DIFFERENTIAL GEOMETRY COURSE

Plan: Fibrados
" vectoriales
" principales } + conexiones curvatura

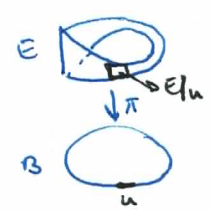
1) FIBER BUNDLES

1.1. Fiber bundles

DEF: Let S be a manifold. A fiber bundle with typical fiber S is a smooth map $\pi: E \to B$, such that; $\forall x \in B$ $\exists$ neighborhood U and a diffeomorphism $\varphi: E|_U \to U \times S$ making this diagram commute.

$E|_U = \pi^{-1}(U)$

$$\begin{tikzcd} E|_U \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & U\times S \arrow[dl, "pr_1"] \\ & U & \end{tikzcd}$$



Remark: π is a surjective submersion $\forall x \in B$ $\pi^{-1}(x) \cong S$

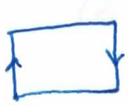
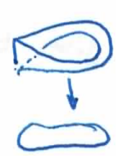
Remark: E is called total space, B base, φ local trivialization, $\pi^{-1}(x)$ ($x \in B$) fiber. A section is a smooth $f: B \to E$ st $\pi \circ f = \mathrm{Id}_B$

Ex. Given S, B take $E = B \times S$ (the trivial bundle). All fiber bundle are locally trivial bundle.
$\downarrow \pi = pr_1$
B

Ex (mapping torus) Given S and $\phi \in \mathrm{Diff}(S)$ we obtain a fiber bundle with typical fiber S.

$$([0,1] \times S)/_{(0,p) \sim (1, \phi(p))}$$
$$\downarrow$$
$$S^1 = ([0,1]/0 \sim 1)$$

For ex., with $S = (-\varepsilon, \varepsilon)$ and $\phi = -\mathrm{Id}$, set

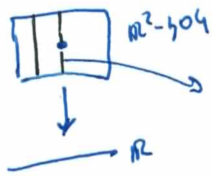
 $\simeq$ Moebius strip 

Ex One can show: if $p: N \to M$ a proper surjective submersion and M is connected then it's a fiber bundle.

(Recall: p proper $=$ $\forall$ compact $K \subset M$ $p^{-1}(K)$ is compact, satisfied if N compact)

(Submersion: $\forall x \in N$, $d_x p: T_x N \to T_{p(x)} M$ surjective)

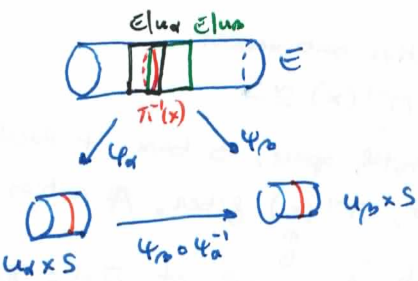
Non-ex: $\pi: \mathbb{R}^2 - \{0\} \to \mathbb{R}$ is not a fiber bundle
$(x, y) \to x$



Disconnected (no diffeomorphic) $\to$ no fiber bundle
(All the fiber must "look the same")

$\Rightarrow$ Fix S. Let $\pi: E \to B$ be a fiber bundle with fiber S.
Take a cover $[U_\alpha]_{\alpha \in A}$ of B. and local trivializations $\psi_\alpha: E|_{U_\alpha} \to U_\alpha \times S$.
Then $\forall \alpha, \beta \in A$ get $(U_\alpha \cap U_\beta) \times S \xrightarrow{\psi_\alpha^{-1}} E|_{U_\alpha \cap U_\beta} \xrightarrow{\psi_\beta} (U_\alpha \times U_\beta) \times S$

It's of the form $(\psi_\beta \circ \psi_\alpha^{-1})(x, s) = (x, (\psi_{\alpha\beta}(x))(s))$
for some $\psi_{\alpha\beta}: U_\alpha \cap U_\beta \to \mathrm{Diff}(S)$ (called transition maps).



$$\left.\begin{aligned} &\text{Clearly } \psi_{\alpha\alpha}(x) = Id_S \quad \forall x \in U_\alpha. \\ &\psi_{\alpha\beta}(x) = (\psi_{\beta\alpha}(x))^{-1} \quad \forall x \in U_\alpha \cap U_\beta \\ &\psi_{\alpha\beta}(x) \circ \psi_{\beta\gamma}(x) = \psi_{\alpha\gamma}(x) \quad \forall x \in U_\alpha \cap U_\beta \cap U_\gamma. \end{aligned}\right\} (*)$$

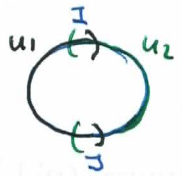
Remark: viceversa, given a manifold B with an open cover $\{U_\alpha\}_{\alpha \in A}$ and $\forall \alpha, \beta$ "smooth" maps $\psi_{\alpha\beta}: U_\alpha \cap U_\beta \to \mathrm{Diff}(S)$ satisfying $(*)$ we obtain an equivalence relation on:

$$\coprod_{\alpha \in A} (U_\alpha \times S) \quad \text{given by} \quad \underset{U_\alpha \times S}{(x, p)} \sim \underset{U_\beta \times S}{(y, q)} \iff \begin{cases} x = y \\ q = (\psi_{\alpha\beta}(x))(p) \end{cases}$$

disjoint union

Further $\coprod_{\alpha \in A} (U_\alpha \times S)/\sim$ is a fiber bundle (with fiber S).

Ex: Let $S = (-\varepsilon, \varepsilon)$, $B = S^1$

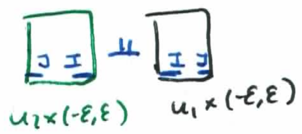


(U_1, U_2 two intervals that cover S^1)

Let $\varphi_{12} : \underset{\substack{\shortparallel \\ I \cup J}}{U_1 \cap U_2} \longrightarrow \mathrm{Diff}(-\varepsilon, \varepsilon)$

$x \in I \longrightarrow Id$ (that is, $\varphi_{12}(x \in I) = Id$)

$x \in J \longrightarrow -Id$ ('' $\varphi_{12}(x \in J) = -Id$)



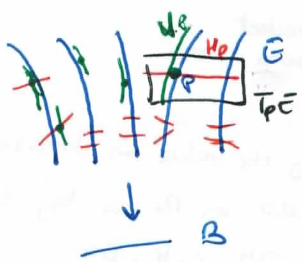
$/ \sim$ = Moebius band

1.2. Connections

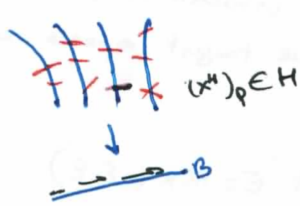
Let $\pi : E \to B$ be a fiber bundle.

DEF: A connection is a smooth choice $\forall p \in E$ of subspace H_p st. $H_p \oplus V_p = T_p E$,

Here $V_p = \ker(d_p \pi : T_p E \to T_{\pi(p)} \cancel{M}B)$ (the vertical bundle).



If $x \in \mathfrak{X}(B)$ (vector fields on B) then we obtain a "horizontal" vector field X^H on E, by $(X^H)_p =$ (unique vector on H_p which projects to X_p)



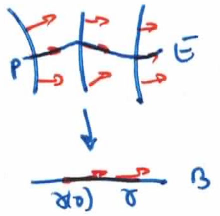
If $\gamma : (0,1) \to B$ is an injective curve, extend its velocity $\dot{\gamma}$ to a vector field on B.

(injective curve = ~~its velocity is never zero~~)

$\Rightarrow$ No injective (we can't extend it to a vector field)

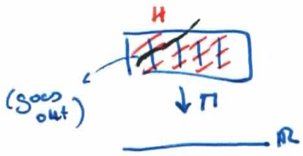
Given $p \in E_{\gamma(0)}$, let τ_p be the integral curve of X^H with $\tau_p(0) = p$.

