09-05-2013

REM: A covariant derivative ∇ on E induces covariant derivative on VB's constructed out of E.

- On E^* we obtain ∇^* by defining:

$$\langle \nabla_X e, \xi \rangle + \langle e, (\nabla^*)_X \xi \rangle = X \langle e, \xi \rangle.$$

$\forall e \in \Gamma(E), \xi \in \Gamma(E^*), X \in \mathfrak{X}(B)$.

- On $E_1 \otimes E_2$ we define:

$$\nabla^{E_1 \otimes E_2}(e_1 \otimes e_2) = (\nabla^{E_1} e_1) \otimes e_2 + e_1 \otimes (\nabla^{E_2} e_2).$$

§4.2 Connection and Curvature

- Fix a VB $\pi : E \to B$. Fix a covariant derivative ∇.

Proposition: Let $f : M \to B$. There is a unique covariant derivative $\bar{\nabla}$ on $f^*E \to M$ such that:

$$(\bar{\nabla}_X (f^* e))_Y = (\nabla_{f_* X} e)_{f(Y)}$$

$\forall e \in \Gamma(E), Y \in M$ and $X \in T_Y M$.

- Let $\gamma : [0,1] \to B$ be a curve. Denote by $\frac{\nabla}{dt}$ the pullback covariant derivative on $\gamma^* E \to [0,1]$

REM Notice: a section of $\gamma^* E$ is giving a choice of element of $E_{\gamma(t)}$ $\forall t \in [0,1]$.

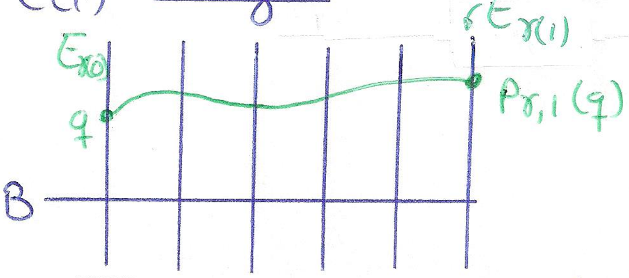
- $\frac{\nabla}{dt}$ is the unique covariant derivative on $\gamma^* E$ (ie, $\frac{\nabla}{dt}(f(t), s(t)) = f'(s) \cdot s(t) + f(t) \frac{\nabla}{dt} s(t)$) with $\forall e \in \Gamma(E)$

$$\frac{\nabla}{dt}(\underbrace{e|_{\gamma(t)}}_{\in \Gamma(\gamma^* E)}) = (\nabla_{\dot\gamma(t)} e)_{\gamma(t)}.$$

Proposition: $\forall q \in E_{\gamma(0)}$ $\exists$ a unique $e \in \Gamma(\gamma^* E)$ such that:

$$\begin{cases} e(0) = q \\ \frac{\nabla}{dt} e = 0 \end{cases}$$

DEF: $\forall t \in [0,1]$ consider the linear isomorphism
$$P_{\gamma,t} : E_{\gamma(0)} \longrightarrow E_{\gamma(t)}, \quad q \longmapsto e(t)$$
called the parallel trans. along γ using ∇.



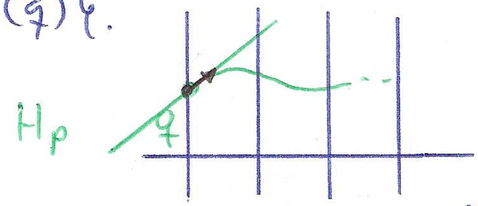
⊗ Parallel transport or Parallel translation.

REM: ∇ is recovered from the parallel transport maps.
Let $x \in T_y B$, $e \in \Gamma(E)$, take a curve γ is B with $\dot{\gamma}(0) = x$. Then:

$$(\nabla_x e)_y = \frac{d}{dt}\Big|_{t=0} [\underbrace{(P_{\alpha,t})^{-1}(e_{\gamma(t)}) - e_{\gamma(0)}}_{\in E_{\gamma(0)}}]$$

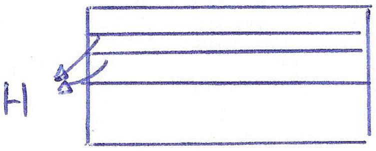
REM: From ∇ we obtain a connection H on the fiber bundle E.

- $\forall q \in E$ define $H_q := \{ \frac{d}{dt}\big|_0 P_{\gamma,t}(q)$ γ is a curve in B starting at $\pi(q) \}$.



- Notice: $\forall$ curve γ in B, the parallel transport along γ using H (or ∇) is a linear $P_{\gamma,1} : E_{\gamma(0)} \longrightarrow E_{\gamma(1)}$

$\boxed{\text{Ex}}$ On $B\times\mathbb{R}^n$ take $\nabla = d$ (recall $\Gamma(B\times\mathbb{R}^n) = C^\infty(B,\mathbb{R}^n)$).
$\downarrow$ we have the constant
B functions



DEF The curvature of ∇ is defined by

$$R(X,Y)e = \nabla_X\nabla_Y e - \nabla_Y\nabla_X e - \nabla_{[X,Y]}e \in \Gamma(E).$$

$\forall X,Y\in\mathfrak{X}(B)$, $e\in\Gamma(E)$.

