Differentiable Manifolds: Homework 4

1 (4 pts)**.** Let X, Y be vector fields on M and $\psi : M \to N$ a diffeomorphism. Show by an explicit computation that

$$[\psi_* X, \psi_* Y] = \psi_*[X, Y].$$

2 (4 pts)**.** On $\mathbb{R}^n$ consider the vector field $X = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$.
a) What is the flow of X? Is it defined for all times t?
b) Restrict the vector field X to $M = \mathbb{R}^n - \{0\}$. Is the flow defined for all times t?

3 (3 pts)**.** Consider the submanifold $S^1 = \{(x, y) : x^2 + y^2 = 1\}$ of $\mathbb{R}^2$.
a) For each $(x, y) \in S^1$ compute the tangent space $T_{(x,y)}S^1$.
b) Show that the vector bundle TS^1 is isomorphic to a trivial vector bundle.

4 (4 pts)**.** Let V be a vector space of dimension $2n$, and fix a basis $e_1, f_1, e_2, f_2, \dots, e_n, f_n$ of V^*. Let $\omega := e_1 \wedge f_1 + \dots + e_n \wedge f_n \in \wedge^2 V^*$. Show that

$$\omega \wedge \omega \wedge \dots \wedge \omega = n!(e_1 \wedge f_1 \wedge e_2 \wedge f_2 \wedge \dots \wedge e_n \wedge f_n),$$

where on the left hand side we are taking the wedge product of n copies of ω.
Hint: Prove the statement by induction over n, and use that $\alpha \wedge \alpha = 0$ for all $\alpha \in V^*$.

5 (5 pts)**.** Let S^2 be the unit sphere in $\mathbb{R}^3$. The group $O(3) := \{A \in Mat(3, \mathbb{R}) : AA^T = I\}$ acts on $\mathbb{R}^3$ by usual matrix multiplication.
a) The action of $O(3)$ on $\mathbb{R}^3$ preserves S^2.
b) The induced action of $O(3)$ on S^2 is transitive, i.e. for all $p, q \in S^2$ there is a $A \in O(3)$ such that $Ap = q$.
c) Let $\omega \in \Omega^2(S^2)$. If ω is invariant under the $O(3)$ action (i.e. $A^*(\omega) = \omega$ for all $A \in O(3)$) then $\omega = 0$.
Remark for c): A^* denotes the pullback of differential forms by the diffeomorphism $S^2 \to S^2, q \mapsto Aq$.
Hint for c): Show that $\omega_{(0,0,1)} = 0$ first, then use b).