7 we would obtain the Jacobi structure on $S^*G/S^*G = S(\mathfrak{g}^*)$ for which $\mathbf{s} : (S^*G, \theta_M) \to S(\mathfrak{g}^*)$ is a Jacobi map, i.e. the one whose Poissonisation is $\mathfrak{g}^* - 0$ with the Lie-Poisson structure (see Example 2.3). Notice that the Jacobi structure of $S(\mathfrak{g}^*)$ is induced by the Poisson structure on its Poissonisation through the embedding as a unit sphere [8]. Let ω_ξ be the KKS form on O_ξ, then $\lambda\omega_\xi = \omega_{\lambda\xi}$. Therefore, by choosing $F = -\|\xi\|^{-1}$, we obtain that $M_{[\xi]}$ is symplectomorphic to O_ξ endowed with the negative of the KKS form, which proves ii). With this choice for F and the requirement that $d\theta_W$ is the pull-back of ω_ξ, by a similar analysis as in Theorem 5.4, Willett's reduced contact form on M_ξ^W is

$$\theta_W = \frac{\|\xi\|}{\sqrt{n_1^2 + ... + n_k^2}} \theta_c, \tag{20}$$

where θ_c is the connection 1-form of the S^1-principal bundle $M_\xi^W \to M_{[\xi]}$ obtained as in Theorem 5.4.

If ω_ξ is integral, following Kostant, one can construct a prequantization bundle L of $O_\xi \cong M_{[\xi]}$. Construct a morphism between the two S^1-principal bundles over $M_{[\xi]}$,

$$\psi : M_\xi^W = G/K_\xi \to L = G \times S^1/G_\xi, \text{ by } [g] \mapsto [(g, 1)].$$

It is well-defined, since $\mathfrak{k}_\xi = \ker 2\pi i \xi|_{\mathfrak{g}_\xi}$, which implies $K_\xi \subset \ker \chi$. Since G_ξ acts on S^1 transitively via χ, ψ is surjective. The quotient group $\ker \chi / K_\xi$ as a subgroup of $G_\xi/K_\xi = S^1$ is closed, therefore it is $\mathbb{Z}_n$ for some integer n. So $\ker \chi = K_\xi \times \mathbb{Z}_n$, and ψ is a n-covering map.

Moreover it is not hard to see that ψ is S^1-equivariant (here we "identify" G_ξ/K_ξ and S^1 via χ), therefore $T\psi$ takes the infinitesimal generator of the first copy of $S^1(= G_\xi/K_\xi)$

to n times the generator of the other S^1, and ψ induces the identity map on the base $M_{[\xi]}$. Hence, we have

$$\psi^*\theta_L = n \cdot \theta_c. \tag{21}$$

Moreover, notice that $d\theta_W$ is the pullback of ω_ξ via projection $M_\xi^W \to M_{[\xi]}$, and that ω_ξ is the curvature form of L. So we have $d\theta_W = d\psi^*\theta_L$. Combining with (20) and (21), we have

$$\theta_W = \psi^*\theta_L, \text{ and } n = \frac{\|\xi\|}{\sqrt{n_1^2 + ... + n_k^2}}. \tag{22}$$

Since n is an integer, $\xi = n \cdot (n_1, ..., n_k)$ is an integer point and obviously $n = gcd(n \cdot n_1, ..., n \cdot n_k)$. Moreover $M_\xi^W/\mathbb{Z}_n$ is a $(G_\xi/K_\xi)/\mathbb{Z}_n = S^1$ principal bundle, and the morphism ψ induces an isomorphism of principal bundles

$$\tilde{\psi} : M_\xi^W/\mathbb{Z}_n \to L.$$

The one form θ_W on M_ξ^W descends to a one form on $M_\xi^W/\mathbb{Z}_n$, and the first equation in (22) shows that $\tilde{\psi}$ is an isomorphism between the S^1 principal bundle $M_\xi^W/\mathbb{Z}_n$ (equipped with this one form) and Kostant's prequantization bundle L. This proves iii) and one direction of i).

For the converse direction in i), suppose that $\xi = (d_1, ..., d_k) = n \cdot (n_1, ..., n_k)$ is an integer point. Then

$$\frac{\|\xi\|}{\sqrt{n_1^2 + ... + n_k^2}} = n = gcd(d_1, ..., d_k).$$

By (20), $M_\xi^W/\mathbb{Z}_n$ is a prequantization of $M_{[\xi]} = O_\xi$, where the $\mathbb{Z}_n$ action is induced by the one of S^1. Therefore the symplectic form on O_ξ is integral. □

Remark 5.9. To remove the condition on the connectedness of G_ξ we can replace the subgroup K_ξ used in Willett's reduction by $\ker\chi$. This is a good choice not only because Willett's contact reduction procedure still goes through with this replacement, but also because the analogs of Theorems 5.4 and 5.7 can be proven without the extra assumption of G_ξ being connected.

Example 5.10. $[G = U(2)]$ Let $G = U(2)$ and $\xi = \frac{1}{\sqrt{5}}\left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Under a bi-invariant inner product $(v_1, v_2) = tr(v_1 v_2^*)$, one can identify $u^*(2)$(Hermitian matrices) with $u(2)$ by $\xi \mapsto -i\xi$. Then $G_\xi = S^1 \times S^1$ is the maximal torus embedded as diagonal matrices in $U(2)$. It is not hard to see that

$$K_\xi = \{\begin{pmatrix} a & 0 \\ 0 & a^{-2} \end{pmatrix} : \|a\| = 1\}.$$

Now let G act on $M = S^*G$ as described at the beginning of this section. Using the identification

$$U(2) \cong S^3 \times S^1, \begin{pmatrix} a & \gamma\bar{b} \\ b & -\gamma\bar{a} \end{pmatrix} \mapsto \left(\begin{pmatrix} a \\ b \end{pmatrix}, \gamma\right) \tag{23}$$

we easily compute that the groupoid reduction is $M_{[\xi]} = U(2)/(S^1 \times S^1) = S^2$ and Willett's reduction is $M_\xi^W = U(2)/K_\xi = S^3$. If we choose the reduction function $F = -\sqrt{5}^{-1}$, then the symplectic form on $M_{[\xi]}$ is the area form, and $M_\xi^W = S^3$ is exactly the prequantization of S^2, which verifies Theorem 5.4.