Then τ_p is a horizontal lift of γ, ~~if~~ i.e $\pi \circ \tau_p = \gamma$ and $\frac{d}{dt} \tau_p(t) \in H_{\tau(t)}$ $\forall t$.



Remark: Γ_p might not be defined on the whole of $(0,1)$. If $\forall$ curves $(0,1) \to B$ all lifts (using H) are defined on $(0,1)$, the connection is called complete or Ehresmann connection.

Non-ex: This connection $\mathbb{R} \times (-\varepsilon, \varepsilon) \to \mathbb{R}$ is not complete.

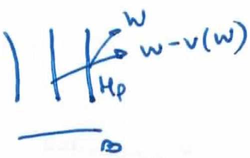


1.3. Curvature

DEF: Let H be a conn. on $\pi : E \to B$. Denote $v : TE \to V$ the projection onto the ~~p~~ vectorial bundle with kernel H.

The curvature of H: $\Omega \in \Omega^2(E, V)$

$$\underbrace{\Omega(w, z)}_{\text{tangent vectors}} = v\,\underbrace{[\underbrace{w - v(w)}_{\in H},\ z - v(z)]}_{\text{Lie bracket of vector fields}} \in V$$



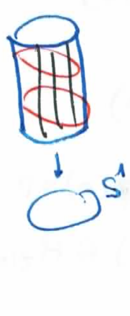
It is determined by the value on horizontal lifts of vector fields on B, ie by $(x, y \in \mathfrak{X}(B))$

$$\Omega(x^H, y^H) = [x, y]^H - [x^H, y^H]$$

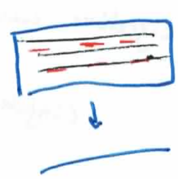
The connection is flat $\iff \Omega = 0$.

Remark: H flat $\iff$ the distribution H on E is involutive. (ie $\forall x, y \in \mathfrak{X}(E)$ lying in H then $[x, y]$ lies in H)
$\iff \exists$ a foliation on E whose tangent spaces are given by H.
(Frobenius theorem)

Ex: Let $E = S^1 \times \cancel{\mathbb{R}} (-\varepsilon, \varepsilon) \to S^1$

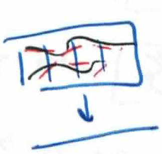


Ex: let $E = \mathbb{R} \times (-\varepsilon, \varepsilon)$



(The horizontal copies of $\mathbb{R}$ are tangent of $\mathbb{R}$)

or



Actually, in this case any connection is flat because the connection is 1-dim and the curvature is a 2-form.

DEF: Let $\pi: E \to B$ be a fibre bundle $f: M \to B$ a map.

The pullback bundle is $\underset{\downarrow \atop M}{f^*E} = \{(e,y) : E\times M ;\ \pi(e) = f(y)\}$

Remark: Notice that $(f^*E)_y = E_{f(y)}$ $\forall y \in M$.

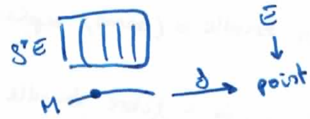
We have a canonical map $F: f^*E \to E$, $(e,y) \mapsto e$

The diagram:

$$\begin{tikzcd} f^*E \arrow[r, "F"] \arrow[d] & E \arrow[d] \\ M \arrow[r, "f"] & B \end{tikzcd}$$

commutes.

Ex: If $B = pt$. Then $f^*E = M \times E$



DEF: Let H be a conn. on E. The pullback connection K on f^*E is defined at every point $q \in f^*E$ by $K_p = \left(\text{premage of } H_{F(p)} \text{ under } d_qF : T_q(f^*E) \to T_{F(q)}E\right)$

Ex: If $B = \{pt.\}$ $\exists$ a unique conn. on E: $H_p = \{0\} \in T_pE$. Given $f: M \to \{pt\}$ we have $K_q = T_{\pi(q)}M \oplus \{0\}$

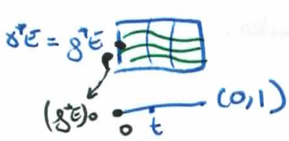
(Geodesics: $\frac{\nabla}{dt}\dot\gamma = 0$)

Prop: Let E be a fiber bundle with conn. H. Let $\gamma: (0,1) \to$ be any curve and $p \in E_{\gamma(0)}$. Then $\exists$ a unique horizontal lift τ_p of γ with $\tau_p(0) = p$.

Consider the pullback bundle

$$\begin{tikzcd} \gamma^*E \arrow[r, "F"] \arrow[d] & E \arrow[d] \\ (0,1) \arrow[r, "\gamma"] & B \end{tikzcd}$$

The pullback connection is 1-dimensional (in general $\dim(H_p) = \dim B$), so it's flat.



We obtain a horizontal curve σ in $\gamma^*(E)$ with $\sigma(0) = p$. Define $\tau_p = F\circ\sigma$.

DEF: The map $E_{\gamma(0)} \to E_{\gamma(1)}$, $p \mapsto \tau_p(1)$ is called parallel transport along γ using H.

2. VECTOR BUNDLES

1.1. Vector bundles

DEF: Fix $n \in \mathbb{N}$. A (real) vector bundle is a smooth $\pi: E \to B$ st.:

- $\forall x \in B$ the fibre $E_x := \pi^{-1}(x)$ is an n-dim (real) vector space.
- $\forall x \in B$ $\exists$ neighbourhood U and a diffeom.

$$\begin{tikzcd} E|_U \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^n \arrow[dl, "pr_1"] \\ & U & \end{tikzcd}$$

- $\forall x \in B$ $\varphi|_{E_x}: E_x \to \mathbb{R}^n$ is a vector space isomorphism.

A complex vector bundle = (same, replacing $\mathbb{R}$ by $\mathbb{C}$)

Rem: Equivalently: a real vector bundle is a fibre bundle with typical fibre $\mathbb{R}^n$ and structure group $GL(n,\mathbb{R})$ (ie: the transition maps $\varphi_{\alpha\beta}: U_\alpha \cap U_\beta \to Diff(\mathbb{R}^n)$ take values in $GL(n,\mathbb{R})$).

Rem: $\Gamma(E) := \{\text{sections of } E\}$ is a vector space.
$\exists$ canonical embedding of B in E: $B \to E$, $x \to$ (zero vector in E_x)

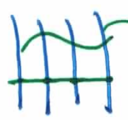


image ($s: B \to E$)

(sections $\equiv$ fields)

Ex: Given B, take $B \times \mathbb{R}^n \xrightarrow{pr_1 = \pi} B$

Remark: A rank n VB is isomorphic to the trivial VB $\Leftrightarrow$ $\exists$ section s_1, s_n which are linearly independent at every $x \in B$.

Ex: Given B, take $TB \to B$ (the tangent bundle).

DEF: Let $\pi_i: E_i \to B_i$ be VB ($i=1,2$). A morphism is a smooth $F: E_1 \to E_2$ that sends fibres to fibres by linear maps, i.e.:

$\exists$ a map $f: B_1 \to B_2$ st.
$$\begin{tikzcd} E_1 \arrow[r, "F"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\ B_1 \arrow[r, "f"] & B_2 \end{tikzcd}$$
commutes.

$\forall x \in B$, $F|_{(E_1)_x}: (E_1)_x \to (E_2)_{f(x)}$ linear.

DEF: Let $\pi_1: E_1 \to B$, $\pi_2: E_2 \to B$ VB. Then Whitney sum is the VB $E_1 \oplus E_2 \to B$ with fibre at $x \in B$ given by $(E_1)_x \oplus (E_2)_x$.

