Differentiable Manifolds: Homework 2

1 (2 pt)**.** Denote by $\mathbb{R}_+$ the positive real numbers. Is

$$p : \mathbb{R}_+ \to S^1, t \mapsto e^{2\pi i t}$$

a covering map? Why?

2 (4 pt)**.** Consider the the universal cover of the circle S^1, i.e. $p : \mathbb{R} \to S^1, t \mapsto e^{2\pi i t}$. Compute explicitly the group G of deck transformations of this covering space, and construct an explicit group isomorphism from G to $\mathbb{Z}$.
Remark From the lecture we just know that this group isomorphism exists, because we know that $G \cong \pi_1(S^1) \cong \mathbb{Z}$; now I am asking you to construct one.

3 (4 pt)**.** a) Show that $(\mathbb{R}, Id_{\mathbb{R}})$ is a chart for the topological space $\mathbb{R}$, where $Id_{\mathbb{R}}$ is the identity $\mathbb{R} \to \mathbb{R}, x \mapsto x$.
b) Show that $(\mathbb{R}, \psi)$ is a chart for the topological space $\mathbb{R}$, where $\psi : \mathbb{R} \to \mathbb{R}, x \mapsto x^3$.

From a) it follows that $\mathbb{R}$, together with the maximal smooth atlas containing $(\mathbb{R}, Id_{\mathbb{R}})$, is a differentiable manifold. Similarly from b) it follows that $\mathbb{R}$, together with the maximal smooth atlas containing $(\mathbb{R}, \psi)$, is a differentiable manifold.
c) Are these two differentiable manifolds the same or are they different? Why?
Hint You will need to use that the map ψ^{-1} is continuous but not differentiable.

4 (4 pt)**.** Let M and N be differentiable manifolds, $p_0 \in M$, $q_0 \in N$. Let

$$i : M \to M \times N, p \mapsto (p, q_0),$$

$$j : N \to M \times N, q \mapsto (p_0, q).$$

Show that $i_*(p_0)$ and $j_*(q_0)$ are injective, and that

$$T_{(p_0,q_0)}(M \times N) = i_*(p_0)(T_{p_0}M) \oplus j_*(q_0)(T_{q_0}N).$$

5 (3 pt)**.** Consider $M := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z = 0\}$. Give explicit expressions for bases of T_pM and T_qM, where $p := (0, 0, 0) \in M$ and $q := (1, 3, 10) \in M$.
Hint: M lies in $\mathbb{R}^3$, so $T_pM \subset T_p\mathbb{R}^3 \cong \mathbb{R}^3$. Therefore, if we denote by e_1, e_2, e_3 the standard basis of $\mathbb{R}^3$, any element of T_pM will be of the form $ae_1 + be_2 + ce_3$ for suitable $a, b, c \in \mathbb{R}$.

6 (3 pt)**.** Prove the following (which was stated in class without proof):
Let $f : M \to N$ be a differentiable map between manifolds, and $p \in M$ so that $f_*(p) : T_pM \to T_{f(p)}N$ is an isomorphism. Then there exists a neighborhood W of p in M such that $f|_W : W \to f(W)$ is a diffeomorphism.
Hint: Use the "inverse function theorem for $\mathbb{R}^n$" stated in class.