AN EXAMPLE OF COISOTROPIC SUBMANIFOLDS C^1-CLOSE TO A GIVEN COISOTROPIC SUBMANIFOLD

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Abstract. We discuss a simple example of coisotropic submanifold M of a symplectic manifold, and show that the set of coisotropic submanifolds which are C^1-close to M does not have a manifold structure.

1. Introduction

In this note we consider the symplectic manifold

$$(N, \Omega) := (\mathbb{T}^4 \times \mathbb{R}^2, dx_1 \wedge dx_2 + dx_3 \wedge d\xi_3 + dx_4 \wedge d\xi_4),$$

where $(x_1, \ldots, x_4)$ and (ξ_3, ξ_4) are canonical coordinates on $\mathbb{T}^4$ and $\mathbb{R}^2$ respectively. We regard N as a vector bundle over $\mathbb{T}^4$. Observe that the zero section $M = \mathbb{T}^4 \times \{0\}$ is coisotropic in N.

We study certain aspects of the set $\mathcal{C}$ of coisotropic submanifolds which are C^1-close to M. First, we characterize elements of $\mathcal{C}$ by means of a certain nonlinear relation. Then in Prop. 2.1 we show that arbitrarily small coisotropic deformations of M have characteristic foliations which are not homeomorphic to that of M.

This suggests that set of coisotropic submanifolds which are C^1-close to M does not have a nice structure. Indeed, in Cor. 2.3 we prove that $\mathcal{C}$ is not a manifold. This shows in particular that the formal coisotropic deformation problem for M in N is obstructed.

2. Coisotropic submanifolds close to M

The submanifolds of N which are C^1-close to M are of the form $M_{f,g} := graph(f, g)$, where f and g are (C^1-small) elements of $C^\infty(\mathbb{T}^4)$. Let $i : M_{f,g} \to N$ be the inclusion. The 2-form $i^*\Omega$, at each point of $M_{f,g}$, can have rank 2 or rank 4. The first case occurs exactly at points where the determinant of $i^*\Omega$ is zero, i.e.

$$f_4 - g_3 = f_1 g_2 - f_2 g_1 \tag{1}$$

where the subscripts denote partial derivatives. We conclude that $M_{f,g}$ is coisotropic iff (f, g) belongs to $\mathcal{K}$, the set of C^1-small elements of $C^\infty(\mathbb{T}^4) \times C^\infty(\mathbb{T}^4)$ satisfying (1). The correspondence $M_{f,g} \leftrightarrow (f, g)$ gives an identification between $\mathcal{C}$ and $\mathcal{K}$.

Recall that the *characteristic foliation* of a coisotropic submanifold $\bar{M}$ of (N, Ω) is the foliation integrating $T\bar{M}^\Omega$, the kernel of the pullback of Ω to $\bar{M}$.

Proposition 2.1. *Arbitrarily C^1-close to M there exist coisotropic submanifolds of (N, Ω) with characteristic leaves not homeomorphic to those of M.*

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Proof. Let $(f,g)\in\mathcal{K}$, so that $M_{f,g}:=\mathrm{graph}(f,g)$ is a coisotropic submanifold of $\mathbb{T}^4\times\mathbb{R}^2$. Instead of working with $(M_{f,g},i^*\Omega)$ we use the diffeomorphism induced by the section $(f,g):\mathbb{T}^4\to M_{f,g}\subset\mathbb{T}^4\times\mathbb{R}^2$ and work with $(\mathbb{T}^4, dx_1\wedge dx_2+dx_3\wedge df+dx_4\wedge dg)$. Its characteristic distribution $E_{f,g}$ is spanned by $(-f_2,f_1,1,0)$ and $(-g_2,g_1,0,1)$. Applying [1](Prop. 1, Ch. V.2) we see that the restriction of the projection $pr:\mathbb{T}^4\to\mathbb{T}^2,(x_1,\cdots,x_4)\mapsto(x_3,x_4)$ to any leaf of $E_{f,g}$ is a covering map. In particular the leaves are homeomorphic to either $\mathbb{R}^2$, $S^1\times\mathbb{R}$ or $\mathbb{T}^2$. When $f=g=0$, i.e. when $M_{f,g}=M$, the characteristic leaves are all homeomorphic to $\mathbb{T}^2$.