If $\varphi_1: E_1|_U \xrightarrow{\sim} U \times \mathbb{R}^{n_1}$ and $\varphi_2: E_2|_U \xrightarrow{\sim} U \times \mathbb{R}^{n_2}$ are local trivializations, then $(E_1 \oplus E_2)_U \xrightarrow{\sim} U \times \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}$

$$(E_1 \oplus E_2)_U \xrightarrow{\sim} U \times \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}$$

$$(e_1, e_2) \longrightarrow (x, \varphi_1(e_1), \varphi_2(e_2)) \in \{x\} \times (\mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2})$$

$\cap$ $E_1|_x$, $\cap$ $E_2|_x$ $\qquad$ $\cap$ $[x] \times \mathbb{R}^{n_1}$, $\cap$ $[x] \times \mathbb{R}^{n_2}$

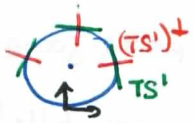
Ex: Let $S^n \subset \mathbb{R}^{n+1}$ be the unit sphere.

Then $TS^n \oplus \underbrace{(TS^n)^\perp}_{\text{normal bundle}} \simeq S^n \times \mathbb{R}^{n+1}$ $\Big\}$ Trivial VB

$\downarrow$

S^n

$n=1$:



For $n > 1 \Rightarrow \underbrace{TS^n}_{\text{(not trivial) in general)}} \oplus \overset{\text{(trivial)}}{(TS^n)^\perp} = \underbrace{\text{Something}}_{\text{trivial}} \; (S^n \times \mathbb{R}^{n+1})$

For $n=1$: TS^n is already trivial

Rem: If $\pi: E \to B$ is a VB then we can construct the dual bundle $E^* \to B$, the tensor bundle $E \otimes E \to B$ etc...

$\Lambda^n E^*$

Ex: $(TB)^*$ is the cotangent bundle T^*B. (The sections are 1-forms)

Rem: Let $\pi: E \to B$ be a VB.

$f: \pi \to B$ smooth. Consider the pullback VB

$$f^*E := \{(e, y) \in E \times M : \pi(e) = f(y)\}$$

$\downarrow$

M

There is an induced morphism F:

$$\begin{tikzcd} f^*(E) \arrow[r, "F"] \arrow[d] & E \arrow[d, "\pi"] \\ M \arrow[r, "f"] & B \end{tikzcd}$$

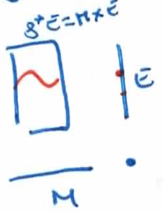
$y \qquad f(y)$

There is an induced map of sections

$$\Gamma(E) \longrightarrow \Gamma(f^*E)$$

$$s \longrightarrow s \circ f$$

Ex:



($\Gamma(f^*E) \to \Gamma(E)$ this direction doesn't make sense)

(Notice: $\forall y \in M$: $(s \circ f)(y) \in E_{f(y)} = (f^*E)_y$).

Ex: Let B_i be manifolds and $\pi_i: TB_i \to B_i$ be the tangent bundles.

Then $T(B_1 \times B_2) \cong (pr_1)^* TB_1 \oplus pr_2^*(TB_2)$ $\qquad$ where $pr_1: B_1 \times B_2 \to B_1$

$pr_2: B_1 \times B_2 \to B_2$

$\forall (x, y) \in B_1 \times B_2$

$$T_{(x,y)} B_1 \times B_2 = T_x B_1 \oplus T_y B_2$$

2.2. Bundle metrics

DEF: Let $\pi: E \to B$ be a VB. A family of section $e_1, \dots, e_n$ is a frame $\Leftrightarrow \forall x \in B$. $\{(e_1)_x, \dots, (e_n)_x\}$ is a basis of E_x.

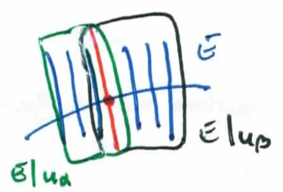
DEF: A bundle metric is a choice of scalar product g_x on E_x $\forall x \in B$, varying smoothly with x.

Ex: On $E = TB$ it's the same as a Riemanian metric on B.

$V^* \otimes V^* = \{\text{bilinear maps } V \times V \to \mathbb{R}\}$
$\cup$
$S^2 V^*$

bundle metric: section of $\underset{\downarrow \atop B}{S^2 E^*}$ taking values in pos def forms.

PROP: Every VB admits a bundle metric. Take a cover $\{U_\alpha\}_{\alpha \in A}$ of B and local trivialization $\psi_\alpha: E|_{U_\alpha} \xrightarrow{\sim} U_\alpha \times \mathbb{R}^n$. Define a bundle metric on $E|_{U_\alpha}$ by $g_\alpha(e_1, e_2) = \psi_\alpha(e_1) \cdot \psi_\alpha(e_2)$.
↳ dot product on $\mathbb{R}^n$

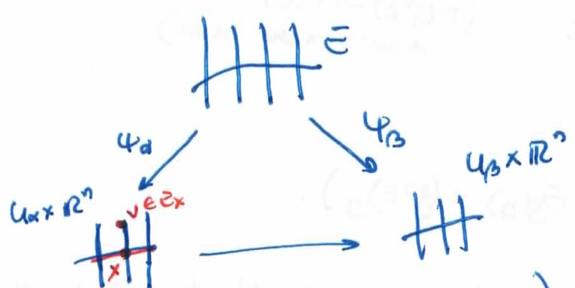


Choose a partition of unity $\{e_\alpha\}_{\alpha \in A}$ on B with $\text{supp}(e_\alpha) \subset U_\alpha$.
Define $\forall x \in B$, $\forall e_1, e_2 \in E_x$:

$$g(e_1, e_2) = \sum_{\alpha \in A} e_\alpha(x)\, g_\alpha(e_1, e_2)$$

[Partition of unity: $e_\alpha: B \to [0,1]$ st $\sum_{\alpha \in A} e_\alpha \equiv 1$]

Recall: Given a VB $\pi: E \to B$ choosing loc. trivializations $\psi_\alpha: E|_{U_\alpha} \xrightarrow{\sim} U_\alpha \times \mathbb{R}^n$, get transition maps $\psi_{\alpha\beta}: U_\alpha \cap U_\beta \to GL(n, \mathbb{R})$



$$(x, v) \longrightarrow (x, \underset{\underset{GL(n,\mathbb{R})}{\cap}}{\psi_{\alpha\beta}(x)} \cdot \underset{\underset{\mathbb{R}^3}{\cap}}{v})$$

The vector bundle has structure group $O(n)$ $\Leftrightarrow$ we can choose the ψ_α with transition maps $\psi_{\alpha\beta}: U_\alpha \cap U_\beta \to O(n)$
$\cap$
$GL(n, \mathbb{R})$

PROP: Every rank n VB has structure group $O(n)$.

Put a bundle metric on E. Take trivializations $\varphi_\alpha: E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^n$

$\exists$ orthonormal frame w_1, w_n of $E|_{U_\alpha}$.

Define new trivializations:

$\hat{\varphi}_\alpha: E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^n$

$w_i \longrightarrow e_i = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ st. respects the bundle metric, so $\hat{\varphi}_\beta \circ \hat{\varphi}_\alpha^{-1}$ also respects the bundle metric.

Rem: Topological K-theory.

Fix a manifold B. For every complex VB $E \to B$, denote its isomorphism class by $[E]$. Let

$$I := \{[E] : E \text{ is a VB over } B\}$$

The Whitney sum $\oplus$ makes I into a commutative monoid.

On $I \times I$ define the equivalence relation

$$([E_1],[E_2]) \sim ([F_1],[F_2]) \Leftrightarrow \forall \text{VB } H \text{ st } E_1 \oplus F_2 \oplus H \simeq F_1 \oplus E_2 \oplus H$$

$\hookrightarrow$ (Think of "$[E_1] - [E_2]$")

The quotient $K_0(B) = (I \times I)/\sim$ is a abelian group, called zero-th K-theory group.

