Marco Zambon due 15/12/2008

Differentiable Manifolds: Homework 5

1 (4pts)**.** The determinant is a map from $Mat(n, \mathbb{R})$ into $\mathbb{R}$. Composing with the isomorphism

$$\mathbb{R}^n \times \cdots \times \mathbb{R}^n \to Mat(n, \mathbb{R}), (v_1, \dots, v_n) \mapsto (\text{the matrix with } v_i \text{ as } i\text{-th column})$$

we obtain a map $DET : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$.
a) Show that DET is an element of $\wedge^n(\mathbb{R}^n)$.
b) What is the geometric interpretation of DET?

2 (4pts)**.** Let M be a manifold and $\omega \in \Omega^k(M)$ and $X_1, \dots, X_{k+1}$ vector fields on M. Denote by "RHS" the right hand side of the formula for $d\omega(X_1, \dots, X_{k+1})$ given in class (i.e. the expression in Def. 5.2.3 of the book of Thomas/Barden). Show that for any $f \in C^\infty(M)$

$$RHS(fX_1, \dots, X_{k+1}) = f \cdot RHS(X_1, \dots, X_{k+1}).$$

Remark: This is the main step to show that RHS, evaluated at some point p, depends only on the vectors $X_1(p), \dots, X_{k+1}(p)$.

3 (4pts)**.** Let M be a manifold, $p \in M$, X a vector field on M with local flow ϕ_t. Let $\omega \in \Omega^k(M)$. In class we claimed that

$$(\mathcal{L}_X\omega)_p = \frac{d}{dt}|_{t=0}[\phi_t^*(\omega_{\phi_t(p)})].$$

Prove this formula when $k = 0$, i.e. when ω is a function on M.

4 (8 pts)**.** Consider the 1-form on $\mathbb{R}^2 - \{0\}$ given by

$$\alpha := \frac{x \cdot dy - y \cdot dx}{x^2 + y^2}.$$

Let $i : S^1 \to \mathbb{R}^2 - \{0\}$ be the inclusion where S^1 is the unit circle.
a) Compute $\int_{S^1} i^*\alpha$
b) Does there exist $f \in C^\infty(\mathbb{R}^2 - \{0\})$ such that $df = \alpha$?
c) Show that $d\alpha = 0$
d) Let γ be a smooth curve in $\mathbb{R}^2 - \{0\}$ very close to the origin which goes once around the origin clockwise, and assume that the image of γ is a submanifold N of $\mathbb{R}^2 - \{0\}$. Let us denote by $j : N \to \mathbb{R}^2 - \{0\}$ the inclusion. Compute $\int_N j^*\alpha$.
Hint for a): Notice that $i^*\alpha = i^*(x \cdot dy - y \cdot dx)$ and use Stokes theorem.
Hint for b): Use Stokes theorem.
Hint for d): Use Stokes theorem and a).