Now, for any fixed $t>0$, $(f,g):=(t\sin(2\pi x_1),0)$ clearly satisfies equation (1). Let γ be any curve in $\mathbb{T}^4$ tangent to $E_{f,g}$ and $(\bar{x}_1,\bar{x}_2,\bar{x}_3,\bar{x}_4)=\gamma(0)$. Using the fact that $E_{f,g}$ is always orthogonal to $\frac{\partial}{\partial x_1}$ we have

$$\dot{\gamma}(s)=\alpha(s)\begin{pmatrix}0\\ 2\pi t\cos(2\pi\bar{x}_1)\\ 1\\ 0\end{pmatrix}+\beta(s)\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

for some functions α,β. Now we have

$$\begin{aligned}&\gamma\text{ is a closed curve}\\ \Leftrightarrow\quad&\exists s_0:\gamma(s_0)-\gamma(0)=\int_0^{s_0}\dot{\gamma}(s)ds\in\mathbb{Z}^4\\ \Leftrightarrow\quad&\exists s_0:\int_0^{s_0}\alpha(s)ds\in\mathbb{Z},\int_0^{s_0}\beta(s)ds\in\mathbb{Z},\Big(\int_0^{s_0}\alpha(s)ds\Big)\cdot 2\pi t\cos(2\pi\bar{x}_1)\in\mathbb{Z}.\end{aligned}$$

Suppose that the characteristic leaf L in which γ lies is homeomorphic to $\mathbb{T}^2$. Then, since the covering map $pr:L\to\mathbb{T}^2$ induces an injection at the level of fundamental groups, we can find a loop in $\mathbb{T}^2$ through $(\bar{x}_3,\bar{x}_4)$ whose class lies in the image of $\pi_1(L)$ and which "winds around in x_3-direction" a non-zero number of times. The lift of this curve is a loop in L with $\int_0^{s_0}\alpha(s)ds\neq 0$. So the above conditions imply that $2\pi t\cos(2\pi\bar{x}_1)\in\mathbb{Q}$.

Therefore leaves through points $\bar{x}$ of $\mathbb{T}^4$ with $2\pi t\cos(2\pi\bar{x}_1)\notin\mathbb{Q}$ must be homeomorphic to $S^1\times\mathbb{R}$ (they cannot be homeomorphic to $\mathbb{R}^2$ because the curve $s\mapsto(\bar{x}_1,\bar{x}_2,\bar{x}_3,\bar{x}_4)+s(0,0,0,1)$ in L is closed and not contractible). All other leaves are easily seen to be homeomorphic to $\mathbb{T}^2$. Making t arbitrarily small finishes the argument. □

Now we consider the "space of tangent vectors to $\mathcal{C}$ at M", which using the identification between $\mathcal{C}$ and $\mathcal{K}$ is

$$T_0\mathcal{K}:=\Big\{\frac{d}{dt}\Big|_0\beta^t:\beta:(-\epsilon,\epsilon)\to\mathcal{K}\text{ is smooth},\beta^0=0\Big\}\subset C^\infty(\mathbb{T}^4)\times C^\infty(\mathbb{T}^4).$$

Proposition 2.2. *$T_0\mathcal{K}$ is* not *a vector subspace of $C^\infty(\mathbb{T}^4)\times C^\infty(\mathbb{T}^4)$.*