Ex: Let $B = \{pt\}$. Get $I \cong \mathbb{N}_{\geq 0}$ (two vector spaces are isomorphic if they have the same dimension, so we have a class for each dimension)

$$(a,b) \sim (c,d) \Leftrightarrow a+d = c+b$$

$$K_0(\{pt\}) = \mathbb{Z}$$

3. PRINCIPAL BUNDLES

3.1. Actions

DEF: Let M be a manifold, G a Lie group. A right action of G on M is a smooth $\varphi: M \times G \to M$ st. $\begin{cases} m \cdot e = m & \forall m \in M \\ (m \cdot g) \cdot h = m \cdot (gh) & \forall g, h \in G \end{cases}$

(Here $m \cdot g = \varphi(m, g)$, $e \in G$ neutral element

A left action: $G \times M \to M$ st $\begin{cases} em = m \\ g(hm) = (gh)m \end{cases}$

Rem: If ψ is a right action then $\phi(g, m) := \psi(m, g^{-1})$ is a left action.

DEF: The corresponding infinitesimal action is the Lie algebra morphism.

$$T_eG \longrightarrow \underbrace{X(M)}_{\substack{\text{vector}\\ \text{fields}}}, \quad v \longrightarrow v_M$$

where $v_M(m) = \frac{d}{dt}\Big|_0 \underbrace{(m \cdot \underbrace{(\exp(tv))}_{\text{curve in } G})}_{\text{curve in } M}$

DEF: ψ is free $\Leftrightarrow$ $\forall g \neq e$
$\psi(\cdot, g)$ $M \to M$ has no fixed points.

ψ is transitive $\Leftrightarrow$ $\forall x, y \in M$ $\exists g \in G$, $x \cdot g = y$

ψ is proper $\Leftrightarrow$ $M \times G \to M \times M$ proper
$(m, g) \to (mg, m)$

Ex: $\phi: SO(2, \mathbb{R}) \times \mathbb{R}^2 \to \mathbb{R}^2$, $\phi(A, v) = A \cdot v$
$\quad \| $
$U(1) \simeq S^1$

$(G \cdot p)$ orbits

(rotations on the plane)

Not free (all fix the zero origin)
Not transitive, proper.
↳ (points with different distance of (0,0) are not connected)

Ex: Let G be a Lie Group. A left action of G on itself is

$$G \to \underbrace{\mathrm{Aut}(G)}_{\substack{\text{automorphisms =}\\ = \{\text{isomorphisms } G \to G\}}}, \quad g \to C_g$$

where $C_g(h) = ghg^{-1}$.

A left action of G on T_eG is the adjoint action.

Ad: $G \to GL(T_eG)$, $g \to d_e(C_g)$

Subex: if $G = GL(n, \mathbb{R})$ then $\mathrm{Ad}(\underset{GL(n,\mathbb{R})}{\underset{\cap}{A}})(\underset{T_eGL(n,\mathbb{R}) = \mathrm{Mat}(n,\mathbb{R})}{\underset{\cap}{B}}) = ABA^{-1}$

3.2. Principal bundles

DEF: Fix a Lie group G. A (principal) G-bundle is a smooth $\pi: P \to B$ with a right action $P \times G \to P$ with the property:

$\forall x \in B$ $\exists$ neighborhood U and a diffeomorphism $\psi: \underbrace{P|_U}_{:=\pi^{-1}(U)} \to U \times G$ st.

$$\begin{tikzcd} P|_U \arrow[r, "\psi"] \arrow[dr, "\pi"'] & U \times G \arrow[d] \\ & U \end{tikzcd}$$ commutes

and ψ is G-equivariant (ie, $\psi(p\cdot g) = \underbrace{\psi(p)\cdot g}_{\text{mult. in } G}$ $\forall p \in P$ $\forall g \in G$).

Rem: $\forall x \in B$, G acts on $P_x = \pi^{-1}(x)$ freely and transitively.

Ex: Given B and G: $\begin{array}{c} B \times G \\ \downarrow \\ B \end{array}$ is the trivial G-bundle.

Rem: If a principal G bundle $\pi: P \to B$ admits a (global) section $s: B \to P$, then it is isomorphic to the trivial G-bundle, vie

$$B \times G \xrightarrow{\sim} P$$
$$(x, g) \longmapsto s(x) \underset{\text{action}}{\cdot} g$$

Ex: Consider the action of $U(1) = \{z \in \mathbb{C} : |z| = 1\}$ on $S^3 =$ unit sphere in $\mathbb{C}^2$ by

$$S^3 \times U(1) \to S^3$$
$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \times \lambda \to \begin{pmatrix} z_1 \lambda \\ z_2 \lambda \end{pmatrix}$$

Then $\pi: S^3 \to S^3/U(1) \simeq S^2$ is a principal $U(1)$-bundle, called "Hopf fibration".

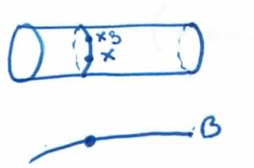
Rem: Consider G and the right action of G on itself by right multiplication.
Let $\phi: G \to G$ be G-equivariant, ie $\phi(hg) = \phi(h)\cdot g$ $\forall h, g \in G$.
Then $\exists$ $g_0 \in G$ st $\phi =$ (left mult by g_0) $(\forall g \in G$ $\phi(\underbrace{eh}_{h}) = \underbrace{\phi(e)}_{:=g_0}\cdot h)$.

so $\{G\text{-equivariant maps } G \to G\} \equiv G$.

A principal G-bundle is in particular a fibre bundle with fibre bundle G.
Given local trivializations $\psi_\alpha: P|_{U_\alpha} \to U_\alpha \times G$ we get G-equivariant maps

$$\psi_\beta \circ \psi_\alpha^{-1}: (U_\alpha \cap U_\beta) \times G \to (U_\alpha \cap U_\beta) \times G$$

of the form $(x, g) \longmapsto (x, \psi_{\alpha\beta}(x)(g))$
where $\psi_{\alpha\beta}(x) \in \{G\text{-equiv. diff. } G \to G\} \equiv G$.



(When you choose a point x you have an identification of the fibre and the Lie group, but not before)

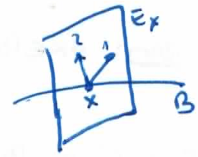
PROP: Let P be a manifold with a right G-action which is free and proper. Then P/G has a unique smooth structure st. $\pi: P \to P/G$ submersion and $\pi: P \to P/G$ is a principal G-bundle.

Rem: All principal bundles arise like them.

PROP: Set $\pi: P \to B$ a surjective submersion, and $P \times G \to P$ a free right action whose orbits are the fibres of π. Then $\pi: P \to B$ is a G-bundle.

3.3. Frame bundles and associated bundles

DEF: Let $\pi: E \to B$ be a rank n VB. Then



$$\begin{array}{l} F(E) := \bigcup\limits_{x \in B} \{\text{ordered basis of } E_x\} \\ \downarrow \\ B \end{array}$$

is a principal $GL(n,\mathbb{R})$ bundle, called frame bundle of E.

The right $GL(n,\mathbb{R})$ action is as follows:

$$\underbrace{(v_1, \dots, v_m)}_{\in F(E)} \cdot \underset{\substack{\cap \\ GL(n,\mathbb{R})}}{A} = \left(\sum_j a_{j1} v_j, \dots \sum_j a_{jm} v_j\right)$$

DEF: Consider the tangent bundle TB and its frame bundle $F(TB) \to B$. Let H be a subgroup of $GL(n,\mathbb{R})$. An H-structure on B is principal H-subbundle of $F(TB)$.

