Metodos matematicos da fisica: Homework 5

1 (4 pt)**.** Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ a Lie ideal, i.e. a subspace such that $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{h}$. Show that the quotient space $\mathfrak{g}/\mathfrak{h}$ admits a unique Lie algebra structure such that the projection map $\pi \colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ is a Lie algebra morphism.

2 (10 pt)**.** Let $\mathfrak{sl}(2,\mathbb{R})$ be the Lie algebra of all real 2x2-Matrices with trace zero. Show that $\exp : \mathfrak{sl}(2,\mathbb{R}) \to SL(2,\mathbb{R})$ is not surjective.

Hint 1: Relate the (*complex*) eigenvalues of $A \in \mathfrak{sl}(2,\mathbb{R})$ with those of $\exp(A)$.

Hint 2: A different, more computational approach is this. Clearly the following (μ_1, μ_2, μ_3) are a basis of $\mathfrak{sl}(2,\mathbb{R})$:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

One can show that

$$\mathrm{tr}(e^{x_1\mu_1 + x_2\mu_2 + x_3\mu_3}) = 2\cosh\sqrt{|a_1^2 - a_2^2 + a_3^2|},$$

and that there exists $B \in SL(2,\mathbb{R})$ with $\mathrm{tr}(B) < 2$.

3 (8 pt)**.** Let V be a vector space, and consider $V^{\otimes 2} := V \otimes V$. Define S^2V to be the subspace of $V^{\otimes 2}$ spanned by elements of the form $\{v_1 \otimes v_2 + v_2 \otimes v_1 : v_1, v_2 \in V\}$.

a. Let $A \colon V \to V$ a linear map. Show that the subspace S^2V is invariant under

$$A^{\otimes 2} \colon V^{\otimes 2} \to V^{\otimes 2}, v_1 \otimes v_2 \mapsto Av_1 \otimes v_2 + v_1 \otimes Av_2.$$

b. If $v_i \in V$ is an eigenvector of A with eigenvalue λ_i, $i = 1, 2$, show that$Av_1 \otimes v_2 + v_1 \otimes Av_2$ is an eigenvector of $A^{\otimes 2}$ with eigenvalue $\lambda_1 + \lambda_2$.

c. Let (ρ_3, V_3) denote the 3-dimensional irreducible complex representation of $SU(2)$ as in class. There is an induced representation Γ of $SU(2)$ on S^2V_3 (obtained restricting the representation on $(V_3)^{\otimes 2}$). What is the decomposition of Γ into irreducible subrepresentations?

Hint: Pass from $SU(2)$ to $\mathfrak{sl}(2,\mathbb{C})$ and use b).

4 (8 pt)**.** Consider the simple complex Lie algebra $\mathfrak{sl}(3,\mathbb{C})$ and the (2-dimensional) Cartan subalgebra $\mathfrak{h}$ given by the diagonal matrices in $\mathfrak{sl}(3,\mathbb{C})$. One can show that the Killing form restricted to $\mathfrak{h}$ is given by

$$B(diag(a_1, a_2, a_3), diag(b_1, b_2, b_3)) = 6\sum_{i=1}^{3} a_i b_i,$$

where $diag(a_1, a_2, a_3)$ denotes the diagonal matrix with entries a_1, a_2, a_3. Consider the isomorphism

$$\Phi \colon \mathfrak{h} \cong \mathfrak{h}^*, H \mapsto B(H, \cdot),$$

and for any $\alpha \in \mathfrak{h}^*$ denote by $H_\alpha := \Phi^{-1}(\alpha) \in \mathfrak{h}$. For

$$L_{12} \in \mathfrak{h}^*, L_{12}(diag(a_1, a_2, a_3)) = a_1 - a_2$$

and

$$L_{23} \in \mathfrak{h}^*, L_{23}(diag(a_1, a_2, a_3)) = a_2 - a_3$$

compute

a. $B(H_{L_{12}}, H_{L_{12}})$,

b. $B(H_{L_{23}}, H_{L_{23}})$,

c. $B(H_{L_{12}}, H_{L_{23}})$.