Moreover, by taking different values of ξ, one recovers all S^1 principal bundles over S^2. Suppose $\xi = \frac{1}{\sqrt{m^2+n^2}}\left(\begin{smallmatrix} m & 0 \\ 0 & n \end{smallmatrix}\right)$, where $m \neq n$ are in $\mathbb{Z}$ and have greatest common divisor 1. Then following exactly the same method above, one sees that M_ξ^W is a lens space, namely the quotient $L(|m-n|, 1)$ of S^3 by the diagonal $\mathbb{Z}_{|m-n|}$ action.

5.3 Relation to Albert's reduction

Given a Hamiltonian contact action of G on M, one can also perform Albert's reduction [1], which we now review. For any regular value $\xi \in \mathfrak{g}^*$ of ϕ, let $\mathfrak{g}_\xi$ act on $Z := \phi^{-1}(\xi)$ by

$$\mathfrak{g}_\xi \to \chi(Z), \quad v \mapsto v_M - \langle \xi, v\rangle E, \tag{24}$$

where $v \in \mathfrak{g}_\xi$, v_M is the infinitesimal action of $\mathfrak{g}$ on M, and E is the Reeb vector field on M. By Proposition 3.1 in [19] we have for all $v \in \mathfrak{g}$

$$d\langle \phi, v\rangle = -i(v_M) d\theta_M.$$

From this, it is easy to see that E is tangent to the ϕ-level sets. So the above action is a Lie algebra action. Assume the Reeb vector field is complete. Then on an open neighborhood of the identity in G_ξ, one has a new action $\cdot_n$ on Z,

$$x \cdot_n \exp v = \varphi_{-\langle \xi, v\rangle}(x \cdot \exp v),$$

where φ_t is the flow of E and $x \cdot \exp v$ is the old action of G on M. For simplicity, let us assume this action is free and proper and G_ξ is connected. Then one can extend the new action to the whole of G_ξ by multiplication in G_ξ ([3]). **Albert's reduction** is defined as[13]

$$M_\xi^A := Z/G_\xi,$$

with the contact structure inherited from M.

Now we show the relation between Albert's reduced spaces and ours. First of all, with the same set-up as for Albert's reduction and using the notation of Example 2.4, we have

Proposition 5.11. *The action of $T^*G \times \mathbb{R}$ on (M, θ_M) given by*

$$m \cdot (\xi, g, r) = \varphi_r(m \cdot g),$$

is a contact groupoid action with moment map ϕ, where φ_r is the time-r flow of the Reeb vector field E on M. Here we identify $T^*G \times \mathbb{R}$ and $\mathfrak{g}^* \rtimes G \times \mathbb{R}$ by right translation.

Proof. Since the G action preserves E (because it preserves θ_M), we have $\varphi_r(m \cdot g) = \varphi_r(m) \cdot g$. So,

$$\phi(m \cdot (\xi, g, r)) = \phi(\varphi_r(m) \cdot g) = \phi(\varphi_r(m)) \cdot g = \phi(m) \cdot g = \mathbf{s}(\xi, g, r).$$

It is not hard to verify that the other conditions in the definition of groupoid action are satisfied. Furthermore, using the fact that θ_M is preserved by both φ_r and the G action, it is easy to check (4). Therefore the given action is a contact groupoid action. □

[13] It coincides with $Z/\tilde{G}_\xi$, where $\tilde{G}_\xi$ is the simply connected group covering G_ξ acting on Z by the lift of the action $\cdot_n$.

Notice that the Lie algebra action (24) sits inside the bigger Lie algebra action

$$\mathfrak{g}_\xi \times \mathbb{R} \to \chi(Z), \quad (v, r) \mapsto v_M + rE$$

via the Lie algebra morphism $i : \mathfrak{g}_\xi \hookrightarrow \mathfrak{g}_\xi \times \mathbb{R}$ defined by $v \mapsto (v, -\langle \xi, v \rangle)$.

The isotropy group of $T^*G \times \mathbb{R}$ at ξ is $G_\xi \times \mathbb{R}$, and its action corresponds exactly to the infinitesimal action above. If this action is free, then the reduction via contact groupoids

$$M_\xi = Z/(G_\xi \times \mathbb{R})$$

is a symplectic manifold. Let $\tilde{G}_\xi$ be the simply connected Lie group covering G_ξ. Then, the above embedding i gives a Lie group morphism (not necessarily injective any more)

$$\bar{i} : \tilde{G}_\xi \to G_\xi \times \mathbb{R}$$

Then $H := \mathbb{R}/\bar{i}(\tilde{G}_\xi) \cap \mathbb{R}$ acts on Z/G_ξ freely. The quotient H can be very singular if $\bar{i}(\tilde{G}_\xi) \cap \mathbb{R}$ is not discrete. If it is discrete, then H is either $\mathbb{R}$ or S^1. In this case, we will have a H-principal bundle $\pi : M_\xi^A \to M_\xi$.

The contact 1-form θ_ξ on M_ξ^A and the symplectic 2-form ω_ξ on M_ξ are induced by θ_M and $d\theta_M$ on Z with $F = -1$. Hence $\pi^*\omega_\xi = d\theta_\xi$. The Reeb vector field on M descends to the Reeb vector field on M_ξ^A. Since $\mathbb{R}$ acts by Reeb flows, the generator of H is a multiple of the Reeb vector field on M_ξ^A. Therefore if $H \cong S^1$, similarly to the discussion of Willett's reduction, one can rescale the reduction function F suitably to make M_ξ^A a prequantization of M_ξ. If $H \cong \mathbb{R}$, then M_ξ^A, being a $\mathbb{R}$-principal bundle over M_ξ, is simply $M_\xi \times \mathbb{R}$. Summarizing we obtain:

Theorem 5.12. *Let M_ξ be the contact groupoid reduction via $T^*G \times \mathbb{R}$ at the point ξ, let M_ξ^A be the Albert reduction space at ξ and H the group defined above. If the groupoid action of $T^*G \times \mathbb{R}$ is free and H is either $\mathbb{R}$ or S^1, then*

1. *M_ξ^A is a prequantization of M_ξ if $H = S^1$;*
2. *$M_\xi^A = M_\xi \times \mathbb{R}$ if $H = \mathbb{R}$.*

6 Examples

In this section we will exhibit some examples of contact groupoid reduction using Theorem 4.1. We start by describing the general strategy we use to apply the above theorem.