Ex: $\exists$ a bijection

$$\begin{array}{ccc} \{\text{Riemanian metrics on } B\} & \longleftrightarrow & O(n)\text{-structures on } B \\ g & \longrightarrow & \bigcup\limits_{x \in B} \{\text{ordered orthonormal basis of } T_x B\} \end{array}$$

Similarly $\{\text{orientation on } B\} \longleftrightarrow GL(n,\mathbb{R})_+\text{-str. on } B$

DEF: A representation of a Lie group G on a vector space V is a left action $G \times V \to V$ by linear maps, ie. a Lie group morphism $G \to GL(V)$.

DEF: Given a G-bundle $\pi: P \to B$ and a representation V of G the associated vector bundle is

$$\begin{array}{l} E := (P \times_G V) := P \times V /_{(p \cdot v) \sim (p \cdot g,\, g^{-1} \cdot v)} \\ \downarrow \\ B \end{array}$$

$\hookrightarrow$ represent.

The fibre over $x \in B$ is

$$(P_x \times V)/_{(p,v) \sim (pg,\, g^{-1}v)} \cong V \quad \text{(because the action of } G \text{ on } P_x \text{ is transitive)}$$

(For any $p_0 \in P_x$, an isomorphism is $[(p_0, v)] \longrightarrow v$).

PROP: Fix a manifold B and $n \in \mathbb{N}$. There is a bijection.

$$\left\{ \begin{array}{c} \text{rank } n \text{ vector bundles} \\ \text{over } B \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} GL(n,\mathbb{R})- \\ \text{bundles over } B \end{array} \right\}$$

$$E \longrightarrow F(E)$$

$$\underbrace{P \times_{GL(n,\mathbb{R})} \mathbb{R}^n}_{} \longleftarrow P$$

(using the representation of $GL(n,\mathbb{R})$ on $\mathbb{R}^n$ by matrix mult.)

4. CONNECTIONS ON VECTOR BUNDLES

4.1. Covariant derivatives

Let $\pi : E \to B$ be a VB.

Rem: Consider the dual bundle $E^* \to B$

Given $\phi \in \Gamma(E^*)$ we get a linear map

$$\Gamma(E) \longrightarrow C^\infty(B)$$
$$e \longrightarrow \langle \phi, e \rangle \text{ where } \langle \phi, e\rangle_x = \langle \underset{E_x^*}{\phi(x)}, \underset{E_x}{e(x)} \rangle \in \mathbb{R}$$

Clearly $\forall f \in C^\infty(B)$: $\langle \phi, fe \rangle = f \langle \phi, e\rangle$

Viceversa any linear map $\varphi : \Gamma(E) \to C^\infty(B)$ which is $C^\infty(B)$-linear (ie, tensorial) comes from an element of $\Gamma(E)$.

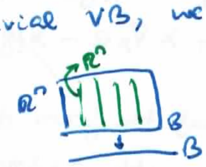
Non-ex: The Lie bracket of vector fields is not a tensor $[x, fy] \neq f[x,y]$

DEF: A covariant derivative is a $\mathbb{R}$-bilinear map $\nabla : \underbrace{\Gamma(TB)}_{= X(B)} \times \Gamma(E) \longrightarrow \Gamma(E)$
$$(x, e) \longrightarrow \nabla_x e$$

st. $f \in C^\infty(B)$

$$\nabla_{fx} e = f \nabla_x e$$
$$\nabla_x (fe) = \underbrace{X(f)}_{\substack{df(x) \\ \in C^\infty(B)}} e + f \nabla_x e$$

Ex: If $E = B \times \mathbb{R}^n$ trivial VB, we have $\Gamma(E) = \{\text{smooth functions } B \to \mathbb{R}^n\}$
$\downarrow$
B



An example of covariant deriv. is the "usual derivative".

$\nabla_x e := de(x)$ (ie. the map $\begin{array}{l} B \to \mathbb{R}^n \\ x \to (d_x e)(X_x) \end{array}$).

LEMMA: Let $y \in B$, $x \in X(B)$, $e \in \Gamma(E)$. Then

$(\nabla_x e)_y \in E_y$ depends only on X_y, on e in a neighborhood of y and along any curve $\gamma : (0,1) \to B$ with $\dot\gamma(0) = X_y$.

1) We show: if $x_y = 0$, then $(\nabla_x e)_y = 0$.

In coordinates $x = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i}$ with $f_i(y) = 0 \ \forall x_i$.

So $\nabla_x e = \nabla_{\sum f_i \frac{\partial}{\partial x_i}} e = \sum_i f_i \left(\nabla_{\frac{\partial}{\partial x_i}} e\right)$ vanishes at y.

2) We show: if in a neighbourhood U of y we have $e|_U = 0$, then $(\nabla_x e)_y = 0$.

Let $\rho: M \to [0,1]$ be a function with $\mathrm{supp}(\rho) \subset U$, $\rho(y) = 1$.

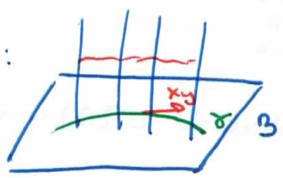
Then $0 = \nabla_x(\rho e) = \underbrace{x(\rho)}_{\substack{= \\ 0 \text{ outside } U}} \cdot \underbrace{e}_{\substack{= \\ 0 \text{ on } U}} + \rho \nabla_x e$. So $0 = \underbrace{\rho(y)}_{=1} (\nabla_x e)_y$

[Recall: If $g: M \to \mathbb{R}$, $y \in T_y M$, then $(d_y g)(\underset{\dot\gamma(0)}{y}) = \frac{d}{dt}\Big|_0 (g \circ \gamma)(t)$
$\gamma: (0,1) \to M$ with $\dot\gamma(0) = y$]

3) Let γ be a curve on B with $\gamma(0) = y$, $\dot\gamma(0) = x_y$. We show:

If $e|_{\gamma(t)} = 0 \ \forall t$, then $(\nabla_x e)_y = 0$. (proof omitted)

$(\nabla_x e)_y$:



DEF: Let $E \to B$ be a VB.

$\Omega(B, E) := \Omega(B) \otimes_{C^\infty(B)} \Gamma(E)$ (tensor product) are the diff. forms on B with values on E:

they assign to $x_1, \dots x_m \in T_y B$ an element of E_y.

PROP: Let ∇^0 a cov. derivative on $E \to B$.

Then $\{\text{cov deriv. on } E\} = \{\nabla^0 + A : A \in \Omega^1(B, \mathrm{End}(E)) = \Gamma(T^*B \otimes E^* \otimes E)\}$ ($\{\text{maps } E \to E\} = E^* \otimes E$)

Proof of the inclusion "$\subset$": [Remind: $\{V \to W\} = V^* \otimes W$]

Let ∇ be a cov. der. Then:

$(\nabla - \nabla^0)_x fe = \nabla_x fe - \nabla^0_x fe = \cancel{x(f)} e + x \nabla_x e - \cancel{x(f)} e - f \nabla^0_x e = f(\nabla - \nabla^0)_x e$ (*)

Ex: If $E = B \times \mathbb{R}^n$ trivial. Any cov. deriv. is of the form $d + A$, where $A \in \Omega^1(B, \mathrm{Mat}(n, \mathbb{R})) \cong \{n \times n \text{ matrices with } 1\text{-forms as entries}\}$.

(*) This difference is tensorial and ~~can be mapped to~~ takes values in $\mathrm{End}(E)$.

REM: A cov. der. ∇ on E induces a cov. der. on VB's constructed out of E.