$\otimes$ It is a tensor and $R\in\Omega^2(M,\mathrm{End}(E))$.

DEF Define the map D by

$$\Gamma(E)\xrightarrow{D_0}\underbrace{\Omega^1(B,E)}_{\Omega^1(B)\otimes_{C^\infty(B)}\Gamma(E)}\xrightarrow{D_1}\Omega^2(B,E)\xrightarrow{D_2}\cdots$$

by $D_n(\omega\otimes e) = d\omega\otimes e + (-1)^n\,\omega\wedge\nabla.e$
$\omega \in \Omega^n(B)$, $\nabla.e \in \Omega^1(B)\otimes\Gamma(E)$.

DEF The curvature of D is

$$F := D_1\circ D_0 : \Gamma(E)\longrightarrow\Omega^2(B,E).$$

Proposition: The 2 definitions agree

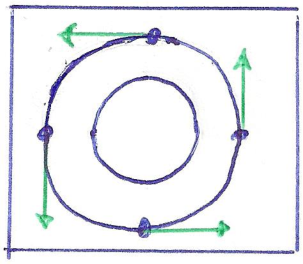
$$F = R\in\Omega^2(B,\mathrm{End}(E))$$

REM: We saw that the curvature of the Ehresmann connection H corresponding to ∇ is determinated by:

$$\Omega(X^H,Y^H) = [X^H,Y^H] - [X,Y]^H \qquad \forall X,Y\in\mathfrak{X}(E)$$

$\Omega(X^H,Y^H) \in$ {Vertical vector fields on E}.

$\boxed{\text{Ex}}$ On $\mathbb{R}^n$ we have $\mathrm{End}(\mathbb{R}^n) = \{$linear vector fields on $\mathbb{R}^n\}$



$$A = (a_{ij}) \longmapsto \sum_{i,j} a_{ij} x_i \frac{\partial}{\partial x_j}$$

- So we deduce $\Gamma(\mathrm{End}(E)) =$
  $= \left\{ \begin{array}{l} \text{vertical vector fields on } E \\ \text{which are linear on every fiber} \end{array} \right\}$

The curvatures of H and ∇ agree under the above identifications

$\boxed{5}$ CONNECTION ON PRINCIPAL BUNDLE.

- Let $\pi : P \to B$ be a G-bundle. Write $\mathfrak{g} = T_e G$. $\forall g \in G$, we write R_g for the corresponding diffeomorphism of P.

REM $\forall q \in P$ we get a canonical isomorphisme:

$$\mathfrak{g} \longrightarrow T_q(P_{\pi(q)})$$
$$v \longmapsto v_p(q)$$
$$v_p(q) = \frac{d}{dt}\Big|_{t=0} (q \cdot \exp(tv))$$

§5.1 CONNECTION

DEF A connection on P is an Ehresmann connection H on P which is G-invariant (ie, $(R_g)_*(H_p) = H_{pg}$ $\forall p \in P, g \in G$

DEF Connection 1-form on P is $\omega \in \Omega^1(P, \mathfrak{g})$ st:

$\omega(v_p) = v \quad \forall v \in \mathfrak{g}$.

$(R_g)^* \omega = Ad_{g^{-1}} \omega \quad \forall g \in G$

(ie, $\underset{\overset{\shortparallel}{\omega_{pg}((R_g)_* X)}}{[(R_g)^*\omega]_p(x)} = Ad_{g^{-1}} \underbrace{\omega_p(x)}_{\in \mathfrak{g}}$

REM The 2 notions are equivalent. Given H, define

ω by $\begin{cases} \omega_q(x) = 0 & \text{if } x \in H_q \\ \omega_q(v_p(q)) = v & \forall v \in \mathfrak{g} \end{cases}$

Given ω we defined H by:

$$H_p = \ker(\omega_q : T_q P \to \mathfrak{g}).$$

DEF
- $\alpha \in \Omega(P)$ is horizontal $\Leftrightarrow$ $i_{v_p}\alpha = 0 \quad \forall v \in \mathfrak{g}$
- $\alpha \in \Omega(P)$ is basic $\Leftrightarrow$ there is a form β on B with $\pi^*(\beta)=\alpha$

  $\Leftrightarrow \alpha$ is horiz. and $(R_g)^*\alpha = \alpha \quad \forall g \in G.$ (2),

