Metodos matematicos da fisica: Exame II

1 (10 pt)**.** a) Consider the hamiltonian system

$$(\mathbb{R}^4, \omega := dx_1 \wedge dx_2 + dx_3 \wedge dx_4, H = x_1 + x_3).$$

Let $f := x_2 - x_4$ and $g := x_3^2$. Compute explicitly $X_f, X_g, [X_f, X_g]$, and show that all three are symmetries of the hamiltonian system.

b) Consider a hamiltonian system $(\mathbb{R}^{2n}, \omega, H)$. Let $f, g \in C^\infty(\mathbb{R}^{2n})$ such that their hamiltonian vector fields X_f,X_g are symmetries of the hamiltonian system. Is the Lie bracket $[X_f, X_g]$ also a symmetry of the hamiltonian system?
Hint: Use $[X_f, X_g] = -X_{\{f,g\}}$.

2 (9 pt)**.** Consider the Lie group

$$G := \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a, b \in \mathbb{C}, |a| = 1 \right\}$$

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a) Is G compact?

b) Consider the natural representation of G on $\mathbb{C}^2$ (by matrix multiplication). Is it irreducible? Is it completely reducible?

3 (8 pt)**.** How many 4-dimensional complex representations of $SU(2)$ are there, up to isomorphism? For each isomorphism class, write down a representative in terms of the V_n's , where V_n ($n \geq 1$) denotes the n-dimensional irreducible representation of $SU(2)$ discussed in class.
Remark: By "4-dimensional complex representation" of course I mean a representation (ρ, V) where $dim_{\mathbb{C}}(V) = 4$.

4 (8 pt)**.** Let V be a real vector space and

$$\rho \colon (\mathbb{R}, +) \to GL(V)$$

be a representation of the group $(\mathbb{R}, +)$ (real numbers with addition), with the following property: for all integers $k \in \mathbb{Z}$, $\rho(k) = Id_V$. Show that ρ is a completely reducible representation.