On E^* we obtain ∇^* by defining:

$$\langle \nabla_X e, \xi \rangle + \langle e, (\nabla^*)_X \xi \rangle = X \langle e, \xi \rangle$$

$\forall e \in \Gamma(E)$, $\xi \in \Gamma(E^*)$, $X \in \mathfrak{X}(B)$.

On $E_1 \otimes E_2$ we define

$$\nabla^{E_1 \otimes E_2} (e_1 \otimes e_2) = \nabla^{E_1} e_1 \otimes e_2 + e_1 \otimes \nabla^{E_2} e_2$$

4.2. Connection and curvature

Fix a VB $\pi: E \to B$. Fix a cov. der. ∇.

PROP: Let $g: M \to B$. There is a unique cov. der. $\overline{\nabla}$ on $g^*E \to M$ such that

$$(\overline{\nabla}_X (g^* E))_y = (\nabla_{g_* X} e)_{g(y)} \quad \forall e \in \Gamma(E),\ y \in M,\ X \in T_y M$$

$$\begin{tikzcd} g^*E \arrow[r, "\overline{g}"] \arrow[d] & E \arrow[d] \\ M \arrow[r, "g"] & B \end{tikzcd}$$

Let $\gamma: [0,1] \to B$ be a curve. Denote by $\frac{\nabla}{dt}$ the pullback cov. der. on $\gamma^* E \to [0,1]$.

REM: Notice: a section of $\gamma^* E$ is just a choice of element of $E_{\gamma(t)}$ $\forall t \in [0,1]$.

$\frac{\nabla}{dt}$ is the unique cov. der. on $\gamma^* E$ $\left(\text{ie, } \frac{\nabla}{dt}(f(t) \cdot s(t)) = f'(t) \cdot s(t) + f(t) \frac{\nabla}{dt} s(t) \right)$

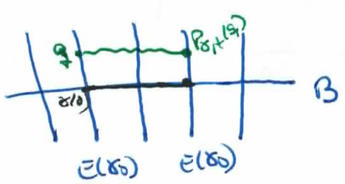
which $\forall e \in \Gamma(E)$ $\frac{\nabla}{dt} \big(\underbrace{e|_{\gamma(t)}}_{\in \Gamma(\gamma^* E)} \big) = (\nabla_{\dot\gamma(t)} e)_{\gamma(t)}$

a unique

PROP: $\forall q \in E_{\gamma(0)}$, $\exists e \in \Gamma(\gamma^* E)$ st. $\begin{cases} e(0) = q \\ \frac{\nabla}{dt} e = 0 \end{cases}$

[Use existence and uniqueness theorem for ODE]

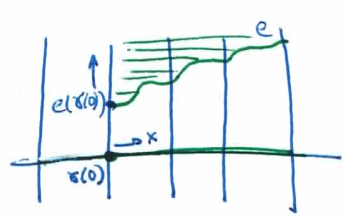
DEF: $\forall t \in [0,1]$ consider the linear isomorphism $P_{\gamma,t}: E_{\gamma(0)} \to E_{\gamma(t)}$, $q \mapsto e(t)$



$P_{\gamma,t}$ is called parallel transport along γ using ∇.

REM: ∇ is recovered from the parallel transport maps: let $X \in T_y B$, $e \in \Gamma(E)$, take a curve γ in B with $\dot\gamma(0) = X$.

Then $(\nabla_X e)_y = \frac{d}{dt}\Big|_{t=0} \underbrace{\left[(P_{\gamma,t})^{-1} (e_{\gamma(t)}) - e_{\gamma(0)} \right]}_{\in E_{\gamma(0)}}$

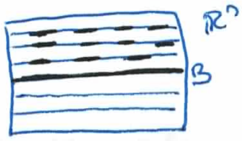


REM: From ∇ we obtain a connection H on the fibre bundle E. $\forall q \in E$ define

$$H_q = \Big\{ \underbrace{\frac{d}{dt}\Big|_{t=0} P_{\gamma,t}(q)}_{\substack{\text{unique section starting at } q \\ \text{with } \frac{\nabla}{dt}=0}} \Big\} \quad \gamma \text{ is a curve in } B \text{ starting at } \pi(q).$$

Notice: $\forall$ curve γ in B, the parallel transport along γ using H (or ∇) is a linear map. $P_{\gamma,1} : E_{\gamma(0)} \longrightarrow E_{\gamma(1)}$

Ex: On $B \times \mathbb{R}^n \to B$ take $\nabla = d$ (recall: $\Gamma(B \times \mathbb{R}^n) = C^\infty(B, \mathbb{R}^n)$). Then

$H =$ 

DEF: The curvature of ∇ is defined by $R(X,Y)_e = \nabla_X \nabla_Y e - \nabla_Y \nabla_X e - \nabla_{[X,Y]} e$ $\forall X, Y \in \mathfrak{X}(B)$, $e \in \Gamma(E)$, such that is a tensor (depends only on X, Y and e is a point) and skew symmetric in X, Y, ie $R \in \Omega^2(M, \mathrm{End}(E))$.

DEF: Define the maps D by

$$\Gamma(E) \xrightarrow{D_0} \underbrace{\Omega^1(B,E)}_{= \Omega^1(B) \otimes_{C^\infty(B)} \Gamma(E)} \xrightarrow{D_1} \Omega^2(B,E) \xrightarrow{D_2} \cdots$$

by $D_n(\underset{\Omega^n(B)}{\omega} \otimes \underset{\Gamma(E)}{e}) = d\omega \otimes e + (-1)^n \omega \wedge \underbrace{\nabla e}_{\in \Omega^1(B) \otimes \Gamma(E)}$ (so $D_0 = \nabla$).

The curvature of ∇ is $F := D_1 \circ D_0$ so $F : \Gamma(E) \longrightarrow \Omega^2(B,E)$.

PROP: The two definitions above agree: $F = R \in \Omega^2(B, \mathrm{End}(E))$

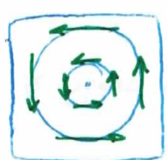
REM: we saw that the curvature Ω of the Ehresmann connection H corresponding to ∇ is determined by $\forall X, Y \in \mathfrak{X}(B)$

$$\Omega(X^H, Y^H) = [X^H, Y^H] - [X,Y]^H \in \{\text{vertical vector fields on } E\}$$

On $\mathbb{R}^n$ we have

$$\mathrm{End}(\mathbb{R}^n) = \{\text{linear vector fields on } \mathbb{R}^n\}$$

$$A = (a_{IJ}) \longmapsto \sum_{I,J} a_{IJ}\, x_i \frac{\partial}{\partial x_J}$$

$x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}$ 

So we deduce $\Gamma(\mathrm{End}(E)) = \Big\{$ vertical vector fields on E which are linear on every fibre $\Big\}$

Recall also $F = R \in \Omega^2(B, \mathrm{End}(E))$, the curvature of H and ∇ agree under that identification

5. CONNECTIONS ON PRINCIPAL BUNDLES

Let $\pi : P \to B$ be a G-bundle. ~~with~~ Write $\mathfrak{g} := T_eG$. $\forall g \in G$, we write R_g for the corresponding diffeomorphism of P.