1. Given a contact manifold (M, θ_M) and an integrable Jacobi manifold Γ_0, choose a complete Jacobi map $J : M \to \Gamma_0$.
2. Let Γ be the $\mathbf{t}$-simply connected contact groupoid of Γ_0. For any choice of x lying in a contact leaf of Γ_0, restricting the Lie algebroid action $J^*(\ker \mathbf{t}_*|_{\Gamma_0}) \to TM, X_{\mathbf{s}^*u} \mapsto X_{J^*u}$, obtain the Lie algebra action of $T_x\Gamma_x$ on $J^{-1}(x)$.
3. Integrating determine the Lie group action of Γ_x on $J^{-1}(x)$.
4. Choose an f-multiplicative function F on $J^{-1}(x)$ (or an open subset thereof).

5. If the quotient of $J^{-1}(x)$ (or an open subset thereof) by Γ_x is a manifold, then it is a contact manifold equipped with the one form induced by $-F^{-1}\theta_M$.

We wish to explain in detail how to obtain the Lie algebra action of $T_x\Gamma_x$ on $J^{-1}(x)$ in (2). By Theorem 3.8 the map J in (1) induces a (contact) groupoid action on Γ on M. From the construction in Theorem 3.8 it is clear that the induced Lie algebroid action [14] is $J^*(\ker \mathbf{t}_*|_{\Gamma_0}) \to TM, (X_{\mathbf{s}^*u}(J(m)), m) \mapsto X_{J^*u}(m)$. Here u is a smooth function on Γ_0. Restricting to $T_x\Gamma_x = \ker(\mathbf{t}_*)_x \cap \ker(\mathbf{s}_*)_x$ we obtain a map $J^*(T_x\Gamma_x) \to TJ^{-1}(x)$, i.e. a map

$$T_x\Gamma_x \to \chi(J^{-1}(x)) \ , \ X_{\mathbf{s}^*u}(x) \mapsto X_{J^*u}|_{J^{-1}(x)}.$$

Being obtained by restriction, this will be the infinitesimal action associated to the Lie group action of Γ_x on $J^{-1}(x)$. Therefore, to obtain explicitly the Γ_x-action, all we have to do is to integrate the above Lie algebra action. If the group action of Γ_x on $J^{-1}(x)$ is free and proper, then a similar proof as in Lemma 3.10 ensures the existence on a function F as above on $J^{-1}(x)$ and the quotient $J^{-1}(x)/\Gamma_x$ will be smooth.

Remark 6.1. In the first three examples below we will have $\Gamma_0 = (\mathbb{R}, dt)$. Let us describe explicitly its $\mathbf{t}$-simply connected contact groupoid Γ (see [12] for the case where Γ_0 is a general contact manifold). We have

$$(\Gamma = \mathbb{R}\times\mathbb{R}\times\mathbb{R}, \theta_\Gamma = -e^{-s}dp + dq, f = e^{-s})$$

where we use coordinates (p, q, s) on Γ. Therefore the Reeb vector field is $E_\Gamma = \frac{\partial}{\partial q}$ and $\Lambda_\Gamma = \frac{\partial}{\partial s} \wedge (e^s\frac{\partial}{\partial p} + \frac{\partial}{\partial q})$. The groupoid structure is given by $\mathbf{t}(p,q,s) = p$, $\mathbf{s}(p,q,s) = q$ and $(p,q,s)(\tilde{p},\tilde{q},\tilde{s}) = (p,\tilde{q},s+\tilde{s})$ when $q = \tilde{p}$, so the isotropy groups are given by $\Gamma_x = \{x\}\times\{x\}\times\mathbb{R}$.

Example 6.2. On $M = \mathbb{R}^{2n+1}$ we choose standard coordinates $(x_1, \cdots, x_n, y_1, \cdots, y_n, z)$, concisely denoted by (x_i, y_i, z). Consider

$$J : (\mathbb{R}^{2n+1}, \sum_{i=1}^{n} x_i dy_i - y_i dx_i + dz) \to (\mathbb{R}, dt) \ , \ (x_i, y_i, z) \mapsto z.$$

Notice that this is indeed a Jacobi map since $E_M = \frac{\partial}{\partial z}$ and $\Lambda_M = \frac{1}{2}\sum(\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial z}) \wedge (\frac{\partial}{\partial y_i} - x_i\frac{\partial}{\partial z})$. Therefore the Lie algebroid action (or rather the induced map from sections of $\ker \mathbf{t}_*|_{\Gamma_0}$ to vector fields on M) is given by

$$X_{\mathbf{s}^*u} = u\cdot\frac{\partial}{\partial q} - u'\cdot\frac{\partial}{\partial s} \mapsto X_{J^*u} = u(z)\frac{\partial}{\partial z} + \frac{1}{2}u'(z)\sum x_i\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial y_i}.$$

Notice that the formula for X_{J^*u} implies that J is a complete map. Indeed, if u is a compactly supported function on Γ_0, then we have $|X_{J^*u}(m)| \le C\cdot r$ at all $m \in \mathbb{R}^{2n+1}$, where r is the distance of m from the origin and C some constant. Therefore at time t the

[14] Given any Lie groupoid $\Gamma \rightrightarrows \Gamma_0$ the associated Lie algebroid is $\ker \mathbf{t}_*|_{\Gamma_0} \to \Gamma_0$, and any groupoid action of Γ on a map $J : M \to \Gamma_0$ induces a Lie algebroid action of $\ker \mathbf{t}_*|_{\Gamma_0}$ by differentiating curves $m\cdot g(t)$, where $m \in M$ and $g(t)$ is a curve in $\mathbf{t}^{-1}(J(m))$ passing through $J(m)$ at time zero (see [6]). Above $J^*(\ker \mathbf{t}_*|_{\Gamma_0})$ denotes the vector bundle on M obtained by pullback via J.

integral curve of X_{J^*u} passing through m_0 will have distance at most $|m_0|e^{Crt}$ from the origin, and hence it will be defined for all time.

