Differentiable Manifolds: Homework 1

Note: Every time an exercise consists of a question (like Ex. 6), of course you should justify your answer with a good argument.

1 (3 pt)**.** Let (X, σ) be a Hausdorff topological space and $x \in X$. Prove that $\{x\}$ is a closed subset of X.

2 (5 pts)**.** Let $k, n \in \mathbb{N}$, consider the topological group

$$G = O(n+k) = \{A \in Mat(k+n, \mathbb{R}) : A^T = A^{-1}\}$$

and its subgroup

$$H = \left\{ \left(\begin{array}{c|c} B & 0 \\ \hline 0 & C \end{array} \right) : B \in O(k), C \in O(n) \right\}.$$

Show that there is a bijection between G/H and $Gr(k, n+k)$, the set of all k-dimensional subspaces of $\mathbb{R}^{k+n}$.
Hint: Use the fact that there is a natural action of G on $Gr(k, n+k)$ and compute the isotropy group at the element $\mathbb{R}^k \times \{0\}$ of $Gr(k, n+k)$.

3 (2pt)**.** Prove that a path-connected topological space is automatically connected.

4 (4 pt)**.** Consider

$$SU(2) := \{A \in Mat(2, \mathbb{C}) : AA^* = I, det(A) = 1\}$$

(with the topology induced from $Mat(2, \mathbb{C}) = \mathbb{C}^4$). Here A^* denotes the transpose of the complex conjugate to A, i.e. $A^* = \bar{A}^T$.)
Prove that $SU(2)$ is homeomorphic to the sphere S^3 (with the topology induced from $\mathbb{R}^4$).

Hint: Write down explicitly the entries of a matrix in $SU(2)$...

5 (2 pt)**.** Let X be a topological space, $f : [0,1] \to X$ a (continuous) path. Show that any reparametrization of f is homotopic with fixed endpoints to f.
(Recall that a reparametrization of f is $f \circ \phi$ where $\phi : [0,1] \to [0,1]$ is continuous with $\phi(0) = 0, \phi(1) = 1$).

6 (4 pt)**.** Answer the following:
(a) Is S^1 homotopic equivalent to $S^1 \times S^1$?
(b) Is S^1 homotopic equivalent to $S^1 \times \mathbb{R}$?
(c) Let $D^2 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ and denote by A the boundary of D^2. Is there a deformation retraction $D^2 \to A$?