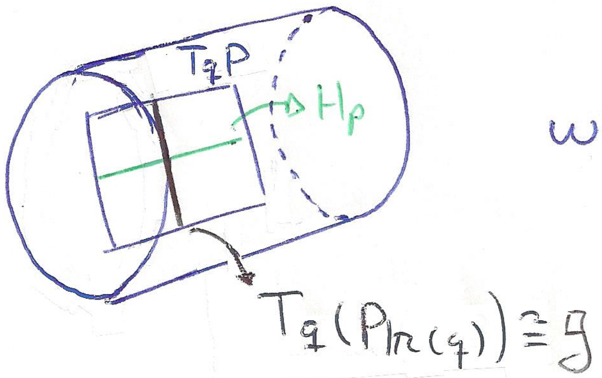
REM Let ω_0 be a conn 1-form on P. Then:

$$\{\text{conn } 1\text{-forms}\} = \omega_0 + \{\alpha \in \Omega^1_{HOR}(P, \mathfrak{g}) : (R_g)^*\alpha = Ad_{g^{-1}}\alpha \ \forall g \in G\}$$

$\boxed{\text{Ex}}$ Let $G = U(1)$. Then the Lie alg of $U(1)$ is $i\mathbb{R} \cong \mathbb{R}$

A conn 1-form is $\omega \in \Omega^1(P)$ $\omega(1_p) = 1$ and ω is $U(1)$-invariant (ie, $\omega((R_g)^* X) = \omega(X)$ $\forall X \in TP$, $g \in U(1)$).

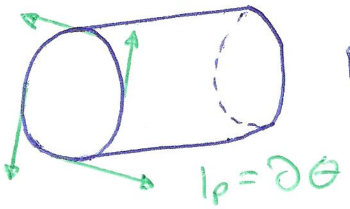
Now let $P = B \times U(1)$ be the trivial bundle.



$$\omega_p : T_q P \longrightarrow \mathfrak{g}$$

$\boxed{\text{Ex}}$

- Denote by θ the "angle coordinates" on $U(1)$, so $1_p = \frac{\partial}{\partial\theta}$



$B \times U(1)$; $d\theta\left(\frac{\partial}{\partial\theta}\right) = 1$

- Then $d\theta$ is a conn. 1-form on P. An arbitrary connection is $d\theta + \underbrace{\pi^*\beta}_{\text{a basic 1-form on } P}$. Here $\beta \in \Omega^1(B)$.

$\boxed{\text{Sub-ex}}$ $B = \mathbb{R}$ (with coord x)

- An arbitrary conn 1-form on $P = \mathbb{R} \times U(1)$ is $d\theta + f(x)\,dx$
- Its kernel is $H = \operatorname{Span}\left\{\frac{\partial}{\partial x} - f(x)\frac{\partial}{\partial\theta}\right\}$

§5.2 CURVATURE

- Let $\alpha \in \Omega^1(P, \mathfrak{g})$ be a conn. 1-form on P.

  Define: $[\alpha \wedge \alpha] \in \Omega^2(P, \mathfrak{g})$ by

$$[\alpha\wedge\alpha](\underset{TP}{X}, \underset{TP}{Y}) = [\alpha(X), \alpha(Y)] - [\alpha(Y), \alpha(X)] = 2[\alpha(X), \alpha(Y)].$$

<u>DEF</u> <u>The curvature</u> of α is $F = d\omega + \frac{1}{2}[\alpha\wedge\alpha] \in \Omega^2(P, \mathfrak{g})$

<u>REM</u>: The vertical bundle V of $P \xrightarrow{\pi} B$ is identified canonically with the trivial VB, $\overset{P}{\cancel{B}} \times \mathfrak{g} \to \overset{P}{\cancel{B}}$

<u>REM</u> F is the pullback by π of element of $\Omega^2(B, P \times \mathfrak{g})$

$\boxed{\text{Ex}}$ Let $G = U(1)$.

As we saw, a conn 1-form is $\omega \in \Omega^1(P)$ st $\omega(1_p) = 1$. Its curvature is $F = d\omega$.

$\boxed{\text{Ex}}$

- F is basic $i_{1_p} d\omega = \underbrace{\mathcal{L}_{1_p}\omega}_{=0} - d(i_{1_p}\omega)$

  $i_{1_p} d\omega = d(i_{1_p}\omega) = 0.$

  So $\exists c \in \Omega^2(B)$ st $\pi^* c = F.$

- c is closed $\pi^* dc = d(\overbrace{\pi^* c}^{=d\omega}) = 0$ and π^* injective. So get a cohomology. Class $[c] \in H^2_{dR}(M, \mathbb{R})$.

- Let $\tilde{\omega}$ be another connection 1-form on P. Then $\omega - \tilde{\omega} \in \Omega^1(P)$, horizontal and satisfies $(*)$, ie basic, ie $\exists \alpha \in \Omega^1(B)$ st $\pi^*\alpha = \omega - \tilde{\omega}$.

- Hence $\pi^*(c - \tilde{c}) = F - \tilde{F} = d(\omega - \tilde{\omega}) = \pi^*(d\alpha)$

  $\Rightarrow c - \tilde{c} = d\alpha \Rightarrow [c] = [\tilde{c}] \in H^2_{dR}(B)$

- The class $[c]$ is independent of the choice of conn 1-form $\left[\frac{-c}{2\pi i}\right]$ is called <u>first Chern class of P.</u>