Metodos matematicos da fisica: Homework 2

1 (6 pt)**.** Consider $F\colon \mathbb{R}^M \to \mathbb{R}^k$ with components $f_1, \dots, f_k$.

i) Show for any $p \in \mathbb{R}^M$ that $d_pF\colon T_p\mathbb{R}^M \to T_{F(p)}\mathbb{R}^k$ is surjective iff $df_1(p), \dots, df_k(p)$ are lineraly independent elements of $T^*_p\mathbb{R}^M$.

Denote $N := F^{-1}(0) := \{q \in \mathbb{R}^M : f_1(q) = \dots = f_k(q) = 0\}$. One can show that if any of the two equivalent conditions above is satisfied for all $q \in N$, then N is a submanifold, and $T_qN = \{v \in T_q\mathbb{R}^M : df_1(q), \dots, df_k(q) = 0\}$.

ii) What is the dimension of N?

Hint: Use that $dim(N) = dim(T_qN)$ for all $q \in N$.

2 (6 pt)**.** Let ω be a symplectic form on $\mathbb{R}^{2n}$, and $f_1, \dots, f_k\colon \mathbb{R}^{2n} \to \mathbb{R}$ such that $df_1(q), \dots, df_k(q)$ are lineraly independent elements of $T^*_q\mathbb{R}^{2n}$ $\forall q \in N$. Prove that:

i) $(T_qN)^\omega = span\{X_{f_1}(q), \dots, X_{f_k}(q)\}$ $\forall q \in N$.

ii) N is a coisotropic submanifold iff $\{f_i, f_j\}(q) = 0$ $\forall q \in N$.

iii) N is a symplectic submanifold iff $det(\{f_i, f_j\}(q)) \neq 0$ $\forall q \in N$.

3 (6 pt)**.** Let ω be a symplectic form on $\mathbb{R}^{2n}$. Show that the Poisson bracket satisfies the Jacobi identity, i.e.,

$$\{f_1, \{f_2, f_3\}\} + \{f_2, \{f_3, f_1\}\} + \{f_3, \{f_1, f_2\}\} = 0.$$

Hint: Use $d\omega = 0$ and the formula for the de Rham differential in terms of Lie brackets of vector fields given in class.

4 (6 pt)**.** Let ω_1, ω_2 be two symplectic forms on $\mathbb{R}^{2n}$. Let $\phi\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be differentiable map such that $\phi^*\omega_2 = \omega_1$.

i) Show that $D_p\phi\colon T_p\mathbb{R}^{2n} \to T_{\phi(p)}\mathbb{R}^{2n}$ is a linear isomorphism, for all $p \in \mathbb{R}^{2n}$.

ii) Show that $graph(\phi) := \{(p, \phi(p)) : p \in \mathbb{R}^{2n}\}$ (which is a submanifold of $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$) is a Lagrangian submanifold of $(\mathbb{R}^{2n} \times \mathbb{R}^{2n}, \omega_1 - \omega_2)$.

iii) For each $p \in graph(\phi)$ construct $2n$ functions $f_1, \dots, f_{2n}$, defined in a little neighborhood U of p in $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$, which are constraints for $graph(\phi)$, i.e., $graph(\phi) \cap U = \{q \in U : f_1(q) = \dots = f_{2n}(q) = 0\}$.

5 (6 pt)**.** Let ω be a symplectic form on $\mathbb{R}^{2n}$, and $f_1, \dots, f_k$ functions s.t. $df_1(p), \dots, df_k(p) \in T^*_p\mathbb{R}^{2n}$ are linearly indepentent at every point $p \in \mathbb{R}^{2n}$.

i) If H is a function such that $\{H, f_i\} = 0$ for all i, then for any $p \in \mathbb{R}^{2n}$ the integral curve of X_H through p must lie inside the submanifold $\{q \in \mathbb{R}^{2n} : f_i(q) = f_i(p)\}$.

ii) If $\{f_i, f_j\} = 0$ for all i, j then necessarily $k \leq n$.