Choosing $\bar{t} = 0 \in \Gamma_0$ we obtain the Lie algebra action[15] $T_{\bar{t}}\Gamma_{\bar{t}} = \mathbb{R} \to J^{-1}(0) = \mathbb{R}^{2n}$ with infinitesimal generator $-\frac{1}{2}\sum(x_i\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial y_i})$, so the Lie group action of $\Gamma_{\bar{t}}$ on $J^{-1}(0)$ is given by $(x_i, y_i)\cdot s = (e^{-\frac{1}{2}s}x_i, e^{-\frac{1}{2}s}y_i)$. Since $f = e^{-s}$ we can choose $F = \sum x_i^2 + y_i^2$. Notice that the action is not free at the origin (not even locally free). Using the fact that each Γ_x-orbit intersects the unit sphere exactly once we see that the quotient of $(\mathbb{R}^{2n} - \{0\}, -\frac{\sum x_i dy_i - y_i dx_i}{\sum x_i^2 + y_i^2})$ by the $\mathbb{R}$-action is

$$(S^{2n-1}, -(\sum x_i dy_i - y_i dx_i)),$$

i.e. up to sign the standard contact form for the unit sphere in $\mathbb{R}^{2n}$.

Remark 6.3. In the above example the groupoid action of Γ on M is given by

$$(x_i, y_i, z)\cdot(p, q, s) = (e^{-\frac{1}{2}s}x_i, e^{-\frac{1}{2}s}y_i, q)$$

whenever $z = p$, and one can check explicitly that formula (4) in the definition of contact groupoid action holds. Also notice that J is a submersion everywhere, however at $m \in \{0\}\times\mathbb{R} \subset \mathbb{R}^{2n+1}$ the tangent space to the J-fiber and $\ker\theta_M$ coincide, so that—as stated in Lemma 3.5—at such points m the groupoid action is not locally free.

Example 6.4. [Cosphere bundle] Let N be any manifold, endowed with a Riemannian metric, and let $M = T^*N \times \mathbb{R}$. Consider

$$J : (T^*N \times \mathbb{R}, \alpha + dz) \to (\mathbb{R}, dt)\ ,\ (\xi, z) \mapsto z.$$

Here α is the canonical one-form on T^*N, i.e. with respect to local coordinates $\{x_i\}$ on N and $\{y_i\}$, which are the coordinates with respect to the dual basis of $\{\frac{\partial}{\partial x_i}\}$ (giving coordinates $\{x_i, y_i\}$ on T^*N) it is just $\sum y_i dx_i$. In local coordinates we have $E_M = \frac{\partial}{\partial z}$ and $\Lambda_M = \sum\frac{\partial}{\partial y_i}\wedge(\frac{\partial}{\partial x_i} - y_i\frac{\partial}{\partial z})$. Therefore the Lie algebroid action is given by

$$X_{\mathbf{s}^*u} = u\frac{\partial}{\partial q} - u'\frac{\partial}{\partial s} \mapsto X_{J^*u} = u(z)\frac{\partial}{\partial z} + u'(z)\sum y_i\frac{\partial}{\partial y_i}.$$

The above expression for $\|X_{J^*u}\|$ ensures that J is a complete map.

Choosing $\bar{t} = 0 \in \Gamma_0$ we obtain as infinitesimal generator of the Lie algebra action the radial vector field $-\sum y_i\frac{\partial}{\partial y_i}$. The Lie group action of $\Gamma_{\bar{t}}$ on $J^{-1}(0)$ is given in local coordinates by $(x_i, y_i)\cdot s = (x_i, y_i e^{-s})$, i.e. by $\xi\cdot s = \xi\cdot e^{-s}$, where $\xi \in T_p^*N$. We choose $F = \|\xi\|$ and notice that the action is free on $T^*N - \{0\}$. Each Γ_0-orbit there intersects the unit cosphere bundle T_1^*N (the set of covectors of length one) exactly once. Since by Theorem 4.1 the one-form $-\frac{\alpha}{\|\xi\|}$ on $T^*N - \{0\}$ is basic w.r.t. the natural projection, we conclude that $T_1^*N \cong (T^*N - \{0\})/\Gamma_{\bar{t}}$ endowed with the one-form $-\alpha|_{T_1^*N}$ is a contact manifold.

Now we present an example where Willett's reduction fails but contact groupoid reduction works.

[15] As usual here $\Gamma_{\bar{t}}$ denotes the isotropy group of Γ at $\bar{t}$.

Example 6.5. [Non-compact group $G = SL(2,\mathbb{R})$] Let G be a Lie group and let G act on $M = (T^*G - G) \times \mathbb{R}$ from the right by $(\xi, g, t)h = (Ad_h^*\xi, gh, t)$. Here we identify T^*G with $\mathfrak{g}^* \times G$ by left translation. By a calculation similar to the one at the beginning of subsection 5.2, we can see that this is a Hamiltonian action with moment map $\phi(\xi, g, t) = \xi$. By Claim 5.1, the cosphere bundle S^*G as a contact groupoid automatically acts on M. Let $G = SL(2,\mathbb{R})$. Then we are actually revisiting Example 3.7 in [19], except that we adapt everything to right actions. In [19] it is shown that Willett's reduction at the point $\xi = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ has four dimensions, therefore it is not a contact manifold.

However, the reduction by contact groupoids is a contact manifold. Using the standard Killing form on $SL(2,\mathbb{R})$, that is $\langle X, Y\rangle = tr(X \cdot Y)$, we identify $sl^*(2,\mathbb{R})$ and $sl(2,\mathbb{R})$. Then the isotropy group $\Gamma_{[\xi]}$ of the groupoid is

$$\Gamma_{[\xi]} = \{\begin{pmatrix} \alpha & \gamma \\ 0 & \alpha^{-1} \end{pmatrix} : \alpha \in \mathbb{R} - 0, \gamma \in \mathbb{R}\},$$

which has one more dimension than the stabilizer group G_ξ. Let B be the Borel subgroup of $SL(2,\mathbb{R})$ embedded as upper triangular matrices. Then B is a normal subgroup of $\Gamma_{[\xi]}$ and $\Gamma_{[\xi]} = \mathbb{R}^+ \times \mathbb{Z}_2 \times B$.

