Metodos matematicos da fisica: Homework 4

1 (3 pt). Show that there a bijection

$$\Phi\colon S^3 \to SU(2), (\alpha, \beta) \mapsto \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}.$$

Here we view S^3 as the set of unit-lenght vectors on $\mathbb{C}^2$.

Remark: This map is actually a homeomorphism. It follows that $SU(2)$ is a simply connected Lie group.

2 (7 pt). Let G, H be Lie groups and $\Phi\colon G \to H$ a Lie group homomorphism. Show that for any element v of the Lie algebra T_eG we have

$$\Phi(exp(v)) = exp(d_e\Phi(v))$$

where exp denotes the exponential map of G or H.

3 (7 pt). For any $n \in \mathbb{Z}$ in class we defined the complex representation

$$\rho_n\colon U(1) \to GL(\mathbb{C}), z \to z^n.$$

Show that any complex, irreducible representation of $U(1)$ is equivalent to ρ_n for some n.

Hint: Exercise 2 could be useful.

4 (3 pt). For any $n \in \mathbb{Z}_+$ consider the complex representation of $SU(2)$ discussed in class

$$\rho_n\colon SU(2) \to GL(V_n)$$

with $\rho_n(A)p = (A^{-1})^*p$, where V_n denotes the space of complex polynomials of degree $n-1$ in 2 variables.

Show that $\rho_n(-A) = (-1)^{n-1}\rho_n(A)$.