REM: $\forall q \in P$ we get a canonical isomorphism $\mathfrak{g} \to T_q(\underbrace{P_{\pi(q)}}_{\pi^{-1}(q)})$

$$v \to v_P(q) := \frac{d}{dt}\Big|_0 (\underbrace{q \cdot \exp(tv)}_{\text{curve in } G})$$

5.1. Connections

DEF: A connection on P is an Ehresmann connection H on P which is G-invariant (ie, $(R_g)_* H_p = H_{pg}$ $\forall p \in P, g \in G$)

DEF: A connection 1-form on P is $\omega \in \Omega^1(P, \mathfrak{g})$ such that $\omega(v_P) = v$ $\forall v \in \mathfrak{g}$, $(R_g)^*\omega = Ad_{g^{-1}}\omega$ $\forall g \in G$ (ie $[(R_g)^*\omega]_p(x) = \omega_{pg}((R_g)_* x) = Ad_{g^{-1}}\omega_p(x)$)

$\omega_q : T_qP \to \mathfrak{g}$

$T_q(\pi^{-1}(x)) \cong \mathfrak{g}$

REM: The two notions are equivalent. Given H, define ω by $\begin{cases} \omega_q(x) = 0 & \text{if } x \in H_q \\ \omega_q(v_P(q)) = v & \forall v \in \mathfrak{g} \end{cases}$

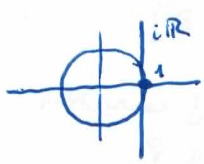
Given ω, define H by $H_q = \ker\{\omega_q : T_qP \to \mathfrak{g}\}$

DEF: $\alpha \in \Omega(P)$ is horizontal $\iff$ $\iota_{v_P}\alpha = 0$ $\forall v \in \mathfrak{g}$

DEF: $\alpha \in \Omega(P)$ is basic $\iff$ α is horizontal and $(R_g)^*\alpha = \alpha$ $\forall g \in G$ $\iff$
$\iff \alpha = \pi^*(\beta)$ for a unique $\beta \in \Omega(B)$.

REM: Let ω_0 be a connection 1-form on P. Then $\{\text{conn. 1 forms}\} =$
$= \{\omega_0 + \alpha : \alpha \in \Omega^1_{hor}(P, \mathfrak{g}),\ (R_g)^* = Ad_{g^{-1}}\alpha \ \ \forall g \in G\}$

Ex: Let $G = U(1)$. The Lie algebra of $U(1)$ is $i\mathbb{R} \cong \mathbb{R}^1$
A conn. 1-form is $\omega \in \Omega^1(P)$ st. $\omega(1_P) = 1$ and ω is $U(1)$-invariant
(ie, $\omega((R_g)_* x) = \omega(x)$ $\forall x \in TP$, $g \in U(1)$).



Now let $P = B \times U(1)$ be the trivial bundle. Denote by θ the "angle coordinate" on $U(1)$ so $1_P = \frac{\partial}{\partial\theta}$

Then $d\theta$ $(d\theta\frac{\partial}{\partial\theta} = 1)$ is a conn. 1-form on P.

An arbitrary connection is $d\theta + \underbrace{\pi^*\beta}$
a base 1-form on P,
here $\beta \in \Omega^1(B)$

Subex: $B = \mathbb{R}$ (with coord. x), an arbitrary conn 1-form on $P = \mathbb{R} \times U(1)$ is $d\theta + f(x)dx$. Its kernel is $H = \mathrm{Span}\left\{ \frac{\partial}{\partial x} - f(x)\frac{\partial}{\partial \theta} \right\}$

5.2. Curvature

Let $\alpha \in \Omega^1(P, \mathfrak{g})$ be a connection 1-form on P. Define $[\alpha \wedge \alpha] \in \Omega^2(P, \mathfrak{g})$ by

$$[\alpha \wedge \alpha](X, Y) = [\alpha(X), \alpha(Y)] - [\alpha(Y), \alpha(X)] = 2[\alpha(X), \alpha(Y)]$$

$X, Y \in TP$

__DEF__: The curvature of α is $F = d\omega + \frac{1}{2}[\alpha \wedge \alpha] \in \Omega^2(P, \mathfrak{g})$.

__REM__: The vertical bundle V of $\pi: P \to B$ is identified canonically with the trivial VB $\cancel{B}\,P \times \mathfrak{g} \to \cancel{B}\,P$ $(T_g(\pi^{-1}(g)) \cong \mathfrak{g})$.
Recall that the curvature of $H := \mathrm{Ker}(\omega)$ is an element of $\Omega^2(P, V)$.
Using the above identification it agrees with F.

__REM__: F is the pullback by π of an element of $\Omega^2(B, P_G \times_{Ad} \mathfrak{g})$

__Ex__: Let $G = U(1)$ as we saw, a connection 1-form is $\omega \in \Omega^1(P)$ st $\omega(1_P) = 1$, ω is invariant.
Its curvature is $F = d\omega$.
F is basic

$$\iota_{1_P} d\omega = \underbrace{\mathcal{L}_{1_P}\omega}_{=0} - d(\underbrace{\iota_{1_P}\omega}_{=1})$$

$$\mathcal{L}_{1_P} d\omega = d\,\mathcal{L}_{1_P}\omega = 0$$

So $\exists c \in \Omega^2(B)$ st $\pi^* c = F$, c is closed $\pi^* dc = d(\underbrace{\pi^* c}_{F = d\omega}) = 0$
and π^* is injective so get a cohomology class

$$[c] \in H^2_{dR}[B, \mathbb{R}]$$

Let $\tilde{\omega}$ be another connection 1-form on P, then $\omega - \tilde{\omega} \in \Omega^1(P)$, horizontal form and invariant (ie, basic), ie $\exists \alpha \in \Omega^1(B)$ st $\pi^*\alpha = \omega - \tilde{\omega}$.

Hence $\pi^*(c - \tilde{c}) = F - \tilde{F} = d\omega - d\tilde{\omega} = d(\omega - \tilde{\omega}) = d(\pi^*\alpha) = \pi^*(d\alpha) \Rightarrow$
(We don't lose information because π^* is injective)

$$\Rightarrow c - \tilde{c} = d\alpha \Rightarrow [c] = [\tilde{c}] \in H^2_{dR}[B, \mathbb{R}]$$

So the class $[c]$ is independent of the choice of connection 1-form
$\left[-\frac{c}{2\pi i}\right]$ is called first Chern class of P.
$\in H^2_{dR}(B, \mathbb{Z})$

6. REMARKS ON CHERN CLASSES

Ex. of complex VB's.

If B is a (real) manifold $\quad B\times\mathbb{C}^3 \to B$

If X is a complex manifold $\quad TX$

If X is a " " of dim 1 (for example a Riemann surface) then

$\forall l$: $TX^{\otimes l} = TX\otimes\ldots\otimes TX$ is a complex vector bundle of rank 1 (= line bundle)

Rem: On a manifold B, $\alpha\in\Omega^k(B)$ closed, is integer form. $\Leftrightarrow$ $\forall$ smooth singular cycle, $\sum \underset{\mathbb{Z}}{\underset{\cap}{\lambda_i}}\underset{\Delta^k\to B}{\underbrace{\sigma_i}}$: $\quad \sum\lambda_i\int_{\Delta^k}(\sigma_i)^*\alpha\in\mathbb{Z}$

6.1. Definitions

DEF: A complex characteristic class c is a map

$$\left\{\begin{array}{c}\text{complex vector bundles}\\ E\to B\end{array}\right\}\Big/_{\text{isomorph.}} \longrightarrow H^*(B,\mathbb{Z})$$

$$\supseteq \{[c]\ c \text{ integer forms}\} \subset H_{dR}(B)$$

st. for all VBs $E\to B$ and continuous maps $f:\tilde{B}\to B$,

$$f^*(c(E)) = c(f^*(E))\in H(\tilde{B},\mathbb{Z})$$

Thm: $\exists$ a unique sequence of maps $c_1, c_2, c_3\ldots$ of maps

$$c_i : \left\{\text{complex VB } \begin{array}{c}E\\\downarrow\\B\end{array}\right\}\Big/_{\text{iso}} \longrightarrow H^{2i}(B,\mathbb{Z})$$

st. a) $c_i(f^*E) = f^*(c_i(E))$

b) $c(E_1\oplus E_2) = c(E_1)\cdot c(E_2)$ where $c = 1 + c_1 + c_2 + \cdots$

c) $c_i(E) = 0$ if $i > rk(E)$

d) For the tautological line bundle

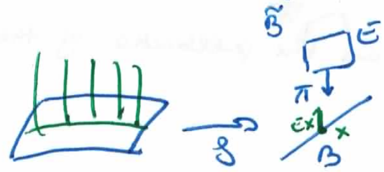
$$\gamma = \{(l,v) : v\in l\} \to \mathbb{CP}^\infty = \{\text{lines } l \text{ in } \mathbb{C}^\infty\}$$

$c_1(\gamma)$ is a generator of $H^2(\mathbb{CP}^\infty,\mathbb{Z})\cong\mathbb{Z}$.

c_i is called i-th Chern class.