We want to quotient out

$$J^{-1}([\xi]) = \{(\lambda\xi, g, t) | \lambda \in \mathbb{R}^+, g \in SL(2,\mathbb{R}), t \in \mathbb{R}\}$$

by Γ_ξ. Notice that $SL(2,\mathbb{R})$ acts on $\mathbb{R}^2 - 0$ transitively with stabilizer B at the point $(1, 0)$. So $SL(2,\mathbb{R})/B = \mathbb{R}^2 - 0$. Therefore, by a more careful examination of the quotient space $J^{-1}([\xi])/\Gamma_\xi$,

$$M_{[\xi]} = ((\mathbb{R}^2 - 0)/\mathbb{Z}_2) \times \mathbb{R} = (\mathbb{R}^2 - 0) \times \mathbb{R}.$$

It is not surprising at all that we get a contact manifold by the groupoid reduction at $[\xi] = [\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)]$, since $[\xi]$ lies in a contact leaf of $S(sl^*(2,\mathbb{R}))$. Indeed, identify $sl^*(2,\mathbb{R})$ with $\mathbb{R}^3$ by a series of new coordinate functions:

$$\begin{aligned} \mu_1 &= \frac{1}{2}(X + Y), \\ \mu_2 &= \frac{1}{2}H, \\ \mu_3 &= \frac{1}{2}(X - Y), \end{aligned}$$

where $X = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $Y = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ and $H = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ are the standard generators of $sl(2,\mathbb{R})$. Then the symplectic leaves of $sl^*(2,\mathbb{R})$ sitting inside $\mathbb{R}^3$ are level surfaces of the Casimir function $\mu_1^2 + \mu_2^2 - \mu_3^2$. That is, they are hyperbolas of two sheets and one sheet as well as symplectic cones. Then $\xi = (1, 0, 1)$ lies inside a symplectic cone, which induces a contact leaf on $S(sl^*(2,\mathbb{R}))$ because the radial vector of the symplectic cone gives exactly the infinitesimal action of $\mathbb{R}^+$, by which we quotient out to get the Jacobi structure on $S(sl^*(2,\mathbb{R}))$.

Remark 6.6. It turns out that every point ξ of a nilpotent adjoint orbit of a semisimple Lie algebra can give rise to a contact manifold as above. This is under further investigation.

Example 6.7. [Variation with non-compact group $G = SL(3,\mathbb{R})$] In Example 6.5, we saw that the action of a group G on the contact manifold $(M = (T^*G - G) \times \mathbb{R}, \theta_c + dt)$ from the right by $(\xi, g, t)h = (Ad_h^*\xi, gh, t)$ is a Hamiltonian action, with moment map $\phi(\xi, g, t) = \xi$. Now we choose $G = SL(3,\mathbb{R})$, and we obtain a Hamiltonian action of $SL(2,\mathbb{R})$ on M by restricting the above action to $SL(2,\mathbb{R}) \subset SL(3,\mathbb{R})$ (the embedding is given by $H \mapsto \left(\begin{smallmatrix} H & 0 \\ 0 & 1 \end{smallmatrix}\right)$). Then, using the Killing form $\langle X, Y \rangle = tr(XY)$ to identify a Lie algebra with its dual and identifying M with $(sl^*(3,\mathbb{R}) - 0) \times SL(3,\mathbb{R}) \times \mathbb{R}$ by left translations, the moment map of the Hamiltonian action reads

$$\phi : (sl^*(3,\mathbb{R}) - 0) \times SL(3,\mathbb{R}) \times \mathbb{R} \to sl^*(2,\mathbb{R}), \left(\left(\begin{smallmatrix} A & b \\ c & d \end{smallmatrix}\right), g, t\right) \mapsto A + \frac{d}{2}\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right).$$

By Claim 5.1 we have an induced action of the contact groupoid of the sphere $S(sl^*(2,\mathbb{R}))$ on M, with moment map $J = [\phi]$. Now we will perform contact groupoid reduction at the point $[\xi] = [\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)]$, which lies in a contact leaf of $S(sl^*(2,\mathbb{R}))$. The reduced space is the quotient of

$$J^{-1}([\xi]) = \Big\{ \left(\begin{pmatrix} -\frac{d}{2} & \lambda & b_1 \\ 0 & -\frac{d}{2} & b_2 \\ c_1 & c_2 & d \end{pmatrix}, g, t \right) \quad : \quad \lambda \in \mathbb{R}_+, b_1, b_2, c_1, c_2, d \in \mathbb{R}; \quad g \in SL(3,\mathbb{R}); t \in \mathbb{R} \Big\}$$

by $\Gamma_{[\xi]} = \{\left(\begin{smallmatrix} \alpha & \gamma \\ 0 & \alpha^{-1} \end{smallmatrix}\right) : \alpha \in \mathbb{R} - 0, \gamma \in \mathbb{R}\}$, which is the isotropy group at $[\xi]$ of the groupoid. Explicitly, the action is given by

$$\left(\begin{pmatrix} A & b \\ c & d \end{pmatrix}, g, t \right) \cdot H = \left(\begin{pmatrix} H^{-1}AH & H^{-1}b \\ cH & d \end{pmatrix}, g \cdot \begin{pmatrix} H & 0 \\ 0 & 1 \end{pmatrix}, t \right)$$

where $\left(\left(\begin{smallmatrix} A & b \\ c & d \end{smallmatrix}\right), g, t\right)$ and $H \in \Gamma_\xi$. As in Example 6.5 we will reduce first by the Borel subgroup $\{\left(\begin{smallmatrix} 1 & \gamma \\ 0 & 1 \end{smallmatrix}\right) : \gamma \in \mathbb{R}\}$ and then by $\{\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right) : \alpha \in \mathbb{R} - 0\}$. To simplify the computation identify $SL(3,\mathbb{R})$ with $U \times \mathbb{R}^2$ by identifying $\begin{pmatrix} | & | & | \\ \mathbf{v} & \mathbf{w} & \mathbf{z} \\ | & | & | \end{pmatrix}$ with $(\mathbf{v}, \mathbf{z}, \nu, \eta)$, where $\mathbf{w} = \frac{\mathbf{v} \times \mathbf{z}}{|\mathbf{v} \times \mathbf{z}|^2} + \nu\mathbf{v} + \eta\mathbf{z}$. Here

$$U = \{\text{pairs of linearly independent vectors in } \mathbb{R}^3\} = (\mathbb{R}^3 - 0) \times (\mathbb{R}^3 - \mathbb{R}).$$

The resulting quotient is

$$(\mathbb{R}^3 - \mathbb{R}) \times \mathbb{R}^3 \times (S^2 \times \mathbb{R}^5)/\mathbb{Z}_2.$$

Since $(S^2 \times \mathbb{R}^5)/\mathbb{Z}_2$ embeds in $(\mathbb{R}^8 - 0)/\mathbb{Z}_2$ (which is an $\mathbb{R}^+ bundleover \mathbb{RP}^7$) as a section of the $\mathbb{R}^+$-bundle defined over $\{[(x_1, \cdots, x_8)] : x_1, x_2, x_3 \neq 0\} \subset \mathbb{RP}^7$, our quotient can be re-written as

$$S^1 \times \mathbb{R}^5 \times (\mathbb{RP}^7 - \mathbb{RP}^4).$$

Remark 6.8. The examples exhibited here are all well known examples of contact manifolds, as one can see using for example Theorem 3.6 in [2].

