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Metodos matematicos da fisica: Exame I

1 (10 pt). Let (V, Ω_V) and (W, Ω_W) be symplectic vector spaces. Let $\Phi\colon V \to W$ be a linear map such that $\Phi^*\Omega_W = \Omega_V$.

a) Is necessarily $dim(W) \leq dim(V)$?

b) Show that $graph(\Phi) := \{(v, \Phi(v)) : v \in V\}$ is an isotropic subspace of $(V \times W, \omega)$, where the linear symplectic form ω is defined as

$$\omega((v_1, w_1), (v_2, w_2)) = \Omega_V(v_1, v_2) - \Omega_W(w_1, w_2).$$

Recall: $\Phi^*\Omega_W$ is defined as $(\Phi^*\Omega_W)(v_1, v_2) = \Omega_W(\Phi(v_1), \Phi(v_2))$ for all $v_1, v_2 \in V$.

2 (9 pt). Let $\mathfrak{g}, \mathfrak{h}$ be real Lie algebras, let G be a Lie group integrating $\mathfrak{g}$, and $\rho\colon G \to GL(\mathfrak{h})$ be a representation of G such that

$$\rho(g)[w_1, w_2] = [\rho(g)(w_1), \rho(g)(w_2)]$$

for all $g \in G, w_i \in \mathfrak{h}$. Show that the Lie algebra representation $d_e\rho\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{h})$ induced by ρ satisfies

$$(d_e\rho(v))[w_1, w_2] = [(d_e\rho(v))w_1, w_2] + [w_1, (d_e\rho(v))w_2]$$

for all $v \in \mathfrak{g}, w_i \in \mathfrak{h}$. Above $[\cdot, \cdot]$ denotes the bracket on $\mathfrak{h}$.

3 (8 pt). Let $n > 1$. Show that

$$\mathfrak{h} := \{A \in \mathfrak{sl}(n, \mathbb{C}) : A \text{ is a diagonal matrix } \}$$

is a Cartan subalgebra of $\mathfrak{g} := \mathfrak{sl}(n, \mathbb{C})$.
Recall: $\mathfrak{h}$ is a Cartan subalgebra iff

- it is abelian (i.e., $[w_1, w_2] = 0$ for all $w_i \in \mathfrak{h}$),
- it is maximal abelian (i.e. if $\mathfrak{h}'$ is an abelian subalgebra of $\mathfrak{g}$ with $\mathfrak{h} \subset \mathfrak{h}'$, then necessarily $\mathfrak{h}' = \mathfrak{h}$)
- $ad_w\colon \mathfrak{g} \to \mathfrak{g}$ is diagonalizable for all $w \in \mathfrak{h}$, where ad denotes the adjoint representation (i.e., $ad_w = [w, \cdot]$).

4 (8 pt). Let G be a group, V a complex vector space and $\rho\colon G \to GL(V)$ an irreducible representation. For $i = 1, 2$ let $\langle \cdot, \cdot \rangle_i$ be a Hermitian product on V which is G-invariant, i.e., $\langle \rho(g)v, \rho(g)w \rangle_i = \langle v, w \rangle_i$ for all $g \in G, v, w \in V$. Show that there exists $\lambda \in \mathbb{C}$ such that

$$\langle v, w \rangle_1 = \lambda \langle v, w \rangle_2$$

for all $v, w \in V$.
Hint: Consider $\phi_i\colon V \to V^*, v \mapsto \langle v, \cdot \rangle_i$ for $i = 1, 2$ and find a way to apply Schur's lemma.
Recall: A Hermitian product $\langle \cdot, \cdot \rangle$ is a map $V \times V \to \mathbb{C}$ which satisfies

$$\langle \lambda v_1 + v_2, w \rangle = \bar{\lambda} \langle\ v_1, w \rangle + \langle v_2, w \rangle,$$

$$\langle w, \lambda v_1 + v_2 \rangle = \lambda \langle w, v_1 \rangle + \langle w, v_2 \rangle,$$

as well as $\langle v_1, v_2 \rangle = \overline{\langle v_2, v_1 \rangle}$ and $\langle v, v \rangle > 0$ whenever $v \neq 0$. Here $\lambda \in \mathbb{C}$ and $v_i, w \in V$.