Proof. Let $(f,g):(-\epsilon,\epsilon)\to C^\infty(\mathbb{T}^4)\times C^\infty(\mathbb{T}^4),t\mapsto(f^t,g^t)$ be any smooth curve in $\mathcal{K}$ with $(f^0,g^0)=(0,0)$. For all t we have

$$0=\int_0^1\int_0^1(f_4^t-g_3^t)dx_3dx_4=\int_0^1\int_0^1(f_1^tg_2^t-f_2^tg_1^t)dx_3dx_4.$$

Applying $\frac{d^2}{dt^2}|_0$ and using the notation $F:=\frac{d}{dt}|_0f^t,G:=\frac{d}{dt}|_0g^t$ we obtain $\int_0^1\int_0^1(F_1G_2-F_2G_1)dx_3dx_4=0$. Therefore all elements (F,G) of $T_0\mathcal{K}$ are subject to the above constraint, which is clearly *non-linear.*

Now consider the curves $(f^t, g^t) = (t\sin(2\pi x_1), 0)$ and $(\tilde{f}^t, \tilde{g}^t) = (0, t\sin(2\pi x_2))$. Both curves lie in $\mathcal{K}$, so their derivatives at time zero (F, G) and $(\tilde{F}, \tilde{G})$ lie in $T_0\mathcal{K}$. But the sum $(F + \tilde{F}, G + \tilde{G})$ does not, because it violates the above constraint:

$$\int_0^1 \int_0^1 ((F + \tilde{F})_1 (G + \tilde{G})_2 - (F + \tilde{F})_2 (G + \tilde{G})_1) dx_3 dx_4 =$$
$$4\pi^2 \int_0^1 \int_0^1 \cos(2\pi x_1) \cos(2\pi x_2) dx_3 dx_4 \not\equiv 0.$$

□

Let $\mathcal{S}(N)$ be the space of all compact submanifolds of N. $\mathcal{S}(N)$ is endowed with the structure of a Fréchet manifold, and each connected component of $\mathcal{S}(N)$ consists of manifolds diffeomorphic to each other ([2], 4.1.7). From Prop. 2.2 we deduce

Corollary 2.3. *The set $\mathcal{C}$ of coisotropic submanifolds of (N, Ω) which are C^1-close to M is* not *a Fréchet submanifold of $\mathcal{S}(N)$.*

Remark 2.4. We have

$$T_0\mathcal{K} \subset \Big\{ (F, G) \mid F_4 - G_3 = 0 \ , \ \int_0^1 \int_0^1 (F_1 G_2 - F_2 G_1) dx_3 dx_4 = 0 \Big\}.$$

The first restriction is obtained by linearizing the equation $f_4 - g_3 = f_1 g_2 - f_2 g_1$ and is equivalent to the fact that $F dx_3 \wedge G dx_4$, viewed as a foliated form along the characteristic foliation of $\mathbb{T}^4$, is closed. The second restriction was derived in the proof of Prop. 2.2 and is exactly the condition that the foliated cohomology class of $F dx_3 \wedge G dx_4$ be in the kernel of the Kuranishi map as defined in section 11 of [3]. This is the primary obstruction to extending the infinitesimal coisotropic deformation $F dx_3 \wedge G dx_4$ to a formal deformation. We refer to section 11 of [3] for a discussion of the formal deformation problem of the coisotropic submanifold M in terms of the L_∞-algebra structure on the space of foliated differential forms along the characteristic foliation of M.

If we restrict ourselves to coisotropic deformations of M whose characteristic foliations are again smooth fibrations with 2-tori as fibers (i.e. so-called integral coisotropic deformations) the deformation problem is unobstructed [4].

Remark 2.5. We saw in equation (1) that the condition for a submanifold C^1-close to M to be an element of $\mathcal{C}$ is a *non-linear* condition. Further we saw in Remark 2.4 that the formal deformation problem of the coisotropic submanifold M is obstructed.

This is in contrast to the case of codimension one or lagrangian submanifolds of any symplectic manifold. Indeed the former are all coisotropic. If L is a lagrangian submanifold, then lagrangian submanifolds C^1-close to L are given exactly by (C^1-small) 1-forms on L which are closed.

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