Appendix I—invariance of contact structures

To prove the invariance of the contact structure on the reduced space, we present in this appendix a "form-free" version (Appendix I, Theorem 1.4) of our main results (Theorem 4.1 and Theorem 4.4). As stated in Section 2, we assume that all contact structures involved in this paper are co-oriented, but the next two definitions make sense even without this assumption.

First, let us recall the definition of conformal contact groupoid [16] from [9].

Definition 1.1. A Lie groupoid Γ together with a contact structure (i.e. a contact hyperplane distribution) $\mathcal{H}_\Gamma$ is called a *conformal contact groupoid* if

i) $(X, Y) \in \mathcal{H}_\Gamma \times \mathcal{H}_\Gamma \Rightarrow X \cdot Y \in \mathcal{H}_\Gamma$, whenever $X \cdot Y$ is defined;

ii) the inversion $i : \Gamma \to \Gamma$ leaves $\mathcal{H}_\Gamma$ invariant.

Definition 1.2. Let $(\Gamma, \mathcal{H}_\Gamma)$ be a conformal contact groupoid and M a manifold with contact structure $\mathcal{H}_M$. A (right) groupoid action Φ of Γ on M is a *conformal contact groupoid action* if

i) $(Y, V) \in \mathcal{H}_M \times \mathcal{H}_\Gamma \Rightarrow \Phi_*(Y, V) \in \mathcal{H}_M$,

ii) $Y \in \mathcal{H}_M, \Phi_*(Y, V) \in \mathcal{H}_M \Rightarrow V \in \mathcal{H}_\Gamma$,

whenever $\Phi_*(Y, V)$ is defined.

Remark 1.3. Condition ii) implies that for the Reeb vector field of any contact one-form θ_Γ with kernel $\mathcal{H}_\Gamma$

$$0 \cdot E_\Gamma \notin \mathcal{H}_M. \tag{25}$$

In fact, it is not hard to deduce from the proof of Lemma 1.7 that (25) is equivalent to condition (ii).

Theorem 1.4. *Let $(M, \mathcal{H}_M)$ be a manifold with a contact structure and let Φ be a conformal contact groupoid action of $(\Gamma, \mathcal{H}_\Gamma)$ on $(M, \mathcal{H}_M)$. Then the point-wise reduced spaces $J^{-1}(x)/\Gamma_x$ inherit naturally a contact or conformal l.c.s. structure, and they are exactly the leaves of the global reduced space M/Γ endowed with the conformal Jacobi structure as in Theorem 4.4.*

We start with a lemma involving only groupoids and not actions:

Lemma 1.5. *Let $(\Gamma, \mathcal{H}_\Gamma)$ be a conformal contact groupoid . Then*

i) there is a multiplicative function f on Γ and a contact form θ_Γ with kernel $\mathcal{H}_\Gamma$ such that the triple $(\Gamma, f, \theta_\Gamma)$ is a contact groupoid.

ii) $(\Gamma, \hat{f}, \hat{\theta}_\Gamma)$ is another such triple if and only if there is a non-vanishing function u on Γ_0 such that $\hat{f} = f\frac{\mathbf{s}^*u}{\mathbf{t}^*u}$ and $\hat{\theta}_\Gamma = \mathbf{s}^*(u)\theta_\Gamma$.

[16] It is known under various names in the literature. Here we use the same name as in [8]

Proof. i) is the remark following Proposition 4.1 in [Da]. We will indicate the proof of ii). Given a contact groupoid $(\Gamma, f, \theta_\Gamma)$, using the fact that $\frac{\mathbf{s}^*u}{\mathbf{t}^*u}$ is multiplicative, it is not hard to verify equation (3) for the triple $(\Gamma, f\frac{\mathbf{s}^*u}{\mathbf{t}^*u}, \mathbf{s}^*u\theta_\Gamma)$, so that it is again a contact groupoid. Conversely suppose that $(\Gamma, \hat{f}, \hat{\theta}_\Gamma)$ is a contact groupoid. Then there exist a multiplicative function ϕ on Γ and a non-vanishing function τ on Γ such that $\hat{f} = \phi f$ and $\hat{\theta}_\Gamma = \tau\theta_\Gamma$. Therefore the multiplication satisfies

$$^*(\tau\theta_\Gamma) = pr_2^*(\phi f) \cdot pr_1^*(\tau\theta_\Gamma) + pr_2^*(\tau\theta_\Gamma).$$

Evaluating this at $(g, h) \in \Gamma_\mathbf{s} \times_\mathbf{t} \Gamma$ and using Lemma 4.1 in [9], we obtain $\tau(gh) = \tau(h) = \phi(h)\tau(g)$. The first equation implies that $\tau = \mathbf{s}^*u$ for some non-vanishing function u on Γ_0, and the second that $\phi = \frac{\mathbf{s}^*u}{\mathbf{t}^*u}$, as claimed. □

Remark 1.6. The change in ii) corresponds to a u^{-1}-conformal change on the base Γ_0 and a $(\mathbf{s}^*u)^{-1}$-conformal change on Γ.

It is not hard to verify that a contact groupoid action is also a conformal contact groupoid action. Now we prove the converse:

Lemma 1.7. *Let $\Phi : M_J \times_\mathbf{t} \Gamma \to M$ be a conformal contact groupoid action. Then*

i) Given a triple $(\Gamma, f, \theta_\Gamma)$ as in Lemma 1.5, there is a unique contact 1-form θ_M on M such that Φ is a contact groupoid action;

ii) $(\Gamma, \hat{f}, \hat{\theta}_\Gamma)$ and $(M, \hat{\theta}_M)$ are another such pair if and only if $\hat{f} = f\frac{\mathbf{s}^*u}{t^*u}$, $\hat{\theta}_\Gamma = \mathbf{s}^*u \cdot \theta_\Gamma$ and $\hat{\theta}_M = J^*u \cdot \theta_M$.

