Marco Zambon due 27/10/2008

Differentiable Manifolds: Homework 3

1 (2 pts)**.** a) Let $f : M \to N$ a differentiable map between manifolds and $y \in N$, and consider the (0-dimensional) submanifold $L := \{y\}$ of N. Express in terms of regular points/values what it means for f to be transversal to L.
b) Is the map

$$\mathbb{R}^2 - \{0\} \to \mathbb{R}, (x, y) \mapsto y$$

a submersion?

2 (3 pts)**.** For what $c \in \mathbb{R}$ is $g^{-1}(c)$ a submanifold of $\mathbb{R}^3$, where

$$g : \mathbb{R}^3 \to \mathbb{R}, (x, y, z) \mapsto x^2 + y^2 - z?$$

Remark: $g^{-1}(0)$ is the submanifold of $\mathbb{R}^3$ we considered in exercise 5 of HW2.

3 (5 pts)**.** Show that

$$SL(n, \mathbb{R}) := \{A \in Mat(n, \mathbb{R}) : det(A) = 1\}$$

is a submanifold of $Mat(n, \mathbb{R})$. What is its dimension?
Hint: For any $A \in SL(n, \mathbb{R})$ consider the curve $t \mapsto (1 + t)A$ into $Mat(n, \mathbb{R})$.

4 (2 pts)**.** Let M be a manifold. Prove that the natural projection $\pi : TM \to M$ is differentiable and that it is a submersion.

5 (2 pts)**.** Let $p \in M^n$, (U, ϕ), (V, ψ) charts for M with $p \in U, p \in V$, and $X \in T_pM$. If the coordinates expression for X under the chart (U, ϕ) is $\sum_{i=1}^n v_i \frac{\partial}{\partial x_i}$ (for some $v_i \in \mathbb{R}$), what is the coordinate expression for X under the chart (V, ψ) (expressed in terms of the v_i's, ϕ and ψ)?
Hint: What you need to compute is the image under $D_{\phi(p)}(\psi \circ \phi^{-1})$ of $\sum_{i=1}^n v_i \frac{\partial}{\partial x_i}$.

6 (6 pts)**.** Let M be a manifold, $f \in C^\infty(M)$ and let X, Y, Z be vector fields on M.
a) Prove that $[X, fY] = f \cdot [X, Y] + X(f) \cdot Y$
b) Show that if Y and Z agree in a neighborhood of a point $p \in M$ then $[X, Y]_p = [X, Z]_p$.
c) If Y and Z agree only at p (i.e. $Y_p = Z_p$), does it follow that $[X, Y]_p = [X, Z]_p$?
d) Show that if $M = \mathbb{R}^n$ and $X = \sum_{i=1}^n a_i(x) \frac{\partial}{\partial x_i}$ and $Y = \sum_{i=1}^n b_i(x) \frac{\partial}{\partial x_i}$ then

$$[X, Y] = \sum_{j=1}^n \left(\sum_{i=1}^n a_i(x) \cdot \frac{\partial b_j}{\partial x_i}(x) - b_i(x) \cdot \frac{\partial a_j}{\partial x_i}(x) \right) \frac{\partial}{\partial x_j}.$$

Hint: A way to prove a) is to use the definition of $[X, Y]$ as $XY - YX$. For b) and d) you need to use a).