Rem: By a), the Chern classes is a measure of how "twisted" the VB is:

Given $f:\tilde{B}\to B$, $\quad \begin{array}{c}f^*E\\\downarrow\\\tilde{B}\end{array}$ is less twisted than $\begin{array}{c}E\\\downarrow\\B\end{array}$



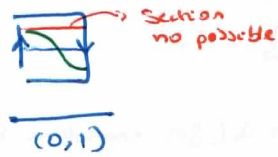
$[(f^*E)_{f^{-1}(x)}$ is trivial$]$

and if $c_i(E)=0$ then $c_i(f^*E) = f^*(\underbrace{c_i(E)}_{0}) = 0$

Rem: Let $n = rk(E)$, then $c_n(E) \in H^{2n}(B, \mathbb{Z})$ is the Euler class of $E_{\mathbb{R}}$ (the underlying real VB).

The Euler class of a real VB satisfies:

- If V (the real VB) has a no-where vanishing section, then $\text{Euler}(V) = 0$.



- If B^k is a compact manifold, taking $V = TB$ then

$$\int_B \underbrace{\text{Euler}(TB)}_{\in H^k(B)} = \underset{\substack{\downarrow \\ \text{Euler} \\ \text{characteristic}}}{\chi(B)}$$

Rem: There is a bijection, $\forall$ manifold B

$$\left\{ \begin{array}{c} \text{complex line bundles} \\ E \to B \end{array} \right\} /_{iso} \longrightarrow H^2(B, \mathbb{Z})$$

$$E \longmapsto c_1(E)$$

DEF: Let $n \in \mathbb{N}$. Define $\underset{\hookrightarrow \text{grassmanian}}{G_n(\mathbb{C}^\infty)} := \left\{ \begin{array}{c} n\text{-dim subspaces} \\ \text{of } \mathbb{C}^\infty \end{array} \right\}$

The rank n vector bundle

$$\gamma^n = \{ (\ell, v) : \ell \in G_n(\mathbb{C}^\infty), v \in \ell \}$$
$$\downarrow$$
$$G_n(\mathbb{C}^\infty)$$

is called universal bundle of rank n.

Rem: $G_1(\mathbb{C}^\infty) = \mathbb{CP}^\infty$

Thm: Fix a manifold B, $n \in \mathbb{N}$. There is a bijection

$$\underbrace{[B, G_n(\mathbb{C}^\infty)]}_{:= \{\text{maps } B \to G_n(\mathbb{C}^\infty)\} /_{\text{homotopy}}} \longleftrightarrow \left\{ \begin{array}{c} \text{rank } n \text{ complex} \\ \text{VBs over } B \end{array} \right\} /_{iso}$$

$$[f] \longrightarrow f^*\gamma^n \qquad \left[\begin{array}{ccc} f^*\gamma^n & \dashrightarrow & \gamma^n \\ \downarrow & & \downarrow \\ B & \longrightarrow & G_n(\mathbb{C}^\infty) \end{array} \right]$$

Rem: In particular, given $E \to B$ of rank n, $\exists f: B \to G_n(\mathbb{C}^\infty)$ st $E \cong f^*\gamma^n$.

Hence $\underset{\substack{\wr\wr \\ f^*\gamma^n}}{c_i(E)} = f^*(c_i(\gamma^n))$

That is, the Chern classes of E are the pullbacks of the Chern classes of the universal bundle.

Thm: $H^*(G_n(\mathbb{C}^\infty), \mathbb{Z}) = \mathbb{Z}[c_1(\gamma^n), \dots c_n(\gamma^n)]$
$\quad = \{\text{polynomials in } c_1(\gamma^n) \dots c_n(\gamma^n)\}$

Rem: So all characteristic classes on $E \to B$ is a polynomial in the Chern classes of E.

6.2. Chern-Weil theory

Fix a complex VB $E \to B$ of rank n. Let ∇ be a cov. der. on E (in particular $\nabla_X f e = X(f)e + f\nabla_X e$, $\forall f \in C^\infty(B,\mathbb{C})$)
Let F be its curvature, so $F \in \Omega^2(B, \mathrm{End}_{\mathbb{C}} E)$. $\quad [\mathrm{Tr}\, F \in \Omega^2(B)]$

Rem: Given trivializations φ_α and φ_β with transition maps $\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n,\mathbb{C})$
we have $\quad \varphi_{\alpha\beta}^{-1} F_\beta \varphi_{\alpha\beta} = F_\alpha$
$\quad = \mathrm{Mat}(n,\mathbb{C})$
Here $F_\alpha \in \Omega^2(U_\alpha, \mathrm{End}(\mathbb{C}^n))$ corresponds to $F|_{U_\alpha} \in \Omega^2(U_\alpha, \mathrm{End}_{\mathbb{C}}(E|_{U_\alpha}))$
under $\varphi_\alpha : E|_{U_\alpha} \xrightarrow{\sim} U_\alpha \times \mathbb{C}^n$.

Fix a polynomial function
$$q : \underset{\mathbb{C}^{n^2}}{\underset{\shortparallel}{\mathrm{Mat}(n,\mathbb{C})}} \longrightarrow \mathbb{C} \quad \text{such that}$$
$$\forall B \in \mathrm{Mat}(n,\mathbb{C}),\ \forall g \in GL(n,\mathbb{C}) : \quad q(g^{-1}Bg) = q(B)$$

Ex: $\forall j = 0, \dots, n$ let q_j be defined by ($\forall B \in \mathrm{Mat}(n,\mathbb{C})$)
$$\det(B + tI) = \underset{\det(B)}{\underset{\shortparallel}{P_n(B)}} + P_{n-1}(B)\,t + \dots + \underbrace{P_1(B)}_{\mathrm{tr}(B)}\,t^{n-1} + \underbrace{P_0(B)}_{\pm 1}\,t^n$$
The P_j satisfy the property above.

Notice that we obtain a well-defined $q(F) \in \Omega^{2\deg(q)}(B,\mathbb{C})$ by defining
$[q(F)]|_{U_\alpha} = $ "$q(F_\alpha)$" (view F_α as a matrix whose entries are 2-forms on U_α)

Prop: $q(F)$ is closed and $[q(F)] \in H^{2\cdot\deg(q)}(B,\mathbb{C})$ is indep of the choice of ∇.

Thm: $[P^j(\frac{i}{2\pi}F)] = \underbrace{c_j(E)}_{j\text{-th Chern class}} \in \Omega^2(B,\mathbb{Z})$

if F is the curv. of a cov. deriv. ∇ which preserves a hermitian metric on E.

Ex: $P^1(\frac{i}{2\pi}F) = \frac{i}{2\pi}\mathrm{Tr}(F) \in \Omega^2(B,\mathbb{Z})$ represents $c_1(E)$ given by
$$\underset{T_yB}{\underset{\cap}{(x,y)}} \longrightarrow \frac{i}{2\pi}\mathrm{Tr}(\underset{\mathrm{End}_{\mathbb{C}}(E_y)}{\underset{\cap}{\underbrace{F(x,y)}}})$$