Proof. Given a triple $(\Gamma, f, \theta_\Gamma)$ as in i), let E_Γ be the Reeb vector field of Γ corresponding to the 1-form θ_Γ. Define a vector field on M by

$$E_M(m) := 0(mg^{-1}) \cdot E_\Gamma(g).$$

This vector field is well-defined since using the f-multiplicativity of θ_Γ one can show that $E_\Gamma(g') = 0(g'g^{-1}) \cdot E_\Gamma(g)$ whenever $\mathbf{s}(g) = \mathbf{s}(g')$. By equation (25) there exists a (unique) contact 1-form θ_M with kernel $\mathcal{H}_M$ and E_M as Reeb vector field. Endowing $M\times\mathbb{R}\times\Gamma\times\mathbb{R}\times M$ with the contact structure as in Lemma 3.3 we obtain as contact hyperplane

$$\begin{aligned}\mathcal{H} \;=\; & (\mathcal{H}_M \times 0 \times \mathcal{H}_M \times 0 \times \mathcal{H}_M) \oplus span\{\frac{\partial}{\partial a}\} \oplus span\{\frac{\partial}{\partial b}\} \\ & \oplus span\{(E_M, 0, 0, 0, fe^{-a}E_M)\} \oplus span\{(0, 0, E_\Gamma, 0, e^{-b}E_M)\}.\end{aligned}$$

Denote the graph of the action Φ by $\mathcal{A}$. By i) in Definition 1.2,

$$\dim\left((\mathcal{H}_M \times 0 \times \mathcal{H}_\Gamma \times 0 \times \mathcal{H}_M) \cap T\mathcal{A}\right) \geq k + n - 1,$$

where $\dim M = k$ and $\dim\Gamma = 2n + 1$. Using again the f-multiplicativity of θ_Γ (Equation (3)) and the fact that $\mathbf{t}$ is $-f$-Jacobi, one can show that

$$E_\Gamma(h) \cdot (X_{-f})_{\mathcal{H}_\Gamma}(g) = f(g)E_\Gamma(hg)$$

whenever $\mathbf{s}(h) = \mathbf{t}(g)$, where $(X_{-f})_{\mathcal{H}_\Gamma}$ is the projection of X_{-f} onto $\mathcal{H}_\Gamma$. This together with the definition of E_M imply that

$$(E_M, 0, (X_{-f})_{\mathcal{H}_\Gamma}, 0, fE_M) \text{ and } (0, 0, E_\Gamma, 0, E_M) \in \mathcal{H} \cap T\mathcal{A}.$$

Therefore with these two more vectors, we have $\dim(\mathcal{H} \cap T\mathcal{A}) \geq k + n + 1$. On the other hand $T\mathcal{A}$ has dimension $k+n+1$, so we have $T\mathcal{A} \subset \mathcal{H}$ and $\mathcal{A}$ is a Legendrian submanifold. By Lemma 3.3, the action is a contact groupoid action. The uniqueness follows because by equation (6) for any contact groupoid action we have $0 \cdot E_\Gamma = E_M$.

To prove ii) notice that the expressions for $\hat{f}$ and $\hat{\theta}_\Gamma$ were derived in Lemma 1.5. By the proof of i) the expression for $\hat{\theta}_M$ is determined by its Reeb vector field $\hat{E}_M := 0 \cdot \hat{E}_\Gamma = 0 \cdot \frac{1}{\mathbf{s}^* u} E_\Gamma = \frac{1}{J^* u} E_M$, where $\hat{E}_\Gamma$ denotes the Reeb vector field of $\hat{\theta}_\Gamma$. □

Now the proof of Theorem 1.4 is straightforward.

Proof of Theorem 1.4. Let $(\Gamma, \mathcal{H}_\Gamma)$ be a contact-structure groupoid. Lemma 1.5 tells us what the "compatible" choices of pairs (θ_Γ, f) are on Γ. Now let $(M, \mathcal{H}_M)$ be a manifold with a contact structure and Φ be a conformal contact groupoid action of $(\Gamma, \mathcal{H}_\Gamma)$ on $(M, \mathcal{H}_M)$. Lemma 1.7 tells us that for each pair (θ_Γ, f) there is a unique choice for θ_M that makes Φ a contact groupoid action. If we make a choice of pair (θ_Γ, f) and consider the corresponding form θ_M, we obtain by Theorem 4.4 a Jacobi structure on M/Γ by requiring that $pr : M \to M/\Gamma$ be a $-F$-conformal Jacobi map, where F is some f-multiplicative function on M.

Let $(\hat{\theta}_\Gamma := \mathbf{s}^* u \cdot \theta_\Gamma, \hat{f} := f \frac{\mathbf{s}^* u}{\mathbf{t}^* u}, \hat{\theta}_M := J^* u \cdot \theta_M)$ be another set of data as above. It is straightforward to check that $\hat{F} := J^* u \cdot F$ is a $\hat{f}$-multiplicative function. The corresponding Jacobi structure on M/Γ is obtained by requiring that pr be a $-\hat{F}$-conformal Jacobi map with respect to the contact form $\hat{\theta}_M = J^* u \cdot \theta_M$, i.e. that it be a Jacobi map with respect to the Jacobi structure on M obtained from the original one [17] twisting by $-\hat{F} \cdot (J^* u)^{-1} = -F$. Therefore the two Jacobi structures on M/Γ obtained above are identical. This shows that the conformal class is independent of all the choices we made. □

Appendix II—On left/right actions and sign conventions

The definition of contact groupoids we adopted (Definition 2.1) allows one to define only *right* actions (Definition 3.1). In this appendix we describe how to switch from such a groupoid to one for which we can naturally define left actions.

We start by describing a setting that includes both kinds of groupoids [9]. Given a conformal contact groupoid $(\Gamma, \mathcal{H}_\Gamma)$ for which the contact structure is co-orientable (see Definition 1.1 in Appendix II), one can choose a corresponding contact form θ and two multiplicative functions $f_L, f_R : \Gamma \to \mathbb{R} - \{0\}$ such that the multiplication satisfies[18]

$$^*(\theta) = pr_2^*(f_R) pr_1^*(\theta) + pr_1^*(f_L) pr_2^* h(\theta). \tag{26}$$

[17] That is, the one corresponding to θ_M

[18] See Proposition 4.1 in [9].

Furthermore Γ_0 can be given a Jacobi structure so that $\mathbf{s}$ is a f_L-Jacobi map and $\mathbf{t}$ an $-f_R$-Jacobi map[19]. Clearly imposing that $\mathbf{s}$ be $-f_L$-Jacobi and $\mathbf{t}$ be f_R-Jacobi endows Γ_0 with a Jacobi structure which is the negative of the above.

