Marco Zambon due 15/12/2008

Differentiable Manifolds: Homework 6

1 (6 pts)**.** Let M be a *compact* manifold and E a rank k vector bundle over M. Prove that there exists *finitely many* sections $s_1, \cdots, s_n$ of E such that $span\{s_1(x), \cdots, s_n(x)\} = E_x$ for all $x \in M$.
Remark: Here "section" of course means global section, i.e. defined on the whole of M.
Hint: To start, choose an open cover $\{U_\alpha\}$ of M such that each $E|_{U_\alpha}$ is a trivial vector bundle, and choose a partition of unity subordinate to this cover. Then construct the s_i's out of sections of $E|_{U_\alpha}$ and the partition of unity.

2 (4 pts)**.** Consider the vector fields $X, Y, Z \in \mathfrak{X}(\mathbb{R}^3)$ given by

$$X = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \qquad Y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \qquad Z = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

Prove that

$$[X, Y] = Z, \qquad [Y, Z] = X, \qquad [Z, X] = Y.$$

3 (4 pts)**.** Let G be the Lie group $SL(2, \mathbb{R})$ and $g = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \in G$. Describe explicitly T_gG (as a subspace of $Mat(2, \mathbb{R})$).
Hint: Use the left multiplication $L_g : G \to G$.

4 (6 pts)**.** Let $(\mathfrak{g}, [\bullet, \bullet])$ be a Lie algebra with $dim(\mathfrak{g}) = 2$. Show that either $[\bullet, \bullet] \equiv 0$ or there is a basis X, Y of $\mathfrak{g}$ such that the bracket is given by

$$[X, Y] = X, \quad [X, X] = 0, \quad [Y, Y] = 0.$$

Hint: Start picking any basis X, Y of $\mathfrak{g}$. We have $[X, Y] = aX + bY$ for some real numbers a, b. Transform suitably X and Y to a new basis with the desired properties.