Differentiable Manifolds: Final Exam 12/01/2009

Each problem is worth 10 points

1. Consider
$$f : \mathbb{R}^3 \to \mathbb{R}, (x, y, z) \mapsto x^2 + y^2 - z^2.$$

a) Which $c \in \mathbb{R}$ are critical values of f?
b) For which $c \in \mathbb{R}$ is $f^{-1}(c)$ a submanifold of $\mathbb{R}^3$?

2. Let M be a manifold of dimension $n > 0$. Is the statement

$$\text{for all vector fields } X \text{ on } M, \text{ for all } f \in C^\infty(M) : \quad [X, fX] = 0$$

true or false? Justify your answer.

3. Let E be a vector bundle over a manifold M.
Prove that for each $x \in M$ one can choose a positive definite, symmetric, bilinear form
$$g_x : E_x \times E_x \to \mathbb{R}$$
which depends *smoothly* on x.
Remark: Recall that a symmetric bilinear form $h : V \times V \to \mathbb{R}$ is positive definite iff $h(v, v) \geq 0$ for all $v \in V$ and $h(v, v) = 0$ only for $v = 0$.
Remark: "g depends smoothly on x" means the following: $g(v, w)$ is a smooth function on M for all smooth sections v, w of E.
Hint: partition of unity.

4. Let M, N be orientable manifolds.
Show that the product manifold $M \times N$ is also orientable.

5. Consider $\mathbb{R}^3$ with coordinates x, y, z and the 1-form

$$\alpha := x\,dy + dz.$$

a) Does there exist $f \in C^\infty(\mathbb{R}^3)$ with $df = \alpha$? Justify your answer.
b) Compute
$$\int_N j^*\alpha$$
where $N := \{(x, y, 0) : x^2 + y^2 = 1\}$ and $j : N \hookrightarrow \mathbb{R}^3$ is the inclusion.

6. Consider a 3-dimensional vector space V with basis e_1, e_2, e_3. Let $v \in V$. Define the anti-symmetric bilinear map $[\ ,\ ] : V \times V \to V$ by

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = v.$$

For which $v \in V$ does $[\ ,\ ]$ define a Lie algebra structure on V?