One can always arrange[20] that either $f_L \equiv 1$ or $f_R \equiv 1$. We will adopt the following conventions for the induced Jacobi structure on Γ_0:

a) If $f_L \equiv 1$ ("right contact groupoid") then $\mathbf{s}$ is a Jacobi map.

b) If $f_R \equiv 1$ ("left contact groupoid") then $\mathbf{t}$ is a Jacobi map.

Notice that convention a) above is the one used by Kebrat and Souici in [12] and the one we followed in this paper (see Definition 2.1).

Now recall that if $\Gamma \rightrightarrows \Gamma_0$ is any Lie groupoid and $\Phi_r : M_J \times_{\mathbf{t}} \Gamma \to M$ is a *right* groupoid action on $J : M \to \Gamma_0$, then by $\Phi_l(g,m) = \Phi_r(m, g^{-1})$ we obtain a *left* groupoid action $\Phi_l : \Gamma_{\mathbf{s}} \times_J M \to M$ on J. Suppose we are given a "right contact groupoid", i.e. a tuple $(\Gamma, \theta_r, 1, f_r)$ satisfying (26), and suppose Φ_r as above is a contact groupoid action on some contact manifold (M, θ_M). Then Φ_l satisfies

$$\Phi_l^*(\theta_l) = pr_\Gamma^*(\theta_l) + pr_\Gamma^*(f_l) pr_M^*(\theta_M), \tag{27}$$

where $\theta_l := i^*\theta_r = -\frac{1}{f_r}\theta_r$ and $f_l := i^* f_r = \frac{1}{f_r}$. The new structure $(\Gamma, \theta_l, f_l, 1)$ satisfies (26), so we can define it to be the "left contact groupoid" associated to $(\Gamma, \theta_r, 1, f_r)$. Furthermore we take (26) to be the defining equation for left contact groupoid actions.

Notice that switching from "right" to "left" contact groupoid does not change the underlying conformal contact groupoid $(\Gamma, \mathcal{H}_\Gamma)$. Furthermore, assuming our conventions a) and b) above, it does not change the Jacobi structure induced on Γ_0 : indeed $\mathbf{s} : (\Gamma, \theta_r = -\frac{1}{f_l}\theta_l) \to \Gamma_0$ is a Jacobi map exactly when $\mathbf{s} : (\Gamma, \theta_l) \to \Gamma_0$ is a $-f_l$-Jacobi map, which happens exactly when $\mathbf{t} : (\Gamma, \theta_l) \to \Gamma_0$ is a Jacobi map.

We conclude this appendix by describing how our conventions a) and b) fit with choices of Lie algebroids for Γ. Recall that a Lie algebroid is a vector bundle $E \to N$ together with a bundle map (the anchor) $E \to TN$ and a Lie bracket on its space of sections satisfying certain conditions (see [6]). Given any Lie groupoid $\Gamma \rightrightarrows \Gamma_0$, there are two associated Lie algebroids: one is $\ker \mathbf{t}_*|_{\Gamma_0}$, with Lie bracket induced by the bracket of left-invariant vector fields on Γ and with anchor $\mathbf{s}_*$. The other one is $\ker \mathbf{s}_*|_{\Gamma_0}$ with anchor $\mathbf{t}_*$. Under the identification $\ker \mathbf{t}_*|_{\Gamma_0} \cong T\Gamma|_{\Gamma_0}/T\Gamma_0 \cong \ker \mathbf{s}_*|_{\Gamma_0}$ (which is given by $-i_*$ for $i : \Gamma \to \Gamma$ the inversion), the two algebroid structures are anti-isomorphic[21]. Notice that this implies that $i_* : \ker \mathbf{t}_*|_{\Gamma_0} \to \ker \mathbf{s}_*|_{\Gamma_0}$ is a Lie algebroid isomorphism, but we will not use this fact.

A *right* action of Γ on a manifold M with moment map $J : M \to \Gamma_0$ clearly induces by differentiation an algebroid action of $\ker \mathbf{t}_*|_{\Gamma_0}$, whereas a *left* groupoid action induces an action of $\ker \mathbf{s}_*|_{\Gamma_0}$. In this sense $\ker \mathbf{t}_*|_{\Gamma_0}$ is the preferred algebroid for "right contact groupoids", and $\ker \mathbf{s}_*|_{\Gamma_0}$ for "left contact groupoids".

Now let $(\Gamma, \theta, f_L, f_R)$ be a groupoid satisfying (26). There are two natural vector bundle isomorphisms[22] from the Lie algebroid $T^*\Gamma_0 \times \mathbb{R}$ of the the Jacobi manifold Γ_0 to the two

[19] See Theorem 4.1ii in [9].

[20] See the proof of Proposition 4.1 of [9].

[21] See Theorem 9.15 in [18].

[22] See Proposition 4.3 and the remarks on page 443 and page 446 in [9]

algebroids of Γ:

$$T^*\Gamma_0 \times \mathbb{R} \to \ker \mathbf{t}_*|_{\Gamma_0}\ ,\ (\varphi_1, \varphi_0) \mapsto \mathbf{s}^*\varphi_0 \cdot X_{f_L} + f_L \cdot \sharp\Lambda\mathbf{s}^*\varphi_1 \tag{28}$$

and

$$T^*\Gamma_0 \times \mathbb{R} \to \ker \mathbf{s}_*|_{\Gamma_0}\ ,\ (\varphi_1, \varphi_0) \mapsto \mathbf{t}^*\varphi_0 \cdot X_{f_R} + f_R \cdot \sharp\Lambda\mathbf{t}^*\varphi_1, \tag{29}$$

and it is a straightforward computation using (26) to show that $-i_* : \ker \mathbf{t}_*|_{\Gamma_0} \to \ker \mathbf{s}_*|_{\Gamma_0}$ intertwines them.

If we endow Γ_0 with a Jacobi structure so that $\mathbf{s}$ is a f_L-Jacobi map and $\mathbf{t}$ a $-f_R$-Jacobi map then the map (28) is an isomorphims of Lie algebroids[23]. Therefore when Γ is a"right contact groupoid" following convention a) we obtain a natural isomorphism between the algebroid of Γ_0 and the preferred algebroid of Γ. The analogous statement for "left contact groupoids" holds as well.

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[23] See the second part of Theorem 4.1 of [9]

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