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Differentiable Manifolds: Voluntary review problems for you to practice if you like

1. Let S^2 be the unit sphere in $\mathbb{R}^3$. Find all the critical points of the "height function"

$$f : S^2 \to \mathbb{R}, (x, y, z) \mapsto z.$$

2. Let M be a smooth manifold and let $X \in \mathfrak{X}(M)$ be a vector field. Let $s : \mathbb{R} \to M$ be an integral curve of X. Assume that there exists a $t_0 \in \mathbb{R}$ such that $\dot{s}(t_0) := \frac{d}{dt}|_{t=t_0} s(t) = 0$. Show that s is constant (i.e. $s(t) = s(t_0)\ \forall t \in \mathbb{R}$).

3. Consider the topological space C given by the boundary of $[0, 1]^n := [0, 1] \times \cdots \times [0, 1]$. (In other words, C is the "surface" of the n-dimensional cube). Show that C can be endowed with the structure of a differentiable manifold.

4. Let M be a manifold and N a *closed* submanifold of M (i.e. N is a closed subset of the topological M viewed as a topological space). Let $f \in C^\infty(N)$.
i) Show that f can be extended to a smooth function on M, i.e. that there exists $F \in C^\infty(M)$ with $F|_N = f$.
ii) Find a counter-example showing that, if one removes the closeness assumption on N, the function f might not be extended to a smooth function on M.
Hint for i): Use partition of unity.